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Liouville Type Theorems for Elliptic Equations with Gradient Terms

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Abstract. In this paper we obtain Liouville type theorems for nonnegative supersolutions of the elliptic problem $-\Delta u + b(x)|\nabla u| = c(x)u$ in exterior domains of $\mathbb{R}^N$. We show that if $\liminf_{x\to\infty} 4c(x) - b(x)^2 > 0$ then no positive supersolutions can exist, provided the coefficients b and c verify a further restriction related to the fundamental solutions of the homogeneous problem. The weights b and c are allowed to be unbounded. As an application, we also consider supersolutions to the problems $-\Delta u + b|x|^\lambda|\nabla u| = c|x|^\mu u^p$ and $-\Delta u + be^{\lambda|x|}|\nabla u| = ce^{\mu|x|}u^p$, where $p > 0$ and $\lambda, \mu \geq 0$, and obtain nonexistence results which are shown to be optimal.

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1. Introduction

It is well-known that nonexistence results for positive solutions of some nonlinear elliptic equations in $\mathbb{R}^N$ are very important in the study of nonlinear partial differential equations. As outstanding applications, they are used for instance to obtain a priori bounds for positive solutions (cf. [16]) or in the analysis of isolated singularities of such solutions, [22]. Most results of this type are related to the model equation

$$-\Delta u = u^p \qquad \text{in } \mathbb{R}^N \tag{1.1}$$

(cf. [15]) and its generalizations. Let us mention the works [2], [4], [5], [6], [9], [10], [11], [13], [14], [18], [19], [21] and [26], where the question of nonexistence of positive solutions has been considered for various differential operators other than the Laplacian and more general nonlinearities than the power. We also refer to the book [27] and the survey [20].

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A very general nonexistence result for a related problem has been recently obtained in [3]. It is shown there, among other things, that the differential inequality

$$-Qu \geq f(u) \qquad \text{in } \mathbb{R}^N \setminus B_{R_0},$$

where $N \geq 3$ and Q is a fully nonlinear operator does not admit positive viscosity solutions provided that f is continuous and positive in $(0, \infty)$ and $\liminf_{s \to 0+} f(s)/s^{\frac{N}{N-2}} > 0$. When transferred to the model problem (1.1), this means that there do not exist positive supersolutions to the equation $-\Delta u = u^p$ in the range $0 < p \leq \frac{N}{N-2}$, not only when the equation is set in all of $\mathbb{R}^N$, but also when it is considered in exterior domains.

The analysis in [3] relies on a Hadamard type property for solutions of the inequality $-Qu \geq 0$, which strongly depends on the homogeneity of the operator Q, that is, $Q(\lambda u) = \lambda Q(u)$ for $\lambda \in \mathbb{R}^+$ (cf. also [13] for this property in the context of fully nonlinear operators). Hence the presence of a nonhomogeneous term in the differential operator, depending for instance on the gradient, might change in principle this regime of nonexistence of supersolutions. This has actually been shown in [1], where the problem

$$-\Delta u + |\nabla u|^q \geq f(u) \qquad \text{in } \mathbb{R}^N \setminus B_{R_0} \tag{1.2}$$

with $q > 1$ was studied. The reference situation there was $f(u) = u^p$, $p > 0$.

It is natural then to ask what happens with problem (1.2) when $q = 1$, so that the operator is homogeneous again but a gradient term is present. As a model equation consider

$$-\Delta u + |\nabla u| \geq \lambda u^p \quad \text{in } \mathbb{R}^N \setminus B_{R_0}, \tag{1.3}$$

where $N \geq 1$, $p > 0$ and $\lambda > 0$. Related problems have been studied in [11] and [12]. However, the results obtained there cannot be applied to problem (1.3) when $p \neq 1$.

Our approach to (1.3) relies in the critical case $p = 1$. Some previous work has been done for instance in [7] and [8], where it has been shown that if b and c are continuos function in $\mathbb{R}^N$, the problem

$$-\Delta u + b(x) \cdot \nabla u \geq c(x)u \quad \text{in } \mathbb{R}^N$$

does not admit any positive solution provided that b and c are bounded and

$$\liminf_{x \to \infty} 4c(x) - |b(x)|^2 > 0. \tag{1.4}$$

It is worth mentioning that these nonexistence results are a consequence of the study of eigenvalue problems in $\mathbb{R}^N$.

A generalization of these theorems to the framework of fully nonlinear elliptic operators has been recently obtained in [24]. As a particular case, it follows that if b, c are *bounded* in $\mathbb{R}^N \setminus B_{R_0}$ and (1.4) holds, then the problem

$$-\Delta u + b(x)|\nabla u| \geq c(x)u \quad \text{in } \mathbb{R}^N \setminus B_{R_0} \tag{1.5}$$

does not admit positive solutions.

Thus our intention in the present work is to prove the nonexistence of positive supersolutions of (1.5) for more general unbounded weights b and c. As a byproduct, we will also be able to completely analyze the issue of existence/nonexistence of supersolutions of (1.3), thereby completing the analysis in [1].

One remarkable property of problem (1.5) is that the presence of the gradient term allows in principle the existence of supersolutions which blow up at infinity, that is,

$$\lim_{x\to\infty} u(x) = +\infty.$$

Therefore to be more concrete in our analysis of nonexistence we will distinguish between supersolutions which blow up at infinity and those which do not.

Let us comment that the proof of our nonexistence results follows earlier works (cf. for instance [13]), since it depends on properties of the function $m(R) = \inf_{|x|=R} u(x)$. However, the corresponding Hadamard type properties for supersolutions are not enough to complete the analysis since the fundamental solutions of the homogeneous equation are typically of exponential growth at infinity, and a slightly different approach must be used. Let us mention that this approach is completely different from that in [24].

