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Calculus of Variations

Harnack inequality for singular fully nonlinear operators and some existence results

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Abstract In this article we further advance in the theory of singular fully nonlinear operators modeled on the q-laplacian proving a Harnack inequality. We provide also several applications of this inequality and the ideas used for proving it. In doing so we have left various open questions, all of them related to the fact that the operator is not sub-linear.

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1 Introduction

In this article we prove a Harnack inequality for positive solutions of a class of singular fully nonlinear elliptic equation of the form

$$F\left(Du, D^2u\right) + b \cdot Du\,|Du|^{\alpha} + cu\,|u|^{\alpha} = f \quad \text{in } \Omega, \tag{1.1}$$

where $\alpha \in (-1, 0)$ and $F : (\mathbb{R}^n \backslash \{0\}) \times \mathcal{S}(n) \to \mathbb{R}$ is a continuous function that satisfies

$(H1)$ $F(tp, \mu X) = t^{\alpha} \mu F(p, X), \quad \forall t, \mu \in \mathbb{R}^+.$

$(H2)$ For all $p \neq 0$ and $M, N \in \mathcal{S}(n)$

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$$|p|^{\alpha}\,\mathcal{M}^{-}_{\lambda,\Lambda}(N) \le F(p, M+N) - F(p, M) \le |p|^{\alpha}\,\mathcal{M}^{+}_{\lambda,\Lambda}(N),$$

where $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ are the Pucci operators with $\Lambda \ge \lambda > 0$.

In Eq. 1.1 we assume that Ω is a domain in $\mathbb{R}^n$, $b : \Omega \to \mathbb{R}^n$ and $c, f : \Omega \to \mathbb{R}$ are continuous functions. We use $\mathcal{S}(n)$ to denote the set of $n \times n$ symmetric matrices.

The study of Eq. 1.1 with $\alpha > -1$, was initiated in a series of papers by Birindelli and Demengel [2–6]. With an appropriate notion of solution an existence theory for the Dirichlet boundary value problem based on Perron's method is developed. A first eigenvalue theory is also studied together with existence results for non-proper operators up to the first eigenvalue. Recently, Patrizi in [37] studies the Neumann problem for this type of operators. One of the difficulties in the study of these equations is that the associated differential operator does not have divergence form and simultaneously it is not sub-linear in any reasonable sense.

In this article we give another step in the study of Eq. 1.1 proving the Harnack inequality in the singular case. In the fully nonlinear case, this inequality is obtained as a consequence of the Alexandroff, Bakelman, and Pucci (ABP) inequality and a sub-linearity property of the operator. However here, even though a version of the ABP inequality has been recently shown to hold, we were not able to follow this path because the operator lacks of sub-linearity properties. See [16,27] and also [25] for a proof of the ABP inequality.

The Harnack inequality is at the center of the mathematical developments of linear elliptic differential equations, potential theory and non-linear elliptic partial differential equations in divergence and non-divergence form. Derived originally by Harnack for two dimensional harmonic functions, it was extended by Moser [35] to general divergence form linear operators, by Serrin [38] to quasilinear divergence form operators including the q-laplacian, and by Trudinger [39] to a more general class including the minimal surface operator. The proof of these results uses in an essential way the divergence form structure of the operator, integrating by parts against appropriate test functions and an iteration procedure. For linear elliptic operators in non-divergence form with general coefficients, the Harnack inequality was proved by Krylov and Safonov [33], opening the way to the general theory of fully nonlinear elliptic operators developed by Caffarelli in [10]. In this situation the proof of the results are based on ABP inequality in connection with a localization property obtained from the sub-linearity of the operator. We refer the reader to the work by Kassmann [29] for a very complete bibliographic review on Harnack inequality.

Now we state our main theorem on Harnack inequality for Eq. 1.1, which may be seen as a fully nonlinear version of the q-laplacian operator in case $\alpha \in (-1, 0)$, corresponding to $1 < q = \alpha + 2 < 2$, that is the singular case. Precisely we have

Theorem 1.1 *Assume $\alpha \in (-1, 0)$ and F satisfies (H1) and (H2). If $u \in C(\Omega)$ is a non-negative viscosity solution of (1.1), with b, c and f continuous functions in Ω, then for every $\Omega' \subset\subset \Omega$ we have*

$$\sup_{\Omega'} u \le C\left\{\inf_{\Omega'} u + \|f\|_{n,\Omega'}^{\frac{1}{\alpha+1}}\right\}, \tag{1.2}$$

where the constant C depends on λ, Λ, α, b, c, n, Ω' and Ω.

Here and in what follows we denote by $\|\cdot\|_{n,A}$ the norm in $L^n(A)$. In Theorem 1.1 the notion of solution is that introduced by Chen et al. in [14] and Evans and Spruck in [22] for singular problems and adopted by Birindelli and Demengel in [2]–[5]. This is a variation of the usual notion of viscosity solution for (1.1), that takes into account that we cannot test functions with vanishing gradient at the testing point. In Sect. 2, we present the precise definition. We refer the reader to [28] for a related work on singular operators.

In Sect. 3 we give a proof of Theorem1.1 based on the ideas presented by Gilbarg and Trudinger [24], which are based on Krylov and Safonov [33] original approach. By taking advantage of the fact that $\alpha \in (-1, 0)$ we reduce the application of ABP to inequalities having the Pucci operator as the second order differential term. After submitting this paper, we learn of other developments on the Harnack inequality in the singular and degenerate case. Birindelli and Demenguel in [7] proved Harnack inequality (1.2) for $\alpha > 0$ in dimension $N = 2$. On the other hand, for a class of degenerate fully nonlinear operators, Imbert in [25] proved a version of a Harnack inequality, see Corollary 1, extending a recent work by Delarue [18]. In his article, Imbert describes how his approach can be adapted to cover also the singular case and to obtain Harnack inequality (1.2) in the full range $\alpha > -1$, at least in some cases like ours, see Theorem 7 in [25]. We should mention that the methods in [7] and [25] are very different than ours.

One of the main applications of Harnack inequality is C^β regularity, $0 < \beta < 1$, for solutions of Eq. 1.1 and the corresponding consequences on compactness properties of solutions. In Sect. 3 we further state and prove a version of Harnack inequality up to the boundary. As an application of these ideas we obtain new estimates for the solutions of a non-homogeneous Dirichlet boundary value problem studied in [5], allowing for weaker regularity assumptions on the boundary data.

Theorem 1.2 *Assume the hypotheses of Theorem 1.1 and additionally that Ω is bounded and satisfies a uniform exterior cone condition, the functions b, c and f are continuous in $\overline{\Omega}$, $c \leq 0$ and $g \in C^\sigma(\overline{\Omega})$, $\sigma \in (0, 1)$, then equation*

$$F\left(Du, D^2u\right) + b \cdot Du\,|Du|^\alpha + cu\,|u|^\alpha = f \;\; in \;\; \Omega, \tag{1.3}$$

$$u = g \;\; on \;\; \partial\Omega, \tag{1.4}$$

possesses at least one solution. Moreover, there are constants $C > 0$ and $\beta \in (0, 1)$ such that

$$\|u\|_{C^\beta(\overline{\Omega})} \leq C\left(\|g\|_{C^\sigma(\partial\Omega)} + \|f\|_{n,\Omega}^{\frac{1}{1+\alpha}}\right). \tag{1.5}$$

When the boundary data g belongs to $W^{2,\infty}(\partial\Omega)$ and the boundary of Ω is C^2, this theorem is a consequence of Theorem 1 in [5]. In Sect. 4 we prove Theorem 1.2 and we discuss some generalizations.

In view of Theorem 1.2 and ABP inequality in [16] is it natural to attempt the extension of the viscosity solution notion to the $W^{2,p}(\Omega)$ setting, developing an L^p theory for Eq. 1.3 and the corresponding boundary value problems (1.3) and (1.4), generalizing the uniformly elliptic theory by Caffarelli et al. in [12]. In case $c \leq 0$ and $f \in L^n(\Omega)$, we could consider a sequence $\{f_m\}$ of continuous functions converging in $L^n(\Omega)$ to f and find a sequence $\{u_m\}$ of solutions of (1.3) and (1.4) (with r.h.s f_m and, say, $g = 0$). This sequence of solutions is precompact so there is a limiting function u, which corresponds to a *good solution* of (1.3) and (1.4), in the definition of Cerutti et al. in [13]. However we are not able to prove that this solution is unique nor we are able to prove its continuity with regard to the right hand side, so we do not know if the problem is well posed. We remark that, assuming that $f \in L^n(\Omega)$ we could rephrase Theorem 1.2 saying that Eqs. 1.3 and 1.4 possesses at least one good solution.

