

## The influence of sources terms on the boundary behavior of the large solutions of quasilinear elliptic equations: the power like case

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**Abstract.** We study the explosive expansion near the boundary of the large solutions of the equation

$$-\Delta_p u + u^m = f \quad \text{in } \Omega$$

where  $\Omega$  is an open bounded set of  $\mathbb{R}^N$ ,  $N > 1$ , with adequately smooth boundary,  $m > p - 1 > 0$ , and  $f$  is a continuous nonnegative function in  $\Omega$ . Roughly speaking, we show that the number of explosive terms in the asymptotic boundary expansion of the solution is finite, but it goes to infinity as  $m$  goes to  $p - 1$ . For illustrative choices of the sources, we prove that the expansion consists of two possible geometrical and nongeometrical parts. For low explosive sources, the nongeometrical part does not exist, and all coefficients depend on the diffusion and the geometry of the domain. For high explosive sources, there are coefficients, relative to the nongeometrical part, independent on  $\Omega$  and the diffusion. In this case, the geometrical part cannot exist, and we say then that the source is very high explosive. We emphasize that low or high explosive sources can cause different geometrical properties in the expansion for a given interior structure of the differential operator. This paper is strongly motivated by the applications, in particular by the non-Newtonian fluid theory where  $p \neq 2$  involves rheological properties of the medium.

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### 1. Introduction

This paper deals with the asymptotic behavior of solutions of the equation

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) + u^m = f \quad \text{in } \Omega \quad (1)$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ ,  $N > 1$ ,  $f \in \mathcal{C}(\Omega)$ ,  $m > 0$  and  $p > 1$ . As it is usual, we denote by  $-\Delta_p$  the leading part of the differential operator. More precisely, our interest is focussed on the solutions with an explosive behavior at the boundary

$$u(x) \rightarrow \infty \quad \text{as } x \rightarrow \partial\Omega, \quad (2)$$

usually called *large* (explosive, boundary blow-up) solutions. A strong motivation of the paper is based on the applications where the values  $p \neq 2$  have a capital role (see below).

As it is well known for the homogeneous case,  $f \equiv 0$ , the large solutions have been studied for several authors provided the extended *Keller–Osserman* condition

$$m > p - 1 \quad (3)$$

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(see, for instance, [11] or [21]). An extensive amount of references, mainly for the uniformly elliptic case  $p = 2$ , is collected in monograph [22] (see also [5] and [6]). Our main comments, in this paper, deal preferably with the general case governed by the nonlinear leading part corresponding to  $p \neq 2$ , strongly motivated by the applications. From the mathematical point of view, the main difference with the linear case is that the differential operator is not uniformly elliptic: degenerate for  $p > 2$  and singular for  $1 < p < 2$ . From the applications point of view, a main difficulty appears: We must know the precise dependence on  $p$  in the boundary behavior of solutions that cannot be deduced from the case  $p = 2$  by a simple way. A first study of problem (1)–(2) was made in [11] where existence, uniqueness and blow-up rate of solutions for certain functions  $f \geq 0$  were proved, under assumption (3) (see also [18, 21], where the previous results were extended to more general nonlinearities in case  $f \equiv 0$ ). We note that these well-known results show that the first term of the expansion near the boundary of large solutions is uniform and independent on the geometry of  $\partial\Omega$  (see again [11] and [21] as well as [13] and the references therein). To the best of our knowledge, we do not know any work on the study the influence of the geometry of the domain on large solutions of (1) when  $f \neq 0$ . Then, an interesting question is to know how nonhomogeneous sources and the geometry of the domain can influence in the asymptotic expansion near the boundary of solutions of (1)–(2).

We recall that for  $p = 2$  and  $f \equiv 0$ , the dependence on the geometry is known from the pioneer work [8] or from [1, 3–7, 17], for instance. In these works, one proves that the geometrical influence appears from the second-order term of the explosive expansion. Our extensions are nontrivial due to the nonlinear nature of the operator with  $p \neq 2$ . Moreover, as in [1], the influence of non-null sources provides a more deep knowledge of the nature of the explosiveness properties of the solutions. We note that large solutions under non-null sources have been also studied in [20, 23] or [24] with different purposes.

As it is usual, the properties near the boundary employ the distance function  $\text{dist}(x, \partial\Omega)$ , here denoted by  $d(x)$ . As it is well known, if the boundary is bounded with  $\partial\Omega \in \mathcal{C}^k$ ,  $k \geq 1$ , it follows from [15] the existence of a positive constant  $\delta_0$ , depending only on  $\partial\Omega$ , such that  $d(\cdot) \in \mathcal{C}^k$  in the parallel strip near the boundary

$$\Omega_{\delta_0} = \{x \in \bar{\Omega} : d(x) < \delta_0\}. \quad (4)$$

Moreover, from the results of [15], also can be deduced the important properties for  $x \in \Omega_{\delta_0}$  as

$$|\nabla d(x)| \equiv 1 \quad \text{and} \quad \Delta d(x) = -(N-1)\mathbf{H}(\bar{x}(x)) + o(1),$$

where  $\bar{x}(x)$  is a point on  $\partial\Omega$  such that  $|x - \bar{x}(x)| = d(x)$  and  $\mathbf{H}(\bar{x}(x))$  denote the mean curvature of  $\partial\Omega$  at  $\bar{x}(x)$ . The simplest geometry occurs on balls, as  $\Omega = \mathbf{B}_R(0)$ , for which

$$\Delta d(x) = -\frac{N-1}{|x|}, \quad |x| < R.$$

These geometrical properties of the domain can take part in the asymptotic expansion near the boundary. Indeed, this influence occurs on secondary terms under more regularity assumptions on the boundary. It is obtained by considering terms containing the mean curvature neglected in the leading coefficient of the expansion.

We emphasize that the existence of large solutions of (1), for  $f \equiv 0$ , is based on the Keller–Osserman condition. This inequality is also a necessary assumption in the nonhomogeneous case  $f \neq 0$ . In order to simplify, the main goal of this paper is to study the influence of sources with the property

$$f(x) \approx f_0(d(x))^{-\alpha_\tau m} \quad \text{as } d(x) \rightarrow 0, \quad f \geq 0,$$

where

$$\alpha_\tau = \frac{p + \tau}{m - p + 1} \quad (\tau \text{ is a nonnegative integer}) \quad (5)$$

(we note that, by construction,  $f_0 \geq 0$ ). As we comment below, we say that the source causes a *low explosion* if  $\tau = 0$ , and the source causes a *high explosion* if  $\tau > 0$  and  $f_0 > 0$ . More precisely, in this paper, we consider continuous and nonnegative sources satisfying

$$f(x) = (d(x))^{-\alpha_\tau m} \left( f_0 + \sum_{n=1}^{M_\tau} f_n (d(x))^n \right), \quad x \in \Omega_{\delta_0}, \tag{6}$$

where  $f_n$ ,  $1 \leq n \leq M_\tau$ , are known constants and

$$M_\tau = \begin{cases} \alpha_\tau - 1 & \text{if } \alpha_\tau \text{ is a positive integer number,} \\ [\alpha_\tau] & \text{otherwise,} \end{cases} \tag{7}$$

for which

$$-\alpha_\tau + n < 0, \quad 1 \leq n < M_\tau, \quad \text{and} \quad -1 \leq -\alpha_\tau + M_\tau < 0.$$

The key of our contributions is based on the construction of a suitable explosive profile given by a master function as

$$V(x) = C_0 (d(x))^{-\alpha_\tau} \left( 1 + \sum_{n=1}^{M_\tau} C_n(x) (d(x))^n \right). \tag{8}$$

Certainly,  $V(x)$  consists of a sum with  $M_\tau + 1$  summands containing all explosive terms. As it will be proved later, when  $2p + \tau - 1 \leq m$  the expansion is very simple, it consists of a unique explosive term (see Lemma 1 in Sect. 2 below). Furthermore, one has

$$\lim_{m \rightarrow p-1} M_\tau = \infty.$$

Preferably, we deal with the condition  $p - 1 < m < 2p + \tau - 1$ . In both cases, we prove in Sect. 2.

