BLOWUP IN HIGHER DIMENSIONAL TWO SPECIES CHEMOTACTIC SYSTEMS

ARTICLE in COMMUNICATIONS ON PURE AND APPLIED ANALYSIS · JANUARY 2013
Impact Factor: 0.71 · DOI: 10.3934/cpaa.2013.12.89

CITATIONS
5

DOWNLOADS
164

VIEWS
146

3 AUTHORS:

Piotr Biler
University of Wroclaw
99 PUBLICATIONS 1,745 CITATIONS
SEE PROFILE

Elio Eduardo Espejo Arenas
National University of Colombia-Medellin
11 PUBLICATIONS 69 CITATIONS
SEE PROFILE

Ignacio Guerra
University of Santiago, Chile
20 PUBLICATIONS 177 CITATIONS
SEE PROFILE

Available from: Elio Eduardo Espejo Arenas
BLOWUP IN HIGHER DIMENSIONAL TWO SPECIES CHEMOTACTIC SYSTEMS

PIOTR BILER
Instytut Matematyczny, Uniwersytet Wrocławski,
pl. Grunwaldzki 2/4, 50–384 Wrocław, Poland

ELIO E. ESPEJO
Departamento de Matemáticas,
Universidad de los Andes, Bogotá, Colombia

IGNACIO GUERRA
Departamento de Matemática y Ciencia de la Computación,
Universidad de Santiago de Chile, Chile

(Communicated by Manuel del Pino)

Abstract. This paper deals with blowup properties of solutions to multicomponent parabolic-elliptic Keller–Segel model of chemotaxis in higher dimensions.

1. Introduction. We consider two Cauchy problems for parabolic-elliptic systems

\[
\begin{align*}
\partial_t u_1 &= \nabla \cdot (\kappa_1 \nabla u_1 - \chi_1 u_1 \nabla v), \\
\partial_t u_2 &= \nabla \cdot (\kappa_2 \nabla u_2 - \chi_2 u_2 \nabla v), \\
-\Delta v &= u_1 + u_2, \\
u_1(x,0) &= u_{10}(x), \quad u_2(x,0) = u_{20}(x),
\end{align*}
\]

and

\[
\begin{align*}
\partial_t u_1 &= \nabla \cdot (\kappa_1 \nabla u_1 - \chi_1 u_1 \nabla v), \\
\partial_t u_2 &= \nabla \cdot (\kappa_2 \nabla u_2 + \chi_2 u_2 \nabla v), \\
-\Delta v &= u_1 - u_2, \\
u_1(x,0) &= u_{10}(x), \quad u_2(x,0) = u_{20}(x),
\end{align*}
\]

describing two components chemotactic systems in the whole space \( \mathbb{R}^d \), \( d \geq 2 \). The first will be called CPI, and the second CPII. Here the positive coefficients \( \kappa_1, \kappa_2 \) and \( \chi_1, \chi_2 \) are related to the diffusion coefficients of the species and the sensitivity of the species to the chemoattractant. These generalizations of the classical parabolic-elliptic Keller–Segel model for one species \( u \) and the density \( v \) of the chemoattractant as well as models for interacting particles (via either electric or gravitational potential) have been proposed in much more general setting by

2000 Mathematics Subject Classification. Primary: 35Q92, 35K.
Key words and phrases. Chemotaxis, multicomponent Keller–Segel model, blowup of solutions.
Wolansky in [22]. When considered in $\mathbb{R}^2$, they have interesting relations to Moser–Trudinger like inequalities for systems, see [21]. Here, however, we do not use those variational formulations.

The system 1–3 (CPI) models either the interaction of two species that both secrete a chemoattractant producing their movement, or the gravitational attraction of a cloud of massive particles of two kinds (e.g. of two different masses of individual particles).

The system 5–7 (CPII) models two species that one of them produces a chemoattractant for this species, the other decomposes the chemical which acts as a chemorepellent for the latter.