We come now to the statement of our main results. Throughout the paper, we are always dealing with classical supersolutions, that is, functions $u \in C^2(\mathbb{R}^N \setminus B_{R_0})$ verifying the corresponding inequality pointwise in $\mathbb{R}^N \setminus B_{R_0}$.

Observe that, by replacing $b(x)$ with $\tilde{b}(x) = \sup_{|y|=|x|} b(y)$, we can always assume that b is radially symmetric, since a function u verifying (1.5) also verifies $-\Delta u + \tilde{b}(x)|\nabla u| \geq c(x)u$. Also, set $h(r) = b(r) + 2/r$ when $N = 1, 2$ and $h(r) = b(r)$ for $N \geq 3$. For some $R_1 > R_0$ define

$$\Phi(R) = \int_R^\infty \frac{1}{s^{N-1}} e^{-\int_{R_1}^s h(\tau)d\tau} ds$$

and

$$\widetilde{\Phi}(R) = \int_1^R \frac{1}{s^{N-1}} e^{\int_{R_1}^s h(\tau)d\tau} ds.$$

These functions are solutions of the equation $-\Delta v + h(|x|)|\nabla v| = 0$ in $\mathbb{R}^N \setminus B_{R_0}$. They will be determinant in obtaining our results. We remark that we replace b by h when $N = 1$ or 2 in order to have the fundamental solution Φ to be well-defined in those cases.

Theorem 1.1. *Let $b, c \in C(\mathbb{R}^N \setminus B_{R_0})$ verify* (1.4)*. Assume the following condition is satisfied: there exists $\theta > 1$ and a sequence $R_n \to \infty$ such that for every sequence x_n with $|x_n| = \theta R_n$:*

$$\lim_{n\to\infty} \frac{1}{R_n^2 \inf_{B_{R_n}(x_n)}(4c - b^2)} \log\left(\frac{\Phi(\theta R_n)}{\Phi((\theta+1)R_n)}\right) < \frac{1}{8N}.$$

Then there are no classical positive supersolutions to (1.5) *which do not blow up at infinity.*

If, similarly, there exists $\theta' > 1$ and a sequence $R'_n \to \infty$ such that for every sequence x'_n with $|x'_n| = \theta' R'_n$ we have:

$$\lim_{n\to\infty} \frac{1}{(R'_n)^2 \inf_{B_{R'_n}(x'_n)}(4c-b^2)} \log\left(\frac{\widetilde{\Phi}(\theta' R'_n)}{\widetilde{\Phi}((\theta'-1)R'_n)}\right) < \frac{1}{8N},$$

then there are no classical positive supersolutions to (1.5) *which blow up at infinity.*

We next consider some particular examples where weights of potential or exponential type are involved.

Corollary 1.2. *Let $b > 0$, $c \in \mathbb{R}$ and $\lambda, \mu > 0$. When $\mu > 2\lambda$ and $c > 0$ or $\mu = 2\lambda$ and $4c - b^2 > 0$, the problem*

$$-\Delta u + b|x|^\lambda |\nabla u| \geq c|x|^\mu u \qquad \text{in } \mathbb{R}^N \setminus B_{R_0}$$

does not admit any positive supersolution. In the remaining cases, a positive supersolution can always be constructed for suitably large R_0.

Corollary 1.3. *Let $b > 0$, $c \in \mathbb{R}$ and $\lambda, \mu > 0$. When $\mu > 2\lambda$ and $c > 0$ or $\mu = 2\lambda$ and $4c - b^2 > 0$, the problem*

$$-\Delta u + b e^{\lambda|x|} |\nabla u| \geq c e^{\mu|x|} u \qquad \text{in } \mathbb{R}^N \setminus B_{R_0}$$

does not admit any positive supersolution. In the remaining cases, a positive supersolution can be always constructed for suitable large R_0.

Let us mention that the results in [12] can also be applied to deal with both problems, but they turn out not to be optimal, since the condition for nonexistence there reads as $c - 3eb^2 > 0$ when $\mu = 2\lambda$, which is not optimal.

These results on the homogeneous case will be used to deal with two slight generalizations of problem (1.3), namely

$$-\Delta u + b|x|^\lambda |\nabla u| \geq c|x|^\mu u^p \tag{1.6}$$

and

$$-\Delta u + b e^{\lambda|x|} |\nabla u| \geq c e^{\mu|x|} u^p \tag{1.7}$$

where $b > 0$, $c \in \mathbb{R}$ and $\lambda, \mu \geq 0$. We refer to Section 5 for statements and related proofs.

As a final comment, we would like to say that most of our proofs can be adapted to obtain similar results for some more general operators of fully nonlinear type, that is, inequalities of the form $-Qu + b(x)|\nabla u| \geq c(x)u$ in $\mathbb{R}^N \setminus B_{R_0}$, where Q is uniformly elliptic and rotationally invariant. For instance, the cases where Q is the $p-$Laplacian Δ_p or a Pucci maximal operator $\mathcal{M}^\pm_{\lambda,\Lambda}$ can be considered.

The rest of the paper is organized as follows: in Section 2 we gather some preliminary properties of the function $m(R) = \inf_{|x|=R} u(x)$. Section 3 deals with the Hadamard type property regarding the function $m(R)$, while Section 4 is dedicated to the proof of Theorem 1.1 and Corollaries 1.2 and 1.3. In the final Section 5, problems (1.6) and (1.7) are considered.

