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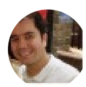

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BLOWUP IN HIGHER DIMENSIONAL TWO SPECIES CHEMOTACTIC SYSTEMS

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Abstract. This paper deals with blowup properties of solutions to multicomponent parabolic-elliptic Keller–Segel model of chemotaxis in higher dimensions.

1. **Introduction.** We consider two Cauchy problems for parabolic-elliptic systems

$$\partial_t u_1 = \nabla\cdot(\kappa_1\nabla u_1 - \chi_1 u_1\nabla v), \tag{1}$$

$$\partial_t u_2 = \nabla\cdot(\kappa_2\nabla u_2 - \chi_2 u_2\nabla v), \tag{2}$$

$$-\Delta v = u_1 + u_2, \tag{3}$$

$$u_1(x,0) = u_{10}(x), \quad u_2(x,0) = u_{20}(x), \tag{4}$$

and

$$\partial_t u_1 = \nabla\cdot(\kappa_1\nabla u_1 - \chi_1 u_1\nabla v), \tag{5}$$

$$\partial_t u_2 = \nabla\cdot(\kappa_2\nabla u_2 + \chi_2 u_2\nabla v), \tag{6}$$

$$-\Delta v = u_1 - u_2, \tag{7}$$

$$u_1(x,0) = u_{10}(x), \quad u_2(x,0) = u_{20}(x), \tag{8}$$

describing two components chemotactic systems in the whole space $\mathbb{R}^d$, $d \geq 2$. The first will be called CPI, and the second CPII. Here the positive coefficients κ_1, κ_2 and χ_1, χ_2 are related to the diffusion coefficients of the species and the sensitivity of the species to the chemoattractant. These generalizations of the classical parabolic-elliptic Keller–Segel model for one species u and the density v of the chemoattractant as well as models for interacting particles (via either electric or gravitational potential) have been proposed in much more general setting by

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Wolansky in [22]. When considered in $\mathbb{R}^2$, they have interesting relations to Moser–Trudinger like inequalities for systems, see [21]. Here, however, we do not use those variational formulations.

The system 1–3 (CPI) models either the interaction of two species that both secrete a chemoattractant producing their movement, or the gravitational attraction of a cloud of massive particles of two kinds (e.g. of two different masses of individual particles).

The system 5–7 (CPII) models two species that one of them produces a chemoattractant for this species, the other decomposes the chemical which acts as a chemorepellent for the latter.

One relevant difference between systems CPI and CPII when $d = 2$ is that the first one presents simultaneous blowup in the radial case, see [13], meanwhile system CPII can present non-simultaneous blowup, see [14]. The simultaneous vs non-simultaneous problem has been proposed for multi-species KS systems in [13] as a way to study the role of chemotaxis in cellular self-organization. Up to the best of our knowledge, in dimension $d \geq 3$ no results for blowup for multi-species KS systems has been developed. It is our aim to give in this paper a first step in this sense by developing blowup criteria in higher dimensional spaces.

Moreover, equations 5–6 in the system CPII resemble drift-diffusion system considered in plasma physics, electrochemistry and semiconductor theory, cf. [6], [18]. However, in this statistical mechanics interpretation the forces between particles of each kind of densities u_1, u_2 here in 5–6 are attractive, and particles of different type repulse each other. Note that for the charged particles generating the Coulombic potential described by the equation $\Delta v = u_1 - u_2$ (note the change of sign compared to 7), the forces are repulsive. This leads to a completely different behavior of solutions with large initial data and the temporal asymptotics, see e.g. [5] for a result on asymptotically diffusive character of evolution for the electric model in $\mathbb{R}^d$, $d > 2$. For the systems motivated by chemotaxis, blowup of solutions with large initial data is expected, and for the two components systems CPII in two space dimensions Kurokiba and Ogawa in [17] showed blowup under a condition of *large discrepancy of masses*: $8\pi(M_1 + M_2) < (M_1 - M_2)^2$, see also 23 below. Their proof of blowup for two-dimensional CPII system does not require any assumption on the existence of moments of solutions, and thus this is a first argument of that type even for the classical one component Keller–Segel system, compare with [1], [8]. However, that proof is actually known for two-dimensional case only. Then [13], [14] studied thorougly blowup of radially symmetric solutions to both systems CPI, CPII. Further refinements of the assumptions on the initial conditions leading to a dichotomic behavior: global in time existence versus finite time blowup (of at least one of the components) are in [10], [11].