In Sect. 5 we consider the equation

$$-F\left(Du, D^2u\right) + |u|^{s-1}u = f \quad \text{in } \mathbb{R}^n, \tag{1.6}$$

where F satisfies (H1) and (H2), $\alpha \in (-1, 0)$, $s > 1 + \alpha$ and f continuous, but without restriction on its growth at infinity. In case of the Laplacian, this problem was studied by

Brezis in [9], where an existence and uniqueness result is obtained with $f \in L^1_{loc}(\mathbb{R}^n)$. This result has been recently extended to the case of fully nonlinear operator by Esteban et al. in [21], see also the paper of Díaz [19] for a previous result in this direction. When the differential operator is in divergence form, even when the operator is singular or degenerate, like the case of q-laplacian, this problem has been studied by Boccardo et al. in [8]. Here we prove

Theorem 1.3 *Assume that $\alpha \in (-1, 0)$ and F satisfies (H1) and (H2). If $s > 1 + \alpha$, then for every $f \in C(\mathbb{R}^n)$, Eq. 1.6 possesses at least one solution.*

In the proof of this theorem we use the compactness property derived from the C^β regularity obtained from the Harnack inequality, together with a local estimate that is proved using some of the ideas of the proof of Theorem 1.1 and the fact that $s > 1+\alpha$. In the earlier works in [9] and [21] uniqueness and positivity of solutions is also proved. Here we are not able to do so since the differential operator is not sub-linear. The Osserman function as in [9] and [21] and the standard uniqueness argument are not available here.

The arguments used to prove Theorem 1.3 are also useful for studying solutions with explosion at the boundary in equations of the form

$$-F\left(Du, D^2u\right) + |u|^{s-1}u = f \quad \text{in } \Omega, \tag{1.7}$$

$$\lim_{x \to \partial\Omega} u(x) = \infty. \tag{1.8}$$

In order to obtain a solution for this equation we need to use a comparison principle proved by Birindelli and Demengel in [3], which is stated in Theorem 6.1 for the reader convenience. We then state our last main theorem

Theorem 1.4 *Assume $\alpha \in (-1, 0)$ and F satisfies (H1) and (H2). Let $s > 1+\alpha$, $f : \Omega \to \mathbb{R}$ continuous and bounded below, with $\Omega \subset \mathbb{R}^n$ an open bounded set with piece-wise C^1 boundary. Then Eqs. 1.7 and 1.8 has at least one solution.*

In the case of uniformly elliptic operators, a theorem of this sort has been recently proved by Esteban et al. in [21], while for divergence form operators the result and various generalizations are known. See, for example, the works by Keller [31], Loewnery Nirenberg [34], Kondratev and Nikishkin [32], Díaz and Letelier [20] and Del Pino and Letelier [17].

This paper is organized in six sections. After the introduction, in Sect. 2 we recall the notion of viscosity solutions given in [2] and we discuss the relation between this definition and the usual one. Section 3 is the heart of the article where we prove Harnack inequality in the interior and the boundary. In Sect. 4 we obtain C^β regularity in the usual way and we find new estimates for solutions of a boundary value problem studied by Birindelli and Demengel [5] and we generalize an existence result. Section 5 is devoted to Theorem 1.3 on the existence of solution to a super-linear coercive problem in $\mathbb{R}^n$ with data having no growth restriction at infinity. The arguments used in Sect. 5 are applied in Sect. 6 to study solutions with explosion at the boundary of a bounded domain.

2 Preliminaries

In this section we start discussing the class of operators to which our results apply. Then we review the notion of solution that is suitable to these operators and we discuss about the relation between this notion of solution and the usual one when both apply.

The operators satisfying our basic hypotheses (H1) and (H2) correspond to the natural extension of the q-Laplacian in the singular range, that is

$$F(p, N) = |p|^{q-2} tr\, N + (q-2)|p|^{q-4}\langle Np, p\rangle,$$

with $q \in (1, 2)$, that appear in the study of non-Newtonian (pseudoplastic) fluids, see [1]. Our class of operators also includes the operator

$$F(p, N) = trN - \frac{\langle Np, p\rangle}{|p|^2},$$

that appears in geometry in the study of surfaces of mean curvature. Parabolic version of this singular class of operators have been studied from the viscosity point of view by Chen et al. [14] and Evans and Spruck in [22]. See also [36,23].

Finally, we mention the basic model operator satisfying our hypotheses

$$F(p, N) = |p|^{\alpha} \mathcal{M}(N),$$

where $\mathcal{M}$ can be $\mathcal{M}^+$ or $\mathcal{M}^-$, which is related to the analysis of equations of the form

$$\mathcal{M}(u) + |Du|^{\beta} f(x) = 0,$$

with $\beta \in (0, 1)$.

Next we review the precise definition of viscosity solution for the singular operators that we are interested in with the general assumptions of Theorem 1.1. We consider $G : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R}$ a function. In case G is continuous in $\Omega \times \mathbb{R} \times (\mathbb{R}^n \backslash \{0\}) \times \mathcal{S}(n)$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ continuous we consider the following definition given in [14], [22] and [2].

Definition 2.1 We say that u is a viscosity super-solution of

$$G(x, u, Du, D^2u) = g(x, u) \text{ in } \Omega, \tag{2.1}$$

in Ω if for every $x_0 \in \Omega$ we have

(i) Either for all $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum at x_0 and $D\varphi(x_0) \neq 0$ we have

$$G(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq g(x_0, u(x_0)). \tag{2.2}$$

(ii) Or there is an open ball $B(x_0, \delta) \subset \Omega$, $\delta > 0$ where u is constant, $u = C$ and

$$0 \leq g(x, C) \quad \forall x \in B(x_0, \delta). \tag{2.3}$$

The definition of viscosity sub-solution is analogous. We say that u is a viscosity solution of the equation if u is a viscosity super-solution and viscosity sub-solution simultaneously.

This definition applies to the general operator that we consider in this article when

$$G(x, u, Du, D^2u) = F\left(Du, D^2u\right) + b \cdot Du\, |Du|^{\alpha} + cu\, |u|^{\alpha}$$

where $\alpha \in (-1, 0)$ and F satisfies (H1) and (H2).

It is interesting (and useful) to see that in case the differential operator is continuous, that is $\alpha \geq 0$, then this definition is equivalent with the usual definition of viscosity solution as given, for example in [15]. Let us recall that definition in case G is continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n)$

Definition 2.2 We say that u is a viscosity super-solution in the usual sense of (2.1) in Ω if for every $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum at x_0, then (2.2) holds. The definition of viscosity sub-solution is analogous.

Next lemma states the compatibility of the two definitions when G is continuous, that will be useful in what follows even though for $\alpha \in (-1, 0)$ our operator is not continuous.

Lemma 2.1 *Let $G : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R}$ be a continuous satisfying*

$$G(x, r, 0, 0) = 0 \quad \textit{and} \quad G(x, r, p, M) \le 0,$$

for all $(x, r) \in \Omega \times \mathbb{R}$, $p \in \mathbb{R}^n$, $M \in \mathcal{S}(n)$, $M \le 0$. Then u is a viscosity super-solution (resp. sub-solution) of (2.1) if and only if u is a viscosity super-solution (resp. sub-solution) of (2.1) in the usual sense.

Proof Let us first assume that u is a super-solution of (2.1) in the usual sense. We only need to consider the case when $u \equiv C$ in $B = B(x_0, \rho)$. We see that u is a test function for all $x \in B$ and it satisfies (2.3), since $G(x, r, 0, 0) = 0$.

Let us assume now u is a super-solution in the sense of Definition 2.1. Let φ and $x_0 \in \Omega$ be such that $u - \varphi$ has a local minimum in $B(x_0, \rho)$ at x_0. We assume, without lose of generality, that x_0 is a strict local minimum. If u is constant in $B(x_0, \rho)$ or a smaller ball centered at x_0 then φ has a local maximum at x_0 and $D^2\varphi(x_0) \le 0$. Thus, by Definition 2.1 inequality (2.3) and the hypothesis on G, we find that (2.2) holds.

From now on we assume that u is not constant in any ball near x_0. If $D\varphi(x_0) \ne 0$ there is nothing to prove, so we assume that $D\varphi(x_0) = 0$. At this point we further assume that φ has exactly one critical point in $B(x_0, \rho)$ and later we see how to treat the general case. Given $y \in B(0, r)$ with $r > 0$ small enough, we define the function $\varphi_y(x) = \varphi(x + y)$ and we see that $u - \varphi_y$ has a local minimum at some point $x_y \in B(x_0, \rho)$.