**Theorem 1.** *Let us consider  $f \in \mathcal{C}(\Omega)$ ,  $f \geq 0$  on  $\Omega_{\delta_0}$ , verifying (6) with*

$$f_0 > 0 \quad \text{when } \tau > 0. \tag{9}$$

*We also assume (3) and  $\partial\Omega \in \mathcal{C}^{2(M_\tau+1)}$ . Then, for coefficients  $C_0, C_1, \dots, C_{M_\tau}$  given in (39) below, the profile function  $V(x)$  defined in (8) satisfies the boundary behavior*

$$-\Delta_p V(x) + (V(x))^m - f(x) = (d(x))^{-\alpha_\tau m} O(d(x)^{1+M_\tau}). \tag{10}$$

On the other hand, as it will be proved in Sect. 3, suitable reasonings on the magnitudes of approximations of  $V(x)$  and a Comparison Principle lead to our main result

**Theorem 2.** *Under the assumptions of Theorem 1 with  $f \geq 0$  in  $\Omega$ , the explosive boundary expansion of the large solution of (1) has the property*

$$u(x) = V(x) + o\left( (d(x))^{-\alpha_\tau + M_\tau} \right).$$

Certainly, sharp computations are required in the proof of Theorem 1. In Remark 6, we have summarized the obtainment of the coefficients. In short, we comment some illustrative properties of the profile  $V(x)$  transferred from the coefficients, as it will be detailed in Sect. 2. First of all, the main term of the expansion

$$C_0 (d(x))^{-\alpha_\tau} \quad (\tau \geq 0)$$

is always governed by a positive coefficient whose dependence on the geometry is based on the norm of the gradient of the distance function. Since  $|\nabla d(x)| \equiv 1$  near the boundary  $\partial\Omega$ , this dependence is

universal; thus, it is independent of the specific geometry of  $\Omega$  [see (28) or Remark 4]. Furthermore, whenever  $2p - 1 + \tau \leq m$ , the explosive profile function is exactly

$$V(x) = C_0(d(x))^{-\alpha_\tau},$$

while inequality  $p - 1 < m < 2p - 1 + \tau$  determines two possible summands in the expansion deduced from the decomposition

$$V(x) = C_0(d(x))^{-\alpha_\tau} \left( 1 + \sum_{n=1}^{\min\{\tau, M_\tau\}} C_n(d(x))^n + \sum_{n=\min\{\tau, M_\tau\}+1}^{M_\tau} C_n(x)(d(x))^n \right).$$

1. Low explosion ( $\tau = 0$ ) The profile function  $V(x)$  is

$$V(x) = C_0(d(x))^{-\alpha_0} \left( 1 + \underbrace{\sum_{n=1}^{M_0} C_n(x)(d(x))^n}_{\text{the geometrical part}} \right),$$

where  $C_n \in \mathcal{C}^2(M_0-n)(\Omega_{\delta_0/2})$ ,  $1 \leq n \leq M_0$ , are the functions obtained through (38) depending on the geometry.

2. High explosion ( $\tau > 0$ ) Here, the profile function always contains a part, independent on the geometry, with a high blow up. Possibly, it also may contain a part where the blow up is more weak. In some sense, the dependence on the geometry provides low explosion due to the influence of the nonlinear diffusion of the differential operator. This influence is neglected whenever the explosion is high. More precisely:

(i) For  $0 < \tau \leq M_\tau$ , the representation becomes

$$V(x) = C_0(d(x))^{-\alpha_\tau} \left( 1 + \underbrace{\sum_{n=1}^{\tau} C_n(d(x))^n}_{\text{the nongeometrical part}} + \underbrace{\sum_{n=\tau+1}^{M_\tau} C_n(x)(d(x))^n}_{\text{the geometrical part}} \right).$$

Here,  $C_n$ ,  $1 \leq n \leq \tau$ , are constants independent on the geometry given by (31) and (34). Now, (37) enables to obtain the coefficients  $C_n \in \mathcal{C}^2(M_\tau-n)(\Omega_{\delta_0/2})$ ,  $\tau + 1 \leq n \leq M_\tau$ , that are functions depending on the geometry.

(ii) If  $0 < M_\tau < \tau$ , all coefficients in the expansion are independent on the geometry. Therefore, one has

$$V(x) = C_0(d(x))^{-\alpha_\tau} \left( 1 + \underbrace{\sum_{n=1}^{M_\tau} C_n(d(x))^n}_{\text{the nongeometrical part}} \right).$$

Here, the last coefficient  $C_{M_\tau}$  is given by (33). We say that this case corresponds to a very high explosion.

It is clear that in the simple case  $\Omega = \mathbf{B}_R(0)$ , the geometrical part is uniform on  $\partial\Omega$  and consequently the expansion is uniform on  $\partial\Omega$ . In general, we may illustrate the results by noting that for two boundary points  $x_0, y_0 \in \partial\Omega$  if

$$|C_n(x_0 - s\vec{\mathbf{n}}_{x_0}) - C_n(y_0 - s\vec{\mathbf{n}}_{y_0})| \rightarrow 0 \quad \text{as } s \rightarrow 0$$

is satisfied for  $\tau + 1 \leq n \leq M_\tau$ , then we deduce

$$|u(x_0 - s\vec{\mathbf{n}}_{x_0}) - u(y_0 - s\vec{\mathbf{n}}_{y_0})| \rightarrow 0 \quad \text{as } s \rightarrow 0;$$

otherwise,

$$|u(x_0 - s\vec{n}_{x_0}) - u(y_0 - s\vec{n}_{y_0})| \rightarrow \infty \quad \text{as } s \rightarrow 0$$

(here  $\vec{n}_{x_0}$  and  $\vec{n}_{y_0}$  denote the relative unit outward vector).

We emphasize that the geometrical properties derived from a given interior structure

$$-\Delta_p u + u^m \quad \text{in } \Omega$$

can change strongly under low or high explosive sources. For instance, under low explosive sources, the second coefficient of the explosive expansion of the large solutions is the first one dependent on the geometry; however, by changing to a high explosive sources, the first presence of the geometry is displaced to lower terms. Even, if we change to a very high explosive source, the influence of the geometry disappears in the explosive expansion. A technical reason is given in Remark 7 [see also the last comments in the Example 2(ii) at the end of Sect. 3].

The couple of papers [17] and [18] is a good example of results for the Laplacian operator extended to  $p$ -Laplacian operator. A first motivation to extend our results of  $p = 2$  to arbitrary  $p > 1$  is based on the applications. As it well known (see [12]), among other applications including image processing and mean curvature flow, the  $p$ -Laplacian operator appears for instance in the variant of the Navier–Stokes equation that describes the motion of non-Newtonian fluids where the velocity gradient depends on the stress tensor as it occurs for instance in glaciology, rheology, nonlinear elasticity and flow through a porous medium. In particular, in studying the laws of motion of fluid media, the shear stress is given by  $\tau = \mu \nabla u$ . This approximation is only available to some fluids not including dispersive media. For non-Newtonian fluids, considered in Rheology, the power rheological law is  $\tau = \mu |\nabla u|^{p-2} \nabla u$ , where  $\mu$  and  $p$  involve rheological properties of the medium. Here,  $p$  has a very important role:  $p > 2$  says that the medium is a dilatant fluid and  $p < 2$  that the medium is pseudoplastic (see [19]). In particular, the knowledge of the case  $p \neq 2$  is very important in the transition  $p \rightarrow 2$  of the behavior from non-Newtonian fluids to Newtonian fluids, mainly for some kind of free boundary phenomena arising when  $p \neq 2$  (see [10] for details).

A second motivation deals with the convergence  $p \rightarrow \infty$ . Very important results were obtained in [14] and [16]. Essentially, in [14], one proves that if  $p^{-1}m(p) \rightarrow Q \in ]1, \infty[$ , as  $p \rightarrow \infty$ , the large solutions of (1), with  $f \equiv 0$ , converge uniformly on compacts subsets to a large viscosity solution of  $\max\{-\Delta_\infty u, -|\nabla u| + u^Q\} = 0$ . If  $Q = 1$ , the solutions go to  $\infty$ , and when  $Q = \infty$ , the solutions converge to 1. In [2], we complete this convergence whenever non-null source terms govern the equation and moreover give a precise approximation to the large solutions of  $-\Delta_\infty u + \beta(u) = f$  not included in [14]. The boundary behavior of the large solution of this fully nonlinear equations was obtained in [9]. We also use a suitable modification of the  $p$ -Laplacian operator in order to go to  $\infty$ -Laplacian operator.