Our goal here is to consider the blowup problem for higher dimensional CPI and CPII which leads to interesting new phenomena compared to a single component classical Keller–Segel model as well as two-dimensional CPI, CPII.

After a brief presentation of the existence of solutions questions, we prove some new sufficient criteria for blowup in the systems CPI, CPII. Compared to existing results in [10], [13]–[14], the novelty of the approach and results is that we do not need radial symmetry assumption on the solutions. Our criteria involve as control parameters total mass and a measure of discrepancy of masses (as was in two dimensional case) as well as (pertinent to the higher dimensional case) concentrations of the initial distributions measured by suitable moments. Here, we do not pursue the generality of reasoning in [7] on the various possible moments presenting just a simple argument leading to the blowup of highly concentrated initial data for which second moments do exist. In the latter reference we used weaker assumptions on the integrability of u's since moments of lower order $\gamma \in (1,2]$ have been considered.

Similarly, we may consider multicomponent species systems with several sensitivity agents but for the clarity of computations we do not look for such a generality, the essential differences being apparent when instead of one component two components case is studied. Our calculations, formally done for $d \geq 3$, are also valid in the case $d = 2$, sometimes with simplifications.

Finally, putting one of the components identically equal to 0 in any of systems CPI, CPII, e.g. $u_2 \equiv 0$, results for the one component Keller–Segel model are easily recovered.

All the integrals with no integration limits are meant $\int_{\mathbb{R}^d}$ — the integrals over the whole space $\mathbb{R}^d$. The norm in $L^p(\mathbb{R}^d)$ space is denoted by $\|\,.\,\|_p$. Inessential constants are denoted by C, even if they may vary from line to line.

2. **Existence of solutions to CPI, CPII — local and global in time.** Single component models of both types: repulsive (electric) and attractive (chemotactic, gravitational) have been studied in many papers. The construction of local in time solutions (irrespective of the type of interaction) can be achieved in different functional frameworks, e.g., $L^p(\mathbb{R}^d)$ spaces with $\frac{d}{2} < p < d$ ([2], [3], [9], [12]), weak $L^p_{\mathrm{w}}(\mathbb{R}^d)$ or Marcinkiewicz spaces with $p = \frac{d}{2}$ ([15], [19]), and more "exotic" spaces like (adapted from Calculus of Variations theories) Morrey spaces $\mathcal{M}^p(\mathbb{R}^d)$, $p = \frac{d}{2}$ ([2], [3]), and pseudomeasures spaces $\mathcal{PM}^a(\mathbb{R}^d)$, $a = d-2$ ([4], [20]). Existence of global in time solutions depends heavily on the type of interactions and the size of initial data.

A convenient notion of a solution is that of *mild* solutions, i.e. those satisfying the integral equation (the Duhamel formula)

$$u_1(t) = \mathrm{e}^{\kappa_1 t\Delta}u_{10} - \chi_1 \int_0^t \nabla\cdot(\mathrm{e}^{\kappa_1(t-s)\Delta}u_1(s)\nabla v(s))ds, \tag{9}$$

$$u_2(t) = \mathrm{e}^{\kappa_2 t\Delta}u_{20} - \varepsilon\chi_2 \int_0^t \nabla\cdot(\mathrm{e}^{\kappa_2(t-s)\Delta}u_2(s)\nabla v(s))ds, \tag{10}$$

$$v(t) = (-\Delta)^{-1}(u_1(t) + \varepsilon u_2(t)), \tag{11}$$

where the symbol $\mathrm{e}^{t\Delta}$ denotes the heat semigroup in $\mathbb{R}^d$ with the convolution kernel $(4\pi t)^{-\frac{d}{2}}\exp\big(-\frac{|x|^2}{4t}\big)$. By $(-\Delta)^{-1}$ we mean the Riesz potential operator with the convolution kernel

$$\frac{1}{(d-2)\sigma_d}|x|^{2-d}, \qquad d > 2. \tag{12}$$

Finally, the sign $\varepsilon = +1$ corresponds to CPI, while $\varepsilon = -1$ — to CPII.