Next we assume that there is a sequence $y_m \searrow 0$, with $D\varphi_{y_m}(x_{y_m}) \ne 0$ for all m. Testing the equation at x_{y_m} we get

$$G\left(x_{y_m}, u\left(x_{y_m}\right), D\varphi_{y_m}\left(x_{y_m}\right), D^2\varphi_{y_m}\left(x_{y_m}\right)\right) \le g\left(x_{y_m}, u\left(x_{y_m}\right)\right),$$

from where (2.2) follows, since G is continuous. On the contrary we must have that for all $y \in B(0, r)$, $x_y + y = x_0$. Then, by definition we find that

$$u\left(x_y\right) - \varphi_y\left(x_y\right) \le u(x) - \varphi_y(x) \quad \forall y \in B(0, r), \; x \in B(x_0, \rho), \tag{2.4}$$

but since $x_y + y = x_0$, (2.4) is transformed into

$$u(x_0 - y) - \varphi(x_0) \le u(x) - \varphi(x + y),$$

for all $y \in B(0, r)$, $x \in B(x_0, \rho)$. If we take here $x = x_0 + h$ and $y = -h - td$ where $t > 0$, d is a unit vector and $h \in B(0, r)$ we get

$$\frac{u(x_0 + h + td) - u(x_0 + h)}{t} \le \frac{\varphi(x_0) - \varphi(x_0 - td)}{t} \quad \forall t > 0, \tag{2.5}$$

and analogously, taking $y = -h$ and $x = x_0 + h + td$ we obtain

$$\frac{u(x_0 + h + td) - u(x_0 + h)}{t} \ge \frac{\varphi(x_0 + td) - \varphi(x_0)}{t} \quad \forall t > 0. \tag{2.6}$$

Taking lim sup in (2.5) and lim inf in (2.6) we find that

$$0 \geq \limsup_{t\to 0} \frac{u\,(x_0+h+td) - u(x_0+h)}{t}$$
$$\geq \liminf_{t\to 0} \frac{u\,(x_0+h+td) - u(x_0+h)}{t} \geq 0,$$

which implies that u has directional derivative near x_0, that vanishes everywhere in the neighborhood. This implies that u is constant around x_0 and hence a contradiction.

We observe that whenever $D^2\varphi(x_0)$ is invertible, x_0 is the only critical point of φ in $B(x_0, \rho)$, by making ρ smaller if necessary. In case $D^2\varphi(x_0)$ is not invertible we consider a symmetric matrix, semi-positive definite such that $D^2\varphi(x_0) - \varepsilon M$ is invertible, for all $\varepsilon > 0$. We then define $\varphi_\varepsilon(x) := \varphi(x) - \frac{\varepsilon}{2}(x - x_0)^t M(x - x_0)$. The function φ_ε is a test function and we may apply the arguments given above to obtain

$$G(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0) - \varepsilon M) \leq g(x_0, u(x_0)).$$

Since this inequality holds for all $\varepsilon > 0$ and G is continuous, inequality (2.2) follows, completing the proof. □

3 Harnack inequality

In this section we prove Harnack inequality as stated in Theorem 1.1. In the proof of the theorem we make an essential use of the fact $\alpha \in (-1, 0)$, which may be informally interpreted as providing a very large ellipticity whenever the gradient of a solution vanishes.

We start with a preliminary lemma.

Lemma 3.1 *Let $\Omega \subset \mathbb{R}^n$ be a domain, $\sigma > 0$ and $\gamma \geq 0$. Assume $f \leq 0$ and u is a viscosity sub-solution of*

$$|Du|^\alpha \mathcal{M}^+\left(D^2u\right) + \gamma\left(|Du|^{\alpha+1} + |u|^{\alpha+1}\right) = f \text{ in } \Omega, \tag{3.1}$$

then u is also a viscosity sub-solution in the usual sense of

$$(|Du| + \sigma)^\alpha \mathcal{M}^+\left(D^2u\right) + \gamma\left(|Du|^{\alpha+1} + |u|^{\alpha+1}\right) = f \text{ in } \Omega. \tag{3.2}$$

Similar statement can be made for viscosity super-solution when $f \geq 0$ and with $\mathcal{M}^-$ instead of $\mathcal{M}^+$.

Proof Assume u is a sub-solution of (3.1) and let $\varphi \in C^2(\Omega)$ be a test function for u in $x_0 \in \Omega$ such that $D\varphi(x_0) \neq 0$, then we have

$$|D\varphi(x_0)|^\alpha \mathcal{M}^+\left(D^2\varphi(x_0)\right) + \gamma\left(|D\varphi(x_0)|^{\alpha+1} + |\varphi(x_0)|^{\alpha+1}\right) \geq f(x_0).$$

Since $\alpha \in (-1, 0)$ and $\sigma \geq 0$ we have $|D\varphi|^\alpha \geq (|D\varphi| + \sigma)^\alpha$. Then, if $\mathcal{M}^+\left(D^2\varphi(x_0)\right) \leq 0$, we see that

$$(|D\varphi(x_0)| + \sigma)^\alpha \mathcal{M}^+\left(D^2\varphi(x_0)\right) + \gamma(|D\varphi(x_0)|^{\alpha+1} + |\varphi(x_0)|^{\alpha+1}) \geq f(x_0).$$

On the contrary, if $\mathcal{M}^+\left(D^2\varphi(x_0)\right) > 0$, then we have

$$(|D\varphi(x_0)| + \sigma)^\alpha \mathcal{M}^+\left(D^2\varphi(x_0)\right) + \gamma(|D\varphi(x_0)|^{\alpha+1} + |\varphi(x_0)|^{\alpha+1}) \geq 0 \geq f(x_0),$$

which finishes the proof, using Lemma 2.1 and noting that the associated operator is continuous. □

The proof of Theorem 1.1 will be done in two steps as in the case of linear elliptic operators presented in [24]. The first step is an estimate of the L^∞ norm in terms of integral norms.

Proposition 3.1 *Let $u \in C(\Omega)$, $f \in C(\Omega)$ and assume that u is a viscosity sub-solution of (1.1). Then, for every ball $B = B_{2R}(y) \subset \Omega$ and $p > 0$ we have*

$$\sup_{B_R(y)} u \le C \left\{ \left(\frac{1}{|B|} \int_B \left(u^+\right)^p \right)^{\frac{1}{p}} + \|f\|_{n,B}^{\frac{1}{\alpha+1}} \right\}, \tag{3.3}$$

where C is a positive constant depending only on n, p, α, Λ, λ, R, b and c.

Proof Since u is a sub-solution of (1.1) we find, by Lemma 3.1, that u satisfies

$$(|Du| + \sigma)^\alpha \mathcal{M}^+\left(D^2 u\right) + \gamma(|Du|^{\alpha+1} + (u^+)^{\alpha+1}) \ge -|f|$$

in Ω, in the usual viscosity sense. Here $\gamma = \max\{\|b\|_\infty, \|c\|_\infty\}$ and $\sigma > 0$. We consider, without loss of generality, that $B = B_1(0)$ and $\lambda = 1$. For $\beta \ge 2$ we define the function $\eta(x) = \left(1 - |x|^2\right)^\beta$ if $x \in B$ and $\eta(x) = 0$ if $x \notin B$ and then we consider the function $v = \eta u$ in Ω. Then the function v is a viscosity solution in the usual sense of

$$\mathcal{M}^+(D^2 v) \ge \gamma \mathcal{A}(x, v, Dv) + \mathcal{B}(x, v, Dv) + \mathcal{C}(x, v, Dv) \tag{3.4}$$

in B, with $v = 0$ on ∂B, and where

$$\mathcal{A}(x, v, Dv) = -\left(2\beta\eta^{-\frac{1}{\beta}}|v| + |Dv| + \sigma\eta\right), \tag{3.5}$$

$$\mathcal{B}(x, v, Dv) = -\left(\eta^{1+\alpha}|f| + \gamma(v^+)^{1+\alpha}\right)(-\mathcal{A})^{-\alpha} \quad \text{and} \tag{3.6}$$

$$\begin{aligned} \mathcal{C}(x, v, Dv) &= -C\left(\eta^{-\frac{2}{\beta}}|v| + \eta^{-\frac{1}{\beta}}|Dv|\right) \\ &\le -\eta\mathcal{M}^+\left(vD^2\eta^{-1} + \overline{Dv \otimes D\eta^{-1}}\right) \end{aligned} \tag{3.7}$$

here and in what follows C denotes a generic constant depending on n, α, β, γ and Λ. We also use the notation

$$\overline{x \otimes y} = x \otimes y + y \otimes x, \quad x, y \in \mathbb{R}^n.$$

In what follows we use an approximation argument similar to the one developed in [12]. We start considering the sup-convolution of v defined, for given $\varepsilon > 0$, as

$$v_\varepsilon(x) = \sup_{y \in B} \left(v(y) - \frac{1}{2\varepsilon}|x - y|^2 \right),$$

in all $\mathbb{R}^n$, whose properties are given in detail in [12]. In particular, from Lemma A.3 in [12], we have that v_ε satisfies

$$\begin{aligned} \mathcal{M}^+\left(D^2 v_\varepsilon(x)\right) &\ge \gamma\mathcal{A}(x^*, v_\varepsilon(x^*), Dv_\varepsilon(x)) + \mathcal{B}(x^*, v_\varepsilon(x^*), Dv_\varepsilon(x)) \\ &\quad + \mathcal{C}(x^*, v_\varepsilon(x^*), Dv_\varepsilon(x)) \quad \forall x \in B_{R_\varepsilon} \ a.e. \end{aligned} \tag{3.8}$$