The paper is organized as follows. In Sect. 2, we construct formal boundary explosive expansions by using several awful straightforward computations. It requires classical explicit expressions as the old formula of Federico Villarreal (1850–1923) on the power of polynomials. These formulas are summarized in a short “Appendix” (see [1, Appendices A and B] for details). In Sect. 3, the formal expansions are applied in order to obtain the whole boundary explosive expansion of the large solution of (1). We include in that section the Examples 1 and 2 in order to illustrate the main contributions of this paper.

## 2. The main properties of the boundary profile function

As it was pointed out in the introduction, we devote this section to construct a profile boundary function of type

$$V(x) = C_0(d(x))^{-\alpha_\tau} \left( 1 + \sum_{n=1}^{M_\tau} C_n(x)(d(x))^n \right), \quad x \in \Omega_{\delta_0/2}, \tag{11}$$

where  $\alpha_\tau$ ,  $M_\tau$  and  $\Omega_{\delta_0}$  are given in (5), (7) and (4), respectively. The coefficients  $C_0$  and  $C_n(x)$  will be chosen later [see (27) below].

First, we relate  $M_\tau$  with the values of  $m$ ,  $p$  and  $\tau$  as follows:

**Lemma 1.** Consider the intervals  $I_0 = [2p + \tau - 1, \infty[$  and

$$I_k \doteq \left[ \frac{(k + 1)(p - 1) + p + \tau}{k + 1}, \frac{k(p - 1) + p + \tau}{k} \right],$$

where  $k$  is a positive integer. Then, the disjoint covering

$$]p - 1, \infty[ = \bigcup_{k=0}^{\infty} I_k, \tag{12}$$

holds. In particular, for the choice  $k = M_\tau$  defined in (7), one has

$$m \in I_{M_\tau}. \tag{13}$$

*Proof.* The covering (12) is obtained by a simple and direct checking. On the other hand, by definition of  $M_\tau$ , one has the inequality  $\alpha_\tau - 1 \leq M_\tau < \alpha_\tau$  that is equivalent to (13).  $\square$

Now, we construct the framework on which we will prove the important boundary property

$$-\Delta_p V(x) + (V(x))^m - f(x) = (d(x))^{-\alpha_\tau m} O(d(x)^{1+M_\tau})$$

[see (10) in Theorem 1]. Two previous lemmas (Lemmas 2 and 3) are proved in order to explain the nature of the expansion of the quasilinear expression

$$-\Delta_p V(x) + (V(x))^m. \tag{14}$$

**Lemma 2.** Let us assume  $C_n \in \mathcal{C}(\Omega_{\delta_0/2})$ . Then, there exist adequate functions  $D_n \in \mathcal{C}(\Omega_{\delta_0/2})$  for which the  $m$ th power of the profile function admits the expansion

$$(V(x))^m = C_0^m (d(x))^{-\alpha_\tau m} \left( 1 + \sum_{n=1}^{M_\tau} D_n(x) (d(x))^n \right) + O\left( (d(x))^{1+M_\tau-\alpha_\tau m} \right), \quad x \in \Omega_{\delta_0/2}. \tag{15}$$

*Proof.* Following classical results, collected in the final ‘‘Appendix’’, one proves that the  $m$ th power of the profile admits a representation as

$$(V(x))^m = C_0^m (d(x))^{-\alpha_\tau m} \left( 1 + \sum_{n=1}^{M_\tau} D_n(x) (d(x))^n + \sum_{n=M_\tau+1}^{\infty} D_n(x) (d(x))^n \right), \tag{16}$$

where

$$D_n(x) = \binom{m}{1} C_n(x) + \sum_{i=2}^n \binom{m}{i} B_{n-i,i}(x), \quad n \geq 1, \tag{17}$$

with

$$B_{n-i,i}(x) = \sum_{j=1}^{n-i} \binom{i}{j} (C_1(x))^{i-j} \sum_{\substack{\ell_1 \cdot \gamma_{\ell_1} + \dots + \ell_j \cdot \gamma_{\ell_j} = n-i+j \\ \gamma_{\ell_1} + \dots + \gamma_{\ell_j} = j \\ 2 \leq \ell_1 < \dots < \ell_j \leq n-i-j+2 \\ \{\gamma_{\ell_k}\}_{k=1}^j \subset \{0, 1, \dots, j\}}} \frac{j!}{\gamma_{\ell_1}! \cdot \dots \cdot \gamma_{\ell_j}!} (C_{\ell_1}(x))^{\gamma_{\ell_1}} \cdot \dots \cdot (C_{\ell_j}(x))^{\gamma_{\ell_j}} \tag{18}$$

for  $i = 2, 3, \dots, n$  [see (48) below]. In Remark 1, we give explicitly the first coefficients  $D_n(x)$ . From the definition of  $D_n(x)$  given in (17), we can deduce that the coefficient  $C_n(x)$  only appears in the first term, while in the remaining terms appear powers and products involving some or all previous coefficients

$C_1(x), C_2(x), \dots, C_{n-1}(x)$ . Therefore,  $D_{M_\tau}$  is the coefficient of (16) where  $C_{M_\tau}$  appears for the first time [see (17)]. It explains the truncation in (16). Then, we conclude (15) by noting that

$$\Psi_m(x; r) = \sum_{n=M_\tau+1}^{\infty} D_n(x)r^n, \quad (x; r) \in \Omega_{\delta_0/2} \times ]0, \delta_0/2[$$

verifies  $\Psi_m \in \mathcal{C}(\overline{\Omega}_{\delta_0/2} \times [0, \delta_0/2])$  with  $\Psi_m(x; r) = O(r^{1+M_\tau})$ . □

**Remark 1.** Provided  $M_\tau \geq 4$ , the first coefficients  $D_n(x)$  are given by

$$D_1(x) = \binom{m}{1} C_1(x),$$

$$D_2(x) = \binom{m}{2} (C_1(x))^2 + \binom{m}{1} C_2(x),$$

$$D_3(x) = \binom{m}{3} (C_1(x))^3 + \binom{m}{2} 2C_1(x)C_2(x) + \binom{m}{1} C_3(x),$$

$$D_4(x) = \binom{m}{4} (C_1(x))^4 + \binom{m}{3} 3(C_1(x))^2 C_2(x) + \binom{m}{2} (2C_1(x)C_3(x) + (C_2(x))^2) + \binom{m}{1} C_4(x). \quad \square$$

The construction of the leading part of (14) is also very laborious. In particular, we have

**Lemma 3.** *Let us assume  $C_n \in \mathcal{C}^2(\Omega_{\delta_0/2})$ . Then, there exist a constant  $A_0$  and some functions  $A_n \in \mathcal{C}(\Omega_{\delta_0/2})$ , for which the following expansion*

$$\Delta_p V(x) = C_0^{p-1} (d(x))^{-\alpha_\tau m} \left( A_0 (d(x))^\tau + \sum_{n=1}^{\max\{M_\tau - \tau, 0\}} A_n(x) (d(x))^{n+\tau} \right) + O\left( (d(x))^{1+\max\{M_\tau, \tau\} - \alpha_\tau m} \right) \quad (19)$$

holds in  $\Omega_{\delta_0/2}$ .

*Proof.* First of all, we obtain

$$\nabla V(x) = C_0 (d(x))^{-(\alpha_\tau + 1)} \left( \overrightarrow{\mathbb{A}}_0(x) + \sum_{n=1}^{M_\tau} \overrightarrow{\mathbb{A}}_n(x) (d(x))^n + \overrightarrow{\mathbb{A}}_{M_\tau+1}(x) (d(x))^{M_\tau+1} \right)$$

with

$$\overrightarrow{\mathbb{A}}_n(x) = \begin{cases} -\alpha_\tau \nabla d(x), & n = 0, \\ (-\alpha_\tau + 1) C_1(x) \nabla d(x), & n = 1, \\ (-\alpha_\tau + n) C_n(x) \nabla d(x) + \nabla C_{n-1}(x), & 2 \leq n \leq M_\tau, \\ \nabla C_{M_\tau}(x), & n = M_\tau + 1. \end{cases} \quad (20)$$

Now, following again the reasonings of the ‘‘Appendix’’, see (51), one has

$$|\nabla V(x)|^2 = C_0^2 (d(x))^{-2(\alpha_\tau + 1)} \left( \alpha_\tau^2 + \sum_{n=1}^{2(M_\tau + 1)} E_n(x) (d(x))^n \right), \quad (21)$$

for certain functions  $E_n(x)$ . As in (15), we focus the attention on the coefficients  $E_n(x)$ ,  $1 \leq n \leq M_\tau$ , given by

$$E_n(x) = \sum_{j=0}^n \langle \vec{A}_j(x), \vec{A}_{n-j}(x) \rangle, \quad 1 \leq n \leq M_\tau \tag{22}$$