One relevant difference between systems CPI and CPII when $d = 2$ is that the first one presents simultaneous blowup in the radial case, see [13], meanwhile system CPII can present non-simultaneous blowup, see [14]. The simultaneous vs non-simultaneous problem has been proposed for multi-species KS systems in [13] as a way to study the role of chemotaxis in cellular self-organization. Up to the best of our knowledge, in dimension $d \geq 3$ no results for blowup for multi-species KS systems has been developed. It is our aim to give in this paper a first step in this sense by developing blowup criteria in higher dimensional spaces.

Moreover, equations 5–6 in the system CPII resemble drift-diffusion system considered in plasma physics, electrochemistry and semiconductor theory, cf. [6], [18]. However, in this statistical mechanics interpretation the forces between particles of each kind of densities $u_1$, $u_2$ here in 5–6 are attractive, and particles of different type repulse each other. Note that for the charged particles generating the Coulombic potential described by the equation $\Delta v = u_1 - u_2$ (note the change of sign compared to 7), the forces are repulsive. This leads to a completely different behavior of solutions with large initial data and the temporal asymptotics, see e.g. [5] for a result on asymptotically diffusive character of evolution for the electric model in $\mathbb{R}^d$, $d > 2$. For the systems motivated by chemotaxis, blowup of solutions with large initial data is expected, and for the two components systems CPII in two space dimensions Kurokiba and Ogawa in [17] showed blowup under a condition of large discrepancy of masses: $8\pi(M_1 + M_2) < (M_1 - M_2)^2$, see also 23 below. Their proof of blowup for two-dimensional CPII system does not require any assumption on the existence of moments of solutions, and thus this is a first argument of that type even for the classical one component Keller–Segel system, compare with [1], [8]. However, that proof is actually known for two-dimensional case only. Then [13], [14] studied thoroughly blowup of radially symmetric solutions to both systems CPI, CPII. Further refinements of the assumptions on the initial conditions leading to a dichotomic behavior: global in time existence versus finite time blowup (of at least one of the components) are in [10], [11].

Our goal here is to consider the blowup problem for higher dimensional CPI and CPII which leads to interesting new phenomena compared to a single component classical Keller–Segel model as well as two-dimensional CPI, CPII.

After a brief presentation of the existence of solutions questions, we prove some new sufficient criteria for blowup in the systems CPI, CPII. Compared to existing results in [10], [13]–[14], the novelty of the approach and results is that we do not need radial symmetry assumption on the solutions. Our criteria involve as control parameters total mass and a measure of discrepancy of masses (as was in two dimensional case) as well as (pertinent to the higher dimensional case) concentrations of the initial distributions measured by suitable moments. Here, we do not pursue
the generality of reasoning in [7] on the various possible moments presenting just a simple argument leading to the blowup of highly concentrated initial data for which second moments do exist. In the latter reference we used weaker assumptions on the integrability of $u$'s since moments of lower order $\gamma \in (1, 2]$ have been considered.

Similarly, we may consider multicomponent species systems with several sensitivity agents but for the clarity of computations we do not look for such a generality, the essential differences being apparent when instead of one component two components case is studied. Our calculations, formally done for $d \geq 3$, are also valid in the case $d = 2$, sometimes with simplifications.

Finally, putting one of the components identically equal to 0 in any of systems CPI, CPII, e.g. $u_2 \equiv 0$, results for the one component Keller–Segel model are easily recovered.

All the integrals with no integration limits are meant $\int_{\mathbb{R}^d}$ — the integrals over the whole space $\mathbb{R}^d$. The norm in $L^p(\mathbb{R}^d)$ space is denoted by $\| \cdot \|_p$. Inessential constants are denoted by $C$, even if they may vary from line to line.

