BUBBLING ALONG BOUNDARY GEODESICS NEAR THE SECOND CRITICAL EXPONENT

MANUEL DEL PINO, MONICA MUSSO, FRANK PACARD

Abstract. The role of the *second critical exponent* $p = \frac{n+1}{n-3}$, the Sobolev critical exponent in one dimension less, is investigated for the classical Lane-Emden-Fowler problem $\Delta u + u^p = 0$, $u > 0$ under zero Dirichlet boundary conditions, in a domain Ω in $\mathbb{R}^n$ with bounded, smooth boundary. Given Γ, a geodesic of the boundary with negative inner normal curvature we find that for $p = \frac{n+1}{n-3} - \varepsilon$, a solution u_ε such that $|\nabla u_\varepsilon|^2$ converges weakly to a Dirac measure on Γ as $\varepsilon \to 0^+$ exists, provided that Γ is non-degenerate in the sense of second variations of length and ε remains away from certain explicit discrete set of values for which a resonance phenomenon takes place.

Contents

1. Introduction and statement of main results

A basic model of nonlinear elliptic PDE is the classical Lane-Emden-Fowler problem [20],

$$\begin{cases} \Delta u + u^p &= 0 \quad \text{in} \quad \Omega, \\ u &> 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^n$ and $p > 1$. While simple looking, the structure of the solution set of this problem is in general very complex and a number of basic questions remain mostly unsolved. Among those, solvability for powers p above the critical exponent $\frac{n+2}{n-2}$ is a especially difficult one. When $1 < p < \frac{n+2}{n-2}$, compactness of Sobolev's embedding yields a solution as a minimizer of the variational problem

$$S(p) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{p+1}\right)^{\frac{2}{p+1}}}. \tag{1.2}$$

For $p \geq \frac{n+2}{n-2}$ this approach fails and essential obstructions to existence arise: Pohozaev [26] found that no solution to (1.1) exists if the domain is star-shaped. In contrast, Kazdan and Warner [22] observed that if Ω is a symmetric annulus then compactness holds for any $p > 1$ within the class of radial functions, and a solution can again always be found by the above minimizing procedure. Compactness in the minimization is also restored, without symmetries, by the addition of suitable linear perturbations exactly at the critical exponent $p = \frac{n+2}{n-2}$, as established by Brezis and Nirenberg [7].

Topology and geometry of the domain are crucial factors for solvability : when $p = \frac{n+2}{n-2}$ it was proven by Bahri and Coron [2] that solutions to (1.1) exist whenever the topology of Ω is non-trivial in suitable sense. For powers larger than critical direct use of variational arguments seems hopeless, and finding general conditions for solvability is a notoriously open issue.

A question raised by Rabinowitz, stated by Brezis in [5] is whether the presence of nontrivial topology in the domain suffices for solvability in the supercritical case $p > \frac{n+2}{n-2}$. Strikingly enough, the answer was found to be negative in dimension $n \geq 4$: Passaseo [24] discovered that for a domain constituted by a thin tubular neighborhood of a copy of the sphere S^{n-2} embedded in $\mathbb{R}^n$, a Pohozaev-type identity yields that no solution exists if $p \geq \frac{n+1}{n-3}$. We call the latter number, which is strictly greater than $\frac{n+2}{n-2}$, *the second critical exponent*.

The purpose of this paper is to construct solutions of (1.1) when p is below but sufficiently close to the (supercritical) second critical exponent. Assuming that $\partial\Omega$ contains a non-degenerate, closed geodesic Γ with strictly negative curvature, we find a solution to (1.1) with a concentration behavior as p approaches $\frac{n+1}{n-3}$ in the form of a *bubbling line*, eventually collapsing onto Γ. One should generically expect that this geometric condition holds if for instance Ω has a convex hole or it is a deformations of a torus-like solid of revolution like Passaseo's domain.

We recall next the familiar notion of "point bubbling" in the slightly subcritical case for problem (1.1),

$$\left\{\begin{array}{rcll} \Delta u + u^{\frac{n+2}{n-2}-\varepsilon} & = & 0 & \text{in } \quad \Omega, \\ u & > & 0 & \text{in } \quad \Omega, \\ u & = & 0 & \text{on } \quad \partial\Omega, \end{array}\right. \tag{1.3}$$

for small $\varepsilon > 0$. The loss of compactness of Sobolev's embedding as $\varepsilon \to 0$ triggers the presence of *bubbling solutions* around special points of the domain, which resemble a sharp extremal of the best Sobolev constant in $\mathbb{R}^n$

$$S_n := \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2}{\left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}}$$

a type of point-concentration behavior extensively considered in the literature. This is precisely the behavior of a solution u_ε of (1.3) which minimizes $S(p)$ in (1.2) for

$$p = p_\varepsilon = \frac{n+2}{n-2} - \varepsilon,$$

see [6, 14, 27, 19]. We have that $S(p_\varepsilon) \to S_n$ and

$$u_\varepsilon(x) \,=\, \mu_\varepsilon^{-\frac{N-2}{2}} w_n(\mu_\varepsilon^{-1}(x-x_\varepsilon)) \,+\, o(1), \qquad \mu_\varepsilon \,\sim\, \varepsilon^{\frac{1}{N-2}}\,,$$

as $\varepsilon \to 0^+$, where w_n is the *standard bubble*,

$$w_n(x) = \left(\frac{c_n}{1+|x|^2}\right)^{\frac{n-2}{2}}, \qquad c_n = (n(n-2))^{\frac{1}{n-2}}, \tag{1.4}$$

a radial solution of

$$\Delta w + w^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n$$

corresponding to an extremal for S_n, [1, 30]. The blow-up point x_ε approaches (up to a subsequences) a harmonic center x_0 of Ω, namely a minimizer for Robin's function of the domain, the diagonal of the regular part of Green's function. The solution concentrates as a Dirac mass at x_0, namely

$$|\nabla u_\varepsilon|^2 \,\rightharpoonup\, S_n^{\frac{n}{2}}\,\delta_{x_0} \quad \text{as } \varepsilon \to 0 \tag{1.5}$$

in the sense of measures. It is found in [27] that actually solutions of (1.3) with this behavior exist, concentrating at any given non-degenerate critical point x_0 of Robin's function. We refer the reader to the works [3, 10, 21] and to the survey [13] for related results on construction of point-bubbling solutions for problems near the critical exponent.

Now, we are interested in problem (1.1) for powers slightly below the second critical exponent, namely

$$\begin{cases} \Delta u + u^{\frac{n+1}{n-3}-\varepsilon} &= 0 \quad \text{in} \quad \Omega, \\ u &> 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.6}$$

We want to find a solution u_ε with a behavior analogous to that just described for (1.3), now concentrating *along a curve*, with a sectional profile given by a scaled standard bubble in one dimension less. This problem is substantially harder than (1.3), in particular because a global variational characterization of the solution does not seem possible in view of its supercritical character. In addition, this solution has formally a large ε-dependent Morse index, and the construction requires us to avoid special values of ε where change of topological type occurs.

We shall assume that $\partial\Omega$ contains a closed geodesic Γ, non-degenerate, which has globally *negative curvature*, and in addition a non-resonance condition of the following form:

$$| \, k^2 \varepsilon^{2\frac{n-2}{n-3}} - \kappa^2 \, | \, > \, \delta \varepsilon^{\frac{n-2}{n-3}} \quad \text{for all } k = 1, 2, \ldots \tag{1.7}$$

where $\kappa > 0$ is given explicitly in terms of Γ by formula (8.9).