(see Remark 3 where the first coefficients  $E_n(x)$  are detailed explicitly). Next, from (21), we may write

$$|\nabla V(x)|^{p-2} = (|\nabla V(x)|^2)^{\frac{p-2}{2}} = C_0^{p-2} (d(x))^{-(\alpha_\tau+1)(p-2)} \Phi \left( \sum_{n=1}^{2(M_\tau+1)} E_n(x) (d(x))^{n-1} \right)$$

where

$$\Phi(s) = (\alpha_\tau^2 + s d(x))^{\frac{p-2}{2}} = \alpha_\tau^{p-2} \sum_{n=0}^{\infty} \binom{\frac{p-2}{2}}{n} \alpha_\tau^{-2n} s^n (d(x))^n.$$

As in (16), we apply the extended Villareal formula (see once again the ‘‘Appendix’’ below) in order to obtain

$$|\nabla V(x)|^{p-2} = C_0^{p-2} (d(x))^{-(\alpha_\tau+1)(p-2)} \alpha_\tau^{p-2} \left( 1 + \sum_{n=1}^{M_\tau} F_n(x) (d(x))^n + \sum_{n=M_\tau+1}^{\infty} F_n(x) (d(x))^n \right) \tag{23}$$

for  $x \in \Omega_{\delta_0}$ , governed by

$$F_n(x) = \binom{\frac{p-2}{2}}{1} \alpha_\tau^{-2} E_n(x) + \sum_{i=2}^n \binom{\frac{p-2}{2}}{i} \alpha_\tau^{-2i} G_{n-i,i}(x), \quad n \geq 1, \tag{24}$$

where

$$G_{n-i,i}(x) = \sum_{j=1}^{n-i} \binom{i}{j} (E_1(x))^{i-j} \sum_{\substack{\ell_1 \cdot \gamma_{\ell_1} + \dots + \ell_j \cdot \gamma_{\ell_j} = n-i+j \\ \gamma_{\ell_1} + \dots + \gamma_{\ell_j} = j \\ 2 \leq \ell_1 < \dots < \ell_j \leq n-i-j+2 \\ \{\gamma_{\ell_k}\}_{k=1}^j \subset \{0,1,\dots,j\}}} \frac{j!}{\gamma_{\ell_1}! \cdot \dots \cdot \gamma_{\ell_j}!} (E_{\ell_1}(x))^{\gamma_{\ell_1}} \cdot \dots \cdot (E_{\ell_j}(x))^{\gamma_{\ell_j}}$$

for  $i = 2, 3, \dots, n$  (see Remark 3 where we give explicitly the first coefficients  $F_n(x)$ ). Here,  $F_{M_\tau}$  is the term where  $C_{M_\tau}$  appears for the first time [see (22) and (24)].

As in the proof of Lemma 2, the coefficient  $E_n(x)$  only appears in the first term of (24), while in the remaining terms appear powers and products involving the previous coefficients  $\{E_1(x), E_2(x), \dots, E_{n-1}(x)\}$ . The above calculations lead to

$$|\nabla V(x)|^{p-2} \nabla V(x) = C_0^{p-1} (d(x))^{-(\alpha_\tau+1)(p-1)} \left( -\alpha_\tau^{p-1} \nabla d(x) + \sum_{n=1}^{\infty} \vec{H}_n(x) (d(x))^n \right)$$

where

$$\vec{H}_n(x) = \begin{cases} \alpha_\tau^{p-2} (p-1) (-\alpha_\tau + 1) C_1(x) \nabla d(x), & n = 1, \\ \alpha_\tau^{p-2} \sum_{j=0}^n F_{n-j}(x) \vec{A}_j(x), & 2 \leq n \leq M_\tau, \\ \alpha_\tau^{p-2} \sum_{j=0}^{M_\tau+1} F_{n-j}(x) \vec{A}_j(x), & n \geq M_\tau + 1, \end{cases}$$

with  $\vec{A}_n(x)$  as in (20) and  $F_{n-j}(x)$  as in (24). Hence,

$$\Delta_p V(x) = C_0^{p-1} (d(x))^{-(\alpha_\tau+1)(p-1)-1} \left( A_0 + \sum_{n=1}^{\infty} A_n(x) (d(x))^n \right)$$



where

$$A_n(x) = \begin{cases} \alpha_\tau^{p-1}(\alpha_\tau + 1)(p - 1), & n = 0, \\ \alpha_\tau^{p-2}((\alpha_\tau - 1)(p - 1)(\alpha_\tau(p - 1) + p - 2)C_1(x) - \alpha_\tau \Delta d(x)), & n = 1, \\ (- (\alpha_\tau + 1)(p - 1) + n) \langle \vec{\mathbb{H}}_n(x), \nabla d(x) \rangle + \operatorname{div} \vec{\mathbb{H}}_{n-1}(x), & n \geq 2. \end{cases} \tag{25}$$

Since

$$\alpha_\tau = \frac{p + \tau}{m - p + 1} \Leftrightarrow (\alpha_\tau + 1)(p - 1) + 1 + \tau = \alpha_\tau m,$$

we may conclude (19) by noting that

$$\Psi_A(x; r) = \sum_{n=\max\{M_\tau-\tau, 0\}+1}^{\infty} A_n(x)r^{n+\tau}, \quad (x; r) \in \Omega_{\delta_0/2} \times ]0, \delta_0/2[.$$

verifies  $\Psi_A \in \mathcal{C}(\overline{\Omega}_{\delta_0/2} \times [0, \delta_0/2])$ , with  $\Psi_A(x; r) = O(r^{1+\max\{M_\tau, \tau\}})$ . We note that the assumed regularity  $C_n \in \mathcal{C}^2(\Omega_{\delta_0/2})$  will be proved in the next section.  $\square$

**Remark 2.** Since  $|\nabla d(x)| \equiv 1$  near the boundary (see [15]), the coefficient  $A_0$  in (25) is independent on the geometry. On the other hand, we note that all functions  $A_n(x)$ ,  $1 \leq n \leq M_\tau + 1$ , depend on the geometry of  $\Omega$  through the distance function  $d(x)$ . More precisely,  $A_1(x)$  depends on the mean curvature. On the other hand, as expected, coefficients  $A_n(x)$  coincide with those found in [1] for the case  $p = 2$ .  $\square$

**Remark 3.** We illustrate the first three terms of (21), provided  $M_\tau \geq 3$ ,

$$\begin{aligned} E_1(x) &= 2\alpha_\tau(\alpha_\tau - 1)C_1(x), \\ E_2(x) &= 2\alpha_\tau((\alpha_\tau - 2)C_2(x) - \langle \nabla C_1(x), \nabla d(x) \rangle) + (\alpha_\tau - 1)^2(C_1(x))^2, \\ E_3(x) &= 2(\alpha_\tau(\alpha_\tau - 3)C_3(x) - \langle \nabla C_2(x), \nabla d(x) \rangle) \\ &\quad + 2(\alpha_\tau - 1)C_1(x)((\alpha_\tau - 2)C_2(x) - \langle \nabla C_1(x), \nabla d(x) \rangle) \end{aligned}$$

and the first two coefficients  $F_n(x)$  of (23), provided  $M_\tau \geq 2$ ,

$$\begin{aligned} F_1(x) &= (p - 2) \left( \frac{\alpha_\tau - 1}{\alpha_\tau} \right) C_1(x), \\ F_2(x) &= (p - 2) \left( \frac{p - 3}{2} \left( \frac{\alpha_\tau - 1}{\alpha_\tau} \right)^2 (C_1(x))^2 - \frac{1}{\alpha_\tau} \langle \nabla C_1(x), \nabla d(x) \rangle + \left( \frac{\alpha_\tau - 2}{\alpha_\tau} \right) C_2(x) \right). \end{aligned}$$

Obviously, equality (23) is irrelevant when  $p = 2$  because  $F_n(x)$ ,  $1 \leq n \leq M_\tau$ , are null functions.  $\square$

Now, we can get to the proof of our main result in this section.