2. Some preliminaries

In this section we collect some preliminary properties which will be needed in proving Theorem 1.1 in Section 4. If u is a solution to (1.5), we first consider some properties of the continuous function

$$m(R) = \min_{|x|=R} u(x), \tag{2.1}$$

which is defined for $R > R_0$. By means of the strong maximum principle in [23] (cf. equation (1.1.9) in Chapter 1 and Chapter 5), we have that $m(R)$ is either identically zero (in which case $u \equiv 0$) or strictly positive. Hence we may assume throughout that m is strictly positive.

We first establish a slight variation of Lemma 5 in [1]. Although it is stated there for functions verifying $-\Delta u + |\nabla u|^q \geq 0$ and $q > 1$, it is easily seen that its proof carries over to the case $q = 1$ and with continuous weights as well. We do not include the full proof but we just sketch it.

Lemma 2.1. *Let $u \in C^2(\mathbb{R}^N \setminus B_{R_0})$ be a positive function verifying $-\Delta u + b(x)|\nabla u| \geq 0$ in $\mathbb{R}^N \setminus B_{R_0}$, where b is continuous and nonnegative. Then there exists $R_1 > R_0$ such that $m(R)$ is monotone for $R > R_1$.*

Sketch of the proof. For $R_2 > R_1$, and applying the comparison principle (which holds by Theorem 3.5.1 in [23]) in the annulus $A(R_1, R_2) = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$, we obtain that $\inf_{A(R_1,R_2)} u = \min\{m(R_1), m(R_2)\}$. Thus the function $\min\{m(R_1), m(R_2)\}$ is increasing in R_1 and decreasing in R_2, so it cannot have a local minimum. The conclusion follows. □

The next result is a variant of Lemma 6 in [1]. Its proof is based on a tool introduced in [13] and refined in [14].

Lemma 2.2. *Let $u \in C^2(\mathbb{R}^N \setminus B_{R_0})$ be a positive solution of* (1.5) *in $\mathbb{R}^N \setminus B_{R_0}$ and $m(R)$ be given by* (2.1). *Assuming*

$$\lim_{x\to\infty} \frac{b(x)}{|x|c(x)} = 0, \qquad \lim_{x\to\infty} \frac{1}{|x|^2 c(x)} = 0, \tag{2.2}$$

then either

(a) *$m(R)$ is decreasing for large R and it converges to zero as $R \to \infty$ or*
(b) *$m(R)$ is increasing for large R and it diverges to infinity as $R \to \infty$.*

Remark 2.3. Observe that the condition $\liminf_{x\to\infty} 4c(x) - b(x)^2 > 0$ implies (2.2). Indeed, if $4c(x) - b(x)^2 \geq 4\eta^2$ for large $|x|$ then $b(x) \leq 2\sqrt{c(x)}$, $c(x) \geq \eta^2$, so that

$$\frac{b(x)}{|x|c(x)} \leq \frac{2}{|x|\sqrt{c(x)}} \leq \frac{2}{\eta|x|} \qquad\qquad \frac{1}{|x|^2 c(x)} \leq \frac{1}{\eta^2|x|^2},$$

hence both limits are zero.

Proof of Lemma 2.2. Notice first that, when m is unbounded, it follows from Lemma 2.1 that m is increasing for large R and then $\lim_{R\to+\infty} m(R) = +\infty$. Thus we may assume for the rest of the proof that m is bounded.

Choose a cut-off function $\phi \in C_0^\infty(\mathbb{R})$ such that $\phi = 0$ in $(0,1) \cup (4,\infty)$ and $\phi = 1$ in $[2,3]$. For large R consider the function

$$v(x) = u(x) - m(2R)\phi\left(\frac{|x|}{R}\right).$$

Notice first that $v > 0$ in $B_R \setminus R_0$ and in $\mathbb{R}^N \setminus B_{4R}$. Since there obviously exists a point x_R with $|x_R| = 2R$ and $u(x_R) = m(2R)$ (so that $v(x_R) = 0$) we conclude that v achieves a nonpositive minimum at some point y_R with $R \leq |y_R| \leq 4R$. Thus $\nabla v(y_R) = 0$, $\Delta v(y_R) \geq 0$, so that

$$\begin{aligned} c(y_R)u(y_R) &\leq -\Delta u(y_R) + b(y_R)|\nabla u(y_R)| \\ &\leq -\frac{m(2R)}{R^2}\Delta\phi\left(\frac{|y_R|}{R}\right) + \frac{m(2R)}{R}b(y_R)\left|\nabla\phi\left(\frac{|y_R|}{R}\right)\right| \\ &\leq Cm(2R)\left(\frac{1}{R^2} + \frac{b(y_R)}{R}\right) \end{aligned}$$

for some positive constant C which only depends on ϕ. Since $u(y_R) \geq \min_{R\leq|x|\leq 4R} u(x) = \min\{m(R), m(4R)\}$, it follows that

$$\min\{m(R), m(4R)\} \leq Cm(2R)\left(\frac{1}{R^2 c(y_R)} + \frac{b(y_R)}{Rc(y_R)}\right). \tag{2.3}$$

According to (2.2) and since m is bounded, the right-hand side of this inequality tends to zero as $R \to \infty$. Hence $m(R) \to 0$ as $R \to +\infty$ and we also obtain from Lemma 2.1 that $m(R)$ is decreasing for large R. This concludes the proof. □

Remark 2.4. It is worth noticing that conditions (2.2) are important in order that Lemma 2.2 holds. To ilustrate this, just consider the function $u(x) = \tanh|x|$, for which $m(R)$ is increasing and bounded, yet it does not converge to zero. It verifies

$$-\Delta u + b(|x|)|\nabla u| = c(|x|)u,$$

where $c(r)$ is arbitrary and $b(r) = c(r)\sinh r\cosh r + 2\tanh r + \frac{N-1}{r}$. It is easily checked that, even if c verifies the second condition in (2.2), the first one does not occur.