Theorem 1.1. *Let $n \geq 8$ and $\Omega \subset \mathbb{R}^n$ be a domain with smooth, bounded boundary $\partial\Omega$, which contains a closed geodesic Γ, non-degenerate with negative inner normal curvature. Then, given $\delta > 0$, we have that for all $\varepsilon > 0$ sufficiently small satisfying condition* (1.7)*, problem* (1.6) *has a solution u_ε that satisfies*

$$|\nabla u_\varepsilon|^2 \rightharpoonup S_{n-1}^{\frac{n-1}{2}} \, \delta_\Gamma$$

as $\varepsilon \to 0$ in the sense of measures, where δ_Γ is the Dirac measure supported on the curve Γ. Besides, u_ε can be described according to formula (1.9) *below.*

Much more precise information on the solution can indeed be gathered as we shall explain later. The condition $n \geq 8$ seems essential for the method used, while we believe the phenomenon described should also be true for lower dimensions.

Theorem 1.1 includes the case of an *exterior domain*, $\Omega \setminus \Lambda$, with Λ bounded. It is worthwhile mentioning that for this case it was established in [8, 9] that Problem (1.1) is actually *always* solvable if $p > \frac{N+2}{N-2}$. In fact a continuum of solutions exist *except* that they are of slow decay (infinite energy). Finding finite-energy (fast decay) solutions for supercritical powers is a much harder question, which is only answered in [9] for p very close from above to $\frac{n+2}{n-2}$. In turns out that a dramatic change of structure in the set of slow decay solutions takes place *precisely* when $p = \frac{n+1}{n-3}$, the second critical exponent.

The line-bubbling phenomenon here discovered is conceptually quite different to point bubbling. In spite of zero boundary data, concentration eventually collapses on the boundary. On the other hand, point concentration is determined by global information on the domain encoded in Green's function, while only local structure of the domain near the curve Γ is relevant to the line-bubbling. In order to describe more precisely the solution we introduce a local system of coordinates near Γ:

For notational simplicity we will write in all what remains of the paper $N = n - 1$, so that the problem is embedded in $\mathbb{R}^{N+1}$.

We consider the metric induced by the Euclidean one on $\partial\Omega$ and denote by $\bar{\nabla}$ the associated connection. We introduce Fermi coordinates in a neighborhood of Γ in $\partial\Omega$. Given $q \in \Gamma$, there is a natural splitting

$$T_q \partial\Omega = T_q\Gamma \oplus N_q\Gamma$$

into the normal and tangent bundle over Γ. We assume that Γ is parameterized by arclength x_0, $x_0 \mapsto \gamma(x_0)$ and denote by E_0 a unit tangent vector to Γ. In a neighborhood of a point q of Γ, assume we are given an orthonormal basis E_i, $i = 1, \ldots, N-1$, of $N_q\Gamma$. We can assume that E_i are parallel transported along Γ which means that

$$\bar{\nabla}_{E_0} E_i = 0$$

for $i = 1, \ldots, N-1$. The geodesic condition for Γ translates precisely into

$$\bar{\nabla}_{E_0} E_0 = 0.$$

To parameterize a neighborhood of a point of Γ in $\partial\Omega$ we define

$$F(x_0, \bar{x}) := \mathrm{Exp}^{\partial\Omega}_{\gamma(x_0)}(x_i\, E_i), \qquad \bar{x} := (x_1, \ldots, x_N),$$

where Exp^{Σ} is the exponential map on $\partial\Omega$ and summation over $i = 1, \ldots, N-1$ is understood. To parameterize a neighborhood of Γ in $\bar{\Omega}$, we consider the system of coordinates $(x_0, x) \in \mathbb{R}^{N+1}$ given by

$$G(x_0, x) = F(x_0, \bar{x}) - x_N\, \mathbf{n}(F(x_0, \bar{x})), \quad x = (\bar{x}, x_N) \in \mathbb{R}^N \tag{1.8}$$

where x is close to 0 and $\mathcal{N}$ designates outward unit normal.

In term of the outward unit normal $\mathbf{n}$, we mean that Γ has globally negative curvature in the sense that

$$\partial^2_{x_0} \gamma = \bar{h}_{00} \mathbf{n},$$

with $\bar{h}_{00}$ a strictly positive function along Γ.

The solution u_ε predicted by the theorem can be described in these coordinates at main order as follows:

$$u_\varepsilon(x_0, x) = \mu_\varepsilon^{-\frac{N-2}{2}}\, w_N\left(\mu_\varepsilon^{-1}(x - d_\varepsilon) \right) + o(1) \tag{1.9}$$

where

$$d_{j\varepsilon}(x_0) \sim \varepsilon \tilde{d}_j(x_0), \quad j = 1, \ldots, N, \quad \mu_\varepsilon(x_0) \sim \varepsilon^{\frac{N-1}{N-2}} \tilde{\mu}(x_0),$$

where $\tilde{d}_j$ and $\tilde{\mu}$ are smooth functions of x_0 with $\tilde{d}_N$ and $\tilde{\mu}$ strictly positive, and w_N is given by (1.4).

Finally, let us make explicit the meaning of nondegeneracy of the geodesic Γ. Let us denote by $\bar{R}$ the Ricci tensor on $\partial\Omega$. Then nondegeneracy of Γ translates exactly into the fact that the linear system of equations

$$-\ddot{\bar{d}}_k + \sum_{j=1}^{N-1} (\bar{R}(E_0, E_j) E_0 \cdot E_k) \bar{d}_j = 0, \quad x_0 \in [-\ell, \ell],\ k = 1, \ldots, N-1, \tag{1.10}$$

has only the trivial 2ℓ-periodic solution $\bar{d} \equiv 0$.

The rest of this paper will be devoted to the proof of Theorem 1.1. We point out that the resonance phenomenon has already been found to arise in the analysis of higher dimensional concentration in other elliptic boundary value problems, in particular for a Neumann singular perturbation problem in [15, 16, 17, 18] and in Schrodinger equations in the plane in [12]. Theorem 1.1 seems to be the first result on higher dimensional concentration phenomena associated to critical exponents. The question of whether one can find concentration results for larger critical exponents, say k-dimensional concentration slightly below $\frac{n+2-k}{n-2-k}$ arises naturally but we will not treat it in this paper.

2. Scheme of the proof of Theorem 1.1

Let us write Problem (1.6) as

$$\begin{cases} \Delta u + u^{p-\varepsilon} & = & 0 & \text{in} & \Omega, \\ u & > & 0 & \text{in} & \Omega, \\ u & = & 0 & \text{on} & \partial\Omega. \end{cases} \tag{2.1}$$

where here and in what follows we label $p = \frac{N+2}{N-2}$. A key element of the proof of Theorem 1.1 consists of the construction of a first approximation of the solution to our problem. The main part of the construction is that close to the geodesic. Let us consider the system of coordinates $(x_0, \bar{x}, x_N)$ introduced in (1.8), which straightens the boundary of Ω in a neighborhood of the geodesic, as the hyperplane $x_N = 0$. In this language the geodesic is represented by the x_0-axis. We recall that x_0 designates arclength of the curve and $x_N > 0$ is the normal coordinate to the boundary. Then for a function u defined on this neighborhood we write

$$\tilde{u}(x_0, x) = u(G(x_0, x)). \tag{2.2}$$

Let 2ℓ represent the total length of the geodesic. Extending $\tilde{u}$ in a 2ℓ-periodic manner in x_0, it is convenient to regard it as a function defined on the infinite half cylinder

$$D = \{(x_0, \bar{x}, x_N) \ / \ |\bar{x}|^2 + |x_N|^2 < a, \quad x_N > 0, \ \}$$

where $a > 0$ is a fixed small number. Equation (2.1) for u reads in terms of $\tilde{u}$ in D as