2. Existence of solutions to CPI, CPII — local and global in time. Single component models of both types: repulsive (electric) and attractive (chemotactic, gravitational) have been studied in many papers. The construction of local in time solutions (irrespective of the type of interaction) can be achieved in different functional frameworks, e.g., $L^p(\mathbb{R}^d)$ spaces with $\frac{d}{2} < p < d$ ([2], [3], [9], [12]), weak $L^p_w(\mathbb{R}^d)$ or Marcinkiewicz spaces with $p = \frac{d}{2}$ ([15], [19]), and more “exotic” spaces like (adapted from Calculus of Variations theories) Morrey spaces $\mathcal{M}^p(\mathbb{R}^d)$, $p = \frac{d}{2}$ ([2], [3]), and pseudomeasures spaces $\mathcal{PM}_a^a(\mathbb{R}^d)$, $a = d - 2$ ([4], [20]). Existence of global in time solutions depends heavily on the type of interactions and the size of initial data.

A convenient notion of a solution is that of mild solutions, i.e. those satisfying the integral equation (the Duhamel formula)

$$u_1(t) = e^{\kappa_1 \Delta} u_{10} - \chi_1 \int_0^t \nabla \cdot (e^{\kappa_1 (t-s) \Delta} u_1(s) \nabla v(s)) ds,$$

$$u_2(t) = e^{\kappa_2 \Delta} u_{20} - \varepsilon \chi_2 \int_0^t \nabla \cdot (e^{\kappa_2 (t-s) \Delta} u_2(s) \nabla v(s)) ds,$$

$$v(t) = (-\Delta)^{-1} (u_1(t) + \varepsilon u_2(t)),$$

where the symbol $e^{\Delta}$ denotes the heat semigroup in $\mathbb{R}^d$ with the convolution kernel $(4\pi t)^{-\frac{d}{2}} \exp \left(-\frac{|s|^2}{4t}\right)$. By $(-\Delta)^{-1}$ we mean the Riesz potential operator with the convolution kernel

$$\frac{1}{(d-2)\sigma_d} |x|^{2-d}, \quad d > 2.$$  

Finally, the sign $\varepsilon = +1$ corresponds to CPI, while $\varepsilon = -1$ — to CPII.

The general scheme of results on the existence of solutions to CPI and CPII in the space $X_T = C^{w}([0, T]; V \times V)$ of (weakly) continuous in time vector valued functions with $u_i(t) \in V$, $V$ is one of the spaces $L^p(\mathbb{R}^d)$ with $\frac{d}{2} < p < d$, $L^p_w$, $\mathcal{M}^2$, $\mathcal{PM}_d^{d-2}$ mentioned above (weak continuity, instead of the norm continuity, marked by the subscript $w$ is relevant in the case of nonseparable spaces $V$), is the following

**Theorem 2.1.**

- if $(u_{10}, u_{20}) \in V \times V$, then there exists $T > 0$, and a local in time solution $u = (u_1, u_2) \in X_T$;
– if the initial data are sufficiently small: \( \|(u_{10}, u_{20})\|_{V \times V} \equiv \|u_{10}\|_V + \|u_{20}\|_V < \delta \) for a suitably small \( \delta > 0 \), then the local in time solution \( u \in \mathcal{X}_T \) can be continued indefinitely in time: \( T = \infty \).

The proofs of these results for two (and more) component systems require no new specific ideas beyond those in the works quoted above.

An alternative scheme of the proof of the existence of solutions to CPI can be proposed in the framework of weak solutions. The result is analogous to the announcement of Theorem 2.1 with \( V = L^\infty(\mathbb{R}^d); \) existence of weak solutions, global in time for sufficiently small initial data \( u \in L^\infty((0, \infty); L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)) \). The main steps are:

i) global in time existence of solutions for the regularized CPI system

\[
\begin{align*}
\partial_t u^n_1 &= \nabla \cdot (\kappa_1 \nabla u^n_1 - \chi_1 u^n_1 \nabla v^n), \\
\partial_t u^n_2 &= \nabla \cdot (\kappa_2 \nabla u^n_2 - \chi_2 u^n_2 \nabla v^n), \\
-\Delta v^n &= (u^n_1 + u^n_2) \ast g_n,
\end{align*}
\]

where \( g_n(x) = n^d \rho(nx) \) with a function \( 0 \leq \rho \in C_0^\infty \), \( \int \rho(x)dx = 1, \) \( n = 1, 2, \ldots \), and the initial data are in \( L^1(\mathbb{R}^d) \);

ii) existence of a local in time solution for CPI with the initial data in \( L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) which can be continued indefinitely to the global in time solutions whenever the data are suitably small in \( L^\frac{d}{d-1}(\mathbb{R}^d) \).