The general scheme of results on the existence of solutions to CPI and CPII in the space $\mathcal{X}_T = \mathcal{C}_{\mathrm{w}}([0,T]; V\times V)$ of (weakly) continuous in time vector valued functions with $u_i(t) \in V$, V is one of the spaces $L^p(\mathbb{R}^d)$ with $\frac{d}{2} < p < d$, $L^{\frac{d}{2}}_{\mathrm{w}}$, $\mathcal{M}^{\frac{d}{2}}$, $\mathcal{PM}^{d-2}$ mentioned above (weak continuity, instead of the norm continuity, marked by the subscript w is relevant in the case of nonseparable spaces V), is the following

Theorem 2.1.
– *if* $(u_{10}, u_{20}) \in V\times V$, *then there exists* $T > 0$, *and a local in time solution* $u = (u_1, u_2) \in \mathcal{X}_T$;

– *if the initial data are sufficiently small:* $\|(u_{10}, u_{20})\|_{V\times V} \equiv \|u_{10}\|_V + \|u_{20}\|_V < \delta$ *for a suitably small* $\delta > 0$*, then the local in time solution* $u \in \mathcal{X}_T$ *can be continued indefinitely in time:* $T = \infty$.

The proofs of these results for two (and more) component systems require no new specific ideas beyond those in the works quoted above.

An alternative scheme of the proof of the existence of solutions to CPI can be proposed in the framework of *weak* solutions. The result is analogous to the announcement of Theorem 2.1 with $V = L^{\frac{d}{2}}(\mathbb{R}^d)$: existence of weak solutions, global in time for sufficiently small initial data 4 with the regularity $u_i \in L^\infty((0,\infty); L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d))$. The main steps are:

i) global in time existence of solutions for the regularized CPI system

$$\begin{aligned}
\partial_t u_1^n &= \nabla\cdot(\kappa_1 \nabla u_1^n - \chi_1 u_1^n \nabla v^n),\\
\partial_t u_2^n &= \nabla\cdot(\kappa_2 \nabla u_2^n - \chi_2 u_1^n \nabla v^n),\\
-\Delta v^n &= (u_1^n + u_2^n) * \varrho_n, \qquad (13)
\end{aligned}$$

where $\varrho_n(x) = n^d \varrho(nx)$ with a function $0 \le \varrho \in C_0^\infty$, $\int \varrho(x)dx = 1$, $n = 1, 2, \dots,$ and the initial data are in $L^1(\mathbb{R}^d)$;

ii) existence of a local in time solution for CPI with the initial data in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ which can be continued indefinitely to the global in time solutions whenever the data are suitably small in $L^{\frac{d}{2}}(\mathbb{R}^d)$.

To prove i) one may apply the Schauder fixed point theorem since v^n are smooth and bounded and, in particular, $\nabla v^n \in L^\infty((0,T)\times\mathbb{R}^d)$ for any fixed $T > 0$. The construction in ii) consists in an application of the Moser–Alikakos iteration method to show an *apriori* estimate of L^p norm of $u^n = (u_1^n, u_2^n)$ independent of n under, of course, assumptions on the initial data. This scheme applies also to CPII (cf. an alternative approach in [17]) since it is independent of the sign of χ_1, χ_2 and of the sign of γ_1, γ_2 in the Poisson equation $-\Delta v = \gamma_1 u_1 + \gamma_2 u_2$ generalizing 3 and 7. One can also compare constructions in [6] in the case of a single component parabolic-elliptic system in bounded domains.

The crucial estimates are those for the L^p norms of approximated solutions with $p \in [\max\{1, \frac{d}{2}-1\}, \frac{d}{2}]$ and $p \in (\frac{d}{2}, \infty)$, uniformly in time. The essential ingredient of the reasoning is an application of the Gagliardo–Nirenberg–Sobolev inequality for a bound on $\|u^n\|_p$ by $\|\nabla (u^n)^{p/2}\|_2$ and $\|u^n\|_{\frac{d}{2}}$. Then $\|\nabla v^n(t)\|_\infty \le C\left(\|u^n\|_{\frac{d}{2}}, \|u^n\|_1\right)$ is obtained with the quantity C independent of time. An application of the compactness criterion of Aubin–Simon concludes the passage to the limit with the approximations u^n, v^n as $n \to \infty$.

The framework of mild solutions is different from that of weak solutions and, in a sense, more flexible since admits as initial data distributions with 'geometric' singularities. We mean by that, e.g., that Morrey spaces have their norms defined rather by geometric quantities while Lebesgue spaces norms involve only measure properties of the density. In particular, questions of regularity of mild solutions can be solved in a way of comparison of the regularizing effects of the heat semigroup and nonlinear effects of transport. For weak solutions, this necessitates the use of recurrent improvement of regularity schemes like DeGiorgi–Moser–Alikakos.