Here $R_\varepsilon = 1 - 2(\varepsilon\|v\|_\infty)^{\frac{1}{2}}$ and $x^* \in B_{R_\varepsilon}$ is a point that satisfies $v_\varepsilon(x) = v(x^*) - \frac{1}{2\varepsilon}|x - x^*|^2$ and $|x - x^*| \le 2(\varepsilon\|v\|_\infty)^{\frac{1}{2}}$. Now consider

$$r_0 = \frac{1}{2}\left(\sup_B v - \sup_{\partial B} v^+\right)$$

and, for $r < r_0$ we define the upper contact set

$$\Gamma_r^+(v) = \{x \in B \,/\, \exists p \in B_r, u(y) \leq u(x) + (p, y - x)\, \forall y \in B\}$$

and similarly the sets $\Gamma_r^+(v_\varepsilon)$, $\varepsilon > 0$. As proved in [12], $v > 0$ and $v_\varepsilon > 0$ in $\Gamma_r^+(v)$ and $\Gamma_r^+(v_\varepsilon)$ respectively, and there is a fixed compact subset of B that contains all sets $\Gamma_r^+(v)$ and $\Gamma_r^+(v_\varepsilon)$ for small ε. By concavity of v_ε in $\Gamma_r^+(v_\varepsilon)$ we obtain the crucial estimate

$$|Dv_\varepsilon(x)| \leq \frac{v_\varepsilon(x)}{R_\varepsilon - |x|}, \quad \forall x \in \Gamma^+(v_\varepsilon)\ a.e. \tag{3.9}$$

From here we obtain estimates for $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, for all $x \in \Gamma_r^+(v_\varepsilon)$ a.e.

$$\mathcal{A}(x^*, v_\varepsilon(x^*), Dv_\varepsilon(x)) \geq -C(\eta^{-\frac{1}{\beta}}(x^*)v_\varepsilon(x^*) + \frac{v_\varepsilon(x)}{R_\varepsilon - |x|} + \sigma\eta(x^*)), \tag{3.10}$$

and similarly for the other terms. On the other hand, we have that for every function $w \in C(\bar{B}) \cap C^2(B)$ which is uniformly close enough to v we have

$$\omega_n r^n = \int_{B_r} \mathrm{d}p \leq \int_{\Gamma_r^+(w)} (\frac{-trD^2 w}{n})^n \mathrm{d}x,$$

where ω_n denotes the volume of the unit ball. Following [12], we apply this formula to a standard mollification v_ε^ρ of the function v_ε and use the properties stated in Lemma A.2 in [12] to obtain

$$\omega_n r^n \leq \int_{\Gamma_r^+(v_\varepsilon)} (\frac{-trD^2 v_\varepsilon}{n})^n \mathrm{d}x.$$

We use (3.8) here and then inequality (3.10) and the corresponding estimates for $\mathcal{B}$ and $\mathcal{C}$. The integral is consider over $\Gamma_r^+(v_\varepsilon)$, which is contained in a fixed compact subset of B, so for $x \in \Gamma_r^+(v_\varepsilon)$ we have

$$\lim_{\varepsilon \to 0} \frac{\eta^{\frac{1}{\beta}}(x)}{R_\varepsilon - |x|} = \lim_{\varepsilon \to 0} (R_\varepsilon + |x|) \frac{\eta^{\frac{1}{\beta}}(x)}{R_\varepsilon^2 - |x|^2} = 1 + |x| \leq 2.$$

Thus, taking limits as $\varepsilon \to 0$ we obtain

$$\sup_B v \leq C \left\| \frac{\eta^{1+\alpha}|f| + (v^+)^{\alpha+1}}{\left(\eta^{-\frac{1}{\beta}} v^+ + \sigma\eta\right)^\alpha} + \eta^{-\frac{1}{\beta}} v^+ + \eta^{\frac{-2}{\beta}} v^+ + \sigma\eta \right\|_{n,B}.$$

We remark that, in case $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ then from Eq. 3.4 and estimate (3.9) in $\Gamma^+(u)$ we can directly use ABP to get this inequality.

Next we take the limit $\sigma \searrow 0$ and we use that $\eta^{\frac{-1}{\beta}} \leq \eta^{\frac{-2}{\beta}}$, which is true since $\eta \leq 1$, to obtain

$$\begin{aligned}\sup_B v &\leq C \left\{ \left\| \frac{\eta^{1+\alpha+\frac{\alpha}{\beta}} f}{(v^+)^\alpha} \right\|_{n,B} + \left\| v^+ \eta^{\frac{-2}{\beta}} \right\|_{n,B} \right\} \\ &\leq C \left\{ \left\| \frac{\eta^{1+\alpha+\frac{\alpha}{\beta}} f}{(v^+)^\alpha} \right\|_{n,B} + \left(\sup_B v^+ \right)^{1-2/\beta} \left\| (u^+)^{\frac{2}{\beta}} \right\|_{n,B} \right\}.\end{aligned} \tag{3.11}$$

Now we choose $\beta = 2n/p$ with $p < n$ and we use Young inequality to get

$$\left(\sup_B v^+\right)^{1-2/\beta} \left\| (u^+)^{\frac{2}{\beta}} \right\|_{n,B} \le C\left(\varepsilon \sup_B v^+ + \varepsilon^{1-\beta/2} \left\| u^+ \right\|_{p,B}\right) \tag{3.12}$$

for $\varepsilon > 0$. Notice that we may take p so that $1 + \frac{\alpha}{\beta} + \alpha \ge 0$, and then by Young inequality again we find

$$\begin{aligned} \left\| \frac{\eta^{1+\alpha+\frac{\alpha}{\beta}} f}{(v^+)^\alpha} \right\|_{n,B} &\le \left(\sup_B v^+\right)^{-\alpha} \|f\|_{n,B} \\ &\le C\left(\varepsilon^{\bar{q}} \sup_B v^+ + \varepsilon^{-q} \|f\|_{n,B}^q\right), \end{aligned} \tag{3.13}$$

for $\varepsilon > 0$ and where $\bar{q} = -1/\alpha$ and $q = \bar{q}/(\bar{q}-1) = 1/(1+\alpha)$. Choosing an adequate ε, the result follows from (3.11), (3.12) and (3.13). □

Continuing with our proof of Theorem 1.1, we prove an estimate of the L^p norm of a non-negative solution in terms of its infimum. This inequality is often known as Weak Harnack Inequality, and is has as a consequence the Hölder regularity of the solution.

Theorem 3.1 *Let $u \in C(\Omega)$, $f \in C(\Omega)$ and assume that u is a positive viscosity super-solution of (1.1). Then, for every ball $B = B_{2R}(y) \subset \Omega$ we have*

$$\left(\frac{1}{|B_R|}\int_{B_R} u^p\right)^{1/p} \le C\left\{\inf_{B_R} u + \|f\|_{n,B}^{\frac{1}{\alpha+1}}\right\}, \tag{3.14}$$

where p, C are positive constants depending on n, α, Λ, λ, R, b and c.

Proof We proceed as in [24] and Proposition 3.1, that is, we find the estimate using the fact that u satisfies

$$|Du|^\alpha \mathcal{M}^-\left(D^2u\right) - \gamma(|Du|^{\alpha+1} + u^{\alpha+1}) \le |f| \tag{3.15}$$

in Ω, where $\gamma = \max\{\|b\|_\infty, \|c\|_\infty\}$. Without loss of generality we assume that $B = B_1(0)$ and $\lambda = 1$. Moreover, along the whole proof we will assume that $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$, understanding that for the general case, we have to use the approximation argument that we described in the proof of Proposition 3.1. For η as in Proposition 3.1 and $\varepsilon > 0$ we define $\bar{u} = u + \varepsilon + \|f\|_{n,B}^{\frac{1}{\alpha+1}}$, $w = -\log(\bar{u})$ and $v = \eta w$. We have

$$Dw = -\frac{1}{\bar{u}} D\bar{u} \quad \text{and} \quad D^2 w = -\frac{1}{\bar{u}} D^2 u + Dw \otimes Dw.$$

Here we again we find an inequality involving $\mathcal{M}^+$. Defining

$$I = \mathcal{M}^-\left(\overline{Dw \otimes D\eta} + wD^2\eta + \eta Dw \otimes Dw\right),$$

we have

$$\mathcal{M}^+\left(D^2 v\right) \ge -\frac{\eta}{\bar{u}}\mathcal{M}^-\left(D^2 u\right) + I.$$