3. The Hadamard property

Next let us deal with the so-called Hadamard property for supersolutions of $-\Delta u + b(x)|\nabla u| = 0$ in exterior domains. The key point is the existence of "fundamental solutions" for a majorant equation. We will see that there actually are two positive fundamental solutions. Let $h(r) = \frac{2}{r} + b(r)$ for $N = 1, 2$, $h(r) = b(r)$ if $N \geq 3$ and define for large fixed R_1:

$$\Phi(R) = \int_R^\infty \frac{1}{s^{N-1}} e^{-\int_{R_1}^s h(\tau)d\tau} ds \tag{3.1}$$

and

$$\widetilde{\Phi}(R) = \int_1^R \frac{1}{s^{N-1}} e^{\int_{R_1}^s h(\tau)d\tau} ds \tag{3.2}$$

(cf. [11] for related fundamental solutions). The first of these functions is clearly well defined for $N \geq 3$, while for $N = 1, 2$ we have

$$\frac{1}{s^{N-1}} e^{-\int_{R_1}^s h(\tau)d\tau} \leq \frac{1}{s^{N-1}} e^{-2\log(s/R_1)} \leq \frac{R_1^2}{s^{N+1}}$$

since $h(r) \geq 2/r$, and the integral is finite. It is then not hard to check that both functions are solutions of the equation $-\Delta v + h(|x|)|\nabla v| = 0$ in $\mathbb{R}^N \setminus \{0\}$. By means of comparison with suitable modifications of these solutions we achieve the next important result.

Theorem 3.1. *Let $u \in C^2(\mathbb{R}^N \setminus B_{R_0})$ verify $-\Delta u + b(x)|\nabla u| \geq 0$ in $\mathbb{R}^N \setminus B_{R_0}$. If u does not blow up at infinity, then for large $R_1 > R_0$ the function*

$$\frac{m(R) - m(R_1)}{\Phi(R) - \Phi(R_1)}$$

is nondecreasing for $R > R_1$. When u blows up at infinity, the function

$$\frac{m(R) - m(R_1)}{\widetilde{\Phi}(R) - \widetilde{\Phi}(R_1)}$$

is nonincreasing for $R > R_1$.

Proof. We only prove the first part, the second part is similar. Observe that, when u does not blow up at infinity, $m(R)$ is decreasing for large R by Lemma 2.2. If we define, for $R_2 > R_1 \gg R_0$, the function

$$\Psi(x) = \frac{m(R_2) - m(R_1)}{\Phi(R_2) - \Phi(R_1)} (\Phi(|x|) - \Phi(R_1)) + m(R_1),$$

it easily follows that the coefficient multiplying $\Phi(|x|)$ is positive, hence Ψ verifies $-\Delta\Psi + h(|x|)|\nabla\Psi| = 0$ in the annulus $A(R_1, R_2) = \{x \in \mathbb{R}^N : \ R_1 < |x| < R_2\}$.

On the other hand, the function u clearly verifies $-\Delta u + h(|x|)|\nabla u| \geq 0$ in $A(R_1, R_2)$, together with $u \geq \Psi$ on $\partial A(R_1, R_2)$. By comparison we obtain that $u \geq \Psi$ in $A(R_1, R_2)$. Choosing x with $|x| = R \in (R_1, R_2)$ yields

$$\frac{m(R) - m(R_1)}{\Phi(R) - \Phi(R_1)} \leq \frac{m(R_2) - m(R_1)}{\Phi(R_2) - \Phi(R_1)} \tag{3.3}$$

(observe that $\Phi(R) - \Phi(R_1) < 0$) as we wanted to show. □

As a corollary of this property we obtain the monotonicity of the function $m(R)/\Phi(R)$, and a similar property for $m(R)/\widetilde{\Phi}(R)$.

Corollary 3.2. *Let $u \in C^2(\mathbb{R}^N \setminus B_{R_0})$ verify $-\Delta u + b(x)|\nabla u| \geq 0$ and assume that u does not blow up at infinity. Then the function $m(R)/\Phi(R)$ is nonincreasing. In particular, for every $\gamma \in (0, 1)$ we have*

$$\frac{m(\gamma R)}{m(R)} \leq \frac{\Phi(\gamma R)}{\Phi(R)} \tag{3.4}$$

for $R > R_0$.

Proof. Take $R > R_1 > R_0$. According to Lemma 2.2, $m(R) \to 0$ as $R \to \infty$, so we can let $R_2 \to \infty$ in (3.3) of Theorem 3.1 to obtain:

$$\frac{m(R) - m(R_1)}{\Phi(R) - \Phi(R_1)} \leq \frac{m(R_1)}{\Phi(R_1)}.$$

This can be rewritten as

$$\frac{m(R_1)}{\Phi(R_1)} \leq \frac{m(R)}{\Phi(R)},$$

so that the function m/Φ is nondecreasing. □

Corollary 3.3. *Let* $u \in C^2(\mathbb{R}^N \setminus B_{R_0})$ *verify* $-\Delta u + b(x)|\nabla u| \geq 0$ *and assume that* u *blows up at infinity. Then for every* $\gamma > 1$ *there exists a constant* $C > 0$ *such that*

$$\frac{m(\gamma R)}{m(R)} \leq C \frac{\widetilde{\Phi}(\gamma R)}{\widetilde{\Phi}(R)} \tag{3.5}$$

for $R > R_0$.