Finally, let us mention that the framework of $L^{\frac{d}{2}}_{\rm w}(\mathbb{R}^d)$, $\mathcal{M}^{\frac{d}{2}}(\mathbb{R}^d)$, $\mathcal{PM}^{d-2}(\mathbb{R}^d)$ spaces allows the consideration of the initial data which are functions homogeneous of degree -2: $u_{i0}(\lambda x) = \lambda^{-2} u_{i0}(x)$ for each $\lambda > 0$. The global in time solutions which emanate from such (suitably small) data are self-similar, i.e. invariant under the scaling

$$\lambda^2 u_i(\lambda x, \lambda^2 t) = u_i(x,t) \ \text{ for each } \ \lambda > 0,$$

and thus they are of the form

$$u_i(x,t) = \frac{1}{t} U_i\left(\frac{x}{\sqrt{t}}\right) \ \text{ for some functions } \ U_i,$$

see [4], [16]. This gives an explanation why the exponent $\frac{d}{2}$ is critical for the systems CPI, CPII in respect to the global existence of solutions.

We assume in the sequel that the initial data u_{10}, u_{20} for CPI and CPII in 4, 8 are nonnegative integrable functions, and we denote by M_1, M_2 — masses of those distributions

$$M_1 = \int_{\mathbb{R}^d} u_{10}(x)dx, \qquad M_2 = \int_{\mathbb{R}^d} u_{20}(x)dx. \tag{14}$$

Let us note that solutions of the systems CPI and CPII enjoy the properties of the conservation of positivity and of mass, the proofs (using, e.g., the Stampacchia truncation method) being essentially the same as for one component systems.

3. **Blowup for CPI.**

Define the second moments

$$m_i(t) = \int_{\mathbb{R}^d} u_i(x,t)|x - x_i|^2 dx, \quad i = 1,\, 2, \tag{15}$$

where $x_i \in \mathbb{R}^d$ is a (in general, variable) vector, in applications we will choose a fixed $x_i = \int_{\mathbb{R}^d} x\, u_{i0}(x)dx$ — the common center of mass of suitable initial data u_{i0}, $i = 1,\, 2$.

Theorem 3.1. *Let $d \geq 3$ and the initial data for the system CPI satisfy the inequality*

$$\left(\frac{\frac{m_1(0)}{\chi_1} + \frac{m_2(0)}{\chi_2}}{M_1 + M_2}\right)^{\frac{d}{2}-1} < \frac{(2\max\{\chi_1,\chi_2\})^{1-\frac{d}{2}}}{2d\sigma_d} \frac{(M_1+M_2)^2}{\frac{\kappa_1}{\chi_1}M_1 + \frac{\kappa_2}{\chi_2}M_2}.$$

Then, the solution u of CPI blows up in a finite time, in the sense $\lim_{t\nearrow T}(\|u_1(t)\|_p + \|u_2(t)\|_p) = \infty$ for some $0 < T < \infty$ and each $p > 1$.

Proof. Let us formally calculate the evolution of the moments, assuming that solutions u_i, $i = 1,\, 2$, are regular enough and decay at infinity sufficiently fast

$$\frac{\mathrm{d}m_i}{dt} = -2\int (\kappa_i \nabla u_i - \chi_i u_i \nabla v)\cdot(x - x_i)dx + 2\int u_i(x - x_i)\dot{x}_i dx. \tag{16}$$

From now on we assume that $x_1 = x_2 =$ const. Using the equation 3 in the form 11 with the Newtonian potential 12, we get

$$\begin{aligned}\frac{\mathrm{d}m_i}{dt} &= 2d\kappa_i M_i - \frac{2\chi_i}{\sigma_d}\int\!\!\int u_i(x,t)(u_1(y,t)+u_2(y,t))\frac{(x-y)\cdot(x-x_i)}{|x-y|^d}dxdy \\ &= 2d\kappa_i M_i - \frac{\chi_i}{\sigma_d}\int\!\!\int \big(u_i(x,t)(u_1(y,t)+u_2(y,t))(x-y)\cdot(x-x_i) \\ &\quad + u_i(y,t)(u_1(x,t)+u_2(x,t))(y-x)\cdot(y-x_i)\big)\frac{1}{|x-y|^d}dxdy,\end{aligned}$$