*Proof of Theorem 1.* From equalities (6), (15) and (19), we can write

$$\begin{aligned} & -\Delta_p V(x) + (V(x))^m - f(x) \\ &= (d(x))^{-\alpha_\tau m} \left[ -C_0^{p-1} \left( A_0(d(x))^\tau + \sum_{n=1}^{\max\{M_\tau-\tau, 0\}} A_n(x)(d(x))^{n+\tau} \right) \right. \\ &\quad \left. + (C_0^m - f_0) + \sum_{n=1}^{M_\tau} (C_0^m D_n(x) - f_n)(d(x))^n + \Upsilon(x; d(x)) \right], \quad x \in \Omega_{\delta_0/2}, \end{aligned} \tag{26}$$

for the remainder

$$\Upsilon(x; d(x)) = -C_0^{p-1}\Psi_A(x; d(x)) + C_0^m\Psi_m(x; d(x))$$

that verifies

$$\Upsilon(x; d(x)) = O((d(x))^{1+M_\tau})$$

for all  $x \in \Omega_{\delta_0}$ . Our goal is clear now: If we make suitable choices of the coefficients  $C_0$  and  $C_n(x)$  such that

$$C_0^{p-1} \left( A_0(d(x))^\tau + \sum_{n=1}^{\max\{M_\tau-\tau, 0\}} A_n(x)(d(x))^{n+\tau} \right) = C_0^m - f_0 + \sum_{n=1}^{M_\tau} (C_0^m D_n(x) - f_n)(d(x))^n, \quad (27)$$

the equality (26) leads to

$$-\Delta_p V(x) + (V(x))^m - f(x) = (d(x))^{-\alpha_\tau m} (\Upsilon(x; d(x))),$$

whence (10) follows.

In order to do it, the value of  $C_0$  is obtained by canceling the constant term in expression (27), that is,

$$C_0^m - f_0 = 0 \quad \text{if } \tau > 0 \quad \text{and} \quad C_0^m - C_0^{p-1}\alpha_0^{p-1}(\alpha_0 + 1)(p - 1) - f_0 = 0 \quad \text{if } \tau = 0 \quad (28)$$

[see (25)]. Therefore,  $C_0$  is independent on the geometry and it is the unique coefficient when  $M_\tau = 0$ , that is,  $m \geq 2p - 1 + \tau$ . We note that when  $\tau = 0$ , we only require  $f_0 \geq 0$ . After obtaining this value  $C_0$ , the rest of coefficients  $C_n(x)$  are determined iteratively from the relation (27) making a balance between the power of  $d(x)$  by canceling the respective coefficients.  $\square$

**Remark 4.** By some conveniences, we may introduce the one–one function  $\phi : [0, 1[ \rightarrow \mathbb{R}_+ \cup \{0\}$

$$\phi(t) = \left( \frac{(m + 1)(p - 1)p^{p-1}}{(m + 1 - p)^p(1 - t)} \right)^{\frac{m}{m+1-p}} t.$$

for which (28) becomes

$$C_0 = \begin{cases} f_0^{\frac{1}{m}} & \text{if } \tau > 0, \\ \left( \frac{(m + 1)(p - 1)p^{p-1}}{(m + 1 - p)^p(1 - \phi^{-1}(f_0))} \right)^{\frac{1}{m+1-p}} & \text{if } \tau = 0. \end{cases} \quad (29) \quad \square$$

As it was pointed out in Introduction, if  $2p - 1 + \tau \leq p$ , the expansion only consists of a unique term governed by  $C_0$  obtained from (29). When  $p - 1 < m < 2p - 1 + \tau$ , the rest of the coefficients  $C_n$  are obtained in order to (27) hold. They depend on the type of explosion, high or low, based on the two possible parts of the explosive expansion

$$V(x) = C_0(d(x))^{-\alpha_\tau} \left( 1 + \underbrace{\sum_{n=1}^{\min\{\tau, M_\tau\}} C_n(d(x))^n}_{\text{the nongeometrical part}} + \underbrace{\sum_{n=\min\{\tau, M_\tau\}+1}^{M_\tau} C_n(x)(d(x))^n}_{\text{the geometrical part}} \right).$$

Thus, the expansion has a possible first part with high explosion and a possible second part whose explosiveness is low due to the influence on the nonlinear diffusion neglected in the previous one. In some sense, the influence of the diffusion is transferred to the influence of the geometry of the domain.

**A. The possible nongeometrical part.** It only appears when  $\tau > 0$ , and consequently, we will require condition (9). Then, we choose  $C_1, \dots, C_{\min\{\tau, M_\tau\}-1}$  from the equalities

$$-C_0^{p-1} \cdot 0 + C_0^m D_n(x) = f_n, \quad 1 \leq n \leq \min\{\tau, M_\tau\} - 1. \tag{30}$$

Certainly, choice  $n = 0$  is also available denoting  $D_0(x) \equiv 1$ , and it implies

$$C_0 = f_0^{\frac{1}{m}}$$

[see (28)]. Hence, in the comments of this part, we may assume  $M_\tau > 0$  or  $\min\{\tau, M_\tau\} \geq 1$ . The representation (17) and the equality (27) lead to

$$C_n = \frac{1}{mf_0} \left( f_n - f_0 \sum_{i=2}^n \binom{m}{i} B_{n-i,i} \right), \quad 1 \leq n \leq \min\{\tau, M_\tau\} - 1. \tag{31}$$

From the properties of  $D_n$ , the coefficients  $C_n$ ,  $1 \leq n \leq \min\{\tau, M_\tau\} - 1$ , are constants independent on  $\Omega$ .

We note that the formulas of Remark 1 lead to

$$C_1 = \frac{1}{mf_0} f_1 \quad \text{and} \quad C_2 = \frac{1}{mf_0} \left( f_2 - \frac{m-1}{2} \frac{1}{mf_0} f_1^2 \right), \tag{32}$$

provided  $\min\{\tau, M_\tau\} \geq 3$ .

The last coefficient of this part,  $C_{\min\{\tau, M_\tau\}}$ , is also independent on the geometry, but it admits two possibilities:

- (i) If  $0 < M_\tau < \tau$ , the expression (30) also provides the last coefficient of the whole explosive expansion given by

$$-C_0^{p-1} \cdot 0 + C_0^m D_{M_\tau}(x) = f_{M_\tau},$$

whence

$$C_{M_\tau} = \frac{1}{mf_0} \left( f_{M_\tau} - f_0 \sum_{i=2}^{M_\tau} \binom{m}{i} B_{M_\tau-i,i} \right). \tag{33}$$

We recall that by construction coefficient,  $C_{M_\tau}$  is the last coefficient of explosive function (11).

- (ii) If  $0 < \tau \leq M_\tau$  from (27), it follows

$$-C_0^{p-1} A_0 + C_0^m D_\tau = f_\tau,$$

whence

$$C_\tau = \frac{1}{mf_0} \left( f_\tau + \frac{(p+\tau)^{p-1}(m+\tau+1)(p-1)}{(m-p+1)^p} f_0^{\frac{p-1}{m}} - f_0 \sum_{i=2}^\tau \binom{m}{i} B_{\tau-i,i} \right). \tag{34}$$

Obviously,  $C_\tau$  is the last coefficient of the whole explosive expansion of the profile function (11) only when  $M_\tau = \tau$ . In general, condition  $M_\tau = \tau$  implies

$$\begin{cases} (m-p)\tau \geq 2p-m-1 & \text{if } \alpha_\tau \text{ is an integer number,} \\ (m-p)\tau > 2p-m-1 & \text{otherwise.} \end{cases} \tag{35}$$

**B. The possible geometrical part.** This part only is possible when  $\tau < M_\tau$ ,  $\tau \geq 0$ , because otherwise it is the nongeometrical part. Consequently,  $\min\{\tau, M_\tau\} = \tau$ .

The study is completed by choosing the coefficients  $C_{\tau+1}(x), \dots, C_{M_\tau}(x)$ ,  $\tau \geq 0$ , from equalities

$$-C_0^{p-1} A_{n-\tau}(x) + C_0^m D_n(x) = f_n, \quad \tau+1 \leq n \leq M_\tau, \tag{36}$$

with  $M_\tau > 0$ , thus  $p-1 < m < 2p+\tau-1$ .

By means of  $A_n(x)$ ,  $\tau+1 \leq n \leq M_\tau$ , these coefficients depend on the geometry of  $\Omega$ . In particular,  $C_{\tau+1}(x)$  depends only on the mean curvature (see Remark 5 below).