To prove i) one may apply the Schauder fixed point theorem since \( v^n \) are smooth and bounded and, in particular, \( \nabla v^n \in L^\infty((0, T) \times \mathbb{R}^d) \) for any fixed \( T > 0 \). The construction in ii) consists in an application of the Moser–Alikakos iteration method to show an apriori estimate of \( L^p \) norm of \( v^n = (u^n_1, u^n_2) \) independent of \( n \) under, of course, assumptions on the initial data. This scheme applies also to CPII (cf. an alternative approach in [17]) since it is independent of the sign of \( \chi_1, \chi_2 \) and of the sign of \( \gamma_1, \gamma_2 \) in the Poisson equation \( -\Delta v = \gamma_1 u_1 + \gamma_2 u_2 \) generalizing 3 and 7. One can also compare constructions in [6] in the case of a single component parabolic-elliptic system in bounded domains.

The crucial estimates are those for the \( L^p \) norms of approximated solutions with \( p \in [\max\{1, \frac{d}{2} - 1\}, \frac{d}{2}] \) and \( p \in (\frac{d}{2}, \infty) \), uniformly in time. The essential ingredient of the reasoning is an application of the Gagliardo–Nirenberg–Sobolev inequality for a bound on \( \|u^n\|_p \) by \( \|\nabla(u^n)^{p/2}\|_2 \) and \( \|u^n\|_{\frac{2}{d}} \). Then \( \|\nabla v^n(t)\|_\infty \leq C \left( \|u^n\|_{\frac{2}{d}}, \|u^n\|_1 \right) \) is obtained with the quantity \( C \) independent of time. An application of the compactness criterion of Aubin–Simon concludes the passage to the limit with the approximations \( u^n, v^n \) as \( n \to \infty \).

The framework of mild solutions is different from that of weak solutions and, in a sense, more flexible since admits as initial data distributions with ‘geometric’ singularities. We mean by that, e.g., that Morrey spaces have their norms defined rather by geometric quantities while Lebesgue spaces norms involve only measure properties of the density. In particular, questions of regularity of mild solutions can be solved in a way of comparison of the regularizing effects of the heat semigroup and nonlinear effects of transport. For weak solutions, this necessitates the use of recurrent improvement of regularity schemes like DeGiorgi–Moser–Alikakos.

Finally, let us mention that the framework of \( L^\frac{d}{d-1}(\mathbb{R}^d), \mathcal{M}^\frac{d}{2}(\mathbb{R}^d),\mathcal{P}\mathcal{M}^{d-2}(\mathbb{R}^d) \) spaces allows the consideration of the initial data which are functions homogeneous of degree \(-2\): \( u_{i0}(\lambda x) = \lambda^{-2}u_{i0}(x) \) for each \( \lambda > 0 \). The global in time solutions
which emanate from such (suitably small) data are self-similar, i.e. invariant under the scaling
\[ \lambda^2 u_i(\lambda x, \lambda^2 t) = u_i(x, t) \quad \text{for each } \lambda > 0, \]
and thus they are of the form
\[ u_i(x, t) = \frac{1}{\ell} U_i \left( \frac{x}{\sqrt{t}} \right) \quad \text{for some functions } U_i, \]
see [4], [16]. This gives an explanation why the exponent \( \frac{d}{2} \) is critical for the systems CPI, CPII in respect to the global existence of solutions.

We assume in the sequel that the initial data \( u_{10}, u_{20} \) for CPI and CPII in 4, 8 are nonnegative integrable functions, and we denote by \( M_1, M_2 \) — masses of those distributions
\[ M_1 = \int_{\mathbb{R}^d} u_{10}(x) dx, \quad M_2 = \int_{\mathbb{R}^d} u_{20}(x) dx. \]  
(14)
Let us note that solutions of the systems CPI and CPII enjoy the properties of the conservation of positivity and of mass, the proofs (using, e.g., the Stampacchia truncation method) being essentially the same as for one component systems.