the last equality being the consequence of the symmetrization $x \mapsto y$, $y \mapsto x$. After summing up the multiples of both moments we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{dt}\left(\frac{1}{\chi_1}m_1 + \frac{1}{\chi_2}m_2\right) =& 2d\left(\frac{\kappa_1}{\chi_1}M_1 + \frac{\kappa_2}{\chi_2}M_2\right) \\
&- \frac{1}{\sigma_d}\iint \left(u_1(x,t)u_1(y,t) + u_2(x,t)u_2(y,t) + u_1(x,t)u_2(y,t)\right. \\
&+ \left. u_1(y,t)u_2(x,t)\right)(x-y)\cdot(x-y)|x-y|^{-d}dxdy \\
=& 2d\left(\frac{\kappa_1}{\chi_1}M_1 + \frac{\kappa_2}{\chi_2}M_2\right) - \frac{1}{\sigma_d}J,
\end{aligned}
$$

where the last term is

$$
J = \int\int (u_1(x,t) + u_2(x,t))(u_1(y,t) + u_2(y,t))|x-y|^{2-d}dxdy. \tag{17}
$$

Now we estimate J as was in [1, p. 232] or [7, (4.16)]

$$
J \ge 2^{1-\frac{d}{2}} \frac{(M_1+M_2)^{\frac{d}{2}+1}}{(m_1+m_2)^{\frac{d}{2}-1}} \ge (2\chi)^{1-\frac{d}{2}} \frac{(M_1+M_2)^{\frac{d}{2}+1}}{\left(\frac{m_1}{\chi_1} + \frac{m_2}{\chi_2}\right)^{\frac{d}{2}-1}}, \tag{18}
$$

for $\chi = \max\{\chi_1, \chi_2\}$, postponing the proof of that inequality to Lemma 3.2 below. Thus, for $w = \frac{m_1}{\chi_1} + \frac{m_2}{\chi_2}$, $w = w(t)$, we arrive at

$$
\frac{\mathrm{d}}{dt}w \le 2d\left(\frac{\kappa_1}{\chi_1}M_1 + \frac{\kappa_2}{\chi_2}M_2\right) - \frac{(2\chi)^{1-\frac{d}{2}}}{\sigma_d}\frac{(M_1+M_2)^{\frac{d}{2}+1}}{w^{\frac{d}{2}-1}}.
$$

This reads as the differential inequality

$$
\frac{2}{d}\frac{\mathrm{d}}{dt}w^{\frac{d}{2}} \le 2d\left(\frac{\kappa_1}{\chi_1}M_1 + \frac{\kappa_2}{\chi_2}M_2\right)w^{\frac{d}{2}-1} - \frac{(2\chi)^{1-\frac{d}{2}}}{\sigma_d}(M_1+M_2)^{\frac{d}{2}+1}.
$$

Since the right hand side of this inequality is an increasing function of w, the condition in the hypotheses of Theorem 3.1 implies that the right hand side is always negative and bounded away from 0. Thus, the combination w of the second moments decreases and assumes negative values in a finite time: a contradiction with the existence of a global in time nonnegative solution. □

Of course, there exist initial data of prescribed masses $M_1 > 0$ and $M_2 > 0$ (even arbitrarily small) satisfying the high concentration condition

$$
\left(\frac{w(0)}{M_1+M_2}\right)^{\frac{d}{2}-1} < \frac{(2\chi)^{1-\frac{d}{2}}}{2d\sigma_d}\frac{(M_1+M_2)^2}{\frac{\kappa_1}{\chi_1}M_1 + \frac{\kappa_2}{\chi_2}M_2}. \tag{19}
$$

Indeed, it suffices to consider, e.g., smooth compactly supported data $u_{i0} \not\equiv 0$ and rescale them taking $\varepsilon^{-d}u_{i0}\left(\frac{x-x_i}{\varepsilon}\right)$ for a sufficiently small $\varepsilon > 0$.

Moreover, the relation 19 implies that $\|u_{10}\|_{\frac{d}{2}} + \|u_{20}\|_{\frac{d}{2}}$ is big enough, which is easily seen from Remark 3.3 on the comparison of the norms and moments. The same inequalities in Remark 3.3 show that the norms $\|u(t)\|_p$ blow up when $t \nearrow T$.

Finally, note that for $d = 2$ the condition in the hypotheses of Theorem 3.1 means that total mass of u_{10} and u_{20} is big enough: $8\pi < \frac{(M_1+M_2)^2}{\frac{\kappa_1}{\chi_1}M_1 + \frac{\kappa_2}{\chi_2}M_2}$.