Certainly, when  $\tau > 0$ , from the properties of  $A_n(x)$  and  $D_n(x)$ , one has the explicit formula

$$C_n(x) = \frac{1}{mf_0} \left( f_n + C_0^{p-1} A_{n-\tau}(x) - f_0 \sum_{i=2}^n \binom{m}{i} B_{n-i,i}(x) \right), \quad \tau + 1 \leq n \leq M_\tau. \tag{37}$$

Whenever  $\tau = 0$ , condition (36) becomes

$$-C_0^{p-1} A_n(x) + C_0^m D_n(x) = f_n, \quad 1 \leq n \leq M_0. \tag{38}$$

From definition of  $D_n(x)$  [see (17)] and  $A_n(x)$  [see (25)], the relative coefficients  $C_n(x)$  chosen in (38) also admit an explicit and hard expression as

$$\mathcal{A}_n C_n(x) = \mathcal{F}(m, p, f_0, \dots, f_n, C_0, C_1(x), \dots, C_{n-1}(x)), \quad 1 \leq n \leq M_0,$$

where

$$\begin{aligned} \mathcal{A}_n &\doteq \lambda m C_0^m - \alpha_0^{p-2} (p-1)(-\alpha_0+n)(-(\alpha_0+1)(p-1)+n) C_0^{p-1} \\ &= C_0^{p-1} \alpha_0^{p-2} (p-1) [(\alpha_0+1)(p+(p-1)n) - n(-\alpha_0+n)] + mf_0 \end{aligned}$$

is a positive constant due to  $-\alpha_0 + n \leq -\alpha_0 + M_0 < 0$ .

The above construction shows that the coefficients  $C_n$  are constants or belong to  $\mathcal{C}^2(\Omega_{\delta_0/2}) \cap L^\infty(\Omega_{\delta_0/2})$ , due to the regularity of the distance function.

**Remark 5.** As it has been pointed out several times, the obtainment of  $C_n(x)$  requires very tedious computations. For example, for  $0 \leq \tau < M_\tau$ , one obtains

$$\begin{aligned} C_{\tau+1}(x) &= \frac{1}{mf_0} \left( f_{\tau+1} - f_0 \sum_{i=2}^{\tau+1} \binom{m}{i} B_{\tau+1-i,i}(x) \right. \\ &\quad \left. + f_0^{\frac{p-1}{m}} \alpha_\tau^{p-2} [(\alpha_\tau-1)(p-1)(\alpha_\tau(p-1)+p-2)C_1(x) - \alpha_\tau \Delta d(x)] \right). \end{aligned}$$

In particular, when  $\tau = f_0 = 0$ , one has

$$C_0 = \left( \frac{(m+1)(p-1)p^{p-1}}{(m-p+1)^p} \right)^{\frac{1}{m-p+1}} \quad \text{and} \quad C_1(x) = \eta(m, p)(\gamma(m, p)f_1 - \Delta d(x)),$$

where

$$\begin{cases} \eta(m, p) = \frac{\alpha_0}{(p-1)[\alpha_0(m+p+1)+p-2]} = \frac{p}{2(p-1)[p(m+1)-(m-p+1)]}, \\ \gamma(m, p) = \frac{1}{C_0^{p-1} \alpha_0^{p-1}} = \left( \frac{(m-p+1)^{m+1}}{(m+1)(p-1)p^m} \right)^{\frac{p-1}{m-p+1}}. \end{cases}$$

Obviously, for  $p = 2$ , the coefficients  $C_0$  and  $C_1(x)$  coincide with those values already obtained for the Laplacian operator (see [1] or [8]). □

**Remark 6.** We summarize the global obtainment of the coefficients as follows. First of all, the constant  $C_0$ , obtained from (28), is the unique coefficient in the expansion whenever  $2p + \tau - 1 \leq m$ , that is,  $M_\tau = 0$ . Otherwise, when  $p - 1 < m < 2p + \tau - 1$ , all coefficients are given by the table

$$\begin{cases} \text{if } 0 < \tau \leq M_\tau \text{ one has} & \begin{matrix} \text{(29)} & \text{(31)} & \text{(37)} & \text{(37)} \\ \overbrace{C_0} & \overbrace{C_1, \dots, C_{\tau-1}} & \overbrace{C_\tau} & \overbrace{C_{\tau+1}, \dots, C_{M_\tau}} \end{matrix} \\ \text{if } 0 < M_\tau < \tau \text{ one has} & \begin{matrix} \text{(29)} & \text{(31)} & \text{(33)} \\ \overbrace{C_0} & \overbrace{C_1, \dots, C_{M_\tau-1}} & \overbrace{C_{M_\tau}} \end{matrix} \\ \text{if } 0 = \tau < M_0 \text{ one has} & \begin{matrix} \text{(29)} & \text{(38)} & \text{(38)} \\ \overbrace{C_0} & \overbrace{C_1, \dots, C_{M_0-1}} & \overbrace{C_{M_0}} \end{matrix} \end{cases} \tag{39}$$

□

**Remark 7.** The presence of the geometry in the expansion is derived exclusively from the functions  $A_n(x)$  [see (25) and (36)]. Then, fixed  $p$  and  $m$ , for different values of  $\tau$ , the Eq. (36) can become (30). Hence, fixed an interior structure of the differential operator,  $p$  and  $m$ , the geometrical properties can appear in different localizations of the explosive expansion. These geometrical properties may even disappear. See the Example 2(ii), at the end of Sect. 3, to an illustration of the above comments.  $\square$

### 3. The boundary asymptotic expansion of the large solution

In this section, we consider the perturbed boundary profile function

$$V_{\mp\delta}(x) = C_0(d(x) \mp \delta)^{-\alpha_\tau} \left( 1 + \sum_{n=1}^{M_\tau} C_n(x)(d(x) \mp \delta)^n \right)$$

defined for  $x \in \Omega$  such that  $d(x) \mp \delta > 0$  with  $\delta > 0$  small enough.

**Proposition 1.** *Under assumptions of Theorem 1, the following behavior*

$$-\Delta_p V_{\mp\delta}(x) + (V_{\mp\delta}(x))^m - f(x) = (d(x))^{-\alpha_\tau m} O\left((d(x))^{1+M_\tau}\right)$$

holds.

*Proof.* The choice of the coefficients  $C_n(x)$  in Theorem 1 leads to

$$C_0^{p-1} \left( A_0(d(x) \mp \delta)^\tau + \sum_{n=1}^{\max\{M_\tau - \tau, 0\}} A_n(x)(d(x) \mp \delta)^{n+\tau} \right) = C_0^m - f_0 + \sum_{n=1}^{M_\tau} (C_0^m D_n(x) - f_n)(d(x) \mp \delta)^n$$

[see (27)]. Consequently, since

$$\begin{aligned} & -\Delta_p V_{\mp\delta}(x) + (V_{\mp\delta}(x))^m - f(x) \\ &= (d(x) \mp \delta)^{-\alpha_\tau m} \left[ -C_0^{p-1} \left( A_0(d(x) \mp \delta)^\tau + \sum_{n=1}^{\max\{M_\tau - \tau, 0\}} A_n(x)(d(x) \mp \delta)^{n+\tau} \right) \right. \\ & \quad \left. + (C_0^m - f_0) + \sum_{n=1}^{M_\tau} (C_0^m D_n(x) - f_n)(d(x) \mp \delta)^n + \Upsilon(x; d(x) \mp \delta) + \Xi(x; \mp\delta) \right] \end{aligned}$$

[see (26)], then

$$-\Delta_p V_{\mp\delta}(x) + (V_{\mp\delta}(x))^m - f(x) = (d(x) \mp \delta)^{-\alpha_\tau m} (\Upsilon(x; d(x) \mp \delta) + \Xi(x; \mp\delta)), \quad x \in \Omega_{\delta_0/2}, \tag{40}$$

for the remainders

$$\begin{cases} \Upsilon(x; d(x) \mp \delta) = -C_0^{p-1} \Psi_A(x; d(x) \mp \delta) + C_0^m \Psi_m(x; d(x) \mp \delta), \\ \Xi(x; \mp\delta) = \left( f_0 + \sum_{n=1}^{M_\tau} f_n(d(x) \mp \delta)^n \right) - \left( \frac{d(x) \mp \delta}{d(x)} \right)^{\alpha_\tau m} \left( f_0 + \sum_{n=1}^{M_\tau} f_n(d(x))^n \right), \end{cases} \tag{41}$$

that verify

$$\lim_{\delta \rightarrow 0} \Upsilon(x; d(x) \mp \delta) = O((d(x))^{1+M_\tau}) \quad \text{and} \quad \lim_{\delta \rightarrow 0} \Xi(x; \mp\delta) = 0$$

for all  $x \in \Omega_{\delta_0}$ .  $\square$

For future purposes, it will be very useful to rewrite (40) as

$$-\Delta_p V_{\mp\delta}(x) + (V_{\mp\delta}(x))^m - f(x) = (d(x) \mp \delta)^{-\alpha_\tau m} \left( \mathcal{P}_\tau(C_0) + \Upsilon(x; d(x) \mp \delta) + \Xi(x; \mp\delta) \right) \quad (42)$$

due to  $C_0$  is the positive root of polynomial

$$\mathcal{P}_\tau(\mu) = \begin{cases} \mu^m - \alpha_0^{p-1}(\alpha_0 + 1)(p - 1)\mu^{p-1} - f_0 & \text{if } \tau = 0, \\ \mu^m - f_0 & \text{if } \tau > 0, \end{cases}$$

[see (28)].