$$\begin{cases} \Delta\tilde{u} + B(\tilde{u}) + \tilde{u}^{p-\varepsilon} & = & 0, \quad u > 0 & \text{in} & D, \\ \tilde{u}(x_0, \bar{x}, 0) & = & 0 & \text{for all} & (x_0, \bar{x}), \\ \tilde{u}(x_0 + 2\ell, \bar{x}, x_N) & = & \tilde{u}(x_0, \bar{x}, x_N) & \text{for all} & (x_0, \bar{x}, x_N). \end{cases} \tag{2.3}$$

where B is a second order linear operator of the form

$$B = b_{lk}(x_0, x)\partial_{lk} + b_l(x_0, x)\partial_l$$

with smooth coefficients, 2ℓ-periodic in x_0, $b_{lk}(x_0, 0) \equiv 0$ which we explicitly find in terms of geometric quantities in §4. If a is sufficiently small, the differential operator involved in (2.3) can be regarded as a small perturbation of the Laplacian inside D. To construct an approximation to a solution of (2.3) with the desired properties the main observation we make is that if

$$\omega(x) := \left(\frac{c_N}{1 + |x|^2} \right)^{\frac{N-2}{2}}, \tag{2.4}$$

then for small numbers $\mu > 0$ and $d = (\bar{d}, d_N) \in \mathbb{R}^N$ the function

$$u_0 = \mu^{-\frac{N-2}{2}} \omega(\mu^{-1}(x - d)) = \left(\frac{c_N \mu}{\mu^2 + |\bar{x} - \bar{d}|^2 + |x_N - d_N|^2} \right)^{\frac{N-2}{2}}$$

satisfies

$$\begin{cases} \Delta u + u^p & = & 0, \quad u > 0 & \text{in} & D, \\ u(x_0 + 2\ell, \bar{x}, x_N) & = & u(x_0, \bar{x}, x_N) & \text{for all} & (x_0, \bar{x}, x_N) \end{cases} \tag{2.5}$$

and can therefore be considered as an approximation of a solution to (2.3). We assume $d_N > 0$ so that the maximum set of u_0 is inside the domain, with value $\sim \mu^{-\frac{N-2}{2}}$. In addition, we want that the boundary values are small compared with this order, which is achieved if $\mu << d_N$. In this case the boundary values are bounded by $\sim \mu^{-\frac{N-2}{2}}(\mu/d_N)^{\frac{N-2}{2}}$. Unfortunately, to obtain a good approximation it does not suffice to choose μ and d just to be constants. We assume instead that they define smooth functions of x_0. As we will see later, a sound choice is to take

$$d_\varepsilon(x_0) = \varepsilon \tilde{d}_\varepsilon(x_0), \qquad \mu_\varepsilon(x_0) = \rho\tilde{\mu}_\varepsilon(x_0), \quad \rho = \varepsilon^{\frac{N-1}{N-2}}, \tag{2.6}$$

where $\tilde{\mu}_\varepsilon$ and $\tilde{d}_\varepsilon$ are uniformly bounded 2ℓ-periodic smooth functions so that, also, $\tilde{\mu}_\varepsilon$, $\tilde{d}_{N\varepsilon}$ are positive and uniformly bounded below away from zero. In particular, observe that $\mu_\varepsilon \sim \varepsilon^{\frac{1}{N-2}} d_{\varepsilon N}$, and we set as an approximation to a solution of (2.3),

$$\tilde{u}_0(x_0, x) = \mu_\varepsilon^{-\frac{N-2}{2}} \omega(\mu_\varepsilon^{-1}(x - d_\varepsilon)).$$

It is natural to consider the further change of variables

$$\tilde{u}(x_0, x) = \mu_\varepsilon^{-\frac{N-2}{2}} v(\rho^{-1}x_0, \mu_\varepsilon^{-1}(x - d_\varepsilon)), \quad v = v(y_0, y), \tag{2.7}$$

under which $\tilde{u}_0$ reads simply as $\omega(y)$. Equation (2.3) is transformed in terms of v into

$$\left\{ \begin{array}{rcl} S(v) := a_0(\rho y_0)\partial_{00}v + \Delta_y v + \tilde{\mathcal{A}}(v) + \mu_\varepsilon^{-\frac{N-2}{2}\varepsilon} v^{p-\varepsilon} & = & 0 \quad \text{in } \mathcal{D}, \\ v(y_0, \bar{y}, -\frac{d_{N\varepsilon}}{\mu_\varepsilon}(\rho y_0)\,) & = & 0, \\ v(y_0 + 2\ell\rho^{-1}, y) & = & v(y_0, y) \end{array} \right. \tag{2.8}$$

where

$$\tilde{\mathcal{A}} = a_{ij}(y_0, y)\partial_{ij} + a_i(y_0, y)\partial_i + c(y_0, y)$$

is again a small operator and now we reduce the original cylinder to take $\mathcal{D}$ as a region of the form

$$\mathcal{D} = \{(y_0, \bar{y}, y_N) \ / \ -\frac{d_{\varepsilon N}}{\mu_\varepsilon}(\rho y_0) < y_N < \frac{\hat{\delta}}{\rho}, \quad |\bar{y}| < \frac{\hat{\delta}}{\rho}\}, \tag{2.9}$$

where $\hat{\delta} > 0$ is a small number which will be further reduced if necessary. Here

$$a_0(x_0) = \rho^{-2}\mu_\varepsilon^2(x_0) = \tilde{\mu}_\varepsilon(x_0)^2, \tag{2.10}$$

and $\tilde{\mathcal{A}}$ is a differential operator with coefficients becoming small with ε, which we will fully identify later. Noting that $\mu_\varepsilon^{-\frac{N-2}{2}\varepsilon} \to 1$ and that the domain $\mathcal{D}$ is expanding into entire $\mathbb{R}^{N+1}$, then we see that $\omega(y)$ indeed approximates a solution to the equation. We will actually take an approximation $\mathtt{w}$ which differs little from ω which in particular satisfies the boundary condition.

Now, setting $v = \mathtt{w} + \phi$ with ϕ small, the equation takes the form

$$L(\phi) := a_0\partial_{00}\phi + \Delta_y\phi + p\omega^{p-1}\phi + \tilde{\mathcal{A}}(\phi) = -S_\varepsilon(\mathtt{w}) - N(\phi)$$

where the operator $N(\phi)$ is of order smaller than linear in ϕ. More precisely

$$N(\phi) = \mu_\varepsilon^{-\frac{N-2}{2}\varepsilon}(\mathtt{w} + \phi)^{p-\varepsilon} - \mu_\varepsilon^{-\frac{N-2}{2}\varepsilon}\mathtt{w}^{p-\varepsilon} - p\omega^{p-1}\phi$$

It is therefore important to understand bounded solvability of a linear equation involving the operator L. This is a rather subtle issue since the limiting L does have a kernel in the space of bounded functions in $\mathbb{R}^{N+1}$. Indeed, the equation

$$\partial_{00}\phi + \Delta_y \phi + p\omega^{p-1}\phi = 0$$

has the bounded solutions Z_i, $1, \ldots, N+1$, and $Z_0(x)\cos(\sqrt{\lambda_1}x_0)$, $Z_0(x)\sin(\sqrt{\lambda_1}x_0)$, where

$$Z_i = \partial_i w, \quad i = 1, \ldots, N, \quad Z_{N+1} = x \cdot \nabla w + \frac{N-2}{2}w, \tag{2.11}$$

and by Z_0, $\lambda_1 > 0$ the first eigenfunction and eigenvalue in $L^2(\mathbb{R}^N)$ of the problem

$$\Delta_y \phi + p\omega(y)^{p-1}\phi = \lambda\phi \quad \text{in} \quad \mathbb{R}^N. \tag{2.12}$$

As we shall show these are *all* the bounded solutions of the equation.