Since u satisfies (3.15), from Lemma 3.1 for super-solutions we find

$$\mathcal{M}^+\left(D^2 v\right) \ge -\eta\frac{|f| + \gamma u^{\alpha+1}}{\bar{u}\,(|Du| + \sigma)^\alpha} - \eta\gamma\frac{|Du| + \sigma}{\bar{u}} + I.$$

Now, we use the concavity of v in $\Gamma^+(v)$ to get

$$|Dw| \leq C\eta^{-1/\beta} w,$$

from where

$$|Du| \leq C\eta^{-1/\beta} w\bar{u}.$$

Thus, in $\Gamma^+(v)$ we have

$$\begin{aligned} \mathcal{M}^+\left(D^2 v\right) \geq -\eta \frac{|f| + \gamma u^{\alpha+1}}{C\bar{u}\left(\eta^{-1/\beta} w\bar{u} + \sigma\right)^{\alpha}} - \eta\frac{\sigma}{\bar{u}} - \gamma\frac{\eta}{4\bar{\varepsilon}} \\ -\gamma\bar{\varepsilon}\eta\,|Dw|^2 + I, \end{aligned} \tag{3.16}$$

where $\bar{\varepsilon} > 0$ is to be chosen later. To continue we estimate the last two terms here. We first notice that $I \geq I_1 + I_2$, where

$$I_1 = w\mathcal{M}^-\left(D^2\eta\right) \quad \text{and} \quad I_2 = \mathcal{M}^-\left(\overline{Dw \otimes D\eta} + \eta Dw \otimes Dw\right).$$

Given $1/2 < \rho < 1$ we enlarge β is necessary to have

$$\beta \geq \frac{1}{2}\left(\frac{n-1+\Lambda}{\rho^2} - n - 1 + \Lambda\right) + 1,$$

which guarantees that $\mathcal{M}^-\left(D^2\eta\right) \geq 0$ outside B_ρ and so

$$w\mathcal{M}^-\left(D^2\eta\right) \geq -Cv\chi_{B_\rho}, \tag{3.17}$$

where χ_A denotes the characteristic function of the set A. In what follows we denote by $\mathcal{S}_{1,\Lambda}$ the set of all $n \times n$ symmetric matrices with eigenvalues in the interval $[1, \Lambda]$. If we consider $N \in \mathcal{S}_{1,\Lambda}$ then for $x, y \in \mathbb{R}^n$, $\langle x, Ny\rangle$ defines and inner product in $\mathbb{R}^n$; the associated norm is denoted by $|\cdot|_N$ and we have that $|x|_N^2 = tr\,(Nx \otimes x)$. Using the Cauchy-Schwarz and Young inequalities we obtain

$$\begin{aligned} \left|tr\left(N\overline{Dw \otimes D\eta}\right)\right| &\leq 2\,|Dw|_N\,|D\eta|_N \\ &\leq s\eta\,|Dw|_N^2 + \frac{1}{s\eta}\,|D\eta|_N^2 \\ &= s\eta\,tr\,(NDw \otimes Dw) + \frac{1}{s\eta} tr\,(ND\eta \otimes D\eta)\,, \end{aligned}$$

where $s > 0$. Moreover, from definition of $\mathcal{M}^-$ we have that

$$\begin{aligned} I_2 &\geq \inf_{N \in \mathcal{S}_{1,\Lambda}} tr\left(N\overline{Dw \otimes D\eta}\right) + \eta\,tr\,(NDw \otimes Dw) \\ &\geq \inf_{N \in \mathcal{S}_{1,\Lambda}} -\left|tr\left(N\overline{Dw \otimes D\eta}\right)\right| + \eta\,tr\,(NDw \otimes Dw) \\ &\geq \inf_{N \in \mathcal{S}_{1,\Lambda}} -\frac{1}{s\eta} tr\,(ND\eta \otimes D\eta) + (1-s)\eta\,tr\,(NDw \otimes Dw) \\ &\geq \mathcal{M}^-\left(-\frac{1}{s\eta}\,(D\eta \otimes D\eta)\right) + (1-s)\eta\mathcal{M}^-\,(Dw \otimes Dw). \end{aligned}$$

Next we claim that

$$J = (1-s)\eta\mathcal{M}^-\,(Dw \otimes Dw) - \gamma\bar{\varepsilon}\eta\,|Dw|^2 \geq 0, \tag{3.18}$$

for an appropriate choice of s and $\bar{\varepsilon}$. In fact, we notice that

$$J = \eta \inf_{N \in \mathcal{S}_{1,\Lambda}} tr\left(\left((1-s)N - \gamma\bar{\varepsilon} I_n\right) Dw \otimes Dw\right),$$

where I_n is the identity matrix. Now we may choose s and ε, depending on Λ and γ, so that the matrix $(1-s)N - \gamma\bar{\varepsilon} I_n$ is positive definite for all $N \in \mathcal{S}_{1,\Lambda}$, proving the claim. On the other hand

$$\frac{1}{\eta}\left(D\eta \otimes D\eta\right)_{i,j} = 4\beta^2 x_i x_j \left(1-|x|^2\right)^{\beta-2},$$

so that

$$\mathcal{M}^-\left(-\frac{1}{\eta}\left(D\eta \otimes D\eta\right)\right) \geq -C, \tag{3.19}$$

where C is a constant only depending on β and n. Finally, from (3.17), (3.18) and (3.19) we reach the following inequality

$$\mathcal{M}^+\left(D^2 v\right) \geq -C\left(\eta \frac{|f| + \gamma u^{\alpha+1}}{\bar{u}\left(\eta^{-1/\beta} w\bar{u} + \sigma\right)^{\alpha}} + \frac{\sigma}{\bar{u}} + v\chi_{B_\rho} + 1\right),$$

where we apply ABP inequality to obtain

$$\sup_B v \leq C\left\{\left\|\eta \frac{|f| + u^{\alpha+1}}{\bar{u}\left(\eta^{-1/\beta} w\bar{u} + \sigma\right)^{\alpha}} + \frac{\sigma}{\bar{u}}\right\|_{n,B} + \left\|v^+\right\|_{n,B_\rho} + 1\right\}.$$

At this point we take $\sigma \searrow 0$ and we obtain

$$\sup_B v \leq C\left\{\left\|\frac{\eta^{1+\alpha/\beta+\alpha} f}{\bar{u}^{1+\alpha} v^{\alpha}}\right\|_{n,B} + \left\|\frac{\eta^{1+\frac{\alpha}{\beta}+\alpha}}{v^{\alpha}}\right\|_{n,B} + \left\|v^+\right\|_{n,B_\rho} + 1\right\},$$

where we used that $u/\bar{u} \leq 1$. Now, taking β larger if necessary to have $\beta \geq \frac{\alpha}{\alpha-1}$, we get

$$\begin{aligned}\sup_B v &\leq C\left\{\left(\sup_B v\right)^{-\alpha}\left(\left\|\frac{f}{\bar{u}^{\alpha+1}}\right\|_{n,B} + \left\|\eta^{1+\alpha/\beta+\alpha}\right\|_{n,B}\right) + \left\|v^+\right\|_{n,B_\rho} + 1\right\} \\ &\leq C\left\{\left(\sup_B v\right)^{-\alpha} + \left\|v^+\right\|_{n,B_\rho} + 1\right\},\end{aligned}$$

and using Young inequality we reach to

$$\sup_B v \leq C\left\{\left\|v^+\right\|_{n,B_\rho} + 1\right\}. \tag{3.20}$$

At this point we can repeat the argument in Theorem 9.22 in [24] to complete the proof. □

Now Theorem 1.1 follows from Proposition 3.1 and Theorem 3.1.

The following result is an extension of Theorem 3.1 to the case when the balls are centered at a point on the boundary of Ω. This weak Harnack inequality is the basis for the regularity theory up to the boundary.

Theorem 3.2 *Let $u \in C(\overline{\Omega})$ be a viscosity solution of*

$$F\left(Du, D^2u\right) + b(x)\cdot Du\,|D(u(x))|^{\alpha} + c(x)u\,|u|^{\alpha} \leq f \ \ in\ \Omega,$$

where $f \in C(\Omega)$. *Assume that* u *is non-negative in* $B \cap \Omega$, *where* $B = B_{2R}(y) \subset \Omega$ *is a ball in* $\mathbb{R}^n$. *If we set* $m = \inf_{B \cap \partial\Omega} u$ *and*

$$u^-(x) = \begin{cases} \inf\{u(x), m\} & \text{for } x \in B \cap \Omega, \\ m & \text{for } x \in B \backslash \Omega \end{cases} \tag{3.21}$$

then

$$\left(\frac{1}{|B_R|} \int_{B_R} (u^-)^p \right)^{1/p} \leq C \left\{ \inf_{\Omega \cap B_R} u + \|f\|_{n, B \cap \Omega}^{\frac{1}{\alpha+1}} \right\}, \tag{3.22}$$

where p, C *are positive constants depending on* n, α, Λ, λ, R, b *and* c.