With all previous results, we get to the proof of our main result.

*Proof of Theorem 2.* In order to apply a comparison argument, we consider the modifications

$$W_{\mp\delta, \pm\varepsilon}(x) = C_0(d(x) \mp \delta)^{-\alpha_\tau} \left( 1 \pm \varepsilon + \sum_{n=1}^{M_\tau} C_n(x)(d(x) \mp \delta)^n \right),$$

where  $\varepsilon > 0$  will be sent to 0. So, we construct the perturbed polynomials

$$\mathcal{P}_{\tau, \pm\varepsilon}(\mu) = \begin{cases} ((1 \pm \varepsilon)\mu)^m - \alpha_0^{p-1}(\alpha_0 + 1)(p - 1)((1 \pm \varepsilon)\mu)^{p-1} - f_0 & \text{if } \tau = 0, \\ ((1 \pm \varepsilon)\mu)^m - f_0 & \text{if } \tau > 0, \end{cases}$$

for which

$$\mathcal{P}_{\tau, +\varepsilon}(C_0) > 0 \quad \text{and} \quad \mathcal{P}_{\tau, -\varepsilon}(C_0) < 0.$$

The reasoning is based on to prove that the functions  $W_{-\delta, +\varepsilon}(x)$  and  $W_{+\delta, -\varepsilon}(x)$  are, respectively, super and sub-solutions in a thin strip near the boundary. Arguing as in Theorem 1, we have

$$-\Delta_p W_{-\delta, +\varepsilon}(x) + (W_{-\delta, +\varepsilon}(x))^m - f(x) = (d(x) - \delta)^{-\alpha_\tau m} \left( \mathcal{P}_{\tau, +\varepsilon}(C_0) + \Upsilon(x; d(x) - \delta) + \Xi(x; -\delta) \right)$$

[see (42)]. We recall that  $\mathcal{P}_{\tau, +\varepsilon}(C_0)$  is a positive constant independent on  $x$  and  $\delta$ , and consequently, (41) proves the inequality

$$\mathcal{P}_{\tau, +\varepsilon}(C_0) + \Upsilon(x; d(x) - \delta) + \Xi(x; -\delta) > 0$$

in a parallel strip  $\delta < d(x) < \delta_1$ , provided  $2\delta_1 < \delta_0$  small enough. Therefore, the inequality

$$-\Delta_p W_{-\delta, +\varepsilon}(x) + (W_{-\delta, +\varepsilon}(x))^m > f(x), \quad \delta < d(x) < \delta_1,$$

holds. Then, Comparison Principle leads to

$$u(x) - W_{-\delta, +\varepsilon}(x) \leq \sup_{d(y)=\delta_1} (u(y) - W_{-\delta, +\varepsilon}(y)), \quad \delta < d(x) < \delta_1$$

or

$$\frac{u(x)}{W_{-\delta, +\varepsilon}(x)} - 1 \leq \frac{\sup_{d(y)=\delta_1} (u(y) - W_{-\delta, +\varepsilon}(y))}{W_{-\delta, +\varepsilon}(x)}, \quad \delta < d(x) < \delta_1.$$

Now, in short, sending  $\delta_1 \rightarrow 0$  and then  $\varepsilon \rightarrow 0$ , we deduce

$$\limsup_{d(x) \rightarrow 0} \frac{u(x)}{V(x)} \leq 1,$$

where  $V$  is our master function given by (8). (In fact, for a more precise way to obtain this inequality, we send  $\delta \rightarrow 0$ , after  $d(x) \rightarrow 0$ , next  $\delta_1 \rightarrow 0$ , and finally  $\varepsilon \rightarrow 0$ .)

Analogously, one obtains

$$-\Delta_p W_{+\delta, -\varepsilon}(x) + (W_{+\delta, -\varepsilon}(x))^m - f(x) = (d(x) + \delta)^{-\alpha_\tau m} \left( \mathcal{P}_{\tau, -\varepsilon}(C_0) + \Upsilon(x; d(x) + \delta) + \Xi(x; +\delta) \right).$$

Since

$$\mathcal{P}_{\tau,-\varepsilon}(C_0) + \Upsilon(x; d(x) + \delta) + \Xi(x; +\delta) < 0$$

in a parallel strip  $0 < d(x) < \delta_1$ , provided  $2\delta_1 < \delta_0$  small enough, inequality

$$-\Delta_p W_{+\delta,-\varepsilon}(x) + (W_{+\delta,-\varepsilon}(x))^m < f(x), \quad 0 < d(x) < \delta_1,$$

holds. Now, by comparing, it follows

$$1 - \frac{u(x)}{W_{+\delta,-\varepsilon}(x)} \leq \frac{\sup_{d(y)=\delta_1} (W_{+\delta,-\varepsilon}(y) - u(y))}{W_{+\delta,-\varepsilon}(x)}, \quad 0 < d(x) < \delta_1.$$

As above, sending  $\delta \rightarrow 0$  and then  $\varepsilon \rightarrow 0$ , we conclude

$$\limsup_{d(x) \rightarrow 0} \frac{u(x)}{V(x)} \leq 1 \leq \liminf_{d(x) \rightarrow 0} \frac{u(x)}{V(x)}. \quad \square$$

**Remark 8.** Certainly, Theorem 2 extends and generalizes Theorem 3.8 of [11]. When  $p = 2$ , Theorem 2 coincides with Theorem 2 of [1] and it extends the results obtained in [5,6] or [8] where only the second explosive term was considered for  $f \equiv 0$ . □

Theorem 2 can be illustrated as follows:

*Example 1. (Low explosive sources)* This is an example without nondiffused part in the expansion of the large solutions. For instance, let us suppose

$$\frac{3p - 2}{2} \leq m < 2p - 1 \tag{43}$$

(or equivalently  $1 < \alpha_0 \leq 2$ ), for which  $M_0 = 1$  and

$$f(x) = f_1(d(x))^{-\frac{pm}{m-p+1}+1}, \quad f_1 \geq 0.$$

If  $\partial\Omega \in \mathcal{C}^4$ , then we obtain

$$u(x) = C_0(d(x))^{-\frac{p}{m-p+1}} \left( 1 + \eta(m,p)[\gamma(m,p)f_1 - \Delta d(x)]d(x) \right) + o\left( (d(x))^{\frac{m-2p+1}{m-p+1}} \right), \tag{44}$$

where  $C_0$ ,  $\eta(m,p)$  and  $\gamma(m,p)$  are given in Remark 5. This example extends the results of [5,6] or [8] obtained for  $p = 2$  and  $f \equiv 0$ . □

*Example 2. (High explosive sources)*

- (i) In order to simplify, we begin by constructing an example without geometrical part in the expansion. For instance, an inequality as  $\tau \geq M_\tau = 1$  requires

$$\begin{cases} M_\tau = 1 & \text{if and only if } \frac{3p + \tau - 2}{2} \leq m < 2p + \tau - 1 & \text{(see Lemma 1)} \\ \tau \geq M_\tau & \text{if and only if } \frac{p(2 + \tau) - 1}{\tau + 1} < m & \text{[see (35)].} \end{cases}$$

Since

$$\frac{(2 + \tau) - 1}{\tau + 1} \leq \frac{3p + \tau - 2}{2} \quad \text{for } \tau \geq 1,$$

both conditions hold when

$$\frac{3p + \tau - 1}{2} < m < 2p + \tau - 1,$$

for which

$$f(x) = (d(x))^{-\alpha_\tau m} (f_0 + f_1 d(x)), \quad f_0 > 0$$

where

$$\alpha_\tau = \frac{p + \tau}{m - p + 1}$$

verifies

$$1 < \alpha_\tau < \frac{2(p + \tau)}{p + \tau + 1}.$$

Theorem 2 proves that the expansion of all explosive terms of the large solution is

$$u(x) = f_0^{\frac{1}{m}} (d(x))^{-\alpha_\tau} \left( 1 + \frac{1}{mf_0} \left( f_1 + \frac{(p + \tau)^{p-1} (m + \tau + 1)(p - 1)}{(m - p + 1)^p} f_0^{\frac{p-1}{m}} \right) d(x) \right) + o\left( (d(x))^{-\alpha_\tau + 1} \right),$$

provided  $\partial\Omega \in C^4$  [see (5), (32), (34) and (35)]. Clearly, both coefficients are independent on the geometry of  $\Omega$ . Here,  $\tau$  is an arbitrary positive integer number.