Proof. By Theorem 3.1 we have for large enough R_1:

$$\frac{m(\gamma R) - m(R_1)}{\widetilde{\Phi}(\gamma R) - \widetilde{\Phi}(R_1)} \leq \frac{m(R) - m(R_1)}{\widetilde{\Phi}(R) - \widetilde{\Phi}(R_1)}.$$

On the other hand, since $m(R) \to \infty$ we have $m(\gamma R) - m(R_1) \geq \frac{1}{2} m(\gamma R)$ for large enough R, and also $\widetilde{\Phi}(R) - \widetilde{\Phi}(R_1) \geq \frac{1}{2}\widetilde{\Phi}(R)$. Thus

$$\frac{m(\gamma R)}{\widetilde{\Phi}(\gamma R)} \leq 4 \frac{m(R)}{\widetilde{\Phi}(R)}$$

if R is sufficiently large. Then (3.5) holds for $R > R_0$ for a suitable constant C (depending on γ). □

4. Proof of the main results

This section is dedicated to the proof of our results. We will only prove the first part of Theorem 1.1, since the arguments to prove the second part are entirely similar (the only significative difference is that Corollary 3.3 replaces Corollary 3.2).

Proof of Theorem 1.1*.* We first observe that the condition $\liminf_{x \to \infty} 4c(x) - b(x)^2 > 0$ implies (2.2) in Lemma 2.2 (cf. Remark 2.3). Thus, according to Lemma 2.2, if u is a supersolution which does not blow up at infinity then $m(R)$ is decreasing for large R and $m(R) \to 0$.

Setting $v = \log u$ we easily see that v verifies $-\Delta v + b(x)|\nabla v| - |\nabla v|^2 \geq c(x)$ in $\mathbb{R}^N \setminus B_{R_0}$. Since $b(x)|\nabla v| - |\nabla v|^2 \leq \frac{b(x)^2}{4}$, we obtain that $-\Delta v \geq c(x) - \frac{b(x)^2}{4}$ in $\mathbb{R}^N \setminus B_{R_0}$.

Let $\theta > 1$ and $R_n \to \infty$ as in our hypothesis. For every n, there exists x_n with $|x_n| = \theta R_n$ and such that $u(x_n) = m(\theta R_n)$. Observe that $B_{R_n}(x_n) \subset \mathbb{R}^N \setminus R_0$ for large n, so we can define

$$\nu(R_n) = \inf_{B_{R_n}(x_n)} c(x) - \frac{b(x)^2}{4} > 0,$$

and then $-\Delta v \geq \nu(R_n)$ in $B_{R_n}(x_n)$. Denoting $M(R) = \log m(R)$, we next consider the function

$$U(x) = -\frac{\nu(R_n)}{2N}(|x - x_n|^2 - R_n^2) + M((\theta + 1)R_n),$$

which verifies $-\Delta U = \nu(R_n)$ in $B_{R_n}(x_n)$, $U = M((\theta+1)R_n)$ on $\partial B_{R_n}(x_n)$. Observe that $\partial B_{R_n}(x_n) \subset B_{(\theta+1)R_n}$, hence using that $M(R)$ is decreasing for large R, we obtain $v \geq U$ on $\partial B_{R_n}(x_n)$.

By comparison we arrive at $v \geq U$ in $B_{R_n}(x_n)$. Taking $x = x_n$:

$$M(\theta R_n) \geq \frac{\nu(R_n)}{2N} R_n^2 + M((\theta + 1)R_n).$$

On the other hand, using Corollary 3.2 with $\gamma = \frac{\theta}{\theta+1}$ and R replaced by $(\theta + 1)R_n$, we arrive at

$$M(\theta R_n) - M((\theta + 1)R_n) \leq \log \left(\frac{\Phi(\theta R_n)}{\Phi((\theta + 1)R_n)} \right).$$

Hence

$$\frac{\nu(R_n)}{2N} R_n^2 \leq \log \left(\frac{\Phi(\theta R_n)}{\Phi((\theta + 1)R_n)} \right)$$

for all sufficiently large n, which is a contradiction to our hypothesis. This concludes the proof. □

Next we will give the proofs of nonexistence for the model problems with potential and exponential weights. Most of them is a direct application of Theorem 1.1.

Proof of Corollary 1.2. In this particular example $b(x) = b|x|^\lambda$, $c(x) = c|x|^\mu$. Thus it is apparent that the condition $\liminf_{x\to\infty} 4c(x) - b(x)^2 > 0$ holds provided that $\mu > 2\lambda$ or $\mu = 2\lambda$ and $4c - b^2 > 0$.

Choose $\theta > 1$ arbitrary. If $R > 0$ is large and x_0 verifies $|x_0| = \theta R$, since $|x| \geq (\theta - 1)R$ in $B_R(x_0)$, we obtain that $\nu(R) \geq CR^\mu$ for large R, where C is a positive constant.

On the other hand, the fundamental solution $\Phi(R)$ is given by

$$\Phi(R) = \int_R^\infty \frac{1}{s^{N+1}} e^{-\frac{b}{\lambda+1}s^{\lambda+1}} ds$$

and it is easily seen by l'Hôpital's rule that $\Phi(R) \sim b^{-1}R^{-\lambda-N-1}e^{-\frac{b}{\lambda+1}R^{\lambda+1}}$ as $R \to \infty$. Therefore

$$\log \left(\frac{\Phi((\theta + 1)R)}{\Phi(\theta R)} \right) \leq C + \frac{b}{\lambda + 1} \left((\theta + 1)^{\lambda+1} - \theta^{\lambda+1} \right) R^{\lambda+1}$$

for large R, where C is another positive constant, not necessarily the same one as before. It follows that

$$\lim_{R\to\infty} \frac{1}{R^2\nu(R)} \log\left(\frac{\Phi((\theta+1)R)}{\Phi(\theta R)}\right) = 0.$$

Hence Theorem 1.1 applies and we conclude the nonexistence of positive supersolutions which do not blow up at infinity.