Lemma 3.2. *Let for a density $0 \le v \in L^1(\mathbb{R}^d, (1+|x|^2)dx)$ the moment and mass be defined by $m = \int v(x)|x|^2dx$ and $M = \int v(x)dx$, resp. Then for the integral*

$$J = \int\int v(x)v(y)|x-y|^{2-d}dxdy$$

the inequality

$$M^{\frac{d}{2}+1} \le J\,(2m)^{\frac{d}{2}-1}$$

holds.

Proof. Using the Hölder inequality we have

$$\begin{aligned} M^2 &= \int\int v(x)v(y)dxdy \\ &\le \left(\int\int v(x)v(y)|x-y|^2dxdy\right)^{1-2/d} \\ &\quad \times \left(\int\int v(x)v(y)|x-y|^{2-d}dxdy\right)^{2/d} \\ &= \left(\int\int v(x)v(y)\left(|x|^2+|y|^2-2x\cdot y\right)dxdy\right)^{1-2/d} J^{2/d} \\ &\le \left(2Mm - 2\left|\int xv(x)dx\right|^2\right)^{1-2/d} J^{2/d} \end{aligned}$$

which leads to the desired relation. □

Remark 3.3. *It is of interest to recall some results on the comparison of the second moments with the norms $\|f\|_p$ and $\|f;\mathcal{M}^p\|$, cf.* [7, Remark 2.6, 2.7].

Namely, we have

$$\|f\|_p \ge CM\left(\frac{M}{m}\right)^{d(1-1/p)/2},$$

and even

$$\|f;\mathcal{M}^p\| \ge CM\left(\frac{M}{m}\right)^{d(1-1/p)/2},$$

where the Morrey norm is defined as

$$\|f;\mathcal{M}^p\| = \sup_{R>0, x_0\in\mathbb{R}^d} R^{d(1/p-1)}\int_{B_r(x_0)}|f(x)|dx,$$

and it is, of course, weaker than the L^p norm of f.

4. **Blowup for CPII.** Our main result for the higher dimensional Kurokiba–Ogawa system is

Theorem 4.1. *Let $d \ge 3$ and the initial data for the system CPII satisfy the inequality*

$$\left(\frac{\frac{m_1(0)}{\chi_1}+\frac{m_2(0)}{\chi_2}}{M_1+M_2}\right)^{\frac{d}{2}-1} < \frac{(2\max\{\chi_1,\chi_2\})^{\frac{d}{2}-1}(M_1+M_2)}{2d\sigma_d\max\left\{\frac{\kappa_1}{\chi_1},\frac{\kappa_2}{\chi_2}\right\}}\left(\frac{|M_1-M_2|}{M_1+M_2}\right)^d.$$

Then, the solution u of CPII blows up in a finite time, in the sense that $\lim_{t\nearrow T}(\|u_1(t)\|_p + \|u_2(t)\|_p) = \infty$ for some $0 < T < \infty$ and each $p > 1$.

Proof. The calculations of the evolution of moments are similar, and lead to

$$
\begin{aligned}
&\frac{\mathrm{d}}{dt}\left(\frac{1}{\chi_1}m_1+\frac{1}{\chi_2}m_2\right)=2d\left(\frac{\kappa_1}{\chi_1}M_1+\frac{\kappa_2}{\chi_2}M_2\right)\\
&\quad -\frac{1}{\sigma_d}\int\int\left((u_1(x,t)u_1(y,t)+u_2(x,t)u_2(y,t))\,|x-y|^2\right.\\
&\quad +u_1(x,t)u_2(y,t)(x-y)\cdot(-x+x_1+y-x_2)\\
&\quad \left.+u_1(y,t)u_2(x,t)(x-y)\cdot(-x_1+y-x+x_2)\right)|x-y|^{-d}dxdy\\
\leq\quad & 2d\left(\frac{\kappa_1}{\chi_1}M_1+\frac{\kappa_2}{\chi_2}M_2\right)\\
&\quad -\frac{1}{\sigma_d}\int\int(u_1(x,t)-u_2(x,t))(u_1(y,t)-u_2(y,t))|x-y|^{2-d}dxdy
\end{aligned}
\tag{20}
$$