Proof First note that by Lemma 3.1 u is a viscosity solution in the usual sense of

$$|Du|^\alpha \mathcal{M}^- \left(D^2 u\right) - \gamma(|Du|^{\alpha+1} + u^{\alpha+1}) \leq |f| \tag{3.23}$$

in Ω, where $\gamma = \max\{\|b\|_\infty, \|c\|_\infty\}$. Now note that since m is positive it is also a solution of (3.23), hence $\inf\{u(x), m\}$ also satisfies the equation, see for example Proposition 2.8 in [11]. From here we proceed as in Theorem 3.1 using the fact that u^- is a solution of (3.23). □

4 Consequences of Harnack inequality in regularity of solutions to the Dirichlet problem

One of the most important consequence of Harnack and weak Harnack inequality (1.2), (3.14) and (3.22) is Hölder regularity estimates and compactness properties of solutions to elliptic equations. Here we obtain this regularity result and we apply it to improve some existence results for the non-homogeneous Dirichlet boundary value problem. We also obtain better estimates for the solutions in terms of the data, since they depend on the L^n norm rather that the L^∞ norm. This last fact allows to define the concept of good solution, however we are not able to prove that the Dirichlet problem is well posed, that is uniqueness and continuity with respect to the data.

Hölder regularity for these problems had already been studied by Birindelli and Demengel, see [2,4,6]. Birindelli and Demengel approach is based on barrier functions, so that our Harnack inequalities provides an alternative way to obtain the regularity results. Moreover, as we mentioned above, our estimates depend on the L^n norm rather that the L^∞ norm. On the other hand we obtain an improvement in the existence theory for the non-homogeneous Dirichlet problem weakening the regularity assumption on the boundary data and obtaining estimates for the solutions depending on weaker norms of the data. In particular, we obtain Hölder norms of the solutions depending on the Hölder norms of the boundary data and the L^n norm of the right hand side of the equation.

Now we state the regularity theorem together with the norm estimates. The proof of this result follows the classical approach.

Theorem 4.1 *Assume that* $u \in C(\Omega)$ *is a viscosity solution of equation*

$$F\left(Du, D^2 u\right) + b(x) \cdot Du\, |D(u(x))|^\alpha + c(x) u\, |u|^\alpha = f \quad \text{in } \Omega, \tag{4.1}$$

where $\alpha \in (-1, 0)$, F satisfies (H1) and (H2) and a, b and f are continuous functions in Ω. Assume further that $R > 0$ is such that $B_R = B_R(y)$, $B_{2R} = B_{2R}(y) \subset \Omega$, then there are constants $C > 0$ and $\beta \in (0, 1)$ depending on n, α, Λ, λ, R, b and c, such that

$$\|u\|_{C^\beta(\bar{B}_R)} \leq C \left(\|u\|_{\infty, B_{2R}} + \|f\|_{n, B_{2R}}^{\frac{1}{1+\alpha}} \right).$$

Moreover, assume that there is a function $\varphi \in C^\sigma(\overline{\Omega})$, $\sigma \in (0, 1)$, such that $u = \varphi$ on $\partial\Omega$ and Ω satisfies a uniform exterior cone condition. Then there are constants $C > 0$ and $\beta \in (0, 1)$ such that

$$\|u\|_{C^\beta(\overline{\Omega})} \leq C \left(\|\varphi\|_{C^\sigma(\partial\Omega)} + \|f\|_{n,\Omega}^{\frac{1}{1+\alpha}} \right).$$

The proof of this theorem is rather standard after the weak Harnack inequality used for $u - m$ and $M - u$ for suitable constant M and m, see Chapter 8 in [24]. The only point to observe is that, in the case of $M - u$ we need to use the operator $H(p, X) = -F(-p, -X)$ that satisfies the same hypothesis as F so weak Harnack inequality also holds for H.

In what follows we discuss the passage to the limit when we deal with a sequence of solutions to equation

$$F\left(Dv, D^2v\right) + b(x) \cdot Dv\, |Dv|^\alpha - \beta(v(x)) = f(x) \text{ in } \Omega. \tag{4.2}$$

This has been studied by Birindelli and Demengel in [4], but we include it here for the reader convenience. The first step in the study is the following lemma which is proved in [4].

Lemma 4.1 *Let v be a super-solution (sub-solution) of (4.2) for continuous functions f, β. Let $q > \sup\left(2, \frac{\alpha+2}{\alpha+1}\right)$ and assume that $\bar{x} \in \Omega$ and C are such that*

$$v(x) + C\,|x - \bar{x}|^q \geq v(\bar{x}), \quad (\leq), \tag{4.3}$$

where $\bar{x}$ is a strict local minimum (maximum) of the left hand side of (4.3). Then

$$-\beta(v(\bar{x})) \leq f(\bar{x}), \quad (\geq). \tag{4.4}$$

With this lemma we prove a proposition on the convergence of solutions that we need. This result is well known and basic for viscosity solutions when the function F is continuous, however in our case a special attention has to be taken. We only sketch a proof that can be found in [4] with slight variations.

Proposition 4.1 *Let u_m be a sequence of viscosity solutions to (4.2) which converges uniformly to a function u in Ω. Then u is also solution of (4.2).*

Proof We assume that u_m is a sequence of viscosity sub-solutions to (4.2) (the case of super-solutions is similar). Assume first that $\bar{x}$ is a point of maximum for $u - \varphi$ and $D\varphi(\bar{x}) \neq 0$. In this case Ishii's proof given in [26] is still true, noticing that it is possible to choose $y_m \to \bar{x}$, such that $u_m - \varphi$ has a maximum at y_m and $D\varphi(y_m) \neq 0$.

Let us look now at the case when $\bar{x}$ is such that $u(\bar{x}) = u(x)$ for all $x \in B(\bar{x}, \delta)$ with $\delta > 0$. For $q > \max\left\{2, \frac{\alpha+2}{\alpha+1}\right\}$ we define φ as

$$\varphi(x) = u(\bar{x}) + \frac{1}{q}\,|x - \bar{x}|^q\,.$$

Then we have that

$$\sup_{x\in B(\bar{x},\delta)} (u-\varphi) = 0$$

is strict and it is achieved at $\bar{x}$. Let ε be such that $4\varepsilon^{\frac{1}{q}} < \delta$ and N sufficiently large so that

$$-\varepsilon \le (u_m - u) \le \varepsilon \ \text{ in } B(\bar{x},\delta),$$

for all $m \ge N$. Then we have that $m \ge N$ and all $x \in B(\bar{x},\delta)$

$$u_m(x) - u(\bar{x}) - \frac{1}{q}|x-\bar{x}|^q \le -\frac{1}{q}|x-\bar{x}|^q + \varepsilon. \tag{4.5}$$

In particular, if $|x-\bar{x}| \ge (2^q(q\varepsilon)+\varepsilon)^{\frac{1}{q}}$, then we have in (4.5)

$$u_m(x) - u(\bar{x}) - \frac{1}{q}|x-\bar{x}|^q \le -2^q\varepsilon \quad \text{for all } m,$$

while for $m > N$ we have $u_m(\bar{x}) - u(\bar{x}) \ge -\varepsilon$. This implies that the maximum of $u_m - \varphi$ is achieved in $B\left(\bar{x}, (2^q(q\varepsilon)+\varepsilon)^{\frac{1}{q}}\right)$ at a point we call y_m. If $y_m \ne \bar{x}$ for infinitely many m then φ would be a test function for u_m at y_m and then

$$F\left(D\varphi(y_m), D^2\varphi(y_m)\right) + b(y_m)\cdot D\varphi(y_m)\,|D(\varphi(y_m))|^\alpha + \beta(\varphi(y_m)) = f(y_m).$$

Noticing that $u_m(y_m) \to u(\bar{x})$ and using the fact that

$$\begin{aligned} &F\left(D\varphi(y_m), D^2\varphi(y_m)\right) + b(y_m)\cdot D\varphi(y_m)\,|D(\varphi(y_m))|^\alpha \\ &= o\left(|y_m-\bar{x}|^{q(\alpha+1)-\alpha-2}\right) = o\left(\varepsilon^{q(\alpha+1)-\alpha-2}\right) \end{aligned}$$

we obtain the desired result.