On the other hand, it is not hard to show that the second fundamental solution $\widetilde{\Phi}(R)$ behaves for large R as $b^{-1}R^{-\lambda-N-1}e^{\frac{b}{\lambda+1}R^{\lambda+1}}$, so that the hypotheses of the second part of Theorem 1.1 hold as before and we conclude that no positive supersolutions which blow up at infinity exist either.

The existence of supersolutions in the remaining cases is easy to establish. Observe first that if $c \le 0$ then positive constants are supersolutions. When $c > 0$ we look for supersolutions of the form $u = e^{-\alpha|x|^{\lambda+1}}$ for $\alpha > 0$ to be chosen. After some algebra we find that it suffices to have

$$-\alpha^2(\lambda+1)^2 + (N-1+\lambda)\alpha|x|^{-\lambda-1} + b\alpha(\lambda+1) \ge c|x|^{\mu-2\lambda}$$

for large $|x|$. When $\mu < 2\lambda$ this is implied by $-\alpha^2(\lambda+1)^2 + b\alpha(\lambda+1) > 0$, which is certainly true if α is small enough since $b > 0$. If $\mu = 2\lambda$ we need

$$-(\alpha(\lambda+1))^2 + b(\alpha(\lambda+1)) - c \ge 0$$

which also holds for some positive α when $4c - b^2 \le 0$. □

Proof of Corollary 1.3. The proof is much the same as that of Corollary 1.2, except that now $b(x) = be^{\lambda|x|}$, $c(x) = ce^{\mu|x|}$. Observe that the condition $\liminf_{x\to\infty} 4c(x) - b(x)^2 > 0$ holds under the same hypotheses on λ, μ, b and c. In particular, $\nu(R) \ge Ce^{\mu(\theta-1)R}$ for large R. Also, by l'Hôpital's rule,

$$\Phi(R) = \int_R^\infty \frac{1}{s^{N+1}} e^{-\frac{b}{\lambda}e^{\lambda s}} ds \sim \frac{1}{R^{N+1}} e^{-\frac{1}{\lambda}e^{\lambda R} - \lambda R}$$

as $R \to \infty$. Hence

$$\frac{1}{R^2\nu(R)} \log\left(\frac{\Phi((\theta+1)R)}{\Phi(\theta R)}\right) \le \frac{Ce^{\lambda(\theta+1)R}}{R^2 e^{\mu(\theta-1)R}} + o(1),$$

as $R \to \infty$. Choosing $\theta > 3$ we find that the limit is zero, so that Theorem 1.1 can be applied and we conclude the nonexistence of positive supersolutions which do not blow up at infinity. Like in Corollary 1.2, it is not hard to show that the previous limit is also zero when $\Phi(R)$ is replaced by $\tilde{\Phi}(R)$.

To prove the part of existence, we observe as before that only the case $c > 0$ needs to be dealt with. We look for a supersolution of the form $u = e^{-\alpha e^{\lambda|x|}}$ for some suitable $\alpha > 0$. It is needed that

$$-\alpha^2\lambda^2 + b\alpha\lambda \ge ce^{(\mu-2\lambda)|x|}$$

which is certainly possible for small α and large $|x|$ when $\mu < 2\lambda$. In the case $\mu = 2\lambda$, provided that $4c - b^2 \le 0$, we can take α as one of the solutions of the equation $-\alpha^2\lambda^2 + b\alpha\lambda - c = 0$. This concludes the proof. □

5. A further nonexistence result

In this final section, we will see how our previous results yield also nonexistence of positive supersolutions for some nonhomogeneous equations. We will consider first a slight generalization of the problem analyzed in Corollary 1.2, namely:

$$-\Delta u + b|x|^{\lambda}|\nabla u| \geq c|x|^{\mu}u^{p} \qquad \text{in } \mathbb{R}^N \setminus B_{R_0} \tag{5.1}$$

where $b > 0$, $c \in \mathbb{R}$, $\lambda, \mu \geq 0$ and $p > 0$. Some related problems have been considered previously in the fully nonlinear setting in [12], but we stress that their results do not apply to (5.1).

Theorem 5.1. *Let $b, c > 0$, $\lambda, \mu \geq 0$ and assume $\lambda < \mu + 1$. When $0 < p < 1$, problem* (5.1) *does not admit positive supersolutions which do not blow up at infinity, while for $p > 1$ no positive supersolutions blowing up at infinity exist.*

Remark 5.2. When $\lambda > \mu + 1$ and $0 < p < 1$ (resp. $p > 1$) positive supersolutions not blowing up at infinity of the form $u = A|x|^{-\alpha}$ (resp. blowing up at infinity of the form $u = A|x|^{\alpha}$) can be constructed, for suitable $A, \alpha > 0$.

It is also worth remarking that, when $p > 1$, supersolutions which do not blow up at infinity can always be constructed in the form $u = Ae^{-\alpha|x|}$ for suitable $A, \alpha > 0$, independently of the values of λ and μ. When $0 < p < 1$, supersolutions blowing up at infinity can also be constructed in the form $u(x) = Ae^{\alpha|x|}$. Thus Theorem 5.1 is almost optimal.