Let us consider a bounded function $h(y_0, y)$ 2ℓ-periodic in y_0 and the following *projected problem* in which we mod out the above functions, and look for bounded functions $c_i(y_0)$ and ϕ such that

$$\left\{\begin{array}{rcl} L(\phi) := a_0\partial_{00}\phi + \Delta_y\phi + p\omega^{p-1}\phi + \tilde{\mathcal{A}}(\phi) & = & h + \sum_{i=0}^{N+1} c_i(y_0)Z_i \quad \text{in } \mathcal{D}, \\ \phi & = & 0 \quad \text{on } \partial\mathcal{D}, \\ \phi(y_0 + 2\ell\rho^{-1}, y) & = & \phi(y_0, y). \\ \int_{\mathcal{D}_{y_0}} \phi(y_0, y)Z_i(y)dy & = & 0 \text{ for all } y_0 \in \mathbb{R}, \quad i = 0, \ldots, N. \end{array}\right. \tag{2.13}$$

As we will see, this problem has a unique solution whenever ε is small enough provided that certain uniform estimates for the parameters involved and its derivatives hold. In addition ϕ satisfies a uniform a priori estimate in L^∞-weighted-norms. We develop this theory in fact in larger generality in §3. Then we consider the projected nonlinear problem

$$\left\{\begin{array}{rcl} L(\phi) & = & -S_\varepsilon(\mathtt{w}) - N(\phi) + \sum_{i=0}^{N+1} c_i(y_0)Z_i \quad \text{in } \mathcal{D}, \\ \phi & = & 0 \quad \text{on } \partial\mathcal{D}, \\ \phi(y_0 + 2\ell\rho^{-1}, y) & = & \phi(y_0, y). \\ \int_{\mathcal{D}_{y_0}} \phi(y_0, y)Z_i(y)dy & = & 0 \text{ for all } y_0 \in \mathbb{R}, \quad i = 0, \ldots, N+1, \end{array}\right. \tag{2.14}$$

where $\mathcal{D}_{y_0} = \{y \,/\, (y_0, y) \in \mathcal{D}\}$, to which we can apply the linear solvability theory and contraction mapping principle to find a unique small solution. Besides, we have that

$$c_i(y_0)\int_{\mathbb{R}^N} Z_i^2 \sim \int_{\mathcal{D}_{y_0}} S_\varepsilon(\mathtt{w})Z_i dy$$

and therefore to have a solution of the original problem (with $c_i \equiv 0$) we need a set of relations that look (approximately!) like

$$\int_{\mathcal{D}_{y_0}} S_\varepsilon(\mathtt{w})Z_i dy = 0, \quad \text{for all } y_0, \quad i = 0, \ldots, N+1. \tag{2.15}$$

At this point we mention that the approximation $\mathtt{w}$ carries as an additive term a function of the form $e_\varepsilon(\rho y_0)Z_0(y)$ where e_ε is another parameter of the form $e_\varepsilon(x_0) = \varepsilon\tilde{e}_\varepsilon(x_0)$. It turns out that adjusting conveniently the $(N+2)$ parameters $\mu_\varepsilon, d_\varepsilon, e_\varepsilon$ we can achieve that the above

$N+2$ relations hold as a system of differential equations for these quantities, which turns out to be solvable because of the non-degeneracy assumptions made. The story is however more involved since the parameters enter the nonlinear relations at different orders so that a further improvement of the approximation $\mathtt{w}$ of the form $\mathtt{W} = \mathtt{w} + \Pi$. This is the main purpose of the work in §5. Π is built upon solving the linear problem (2.13) for $h = -S_\varepsilon(\mathtt{w})$, after identifying the right main order values of the parameters in the solvability conditions (2.15), which turns out to reduce substantially the size of the error of approximation $S_\varepsilon(\mathtt{W})$. Another crucial step is a gluing procedure carried out in §6, where the full problem (2.1), for which a global approximation is built by just multiplying $\mathtt{W}$ by a cut-off function, is reduced to solving an equation similar to (2.14) for $c_i \equiv 0$, just in a neighborhood of the geodesic, but where the operator $N(\phi)$ is replaced by a similar one which includes nonlocal terms in ϕ encoding the information of the rest of the domain. This is what tells us that the influence of geometry of the remaining part of the domain is basically negligible. The corresponding projected version of the nonlinear problem is solved in §7 and the final adjustment of the remaining parts of the parameters is done in §8, thus completing the proof of Theorem 1.1. We devote the rest of this paper to carry out the program outlined above.

3. The linear theory

In this section we will develop a linear theory suitable to solve problem (2.13). Our main result is contained in Proposition 3.1 below, for which we need some preliminaries. Let $\omega(x)$ the function defined in (2.4) as

$$\omega(x) := \left(\frac{c_N}{1+|x|^2} \right)^{\frac{N-2}{2}},$$

where $x \in \mathbb{R}^N$ and $c_N = (N(N-2))^{\frac{1}{2}}$ which is, we recall, an entire solution of the problem

$$\Delta_{\mathbb{R}^N} \omega + \omega^p = 0 \qquad \text{in} \qquad \mathbb{R}^N \tag{3.1}$$

where $p = \frac{N+2}{N-2}$. Let us consider the operator

$$L_0 := \Delta_{\mathbb{R}^N} + p\, \omega^{p-1},$$

which corresponds to the nonlinear operator in (3.1) linearized at ω.

To analyze the point spectrum of this operator, we use the conformal invariance of (3.1). Let us consider on $\mathbb{R}^N$, the metric

$$g_{S^N} := \left(\tfrac{2}{1+|x|^2} \right)^2 dx^2,$$

which is conformal to the euclidean metric dx^2 and corresponds to the standard metric on S^N when parameterized by the inverse of the stereographic projection

$$x \in \mathbb{R}^N \longmapsto \left(\tfrac{2}{1+|x|^2}\, x, \tfrac{1-|x|^2}{1+|x|^2} \right) \in S^N.$$

In polar coordinates, we have the expression of the Laplace-Beltrami operator on S^N given by

$$\Delta_{S^N} = \left(\tfrac{2}{1+r^2} \right)^{-n} r^{1-n} \partial_r \left(\left(\tfrac{2}{1+r^2} \right)^{n-2} r^{n-1} \partial_r \right) + \left(\tfrac{2}{1+r^2} \right)^{-2} r^{-2}\, \Delta_{S^{N-1}},$$

where $r = |x|$. The following identity follows from the conformal invariance of the so called "conformal Laplacian" [?] or can also be obtained by direct computation

$$L = \left(\tfrac{2}{1+|x|^2}\right)^{\frac{N+2}{2}} \, (\Delta_{S^N} + N) \, \left(\tfrac{2}{1+|x|^2}\right)^{\frac{2-N}{2}} .$$

We also have

$$\int_{S^N} Z \, (\Delta + N) \, Z \, dvol_{S^N} = \int_{\mathbb{R}^N} \tilde{Z} \, L \, \tilde{Z} \, dvol_{\mathbb{R}^N} ,$$

where $\tilde{Z}$ and Z are related by

$$\tilde{Z} = \left(\frac{2}{1+r^2}\right)^{\frac{N-2}{2}} \, Z.$$

Now, the operator $\Delta_{S^N} + N$ has a $N+1$ dimensional kernel corresponding to the coordinate functions on S^N (since N is an eigenvalue of $-\Delta_{S^N}$). This implies that the L^2-null space of the operator L is $N+1$ dimensional and spanned by the functions

$$Z_j := \partial_{x_j} \omega, \qquad j = 1, \ldots, N, \quad \text{and} \qquad Z_{N+1} := x \cdot \nabla \omega + \tfrac{N-2}{2} \, \omega$$

(see (2.11)). The fact that $L \, Z_j = 0$ can also be checked directly or can be proved using the fact that (3.1) enjoys some translation and dilation invariance in the sense that, for all $\lambda > 0$ and $a \in \mathbb{R}^N$, the function

$$x \longmapsto \lambda^{\frac{n-2}{2}} \, u(\lambda \, x + a),$$

is a solution of (3.1) whenever u is a solution of (3.1). Differentiation with respect to λ or with respet to a, at $\lambda = 1$ and $a = 0$ directly shows that Z_j are solutions of $L \, Z_j = 0$.