3. Blowup for CPI. Define the second moments
\[ m_i(t) = \int_{\mathbb{R}^d} u_i(x, t) |x - x_i|^2 d\sigma, \quad i = 1, 2, \]
(15)
where \( x_i \in \mathbb{R}^d \) is a (in general, variable) vector, in applications we will choose a fixed \( x_i = \int_{\mathbb{R}^d} x u_{i0}(x) dx \) — the common center of mass of suitable initial data \( u_{i0} \), \( i = 1, 2 \).

**Theorem 3.1.** Let \( d \geq 3 \) and the initial data for the system CPI satisfy the inequality
\[ \left( \frac{m_{10}(0)}{\chi_1} + \frac{m_{20}(0)}{\chi_2} \right)^{\frac{d-1}{d}} < \frac{(2 \max\{\chi_1, \chi_2\})^{1-\frac{d}{2}} (M_1 + M_2)^2}{2d\sigma_d \chi_1^{\frac{1}{\chi_1}} M_1 + \chi_2^{\frac{2}{\chi_2}} M_2}. \]
Then, the solution \( u \) of CPI blows up in a finite time, in the sense \( \lim_{t \to T} (\|u_1(t)\|_p + \|u_2(t)\|_p) = \infty \) for some \( 0 < T < \infty \) and each \( p > 1 \).

**Proof.** Let us formally calculate the evolution of the moments, assuming that solutions \( u_i, i = 1, 2 \), are regular enough and decay at infinity sufficiently fast
\[ \frac{dm_i}{dt} = -2 \int (\kappa_i \nabla u_i - \chi_i u_i \nabla v) \cdot (x - x_i) d\sigma + 2 \int u_i(x - x_i) \cdot dx. \]  
(16)
From now on we assume that \( x_1 = x_2 = \text{const.} \) Using the equation 3 in the form 11 with the Newtonian potential 12, we get
\[ \frac{dm_i}{dt} = 2 \kappa_i M_i - \frac{2\chi_i}{\sigma_d} \int \int u_i(x, t)(u_1(y, t) + u_2(y, t)) \frac{(x - y) \cdot (x - x_i)}{|x - y|^d} dxdy 
= 2 \kappa_i M_i - \frac{\chi_i}{\sigma_d} \int \int (u_i(x, t)(u_1(y, t) + u_2(y, t))(x - y) \cdot (x - x_i) + u_i(y, t)(u_1(x, t) + u_2(x, t))(y - x) \cdot (y - x_i)) \frac{1}{|x - y|^d} dxdy, \]
the last equality being the consequence of the symmetrization \( x \mapsto y, y \mapsto x \). After summing up the multiples of both moments we obtain
\[
\frac{d}{dt} \left( \frac{1}{\chi_1} m_1 + \frac{1}{\chi_2} m_2 \right) = 2d \left( \frac{\kappa_1}{\chi_1} M_1 + \frac{\kappa_2}{\chi_2} M_2 \right)
- \frac{1}{\sigma_d} \iint (u_1(x,t) u_1(y,t) + u_2(x,t) u_2(y,t) + u_1(x,t) u_2(y,t)) (x-y) \cdot (x-y) |x-y|^{-d} dxdy
+ \iint u_1(y,t) u_2(x,t) (x-y) \cdot (x-y) |x-y|^{-d} dxdy
= 2d \left( \frac{\kappa_1}{\chi_1} M_1 + \frac{\kappa_2}{\chi_2} M_2 \right) - \frac{1}{\sigma_d} J,
\]
where the last term is
\[
J = \iint (u_1(x,t) + u_2(x,t))(u_1(y,t) + u_2(y,t)) |x-y|^{2-d} dxdy.
\] (17)
Now we estimate \( J \) as was in [1, p. 232] or [7, (4.16)]
\[
J \geq 2^{-1-\frac{d}{2}} \frac{(M_1 + M_2)^{\frac{d}{2}+1}}{(m_1 + m_2)^{\frac{d}{2}+1}} \geq (2\chi)^{1-\frac{d}{2}} \frac{(M_1 + M_2)^{\frac{d}{2}+1}}{(\frac{m_1}{\chi_1} + \frac{m_2}{\chi_2})^{\frac{d}{2}+1}},
\] (18)
for \( \chi = \max\{\chi_1, \chi_2\} \), postponing the proof of that inequality to Lemma 3.2 below. Thus, for \( w = \frac{m_1}{\chi_1} + \frac{m_2}{\chi_2} \), \( w = w(t) \), we arrive at
\[
\frac{d}{dt} w \leq 2d \left( \frac{\kappa_1}{\chi_1} M_1 + \frac{\kappa_2}{\chi_2} M_2 \right) - \frac{(2\chi)^{1-\frac{d}{2}}}{\sigma_d} \frac{(M_1 + M_2)^{\frac{d}{2}+1}}{w^{\frac{d}{2}-1}}.
\]
This reads as the differential inequality
\[
2 \frac{d}{dt} w^{\frac{d}{2}} \leq 2d \left( \frac{\kappa_1}{\chi_1} M_1 + \frac{\kappa_2}{\chi_2} M_2 \right) w^{\frac{d}{2}-1} - \frac{(2\chi)^{1-\frac{d}{2}}}{\sigma_d} (M_1 + M_2)^{\frac{d}{2}+1}.
\]
Since the right hand side of this inequality is an increasing function of \( w \), the condition in the hypotheses of Theorem 3.1 implies that the right hand side is always negative and bounded away from 0. Thus, the combination \( w \) of the second moments decreases and assumes negative values in a finite time: a contradiction with the existence of a global in time nonnegative solution. \( \square \)