Proof of Theorem 5.1. Let u be a positive supersolution with $\lambda < \mu + 1$. Proceeding as in Lemma 2.2 we obtain (cf. (2.3)) that there exists y_R with $R < |y_R| < 4R$ such that

$$\begin{aligned} \min\{m(2R), m(4R)\}^p &\leq Cm(2R)\left(\frac{1}{R^2|y_R|^{\mu}} + \frac{|y_R|^{\lambda-\mu}}{R}\right) \\ &\leq Cm(2R)\left(\frac{1}{R^{2+\mu}} + \frac{1}{R^{\mu-\lambda+1}}\right) \leq C\frac{m(2R)}{R^{\mu-\lambda+1}}. \end{aligned} \tag{5.2}$$

Assume first that $0 < p < 1$ and u does not blow up at infinity, so that $m(R)$ is bounded and we have $m(R) \to 0$. Hence Lemma 2.1 implies that $m(R)$ is decreasing for large R. Thus replacing R by $R/2$:

$$m(2R)^p \leq C\frac{m(R)}{R^{\mu-\lambda+1}} \tag{5.3}$$

for some positive constant C and all large values of R. We can now iterate this inequality to obtain a good upper bound for $m(R)$, as in [1]. Observe that (5.3) implies

$$m(2R) \leq CR^{-\gamma_0},$$

where $\gamma_0 = (\mu - \lambda + 1)/p$. Taking this inequality in (5.3), we get $m(2R) \leq CR^{-\gamma_1}$ with $\gamma_1 = (\gamma_0 + \mu - \lambda + 1)/p$. Proceeding inductively we find that

$$m(R) \leq CR^{-\gamma_k}$$

for every $k \geq 0$, where $\gamma_k = (\gamma_{k-1} + \mu - \lambda + 1)/p$. Since $0 < p < 1$, it follows that $\gamma_k \to \infty$ as $k \to \infty$, so that $m(R) \leq CR^{-\theta}$ for every $\theta > 0$ (observe that we do not need to keep track of the constants appearing in each step, since for a fixed value of θ we need only finitely many iterations).

Next, for $R_2 > R_1 > R_0$ consider the problem

$$\begin{cases} -\Delta v + b|x|^\lambda |\nabla v| = c|x|^\mu v^p & \text{in } R_1 < |x| < R_2 \\ v = m(R_1) & \text{on } |x| = R_1 \\ v = m(R_2) & \text{on } |x| = R_2. \end{cases}$$

Since u is a supersolution of this problem while $\underline{u} = m(R_2)$ is a subsolution with $u \geq \underline{u}$ we find that the problem has a radial solution v_{R_2} verifying $v_{R_2} \leq u$.

In particular, this provides with local bounds for v_{R_2}, so that it is standard (cf. [17]) to obtain that $v_{R_2} \to v$ in $C^2_{\text{loc}}(\mathbb{R}^N \setminus B_{R_1})$, where v is a radial solution of

$$\begin{cases} -\Delta v + b|x|^\lambda |\nabla v| = c|x|^\mu v^p & \text{in } |x| > R_1 \\ v = m(R_1) & \text{on } |x| = R_1 \end{cases}$$

verifying $v \leq u$. Hence $v(R) \leq m(R) \leq CR^{-\theta}$ for every $\theta > 0$. Thus

$$-\Delta v + b|x|^\lambda |\nabla v| \geq C|x|^{\mu + \theta(1-p)} v \qquad \text{in } |x| > R_1.$$

Choosing a large θ so that $\mu + \theta(1-p) > 2\lambda$ we can apply Corollary 1.2 to reach a contradiction.

Next, let u be a positive supersolution blowing up at infinity with $p > 1$. Since $m(R)$ is now increasing by Lemma 2.1, we find from (5.2) that

$$m(R)^p \leq C \frac{m(2R)}{R^{\mu - \lambda + 1}}.$$

Iterating as before this yields $m(R) \geq CR^\theta$ for every $\theta > 0$. This implies $u(x) \geq C|x|^\theta$ for every θ and $|x| > R_0$. Hence

$$-\Delta u + b|x|^\lambda |\nabla u| \geq C|x|^{\mu + \theta(p-1)} u \qquad \text{in } |x| > R_0,$$

and choosing θ large enough we obtain a contradiction with Corollary 1.2. Thus no positive supersolutions blowing up at infinity can exist if $p > 1$. □

With similar methods, we can also deal with a related problem with exponential weights, namely

$$-\Delta u + be^{\lambda |x|} |\nabla u| \geq ce^{\mu |x|} u^p \qquad \text{in } \mathbb{R}^N \setminus B_{R_0} \tag{5.4}$$

for $\lambda, \mu \geq 0$ and $p > 0$. The proof is completely similar to that of Theorem 5.1, hence we will only sketch it, stressing the main differences.

Theorem 5.3. *Let $b, c > 0$, $\lambda, \mu \geq 0$ and assume $\lambda < \mu$. When $0 < p < 1$, problem (5.4) does not admit positive supersolutions which do not blow up at infinity, while for $p > 1$ no positive supersolutions blowing up at infinity exist.*

Remark 5.4. A similar observation as in Remark 5.2 is in order. That is, when $\lambda > \mu$ and $0 < p < 1$ positive supersolutions not blowing up at infinity can be constructed, while for $p > 1$ supersolutions which blow up also exist.

For $p > 1$, supersolutions which do not blow up at infinity can also be constructed and for $0 < p < 1$, supersolutions blowing up at infinity also exist. This construction does not depend on the parameters λ and μ. All of these supersolutions can be found in the form $u(x) = Ae^{\pm\alpha|x|}$, with $A, \alpha > 0$.