Moreover, the space where the quadratic form

$$\tilde{Z} \longmapsto - \int_{S^N} \tilde{Z} \, (\Delta + N) \, \tilde{Z} \, dvol_{S^N} ,$$

is negative definite is one dimensional, and coincides with the space of constant functions, which implies that the space where

$$Z \longmapsto - \int_{\mathbb{R}^N} Z \, L \, Z \, dvol_{\mathbb{R}^N} ,$$

is negative is also one dimensional. Hence, the operator L_0 has one negative eigenvalue $-\lambda_1 < 0$, and we denote by Z_0 the corresponding eigenfunction (normalized to have L^2-norm equal to 1). See (2.12). We observe that this eigenfunction decays exponentially at infinity with exponential order $O(e^{-\sqrt{\lambda_1} \, |x|})$.

Having understood the point spectrum of the operator L we have the

Lemma 3.1. *Assume that $\xi \notin \{0, \pm\sqrt{\lambda_1}\}$. Then given $h \in L^\infty(\mathbb{R}^N)$, there exists a unique bounded solution of*

$$(L_0 - |\xi|^2) \, \psi = h$$

in $\mathbb{R}^N$. Moreover

$$\|\psi\|_{L^\infty} \leq c_\xi \, \|h\|_{L^\infty}$$

for some constant $c_\xi > 0$ only depending on ξ.

Proof. For all $r > 0$, we denote B_r the ball of radius r in $\mathbb{R}^N$ centered at the origin. We assume that $\xi \notin \{0, \pm\sqrt{\lambda_1}\}$ is fixed. We first prove that, there exists $r_\xi > 0$ (depending on ξ) such that, for all $r \geq r_\xi$, the following *a priori* estimate

$$\|\psi\|_{L^\infty(B_r)} \leq c_\xi \, \|(L - |\xi|^2)\, \psi\|_{L^\infty(B_r)} \tag{3.2}$$

holds for any bounded function ψ vanishing on ∂B_r.

Assume for the time being that this estimate is already proven. Then, for $r \geq r_\xi$, the operator $L_0 - |\xi|^2$ is injective on the ball of radius r (being understood that we consider 0 Dirichlet boundary conditions). Fredholm alternative implies that, for all $r \geq r_\xi$, we can find a unique solution of

$$(L_0 - |\xi|^2)\, \psi_r = h$$

on B_r with $\psi_r = 0$ on ∂B_r. Given a sequence where r_j tending to ∞, the *a priori* estimate (3.2), elliptic estimates and Ascoli-Arzela's theorem allow one to extract from $(\psi_{r_j})_j$ a subsequence which converges (uniformly on compacts) to a function ψ, solution of

$$(L_0 - |\xi|^2)\, \psi = h$$

in $\mathbb{R}^N$. Moreover, passing to the limit in (3.2), we find that $\|\psi\|_{L^\infty} \leq c_\xi \, \|h\|_{L^\infty}$. This completes the proof of the existence of ψ. Uniqueness follow at once from the fact that (3.2) extends to the case where the functions are defined on R^N.

It remains to prove the validity of relation (3.2). First observe that, since $\xi \neq 0$, there exists $\bar{r}_\xi > 0$ such that

$$p\, \omega^{p-1} - |\xi|^2 \leq -\tfrac{1}{2}\, |\xi|^2$$

in $\mathbb{R}^N \setminus B_{\bar{r}_\xi}$. Given $r > \bar{r}_\xi$ and using the constant function as a barrier, we find immediately the estimate

$$\|\psi\|_{L^\infty(B_r \setminus B_{\bar{r}_\xi})} \leq c_\xi \, \left(\|(L_0 - |\xi|^2)\, \psi\|_{L^\infty(B_r \setminus B_{\bar{r}_\xi})} + \|\psi\|_{L^\infty(\partial B_{\bar{r}_\xi})} \right), \tag{3.3}$$

for any bounded function ψ vanishing on ∂B_r.

We now argue by contradiction and assume that (3.2) does not hold. Then there exists a sequence of radii r_j tending to ∞ and functions ψ_j vanishing on ∂B_{r_j}, such that

$$\|\psi\|_{L^\infty(B_{r_j})} = 1$$

while

$$\lim_{j \to \infty} \|(L_0 - |\xi|^2)\, \psi_j\|_{L^\infty(B_{r_j})} = 0$$

Observe that, without loss of generality, we can assume that $r_j \geq \bar{r}_\xi$, and (3.3) implies that that $\|\psi_j\|_{L^\infty(B_{\bar{r}_\xi})}$ remains bounded away from 0 as j tends to ∞.

Elliptic estimates and Ascoli-Arzelá's theorem allow us to extract from $(\psi_j)_j$ a subsequence which converges (uniformly on compacts) to a function ψ, solution of

$$(L_0 - |\xi|^2)\, \psi = 0$$

in $\mathbb{R}^N$. Moreover, ψ is bounded and not identically equal to 0 (since $\|\psi_j\|_{L^\infty(B_{\bar r_\xi})}$ remains bounded away from 0). But, since $\xi \notin \{0, \pm\sqrt{\lambda_1}\}$, this contradicts the classification of the point spectrum of L. The proof of the *a priori* estimate is therefore complete. □

If x is the coordinate in $\mathbb{R}^N$, we denote by (x_0, x) the coordinate in $\mathbb{R} \times \mathbb{R}^N = \mathbb{R}^{N+1}$. We consider the operator

$$\tilde L := \partial_{00} + \Delta_{\mathbb{R}^N} + \tfrac{N+2}{N-2}\, \omega^{\frac{4}{N-2}},$$

The next result classifies the bounded solution of the homogeneous problem $\tilde L\, \phi = 0$ in $\mathbb{R}^{N+1}$.

Lemma 3.2. *The bounded solutions $(x_0, x) \longmapsto \phi(x_0, x)$ of the equation $\tilde L\, \phi = 0$ in $\mathbb{R}^{N+1}$ are all linear combinations of the functions*

$$(x_0, x) \longmapsto Z_j(x),$$

for $j = 1, \ldots, N+1$, and the functions

$$(x_0, x) \longmapsto Z_0(x)\, \cos(\lambda_0\, x_0) \qquad\qquad (x_0, x) \longmapsto Z_0(x)\, \sin(\lambda_0\, x_0).$$

Proof. Assume that ϕ is a bounded solution of $\tilde L\, \phi = 0$ in $\mathbb{R}^{N+1}$. We take Fourier transform in the x_0 variable and define

$$\hat\phi(\xi, x) := \int e^{i x_0 \xi}\, \phi(x_0, x)\, dx_0.$$

Then, $\hat\phi$ is a distribution which depends parametrically on x and which satisfies the equation

$$(L - |\xi|^2)\, \hat\phi = 0 \qquad \text{in} \quad \mathbb{R}^{N+1}. \tag{3.4}$$

The precise meaning of this equation is that

$$\int_{\mathbb{R}^{N+1}} \hat\phi\, \left(L - |\xi|^2\right)\, \psi\, d\xi\, dx = 0. \tag{3.5}$$

for any $\psi \in \mathcal{C}^\infty(\mathbb{R}^{N+1})$ which is rapidly decreasing in ξ (and decreasing enough in x so that $\psi(x_0, \cdot) \in L^1(\mathbb{R}^N)$, for all x_0).