for $x_1=x_2$. To estimate the bilinear integral in the last line of 20 from below, let us represent u_1-u_2 as the difference of two positive functions with disjoint (interiors of) supports: $v=(u_1-u_2)_+$, $z=(u_1-u_2)_-$. Evidently, we have $u_1-u_2=v-z$ and $\{v>0\}\cap\{z>0\}=\emptyset$. Now,

$$
\begin{aligned}
&\int\int(u_1(x,t)-u_2(x,t))(u_1(y,t)-u_2(y,t))|x-y|^{2-d}dxdy\\
\geq\quad & \int\int v(x,t)v(y,t)|x-y|^{2-d}dxdy+\int\int z(x,t)z(y,t)|x-y|^{2-d}dxdy\\
\geq\quad & 2^{1-\frac{d}{2}}\frac{\left((M_1-M_2)^2\right)^{\frac{d}{2}}}{(aA+bB)^{\frac{d}{2}-1}}\geq 2^{1-\frac{d}{2}}\frac{\left((M_1-M_2)^2\right)^{\frac{d}{2}}}{(m_1M_1+m_2M_2)^{\frac{d}{2}-1}},
\end{aligned}
\tag{21}
$$

where $a=\int v(x,t)|x|^2dx$, $A=\int v(x,t)dx$, $b=\int z(x,t)|x|^2dx$, $B=\int z(x,t)dx$, by Lemma 3.2. Indeed, the function $s\mapsto s^q$ is convex for $q=\frac{d}{2}\geq 1$ and all $s\geq 0$ so that

$$
\left(\frac{aA}{aA+bB}\frac{A}{a}+\frac{bB}{aA+bB}\frac{B}{b}\right)^{\frac{d}{2}}\leq\frac{aA}{aA+bB}\left(\frac{A}{a}\right)^{\frac{d}{2}}+\frac{bB}{aA+bB}\left(\frac{B}{b}\right)^{\frac{d}{2}}
$$

holds. Moreover, we used $A^2+B^2\geq(M_1-M_2)^2$. Next, we have with $\chi=\max\{\chi_1,\chi_2\}$

$$
\begin{aligned}
\frac{\mathrm{d}}{dt}\left(\frac{1}{\chi_1}m_1+\frac{1}{\chi_2}m_2\right)\quad&\leq\quad 2d\left(\frac{\kappa_1}{\chi_1}M_1+\frac{\kappa_2}{\chi_2}M_2\right)-\frac{2^{1-\frac{d}{2}}}{\sigma_d}\frac{|M_1-M_2|^d}{(m_1M_1+m_2M_2)^{\frac{d}{2}-1}}\\
&\leq\quad 2d\max\left\{\frac{\kappa_1}{\chi_1},\frac{\kappa_2}{\chi_2}\right\}(M_1+M_2)\\
&\qquad -\frac{1}{\sigma_d(2\chi)^{\frac{d}{2}-1}}\frac{|M_1-M_2|^d}{\left((M_1+M_2)\left(\frac{m_1}{\chi_1}+\frac{m_2}{\chi_2}\right)\right)^{\frac{d}{2}-1}}.
\end{aligned}
\tag{22}
$$

Clearly, the large discrepancy of masses condition (and at the same time a high concentration condition)

$$
\left(\frac{\frac{m_1(0)}{\chi_1}+\frac{m_2(0)}{\chi_2}}{M_1+M_2})\right)^{\frac{d}{2}-1}<\frac{(2\chi)^{\frac{d}{2}-1}(M_1+M_2)}{2d\sigma_d\max\left\{\frac{\kappa_1}{\chi_1},\frac{\kappa_2}{\chi_2}\right\}}\left(\frac{|M_1-M_2|}{M_1+M_2}\right)^d
\tag{23}
$$

is sufficient for a finite time blowup by the same type of arguments as in the proof of Theorem 3.1. The above condition can be rewritten as

$$\left(\frac{m_1(0)+m_2(0)}{M_1+M_2}\right)^{\frac{d}{2}-1} < C(M_1+M_2)\left(\frac{M_1-M_2}{M_1+M_2}\right)^{d} \tag{24}$$

which is reminiscent of [7, (2.4)]

$$\frac{m(0)}{M} < CM^{\frac{2}{d-2}},$$

(with another constant C), a sufficient condition for blowup for the classical (one component) Keller–Segel model. □

It is interesting to note that sufficient conditions for blowup in $d \geq 3$ dimensions involve masses and concentrations, while only total masses are important when $d = 2$, see [17].

In particular, even if $M_1 \approx M_2$ but $M_1 \neq M_2$, one can have blowup, cf. [14, Th. 5] in the two-dimensional case.