Sketch of proof of Theorem 5.3. For small positive $\varepsilon > 0$, we choose a cut-off function $\phi \in C_0^\infty(\mathbb{R})$ such that $\phi = 0$ in $(0, 1-\varepsilon) \cup (1+2\varepsilon, \infty)$ and $\phi = 1$ in $[1, 1+\varepsilon]$. Arguing as in the proof of Lemma 2.2 with $v(x) = u(x) - m(R)\phi(|x|/R)$, we obtain

$$\min\{m((1-\varepsilon)R), m((1+2\varepsilon)R)\}^p \leq Cm(R)e^{(\lambda-\mu)(1-\varepsilon)R}$$

for some positive C and large R. If u is a supersolution which does not blow up at infinity with $0 < p < 1$ and ε is chosen to satisfy $p(1+2\varepsilon) < 1$, we obtain with an iteration as in Lemma 5.1 that $m(R) \leq Ce^{-\theta R}$ for every $\theta > 0$. We conclude as in Lemma 5.1, by choosing θ large enough and using Corollary 1.3.

If u is a supersolution which blows up at infinity with $p > 1$, it follows that $m(R) \geq Ce^{\theta R}$ for every $\theta > 0$ and the proof finishes as before. □

References

[1] S. Alarcón, J. García-Melián, A. Quaas, *Nonexistence of positive supersolutions to some nonlinear elliptic equations*, to appear in J. Math. Pures Appl.

[2] S. N. Armstrong, B. Sirakov, *Liouville results for fully nonlinear elliptic equations with power growth nonlinearities.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **X** (2011), 711–728.

[3] S. N. Armstrong, B. Sirakov, *Nonexistence of positive supersolutions of elliptic equations via the maximum principle*, Comm. Part. Diff. Eqns. **36** (2011), 2011–2047.

[4] C. Bandle, M. Essén, *On positive solutions of Emden equations in cone-like domains.* Arch. Rational Mech. Anal. **112** (4) (1990) 319–338.

[5] C. Bandle, H. Levine, *On the existence and nonexistence of global solutions of reaction-diffusion equations in sectorial domains.* Trans. Amer. Math. Soc. **316** (2) (1989) 595–622.

[6] H. Berestycki, I. Capuzzo-Dolcetta, L. Nirenberg, *Superlinear indefinite elliptic problems and nonlinear Liouville theorems.* Topol. Methods Nonlinear Anal. **4** (1) (1994) 59–78.

[7] H. Berestycki, F. Hamel, N. Nadirashvili, *The speed of propagation for KPP type problems I. Periodic framework*, J. Europ. Math. Soc. **7** (2005), 173–213.

[8] H. Berestycki, F. Hamel, L. Rossi, *Liouville type results for semilinear elliptic equations in unbounded domains.* Ann. Mat. Pura Appl. (4) **186** (2007), 469–507.

[9] M. F. Bidaut-Véron, *Local and global behavior of solutions of quasilinear equations of Emden-Fowler type.* Arch. Rational Mech. Anal. **107** (4) (1989), 293–324.

[10] M. F. Bidaut-Véron, S. Pohozaev, *Nonexistence results and estimates for some nonlinear elliptic problems*. J. Anal. Math. **84** (2001), 1-49.

[11] I. Capuzzo Dolcetta, A. Cutrì, *Hadamard and Liouville type results for fully nonlinear partial differential inequalities*, Commun. Contemp. Math. **5** (2003), no. 3, 435–448.

[12] H. Chen, P. Felmer, *On Liouville type theorems for fully nonlinear elliptic equations with gradient term*, preprint.

[13] A. Cutrì, F. Leoni, *On the Liouville property for fully nonlinear equations*, Ann. Inst. H. Poincaré (C) An. Non Linéaire **17** (2000), 219–245.

[14] P. L. Felmer, A. Quaas. *Fundamental solutions and Liouville type theorems for nonlinear integral operators*. Adv. Math. **226** (2011), 2712–2738.

[15] B. Gidas, J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*. Comm. Pure Appl. Math. **34** (1981), 525–598.

[16] B. Gidas, J. Spruck, *A priori bounds for positive solutions of semilinear elliptic equations*, Comm. Partial Differential Equations **6** (1981), 883–901.

[17] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer Verlag, 1983.

[18] V. Kondratiev, V. Liskevich, V. Moroz, *Positive solutions to superlinear second-order divergence type elliptic equations in cone-like domains*. Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (1) (2005) 25–43.

[19] V. Kondratiev, V. Liskevich, Z. Sobol, *Positive supersolutions to semi-linear second-order non-divergence type elliptic equations in exterior domains*. Trans. Amer. Math. Soc. **361** (2) (2009), 697–713.

[20] V. Kondratiev, V. Liskevich, Z. Sobol, *Positive solutions to semi-linear and quasi-linear elliptic equations on unbounded domains*. In "Handbook of differential equations: stationary partial differential equations", volume **6**, pages 255–273. Elsevier, 2008.

[21] V. Liskevich, I. I. Skrypnik, I. V. Skrypnik, *Positive supersolutions to general nonlinear elliptic equations in exterior domains*. Manuscripta Math. **115** (4) (2004) 521–538.

[22] P. Polacik, P. Quittner, P. Soupplet, *Singularity and decay estimates in super linear problems via Liouville-type theorems. Part I: elliptic equations and systems*, Duke Math. J. **139** (2007), 555–579.

[23] P. Pucci, J. Serrin, *The maximum principle*. Birkhäuser, 2007.

[24] L. Rossi, *Non-existence of positive solutions of fully nonlinear elliptic equations in unbounded domains*, Commun. Pure Appl. Anal. 7 (2008), 125–141.

[25] J. Serrin, H. Zou, *Existence and non-existence results for ground states of quasilinear elliptic equations*, Arch. Rat. Mech. Anal. **121** (1992), 101–130.

[26] J. Serrin, H. Zou. *Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities*. Acta Math. **189** (1) (2002) 79–142.

[27] L. Véron, *Singularities of solutions of second order quasilinear equations*, vol. **353**, Pitman Research Notes in Mathematics Series. Longman, Harlow, 1996.

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