We would like to show that $\hat\phi \equiv 0$. To this aim, we choose $g \in \mathcal{C}^\infty_c(\mathbb{R} \setminus \{0, \pm\lambda_0\})$ and $h \in \mathcal{C}^\infty_c(\mathbb{R}^N)$. We set

$$\zeta(\xi, x) := g(\xi)\, h(x)$$

and define $\psi(\xi, x)$ to be the unique bounded solution of the equation

$$\left(L - |\xi|^2\right)\, \psi = \zeta \quad \text{in } \mathbb{R}^N.$$

Here ξ is considered as a parameter. Using the result of Lemma 3.1, it is easy to check that ψ is well defined (since the function ζ is supported away from 0 and $\pm\lambda_0$), moreover $x \longmapsto \psi(\xi, x)$ is rapidly decreasing (in fact decays exponentially like the function $x \longmapsto e^{-|\xi|\, x}$). Also observe that ψ is compactly supported in the ξ variable. Inserting the function ψ in (3.5), we get

$$0 = \int_{\mathbb{R}^N} h(x)\, \langle\, \hat\phi(\cdot, x), g\rangle_{\mathcal{D}', \mathcal{D}}\, dx$$

Since h is arbitrary, we conclude that

$$\langle \hat\phi(\cdot,x)\,,\,g\,\rangle_{\mathcal{D}',\mathcal{D}}=0 \qquad\qquad \text{for all } x\in\mathbb{R}^N .$$

Since g is chosen arbitrarily, with compact support in $\mathbb{R}\setminus\{0,\pm\lambda_0\}$, by definition the distribution $\hat\phi(\cdot,x)$ has its support contained in this set. From standard distribution theory, this implies that $\hat\phi(\cdot,x)$ is a linear combination (with x-dependent coefficients) of derivatives of Dirac masses at the points $0,\pm\lambda_0$. Taking the inverse Fourier transform and using the fact that ϕ is bounded, we obtain the decomposition of ϕ as

$$\phi(x_0,x)=a(x)+b(x)\,\cos(\lambda_0\,x_0)+c(x)\,\sin(\lambda_0\,x_0).$$

Moreover, the functions a,b,c are bounded solutions of

$$L\,a=0,\qquad L\,b-\lambda_0^2\,b=0,\qquad L\,c-\lambda_0^2\,c=0,$$

in $\mathbb{R}^N$. This immediately implies that the function a must be a linear combination of the functions Z_j, for $j=1,\dots,N+1$, while the functions b,c have to be scalar multiples of Z_0. The proof of the result is complete. $\square$

We shall use the previous result in order to obtain a priori estimates and a solvability theory for problem (2.13). We consider here a slightly more general problem that involves the essential features needed. For a positive smooth function $R(y_0)$ and a constant $M>0$ we consider the domain $\mathcal{D}$ defined as

$$\mathcal{D}=\{(y_0,\bar y,y_N)\in\mathbb{R}^{N+1}\ /\ -R(y_0)<y_N<M,\ |\bar y|<M\}$$

and for functions ϕ defined on $\mathcal{D}$, an operator of the form

$$L(\phi):=b(y_0)\partial_{00}\phi+\Delta_y\phi+p\omega^{p-1}\phi+b_{ij}(y_0,y)\partial_{ij}\phi+b_i(y_0,y)\partial_i\phi+d(y_0,y)\phi$$

where $b_{00}\equiv 0$. Then for a given function h we want to solve the following projected problem.

$$\left\{\begin{array}{rcl} L(\phi) &=& h+\sum_{i=0}^{N+1}c_i(y_0)Z_i(y) \quad \text{in } \mathcal{D},\\ \phi &=& 0 \quad \text{on } \partial\mathcal{D},\\ \int_{\mathcal{D}_{y_0}}\phi(y_0,y)Z_i(y)dy &=& 0 \text{ for all } y_0\in\mathbb{R},\quad i=0,\dots,N, \end{array}\right. \tag{3.6}$$

where

$$\mathcal{D}_{y_0}=\{y\in\mathbb{R}^N\ /\ (y_0,y)\in\mathcal{D}\}.$$

We fix a number $2\le\nu<N$ and consider the following L^∞-weighted norms.

$$\|\phi\|_*=\sup_{\mathcal{D}}(1+|y|^{\nu-2})|\phi(y_0,y)|+\sup_{\mathcal{D}}(1+|x|^{\nu-1})|D\phi(x_0,x)|,$$

$$\|h\|_{**}=\sup_{\mathcal{D}}(1+|y|^{\nu})|h(y_0,y)|\ .$$

We assume that all functions involved are smooth. We will establish existence and uniform a priori estimates for problem (3.6) in the above norms, provided that appropriate bounds for the coefficients hold.

Proposition 3.1. *Assume that $N \geq 7$, $N-2 \leq \nu < N$. Assume that for a number $m > 0$ we have that*

$$m \leq b(y_0) \leq m^{-1} \quad \text{for all } y_0 \in \mathbb{R}.$$

Then there exist positive numbers δ, C such that if, for all i, j

$$\begin{aligned} \|\partial_0 R\|_\infty &+ M\|\partial_{00} R\|_\infty + M\|\partial_0 b\|_\infty + \|b_{ij}\|_\infty \\ &+ \|Db_{ij}\|_\infty + \|(1+|y|)b_i\|_\infty + \|(1+|y|^2)d\|_\infty < \delta, \end{aligned} \tag{3.7}$$

and

$$\delta^{-1} < R(y_0), \quad M^{-1}R(y_0) < \delta \quad \text{for all } y_0 \in \mathbb{R}, \tag{3.8}$$

then for any h with $\|h\|_{**} < +\infty$ there exists a unique solution $\phi = T(h)$ of problem (3.6) *with $\|\phi\|_* < +\infty$ we have*

$$\|\phi\|_* \leq C\|h\|_{**}.$$

Proof. The proof of this result will be carried out in three steps.

Step 1. Let us assume that in Problem (3.6) the coefficients b_i, d, and the functions c_i are identically zero. We will prove that δ, C as in the above statement can then be chosen so that for any h with $\|h\|_{**} < +\infty$ and any solution ϕ of problem (3.6) with $\|\phi\|_* < +\infty$ we have

$$\|\phi\|_* \leq C\|h\|_{**}.$$

To establish this we argue by contradiction, namely we assume the existence of b^n, ϕ_n, h_n, b^n_{ij}, R_n, M_n such that

$$m \leq b^n(y_0) \leq m^{-1} \quad \text{for all } x_0 \in \mathbb{R},$$

$$\|\phi_n\|_* = 1, \quad \|h_n\|_{**} \to 0,$$

$$M_n\|\partial_0 b^n\|_\infty + M_n^{-1}\|R_n\|_\infty + \|\partial_0 R_n\|_\infty + M_n\|\partial_{00} R_n\|_\infty + \|b^n_{ij}\|_\infty \to 0, \quad \inf_{x_0} R_n \to +\infty$$

and satisfy

$$b^n(y_0)\partial_{00}\phi_n + \Delta_y \phi_n + b^n_{ij}\partial_{ij}\phi_n + pw(y)^{p-1}\phi_n = h_n \quad \text{in} \quad \mathcal{D},$$

together with the orthogonality and boundary conditions.