Of course, there exist initial data of prescribed masses \( M_1 > 0 \) and \( M_2 > 0 \) (even arbitrarily small) satisfying the high concentration condition
\[
\left( \frac{w(0)}{M_1 + M_2} \right)^{\frac{d}{2}-1} < \frac{(2\chi)^{1-\frac{d}{2}}}{2d\sigma_d} \frac{(M_1 + M_2)^{\frac{d}{2}+1}}{\frac{m_1}{\chi_1} M_1 + \frac{m_2}{\chi_2} M_2}.
\] (19)
Indeed, it suffices to consider, e.g., smooth compactly supported data \( u_{i0} \neq 0 \) and rescale them taking \( \varepsilon^{-d} u_{i0} \left( \frac{x}{\varepsilon} \right) \) for a sufficiently small \( \varepsilon > 0 \).

Moreover, the relation 19 implies that \( \|u_{i0}\|_2 \leq c \|u_{i0}\|_2 \) is big enough, which is easily seen from Remark 3.3 on the comparison of the norms and moments. The same inequalities in Remark 3.3 show that the norms \( \|u(t)\|_p \) blow up when \( t \nearrow T \).

Finally, note that for \( d = 2 \) the condition in the hypotheses of Theorem 3.1 means that total mass of \( u_{i0} \) and \( u_{20} \) is big enough: \( 8\pi < \frac{(M_1 + M_2)^2}{\frac{1}{\chi_1} M_1 + \frac{1}{\chi_2} M_2} \).
Lemma 3.2. Let for a density $0 \leq v \in L^1(\mathbb{R}^d, (1 + |x|^2)dx)$ the moment and mass be defined by $m = \int v(x)|x|^2dx$ and $M = \int v(x)dx$, resp. Then for the integral

$$J = \int \int v(x)v(y)|x - y|^{2-d}dxdy$$

the inequality

$$M^{\frac{d}{2} + 1} \leq J (2m)^{\frac{d}{2} - 1}$$

holds.