It will be interesting to know whether the blowup of u_1 and u_2 is instantaneous or not, as was studied in the two-dimensional radially symmetric case where both possibilities may occur, see [14].

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REFERENCES

[1] P. Biler, *Existence and nonexistence of solutions for a model of gravitational interaction of particles III,* Colloq. Math., **68** (1995), 229–239.

[2] P. Biler, *The Cauchy problem and self-similar solutions for a nonlinear parabolic equation,* Studia Math., **114** (1995), 181–205.

[3] P. Biler, *Local and global solvability of some parabolic systems modelling chemotaxis*, Adv. Math. Sci. Appl., **8** (1998), 715–743.

[4] P. Biler, M. Cannone, I. Guerra and G. Karch, *Global regular and singular solutions for a model of gravitating particles*, Math. Ann., **330** (2004), 693–708.

[5] P. Biler and J. Dolbeault, *Long time behaviour of solutions to Nernst–Planck and Debye–Hückel drift-diffusion systems*, Ann. Henri Poincaré, **1** (2000), 461–472.

[6] P. Biler, W. Hebisch and T. Nadzieja, *The Debye system: existence and large time behavior of solutions*, Nonlinear Analysis TMA, **23** (1994), 1189–1209.

[7] P. Biler and G. Karch, *Blowup of solutions to generalized Keller–Segel model*, J. Evol. Eq., **10** (2010), 247–262.

[8] A. Blanchet, J. Dolbeault and B. Perthame, *Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions*, Electron. J. Differential Equations, **44** (2006), 1–32.

[9] V. Calvez, L. Corrias and M. Ebde, *Blow-up, concentration phenomenon and global existence for the Keller-Segel model in high dimension,* Comm. Partial Diff. Eq., **37** (2012), 561–584.

[10] C. Conca and E. E. Espejo, *Threshold condition for global existence and blow-up to a radially symmetric drift-diffusion System*, Applied Mathematics Letters, **25** (2012), 352–356.

[11] C. Conca, E. E. Espejo and K. Vilches, *Remarks on the blow-up and global existence for a two-species chemotactic Keller-Segel system in R^2*, Euro. J. Appl. Math, **22** (2011), 553–580.

[12] L. Corrias and B. Perthame, *Asymptotic decay for the solutions of the parabolic-parabolic Keller-Segel chemotaxis system in critical spaces*, Math. Comp. Modelling, **47** (2008), 755–764.

[13] E. E. Espejo, A. Stevens and J. J. L. Velázquez, *Simultaneous finite time blow-up in a two-species model for chemotaxis*, Analysis (Munich), **29** (2009), 317–338.

[14] E. E. Espejo, A. Stevens and J. J. L. Velázquez, *A note on non-simultaneous blow-up for a drift-diffusion model*, Differential Integral Equations, **23** (2010), 451–462.

[15] H. Kozono and Y. Sugiyama, *The Keller-Segel model of parabolic-parabolic type with initial data in weak $L^{n/2}(\mathbb{R}^n)$ spaces with applications to self-similar solutions*, Indiana Univ. Math. J., **57** (2008), 1467–1500.

[16] H. Kozono and Y. Sugiyama, *Global strong solution to the semi-linear Keller-Segel system of parabolic-parabolic type with small data in scale invariant spaces*, J. Differential Eq., **247** (2009), 1–32.

[17] M. Kurokiba and T. Ogawa, *Finite time blow-up of the solution for a nonlinear parabolic equation of drift-diffusion type*, Differential Integral Equations, **16** (2003), 427–452.

[18] M. Kurokiba and T. Ogawa, *Well-posedness for the drift-diffusion system in L^p arising from the semiconductor device simulation*, J. Math. Anal. Appl., **342** (2008), 1052–1067.

[19] A. Raczyński, *Weak-L^p solutions for a model of self-gravitating particles with an external potential*, Studia Math., **179** (2007), 199–216.

[20] A. Raczyński, *Stability property of the two-dimensional Keller–Segel model*, Asymptotic Analysis, **61** (2009), 35–59.

[21] I. Shafrir and G. Wolansky, *Moser-Trudinger and logarithmic HLS inequalities for systems*, J. Eur. Math. Soc., **7** (2005), 413–448.

[22] G. Wolansky, *Multi-components chemotactic system in the absence of conflicts*, European J. Appl. Math., **13** (2002), 641–661.

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