Finally, let us assume that there are infinitely many m such that $y_m = \bar{x}$. Then, using Lemma 4.1 we also have

$$-\beta(u_m(\bar{x})) \ge f(\bar{x}),$$

from where we get the result taking the limit. □

Now we are in a position to prove Theorem 1.2, which generalize a result by Birindelli and Demengel in [5], allowing more general boundary data and obtaining better estimates for the solution. The following existence theorem was proved in [5] (see Proposition 2 in that paper)

Theorem 4.2 *Assume that $g \in W^{2,\infty}(\partial\Omega)$, Ω has a C^2 boundary, f, b continuous in $\overline{\Omega}$, $h : \Omega\times\mathbb{R} \to \mathbb{R}$ is also continuous and $h(x,.)$ is non-increasing. Then there is a solution to equation*

$$F\left(Du, D^2u\right) + b(x)\cdot Du\,|Du|^\alpha + h(x,u) = f \text{ in } \Omega, \tag{4.6}$$

$$u = g \text{ on } \partial\Omega \tag{4.7}$$

Using the estimates given in our Theorem 4.1 and the convergence result we just proved in what follows we give a

Proof of Theorem 1.2 First we take a sequence of function $g_n \in C^2(\Omega)$ such that $g_n \to g$ in $C^\sigma(\omega)$ for any ω compact subset of Ω. Let $\{\omega_j\}$ be a sequence of bounded open sets with C^2 boundary $\partial\omega_j$ such that $\omega_j \subset \overline{\omega}_j \subset \omega_{j+1}$ for all $j \geq 1$, whose union equals Ω. Since Ω satisfies a uniform exterior cone condition we can choose ω_j such that they satisfies the exterior cone condition uniformly in j.

Then we use Theorem 4.2 to find a solution u_j^n to (1.3) in ω_j with $u_j^n = g_n$ on $\partial\omega_j$. So, using Theorem 4.1, we find that $\{u_j^n\}$ has a convergent subsequence and then by Proposition 4.1 we find a solution u_j to Eq. 1.3 in ω_j with $u_j = g$ on $\partial\omega_j$. By the uniform exterior cone condition of ω_j we can extend u_j by g in $\bar{\Omega}\backslash\bar{\omega}_j$ and the extension will be Hölder continuous. We still denote u_j the extension. Since in Theorem 4.1 we can take the constant independent of j and $\|g\|_{C^\sigma(\partial\omega_j)} \leq \|g\|_{C^\sigma(\bar{\Omega})}$ we have

$$\|u_j\|_{C^\beta(\bar{\omega}_j)} \leq C, \tag{4.8}$$

where C is independent of j. Therefore, we can find a subsequence of $\{u_j\}$ (still denoted by u_j) such that $\{u_j\}$ converges uniformly to u in Ω and u satisfies (1.3). Moreover, by (4.8) and the uniform convergence of $\{u_j\}$ we get that u is Hölder continuous in Ω. On the other hand $u_j = g$ on $\partial\Omega$ for all j, then $u = g$ on $\partial\Omega$. So, u satisfies (1.3) and (1.4). Finally again by Theorem 4.1 we obtain (1.5). □

Remark 4.1 In [5] the authors proved an existence theorem allowing for more general terms h that its behavior is related to the first eigenvalue, see Theorem 1 on that paper. We think that this result can be extended also for more general boundary data.

5 Existence of solutions in $\mathbb{R}^n$, without growth assumption on the data

The ideas used to prove the Harnack inequality in Theorem 1.1 can be used to obtain local estimates for solutions of 'super-linear' elliptic problems in $\mathbb{R}^n$. These estimates and Theorem 4.1 allow to obtain solutions to (1.6), the super-linear problem in $\mathbb{R}^n$ without any growth assumption on the right hand side.

In what follows we consider a sequence of problems in $B_m = B(0, m)$, balls centered at 0 and with radius m, and then we take the limit as m goes to infinity.

Remark 5.1 In what follows we consider a simple nonlinearity even though the authors believe that an extension to more general ones can be derived from the same estimates here presented.

Observe that under the hypotheses of Theorem 1.3, for each $m \in \mathbb{N}$ by Theorem 4.2 and Theorem 4.1 there is a solution $u_m \in C^\gamma(B_m)$ of equation

$$\begin{aligned} -F\left(Du_m, D^2u_m\right) + |u_m|^{s-1} u_m = f \ \text{ in } B_m, \\ u = 0 \ \text{ on } \partial B_m. \end{aligned} \tag{5.1}$$

In the analysis of the sequence u_m, the following lemma provides a local estimate which is crucial in the limit process. This lemma will also be the key step in the study of existence of explosive solutions on the boundary of a bounded domain.

Lemma 5.1 *Under the hypotheses of Theorem 1.3 and assuming that g is a continuous function in the open set $\Omega \subset \mathbb{R}^n$. If u is a viscosity solution of*

$$-F\left(Du, D^2u\right) + |u|^{s-1} u \leq g \text{ in } \Omega$$

then, for every $R' > R > 0$ such that $B_{R'} \subset \Omega$ we have

$$\sup_{B_R} u \le C\left(1 + \|g\|_{n,B_{R'}}^{\frac{1}{1+\alpha}}\right), \tag{5.2}$$

where $C = C(s, R, R', n, \lambda, \Lambda)$ does not depend on g.

Proof We use arguments similar to those used in the proof of Proposition 3.1. Using hypotheses (H2) we find that u satisfies

$$-|Du|^{\alpha}\,\mathcal{M}^{+}\left(D^2u\right) + |u|^{s-1}u \le g \text{ in } \Omega.$$

In what follows we assume, without loss of generality, that u is nontrivial, that $R' = 1$, $B = B_1(0)$ and $\lambda = 1$.

We also assume that $u \in C^2(\Omega) \cap C(\bar{\Omega})$, since we may use the approximation procedure as in Proposition 3.1. For $\beta \ge 1$ we define a cut-off function $\eta(x) = (1 - |x|^2)^{\beta}$ if $x \in B$ and $\eta(x) = 0$ if $x \notin B$ and we consider the function $v = \eta u$. Then we have

$$\mathcal{M}^{+}\left(D^2v\right) \ge \eta\mathcal{M}^{+}\left(D^2u\right) + \mathcal{M}^{-}\left(\overline{Du \otimes D\eta} + uD^2\eta\right)$$

and using the estimates obtained in the proof of Proposition 3.1 we find that

$$\mathcal{M}^{+}\left(D^2v\right) \ge C_1\eta\left\{\frac{-|g| + |u|^{s-1}u}{\left(\eta^{-1/\beta}u\right)^{\alpha}}\right\} - C_2v\eta^{\frac{-2}{\beta}} \tag{5.3}$$

in $\Gamma^{+}(v)$, where the constants C_1 and C_2 are universal. Now, since $u > 0$ in $\Gamma^{+}(v)$, we have

$$\eta\frac{-|g| + |u|^{s-1}u}{\left(\eta^{-1/\beta}u\right)^{\alpha}} = -\eta^{1+\alpha/\beta+\alpha}|g|\,v^{-\alpha} + \eta^{\alpha/\beta+1+\alpha-s}v^{s-\alpha}$$

and then (5.3) is transformed into

$$-\mathcal{M}^{+}\left(D^2v\right) + C_1\eta^{\alpha/\beta+1+\alpha-s}v^{s-\alpha} - C_2\eta^{-2/\beta}v \le C_1\eta^{1+\alpha/\beta+\alpha}|g|\,v^{-\alpha}. \tag{5.4}$$

Now we choose β as

$$\beta = \max\left\{\frac{-\alpha}{1+\alpha}, \frac{\alpha+2}{s-1-\alpha}\right\},$$

so that $\eta^{\alpha/\beta+1+\alpha-s} \ge \eta^{-2/\beta}$ and then (5.4) implies

$$-\mathcal{M}^{+}\left(D^2v\right) + \eta^{-2/\beta}v\left(v^{s-\alpha-1} - C\right) \le C_1\eta^{1+\alpha/\beta+\alpha}|g|\,v^{-\alpha}.$$

Then we define $w = \max\left\{v - C^{1/(s-\alpha-1)}, 0\right\}$ as in [21] and we notice that $\Gamma^{+}(w) \subset \Gamma^{+}(v)$ and $\Gamma^{+}(w) \subset \{x \in \Omega \text{ s.t. } w(x) \ge 0\}$. Thus w satisfies

$$-\mathcal{M}^{+}\left(D^2w\right) \le C\eta^{1+\alpha/\beta+\alpha}|g|\,v^{-\alpha} \text{ in } \Gamma^{+}(w).$$

Then we apply ABP inequality to obtain

$$\sup_{B_1} w \le C\left\|\eta^{1+\alpha/\beta+\alpha}gv^{-\alpha}\right\|_{n,B_1},$$

but then

$$\sup_{B_1} v \le \sup_{B_1} w + C^{1/(s-\alpha-1)} \le C\left(1 + \left\|\eta^{1+\alpha/\beta+\alpha}gv^{-\alpha}\right\|_{n,B_1}\right).$$

Finally we notice that, for every $\varepsilon > 0$,

$$\left\|\eta^{1+\alpha/\beta+\alpha} g v^{-\alpha}\right\|_{n,B_1} \leq \left(\sup_{B_1} v\right)^{-\alpha} \|g\|_{n,B_1}$$
$$\leq -\alpha\varepsilon^{-\frac{1}{\alpha}} \sup_{B_1} v + (1+\alpha)\varepsilon^{\frac{-1}{1+\alpha}} \|g\|_{n,B_1}^{\frac{1}{1+\alpha}}$$

so that by choosing ε adequately we conclude that

$$\sup_{B_1} v \leq C\left(1 + \|g\|_{n,B_1}\right).$$

where C does not depend on g.