To achieve a contradiction we will first show that

$$\|\phi_n\|_\infty \to 0. \tag{3.9}$$

If this was not the case then we may assume that there is a positive number γ for which $\|\phi_n\|_\infty > \gamma$. Since we also know that

$$|\phi(y_0, y)| \leq \frac{C}{(1+|y|)^{\nu-2}},$$

we conclude that for some $A > 0$,

$$\|\phi_n\|_{L^\infty(|x| \leq A)} \geq \gamma.$$

Let us fix an y_{0n} such that

$$\|\phi_n(y_{0n}, \cdot)\|_{L^\infty(|y| \leq A)} \geq \frac{\gamma}{2}.$$

By elliptic estimates and compactness of Sobolev embeddings, we see that we may assume that the sequence of functions $\phi(y_0 + y_{0n}, y)$ converges uniformly over compact subsets of $\mathbb{R}^{N+1}$ to a nontrivial, bounded solution of

$$\Delta_y \tilde{\phi} + a_0^\infty \partial_{00} \tilde{\phi} + p\omega(y)^{p-1} \tilde{\phi} = 0 \quad \text{in} \quad \mathbb{R}^{N+1}$$

where a_0^∞ is a positive constant, which with no loss of generality via scaling, we may assume equal to one. By virtue of Lemma 3.2 and the orthogonality conditions assumed, which pass to the limit thanks to dominated convergence, and the assumptions $N \geq 7$, $N-2 < \alpha$, we find then that $\tilde{\phi} \equiv 0$. This is a contradiction that shows the validity of statement (3.9).

Let us conclude now the result of Step 1. Since $\|\phi_n\|_* = 1$, there exists (y_{0n}, y_n) with $r_n := |y_n| \to +\infty$ such that

$$r_n^{\nu-2} |\phi_n(y_{0n}, y_n)| + r_n^{\nu-1} |D\phi_n(y_{0n}, y_n)| \geq \gamma > 0.$$

Let us consider now the scaled function

$$\tilde{\phi}_n(z, x) = r_n^{\nu-2} \phi_n(y_{0n} + r_n z_0, r_n z)$$

defined on $\tilde{\mathcal{D}}$ given by

$$\tilde{\mathcal{D}} = \{(z_0, \bar{z}, z_N) \ / \ -\tilde{R}_n(z_0) < z_N < M_n r_n^{-1}, \ |\bar{z}| < M_n r_n^{-1}\}.$$

with $\tilde{R}_n(z_0) = r_n^{-1} R_n(y_{0n} + r_n z_0)$. Note that $M_n r_n^{-1} \geq 1/\sqrt{2}$. Then we have

$$|\tilde{\phi}_n(z_0, z)| + |z||D\tilde{\phi}(z_0, z)| \leq |z|^{2-\nu} \quad \text{in } \tilde{\mathcal{D}}$$

and for some z_n with $|z_n| = 1$,

$$|\tilde{\phi}_n(0, z_n)| + |D\tilde{\phi}(0, z_n)| \geq \gamma > 0.$$

$\tilde{\phi}_n$ satisfies

$$\tilde{a}_{0n} \partial_{00} \tilde{\phi}_n + \Delta_z \tilde{\phi}_n + o(1) \partial_{ij} \tilde{\phi}_n + O(r_n^{-2}) |z|^{-4} \tilde{\phi}_n = \tilde{h}_n \quad \text{in } \tilde{\mathcal{D}}$$

where,

$$\tilde{h}_n(z_0, z) = r_n^\nu h_n(y_{0n} + r_n z_0, r_n z), \quad \tilde{b}^n(z_0) = b^n(y_{0n} + r_n z_0)$$

Let us observe that from the assumptions made we get

$$\|\partial_0 \tilde{b}^n\|_\infty + \|\partial_0 \tilde{R}_n\|_\infty + \|\partial_{00} \tilde{R}_n\|_\infty \to 0.$$

Then, we may assume that

$$\tilde{b}^n(z_0) \to b_* > 0,$$

and that the function $\tilde{\phi}_n$ converges uniformly, in C^1-sense over compact subsets of $\mathcal{D}_* \setminus \{z = 0\}$ to $\tilde{\phi}$ which satisfies

$$b_* \partial_{00} \tilde{\phi} + \Delta_z \tilde{\phi} = 0 \quad \text{in } \mathcal{D}_* \setminus \{z = 0\}.$$

where either

$$\mathcal{D}_* = \{(z_0, \bar{z}, z_N) \ / \ 0 < z_N < d_*, \ |\bar{z}| < d_*\}$$

with $1 < d_* < +\infty$, or

$$\mathcal{D}_* = \{(z_0, \bar{z}, z_N) \ / \ a_* < z_N\}$$

with $a_* \geq 0$ or

$$\mathcal{D}_* = \mathbb{R}^{N+1}.$$

and $\tilde\phi$ satisfies

$$|\tilde\phi(z_0, z)| + |z||\tilde\phi(z_0, z)| \leq |z|^{2-\nu} \quad \text{in } \mathbb{R}^{N+1}_{d_*} \setminus \{z = 0\}$$

with the value $\tilde\phi = 0$ assumed continuously on the boundary of $\partial\mathcal{D}_* \setminus \{z = 0\}$. Besides, since $\partial_{00}\tilde R_n$ is uniformly bounded, standard elliptic estimates at the boundary yield the presence of a uniform $C^{1,\alpha}$ bound for $\tilde\phi_n$, which thus implies that the limit of the derivative is uniform, therefore $\tilde\phi \not\equiv 0$. With no loss of generality we may assume that $b_* = 1$. If the singular line $z = 0$ lies inside $\mathcal{D}_*$, the fact that $\nu < N$ makes it removable. Indeed, the limit $\tilde\phi$ is easily seen to be weakly harmonic in $\mathcal{D}_*$. This plus boundedness the boundary value zero yields that $\tilde\phi \equiv 0$ in all cases. If the singularity lies on the boundary, this happens on the hyperplane $z_N = 0$. In such a case, an odd reflection reduces us to the case of the interior singularity, so that in any event, $\tilde\phi \equiv 0$. We have obtained a contradiction which concludes Step 1.

Step 2. We claim that the a priori estimate estimate obtained in Step 1 is in reality valid for the full problem (3.6), potentially reducing the value of δ. Let δ be a small number so that the conclusion of Step 1 holds. Now we additionally assume:

$$\|Db_{ij}\|_\infty + \|\,(1+|y|)b_i\|_\infty + \|\,(1+|y|^2)d\|_\infty \;\leq\; \delta \tag{3.10}$$

where δ will be taken smaller if necessary. Then there exist positive numbers δ, C such that if the conditions of Proposition 3.1 and estimate (3.10) hold for all i, j, then for any h with $\|h\|_{**} < +\infty$ and any solution ϕ of problem (3.6) with $\|\phi\|_* < +\infty$ we have that for all i,

$$|c_i|_\infty + \|\phi\|_* \leq C\|h\|_{**}.$$

Besides

$$c_l(y_0)\int_{\mathcal{D}_{y_0}} Z_l^2 = -\int_{\mathcal{D}_{y_0}} h(y_0, y)Z_l(y)\,dy \;+ o(1)\|h\|_{**}$$

where $o(1) \to 0$ as $\delta \to 0$.