Proof. Using the Hölder inequality we have

$$M^2 = \int \int v(x)v(y)dxdy$$

$$\leq \left( \int \int v(x)v(y)|x - y|^2dxdy \right)^{1-2/d} \times \left( \int \int v(x)v(y)|x - y|^{2-d}dxdy \right)^{2/d}$$

$$= \left( \int \int v(x)v(y) \left( |x|^2 + |y|^2 - 2x \cdot y \right) dxdy \right)^{1-2/d} J^{2/d}$$

$$\leq \left( 2Mm - 2 \left| \int xv(x)dx \right|^2 \right)^{1-2/d} J^{2/d}$$

which leads to the desired relation. \hfill \square

Remark 3.3. It is of interest to recall some results on the comparison of the second moments with the norms $\|f\|_p$ and $\|f; M^p\|$; cf. [7, Remark 2.6, 2.7].

Namely, we have

$$\|f\|_p \geq CM \left( \frac{M}{m} \right)^{d(1-1/p)/2},$$

and even

$$\|f; M^p\| \geq CM \left( \frac{M}{m} \right)^{d(1-1/p)/2},$$

where the Morrey norm is defined as

$$\|f; M^p\| = \sup_{R>0,x_0\in\mathbb{R}^d} R^{d(1/p-1)} \int_{B_R(x_0)} |f(x)|dx,$$

and it is, of course, weaker than the $L^p$ norm of $f$.

4. Blowup for CPII. Our main result for the higher dimensional Kurokiba–Ogawa system is

Theorem 4.1. Let $d \geq 3$ and the initial data for the system CPII satisfy the inequality

$$\left( \frac{m_1(0) + m_2(0)}{M_1 + M_2} \right)^{\frac{d}{2} - 1} \leq \left( 2\max\{\chi_1, \chi_2\} \right)^{\frac{d}{2} - 1} \left( M_1 + M_2 \right)^{d} \left( \frac{|M_1 - M_2|}{M_1 + M_2} \right)^d. $$

Then, the solution $u$ of CPII blows up in a finite time, in the sense that

$$\lim_{t \to T} (\|u_1(t)\|_p + \|u_2(t)\|_p) = \infty$$

for some $0 < T < \infty$ and each $p > 1$. 