From here (5.2) follows. □

In order to get useful L^∞ estimates from Lemma 5.1 we need to use a Kato inequality adapted to our class of equations. We obtain such inequality in the following lemma, which may have interest beyond this work.

Lemma 5.2 *Under the assumptions of Theorem 1.3, given a domain $\Omega \subset \mathbb{R}^n$ and continuous functions $u, v, f : \Omega \to \mathbb{R}$, if $u - v$ is a viscosity solution of*

$$-\left|D(u-v)\right|^\alpha \mathcal{M}(D^2(u-v)) + R(x) \leq f \ \ in\ \Omega, \tag{5.5}$$

where $R(x) = |u(x)|^{s-1} u(x) - |v(x)|^{s-1} v(x)$, then $(u-v)^+$ is a viscosity solution of

$$-\left|D(u-v)^+\right|^\alpha \mathcal{M}(D^2(u-v)^+) + R^+(x) \leq f^+ \tag{5.6}$$

in Ω. Here $\mathcal{M}$ denotes $\mathcal{M}^+$ or $\mathcal{M}^-$.

Proof Let us assume first that $(u-v)^+$ is not locally constant. If $x \in \Omega$ is such that $u(x) - v(x) > 0$ then $u - v$ satisfies (5.6) at such an x. If, on the other side, $u(x) - v(x) = 0$ let φ be a test function such that $(u-v)^+ - \varphi$ has a local maximum at x and $D\varphi(x) \neq 0$. But then $(u-v) - \varphi$ has also a local maximum at x and $D\varphi(x) \neq 0$, so we can use (5.5) to obtain

$$-\left|D\varphi(x)\right|^\alpha \mathcal{M}\left(D^2\varphi(x)\right) \leq f^+,$$

and thus (5.6) holds since $R(x) = 0$.

Now we assume that for a constant C we have $(u-v)^+ = C$ in a ball $B(\hat{x}, \rho)$. The we have to prove that

$$R^+(x) \leq f^+(x) \ \ \forall x \in B(\hat{x}, \rho). \tag{5.7}$$

If $C > 0$ then $u(x) - v(x) = (u(x) - v(x))^+$, $R(x) = R^+(x)$ and (5.7) is satisfied since $u - v$ is a viscosity solution of (5.5). If $C = 0$ then $R^+(x) = 0$ and the inequality is trivial. □

From this lemma we obtain the following generalization of Kato inequality for solutions of our equation

Corollary 5.1 *Under the hypotheses of Theorem 1.3, if $f : \Omega \to \mathbb{R}$ continuous and u is a viscosity solution of*

$$-F\left(Du, D^2u\right) + |u|^{s-1} u = f \ \ in\ \Omega, \tag{5.8}$$

then $w = |u|$ *satisfies*

$$-|Dw|^{\alpha}\mathcal{M}^{+}\left(D^2 w\right) + |w|^{s} \leq |f| \ \text{in}\ \Omega. \tag{5.9}$$

Proof First we notice that u satisfies

$$-|Du|^{\alpha}\mathcal{M}^{+}\left(D^2 u\right) + |u|^{s-1} u \leq f \ \text{ in } \Omega,$$

so, by taking $v = 0$ in Lemma 5.2 we obtain that u^{+} is a solution of (5.9) with f^{+} as right hand side. Then we observe that

$$-|D(-u)|^{\alpha}\mathcal{M}^{+}\left(D^2(-u)\right) + |u|^{s-1}(-u) \leq -f^{-} \ \text{ in } \Omega,$$

which, repeating the argument as before, implies that u^{-} is a solution of (5.9) with $-f^{-}$ as the right hand side. We conclude, since $|u| = \max\left\{u^{+}, -u^{-}\right\}$ satisfies (5.9). □

Now we can give a proof to Theorem 1.3.

Proof of Theorem 1.3 Let $\{u_m\}$ be a sequence of solutions of (5.1). Using Corollary 5.1 and Lemma 5.1 we find that for every $0 < R < m$

$$\sup_{B_R} |u_m| \leq C\left(1 + \|f\|_{n,B_R}\right),$$

where C does not depend on f nor m. Then, using Theorem 4.1,

$$\|u_m\|_{C^{\gamma}(B_R)} \leq C,$$

where C does not depend on m. Thus, by a diagonal argument we find a subsequence of solutions of

$$-F\left(Du_m, D^2 u_m\right) + |u_m|^{s-1} u_m = f, \quad \text{in} \quad B_m$$

that we keep calling $\{u_m\}$, such that u_m converges uniformly on any bounded subset of $\mathbb{R}^n$. Using Lemma 4.1 and Proposition 4.1, we find that $u = \lim_{m\to\infty} u_m$ is a solution of (1.6), completing the proof. □

Remark 5.2 If we are given a function $f \in L^n_{\text{loc}}(\mathbb{R}^n)$, we may consider a sequence of smooth functions $\{f_m\}$ such that on every bounded domain Ω we have

$$\lim_{n\to\infty} \int_{\Omega} |f_m - f|^n \, \mathrm{d}x = 0. \tag{5.10}$$

Then we can construct a sequence of solutions to Eq. 5.1, with right hand side f_m. Proceeding as before we find that, up to a subsequence, $\{u_m\}$ converges uniformly to a function u that is a good solution of (1.6).

6 Existence of solutions with explosion on the boundary

The estimate obtained in Lemma 5.1 can be used to construct solutions of super-linear equations in a bounded domain, with explosion on the boundary. We do this in this section and we prove Theorem 1.4.

We proceed as in Sect. 5, but we need some additional arguments so we can prove that the approximating sequence is monotone and thus we guarantee that the limiting solution actually blows-up at the boundary. Under our structural assumptions (H1) and (H2) Birindelli and Demengel proved a comparison theorem, which is what we need in our argument [3]. Actually our statement is slightly different than the original in [3], but the proof is the same.

Theorem 6.1 *Assume that F satisfies (H1) and (H2) and that Ω is a bounded, open subset of $\mathbb{R}^n$ with piecewise C^1 boundary. Further assume that f and g are continuous functions in Ω and β is a continuous real function such that $\beta(0) = 0$.*

If ϕ and σ are viscosity solutions of

$$F\left(D\phi, D^2\phi\right) - \beta(\phi) \leq f \ \text{ in } \Omega,$$
$$F\left(D\sigma, D^2\sigma\right) - \beta(\sigma) \geq g \ \text{ in } \Omega$$

and additionally either:

(i) β is strictly increasing and $f \leq g$, or
(ii) β is non-decreasing and $f < g$

then $\sigma \leq \phi$ on $\partial\Omega$ implies $\sigma \leq \phi$ in Ω.

Now we are in conditions to provide a

Proof of Theorem 1.4 Since f is not necessarily continuous in $\bar{\Omega}$, we consider an increasing sequence $\{f_m\}$ of functions that are continuous in $\overline{\Omega}$ such that

$$\lim_{m\to\infty} \int\limits_{\Omega'} |f_m - f|^n = 0,$$

for every open subset $\Omega' \subset \overline{\Omega'} \subset \Omega$. Then, as a direct consequence of Theorem 1.2, for every $m \in \mathbb{N}$ there is a solution u_m of equation

$$-F\left(Du_m, D^2u_m\right) + |u_m|^{s-1} u_m = f_m \ \text{ in } \Omega,$$
$$u_m = m \ \text{ in } \partial\Omega.$$

Then, by Theorem 6.1 we have $u_{m+1} \geq u_m$ in Ω for all $m \in \mathbb{N}$. Next using the arguments in the proof of Theorem 1.3 (see Lemma 5.1), we obtain that, up to a subsequence, u_m converges uniformly to a solution u of

$$-F\left(Du, D^2u\right) + |u|^{s-1} u = f \ \text{ in } \Omega.$$

To complete the proof we use the monotonicity of $\{u_m\}$ to get that $u \geq u_m$ in Ω, for all $m \in \mathbb{N}$ and consequently

$$\liminf_{x\to\partial\Omega} u(x) \geq m \quad \forall m \in \mathbb{N},$$

so that u solves (1.7) and (1.8). □

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