- (ii) Finally, we construct an example where the expansion has one coefficient dependent on  $\Omega$  plus two coefficients uniform and independent on  $\Omega$ ; therefore,  $\tau = 1$  and  $M_1 + 1 = 3$ . So, Lemma 1 enables us to consider

$$\frac{4p - 2}{3} \leq m < \frac{3p - 1}{2} \tag{45}$$

(or equivalently  $2 < \alpha_1 \leq 3$ ), and for simplicity, we suppose

$$f(x) = f_0 (d(x))^{-\frac{(p+1)m}{m-p+1}} \left( 1 + f_1 d(x) + f_2 (d(x))^2 \right), \quad f_0 > 0.$$

Then, the expansion of all explosive terms of the large solution is

$$u(x) = C_0 (d(x))^{-\frac{p+1}{m-p+1}} \left( 1 + C_1 d(x) + C_2(x) (d(x))^2 \right) + o\left( (d(x))^{\frac{2m-3p+1}{m-p+1}} \right), \tag{46}$$

for coefficients

$$C_0 = f_0^{\frac{1}{m}} \tag{independent on the nonlinear diffusion}$$

$$C_1 = \frac{1}{mf_0} \left( f_1 + \alpha_1^{p-1} (\alpha_1 + 1)(p - 1) f_0^{\frac{p-1}{m}} \right) \tag{dependent on the nonlinear diffusion}$$

and

$$C_2(x) = \frac{1}{mf_0} \left( f_2 - f_0 \frac{m(m-1)}{2} C_1^2 + f_0^{\frac{p-1}{m}} \alpha_1^{p-2} \left( (\alpha_1 - 1)(p-1)(\alpha_1(p-1) + p - 2) C_1 - \alpha_1 \Delta d(x) \right) \right),$$

where  $\alpha_1 = \frac{p+1}{m-p+1}$  provided  $\partial\Omega \in C^6$  (see Remarks 1 and 5).

One last comment derived from conditions (43) and (45). Since the inclusion

$$\left[ \frac{4p - 2}{3}, \frac{3p - 1}{2} \right] \subseteq \left[ \frac{3p - 2}{2}, 2p - 1 \right],$$

holds whenever  $p \geq 2$ , we note that for every

$$m \in \left[ \frac{4p - 2}{3}, \frac{3p - 1}{2} \right], \quad p \geq 2,$$

and  $\partial\Omega \in C^4$ , the first geometrical property appears in the second coefficient of the expansion for the low explosion source

$$f(x) = f_1 (d(x))^{-\frac{pm}{m-p+1} + 1}, \quad f_1 \geq 0$$

[see (44)]. However, if  $\partial\Omega \in C^6$ , and we change to the high explosion source

$$f(x) = f_0 (d(x))^{-\frac{(p+1)m}{m-p+1}} \left( 1 + f_1 d(x) + f_2 (d(x))^2 \right), \quad f_0 > 0,$$



that first geometrical property appears now in the third coefficient [see (46)]. The importance of the kind of sources was commented in Remark 7. □

### Appendix: Expanding the $m$ th power of the asymptotic profile

In Appendix A of [1], the old formula of Federico Villarreal (1850–1923) on the power of polynomials was extended by means of an explicit expression. It was applied in order to obtain representations of the power of polynomials. Here, we sketch the results of Appendix B of [1] related to the formal expansion

$$V(x) = C_0(d(x))^{-\alpha_\tau} \left( 1 + \sum_{n=1}^{M_\tau} C_n(x)(d(x))^n \right)$$

for which

$$(V(x))^m = C_0^m(d(x))^{-\alpha_\tau m} \Phi \left( \sum_{n=1}^{M_\tau} C_n(x)(d(x))^{n-1} \right),$$

where

$$\Phi(s) = (1 + s d(x))^m.$$

Applying Taylor expansion of  $\Phi(s)$ , one obtains

$$(V(x))^m = C_0^m(d(x))^{-\alpha_\tau m} \sum_{n \geq 0} \binom{m}{n} \left( \sum_{k=1}^{M_\tau} C_k(x)(d(x))^{k-1} \right)^n (d(x))^n.$$

On the other hand, we may write

$$\left( \sum_{k=1}^{M_\tau} C_k(x)(d(x))^{k-1} \right)^n = \left( \sum_{k=0}^{M_\tau-1} C_{k+1}(x)(d(x))^k \right)^n = \sum_{i=0}^{(M_\tau-1)n} B_{i,n}(x)(d(x))^i \tag{47}$$

where

$$B_{i,n}(x) = \begin{cases} (C_1(x))^n, & i = 0, \\ \frac{1}{i C_1(x)} \sum_{\ell=0}^{i-1} ((i-\ell)(n+1) - i) C_{i-\ell+1}(x) B_{\ell,n}(x), & 1 \leq i \leq M_\tau - 1, \\ \frac{1}{i C_1(x)} \sum_{\ell=i-M_\tau+1}^{i-1} ((i-\ell)(n+1) - i) C_{i-\ell+1}(x) B_{\ell,n}(x), & M_\tau \leq i \leq (M_\tau - 1)n \end{cases}$$

(for details see [1, Appendix A]).

In general, by means of a transfinite induction argument, we may adjust explicit Villarreal formula (see now [1, Theorem 4]) in order to obtain the explicit expression of  $B_{i,n}(x)$  for  $i \in \{1, 2, \dots, n\}$  [see also (18)]. Then, one has

$$\begin{aligned} (V(x))^m &= C_0^m(d(x))^{-\alpha_\tau m} \sum_{n \geq 0} \binom{m}{n} \sum_{i=0}^{(M_\tau-1)n} B_{i,n}(x)(d(x))^{i+n} \\ &= C_0^m(d(x))^{-\alpha_\tau m} \left( 1 + \sum_{n=1}^{\infty} D_n(x)(d(x))^n \right) \end{aligned} \tag{48}$$

where

$$D_n(x) = \sum_{i=1}^n \binom{m}{i} B_{n-i,i}(x), \quad \text{for all } n. \tag{49}$$

Choosing  $n = 1$  in (47), we deduce

$$B_{i,1}(x) = C_{i+1}(x), \quad 0 \leq i \leq M_{\tau-1},$$

so that, (49) becomes

$$D_n(x) = \binom{m}{1} C_n(x) + \sum_{i=2}^n \binom{m}{i} B_{n-i,i}(x), \quad 1 \leq n \leq M_{\tau}, \quad (50)$$

whence, in (50), each  $C_n(x)$ ,  $1 \leq n \leq M_{\tau}$ , does not appear in  $B_{n-i,i}(x)$ ,  $i \neq 1$ . Certainly, all coefficients  $C_n(x)$ ,  $1 \leq n \leq M_{\tau}$ , are involved in the other  $D_n(x)$ ,  $n \geq M_{\tau} + 1$ .

Clearly, the Taylor expansion is finite when  $m$  is an integer number. In this case, representation (48) becomes

$$(V(x))^m = C_0^m(d(x))^{-\alpha_{\tau}m} \left( 1 + \sum_{n=1}^{m(M_{\tau}-1)} D_n(x)(d(x))^n \right) \quad (51)$$

where coefficients  $D_n(x)$  are given in (49).

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