Proof. The calculations of the evolution of moments are similar, and lead to
\[
\frac{d}{dt} \left( \frac{1}{\chi_1} m_1 + \frac{1}{\chi_2} m_2 \right) = 2d \left( \frac{\kappa_1}{\chi_1} M_1 + \frac{\kappa_2}{\chi_2} M_2 \right) \\
- \frac{1}{\sigma_d} \int \int \left( (u_1(x,t)u_1(y,t) + u_2(x,t)u_2(y,t)) |x-y|^2 \right. \\
+ u_1(x,t)u_2(y,t)(-x + x_1 + y - x_2) \\
+ u_1(y,t)u_2(x,t)(-x_1 + y - x_2) \\
\left. \right) |x-y|^{-d} dx dy
\]
for \( x_1 = x_2 \). To estimate the bilinear integral in the last line of (20) from below, let us represent \( u_1 - u_2 \) as the difference of two positive functions with disjoint (interiors of) supports: \( v = (u_1 - u_2)_+, z = (u_1 - u_2)_- \). Evidently, we have \( u_1 - u_2 = v - z \) and \( \{v > 0\} \cap \{z > 0\} = \emptyset \). Now,
\[
\int \int \left( (u_1(x,t) - u_2(x,t))(u_1(y,t) - u_2(y,t)) |x-y|^2 dx dy \right)
\geq \int \int v(x,t)v(y,t) |x-y|^2 dx dy + \int \int z(x,t)z(y,t) |x-y|^2 dx dy
\geq 2^{1-\frac{d}{2}} \frac{(M_1 - M_2)^2}{(aA + bB)^{\frac{d}{2}-1}} \geq 2^{1-\frac{d}{2}} \frac{(M_1 - M_2)^2}{(m_1 M_1 + m_2 M_2)^{\frac{d}{2}-1}}
\]
where \( a = \int v(x,t) |x|^2 dx, A = \int v(x,t) dx, b = \int z(x,t) |x|^2 dx, B = \int z(x,t) dx, \) by Lemma 3.2. Indeed, the function \( s \mapsto s^q \) is convex for \( q = \frac{d}{2} \geq 1 \) and all \( s \geq 0 \) so that
\[
\left( \frac{aA}{aA + bB} \frac{A}{a} + \frac{bB}{aA + bB} \frac{B}{b} \right)^{\frac{d}{2}} \leq \frac{aA}{aA + bB} \left( \frac{A}{a} \right)^{\frac{d}{2}} + \frac{bB}{aA + bB} \left( \frac{B}{b} \right)^{\frac{d}{2}}
\]
holds. Moreover, we used \( A^2 + B^2 \geq (M_1 - M_2)^2 \). Next, we have with \( \chi = \max\{\chi_1, \chi_2\} \)
\[
\frac{d}{dt} \left( \frac{1}{\chi_1} m_1 + \frac{1}{\chi_2} m_2 \right) \leq 2d \left( \frac{\kappa_1}{\chi_1} M_1 + \frac{\kappa_2}{\chi_2} M_2 \right) - \frac{2^{1-\frac{d}{2}}}{\sigma_d} \frac{|M_1 - M_2|^d}{(m_1 M_1 + m_2 M_2)^{\frac{d}{2}-1}} \\
\leq 2d \max \left\{ \frac{\kappa_1}{\chi_1}, \frac{\kappa_2}{\chi_2} \right\} (M_1 + M_2) \\
- \frac{1}{\sigma_d(2\chi)^{\frac{d}{2}-1}} \left( (M_1 + M_2) \left( \frac{m_1}{\chi_1} + \frac{m_2}{\chi_2} \right) \right)^{\frac{d}{2}-1}
\]
Clearly, the large discrepancy of masses condition (and at the same time a high concentration condition)
\[
\left( \frac{m_1(0)}{\chi_1} + \frac{m_2(0)}{\chi_2} \right)^{\frac{d}{2}-1} \left( \frac{2\chi}{M_1 + M_2} \right)^{\frac{d}{2}-1} \left( \frac{m_1}{\chi_1} + \frac{m_2}{\chi_2} \right) \left( \frac{M_1 - M_2}{M_1 + M_2} \right)^d
\]
is sufficient for a finite time blowup by the same type of arguments as in the proof of Theorem 3.1. The above condition can be rewritten as

\[
\left( \frac{m_1(0) + m_2(0)}{M_1 + M_2} \right)^{\frac{2}{d-1}} < C(M_1 + M_2) \left( \frac{M_1 - M_2}{M_1 + M_2} \right)^d
\]

which is reminiscent of [7, (2.4)]

\[
\frac{m(0)}{M} < CM^{-\frac{2}{d-2}}
\]

(with another constant \(C\)), a sufficient condition for blowup for the classical (one component) Keller–Segel model.

It is interesting to note that sufficient conditions for blowup in \(d \geq 3\) dimensions involve masses and concentrations, while only total masses are important when \(d = 2\), see [17].

In particular, even if \(M_1 \approx M_2\) but \(M_1 \neq M_2\), one can have blowup, cf. [14, Th. 5] in the two-dimensional case.

It will be interesting to know whether the blowup of \(u_1\) and \(u_2\) is instantaneous or not, as was studied in the two-dimensional radially symmetric case where both possibilities may occur, see [14].

Acknowledgments. The preparation of this paper was supported by CAPDE-Anillo ACT-125, the Polish Ministry of Science (MNSzW) grant N201 418839, and the Foundation for Polish Science operated within the Innovative Economy Operational Programme 2007–2013 funded by European Regional Development Fund (Ph.D. Programme: Mathematical Methods in Natural Sciences). The first named author acknowledges a warm hospitality during his visits to Universidad de Chile and Universidad de Santiago de Chile in the framework of the international cooperation within FONDECYT grant 1090470.

REFERENCES


Received July 2011; revised February 2012.

*E-mail address: Piotr.Biler@math.uni.wroc.pl*

*E-mail address: ee.espejo34@uniandes.edu.co*

*E-mail address: ignacio.guerra@usach.cl*