Testing the equation against $Z_l(y)$ and integrating only in y we find

$$c_l(y_0)\int_{\mathcal{D}_{y_0}} Z_l^2 = b(y_0)\int_{\mathcal{D}_{y_0}} \partial_{00}\phi Z_l - \int_{\mathcal{D}_{y_0}} hZ_l + \int_{\mathcal{D}_{y_0}} b_{ij}\partial_{ij}\phi Z_l + \tag{3.11}$$

$$\int_{\mathcal{D}_{y_0}} (b_i\partial_i\phi + d\phi)Z_l + \int_{\mathbb{R}^{N-1}} Z_l(\bar y, R(y_0))\partial_{y_N}\phi(y_0, \bar y, R(y_0))\,d\bar y\,.$$

Now, we have that

$$|\int_{\mathbb{R}^{N-1}} Z(y', R(x_0))\partial_{y_N}\phi(x_0, y', R(x_0))\,dy'\;| \;\leq\; \|\phi\|_*\int_{\mathbb{R}^{N-1}} (|y'| + R(x_0))^{2-N+1-\alpha}\,dy' \leq \delta^\sigma\|\phi\|_*$$

for some $\sigma > 0$ depending on α and N. We immediately find that also

$$|\int_{\mathcal{D}_{y_0}} (b_i\partial_i\phi + c\phi)Z_l| \;\leq\; C\delta\|\phi\|_*,$$

while, integrating by parts in indices carrying the y' variables,

$$|\int_{\mathcal{D}_{y_0}} a_{ij}\partial_{ij}\phi Z_l| = |\int_{\mathcal{D}_{y_0}} \partial_i(a_{ij}Z_l)\partial_j\phi| \le C\delta\|\phi\|_*.$$

and

$$|\int_{\mathcal{D}_{y_0}} hZ_l| \le C\|h\|_{**}.$$

Now, we know that

$$\int_{\mathcal{D}_{y_0}} \phi(y_0,y)Z_l(y)dy = 0$$

and hence, using the boundary value zero,

$$\int_{\mathcal{D}_{y_0}} \partial_0\phi(y_0,y)Z_l(y)dy = 0,$$

or

$$\int_{\mathbb{R}^{N-1}} dy' \int_{-\infty}^{R(y_0)} \partial_0\phi(y_0,\bar{y},t)Z_l(y',t)\,dt \ = \ 0,$$

so that differentiating once more we find

$$0 \ = \ \int_{\mathcal{D}_{y_0}} \partial_{00}\phi Z_l\,dx + \partial_0 R(x_0)\int_{\mathbb{R}^{N-1}} \partial_0\phi(y_0,\bar{y},R(y_0))Z_l(y',R(y_0))\,dy'$$

from where it follows that

$$|\int_{\mathcal{D}_{y_0}} \partial_{00}\phi Z_l\,dy\,| \ \le C\delta^\sigma\|\phi\|_*.$$

Combining the above inequalities into (3.11) we then find the estimate

$$|c_l(y_0)| \le C(\|h\|_{**} + \delta^\sigma\|\phi\|_*)\ . \tag{3.12}$$

On the other hand, Lemma 3.1 implies that

$$\|\phi\|_* \le C[\|h\|_{**} + \sum_i \|c_iZ_i\|_{**}] \le C[\|h\|_{**} + \sum_i \|c_i\|_\infty + \delta\|\phi\|_*].$$

Combining this last inequality and (3.12), reducing the value of δ if necessary, we obtain that c_is are controlled by h,

$$\|c_i\|_\infty \le C\|h\|_{**},$$

and the result of Step 2 readily follows. □

Step 3. We shall discuss next the issue of existence for Problem (3.6), under the assumptions so that the result of Step 2 holds true. We consider first the case of right hand sides $h(y_0,y)$ which are T-periodic in y_0, for and arbitrarily large but fixed T, the same property being valid for the coefficients. This is in reality the assumption we need. We then look for a weak solution ϕ to (3.6) in the space H_T defined as the subspace of functions ψ which are in $H^1(B)$ for any bounded subset of $\mathcal{D}$, which are T-periodic in y_0, such that in addition $\psi = 0$ on $\partial\mathcal{D}$ in the trace sense and so that

$$\int_{\mathcal{D}_{y_0}} \psi(y_0, y) Z_j(y) dy = 0 \quad \text{for all } y_0 \in \mathbb{R}, \quad j = 0, \dots, N+1.$$

Let $\mathcal{D}_T = \{y \in \mathcal{D} \ y_0 \in (-T, T)\}$ and the bilinear form defined in H_T (after one integration by parts)

$$B(\phi, \psi) := \int_{\mathcal{D}_T} \psi L \phi.$$

Then Problem (3.6) gets weakly formulated as that of finding $\phi \in H_T$ such that

$$B(\phi, \psi) = \int_{\mathcal{D}_T} h\psi \quad \text{for all } \psi \in H_T$$

If h is smooth, elliptic regularity yields that a weak solution is a classical one. The weak formulation can be readily be put into the form

$$\phi + K(\phi) = \hat{h}$$

in H_T, where $\hat{h}$ is a linear operator of h and K is compact. The a priori estimate of Step 2 yields that for $h = 0$ only the trivial solution is present. Fredholm alternative thus applies yielding that problem (3.6) is thus solvable in the periodic setting. While this is enough for our purposes, it is worthwhile observing that approximating a general h by periodic functions of increasing period, and using the uniform estimate provided by Step 2, we obtain in the limit a solution to the problem with the desired property. This completes the proof of the proposition. □

4. Geometric setting

We consider the metric induced by the Euclidean one on $\partial\Omega$ and denote by $\bar{\nabla}$ the associated connection. We introduce Fermi coordinates in a neighborhood of Γ in

$$\Sigma := \partial\Omega.$$

Given $q \in \Gamma$, there is a natural splitting

$$T_q\Sigma = T_q\Gamma \oplus N_q\Gamma.$$

into the normal and tangent bundle over Γ. We assume that Γ is parameterized by arclength $x_0 \in (-\ell, \ell)$,

$$x_0 \longmapsto \gamma(x_0),$$

and denote by E_0 a unit tangent vector to Γ. In a neighborhood of a point q of Γ, assume that we are given an orthonormal basis E_i, $i = 1, \dots, N-1$, of $N_q\Gamma$. We can assume that E_i are parallel transported along Γ which means that

$$\bar{\nabla}_{E_0} E_i = 0$$

for $i = 1, \dots, N-1$. The geodesic condition for Γ translates precisely into

$$\bar{\nabla}_{E_0} E_0 \equiv 0.$$

To parameterize a neighborhood of $q \in \Gamma$ in Σ we define

$$F(x_0, \bar{x}) := \mathrm{Exp}^{\Sigma}_{\gamma(x_0)}(\sum_i x_i\, E_i), \qquad \bar{x} := (x_1, \dots, x_{N-1}),$$

where Exp^{Σ} is the exponential map on Σ and summation over $i = 1, \dots, N-1$ is understood. This parameterization induces coordinate vector fields

$$X_a := F_*(\partial_{x_a}),$$

for $a = 0, \dots, N-1$.

By construction $X_a = E_a$ along Γ and

$$\bar{\nabla}_{E_a} E_b = 0 \tag{4.1}$$

Let $\bar{g}$ denote the metric on Σ which is induced by the Euclidean metric. The Fermi coordinates above are defined in such a way that the coefficients of $\bar{g}$

$$\bar{g}_{ab} = X_a \cdot X_b,$$

are equal to δ_{ab} along Γ. We now compute higher terms in the Taylor expansions of the functions g_{ab}. The metric coefficients at $q := F(x_0, \bar{x})$ are given in terms of geometric data at $p := F(x_0, 0)$ and $\bar{x}$.

Notation The symbol $\mathcal{O}(|\bar{x}|^r)$ indicates a smooth function whose Taylor expansion does not involve any term up to order r in the variables x_i, $i = 1, \dots, N-1$.

We now give the expansion of the metric coefficients. The expansion of the $\bar{g}_{ij}$, $i, j = 1, \dots, N-1$, agrees with the well known expansion for the metric in normal coordinates [28], [23] or [31], but we briefly recall the proof here for completeness. We agree that indices $a, b, c, \dots$ run from 0 to $N-1$ while indices $i, j, k, \dots$ run from 1 to $N-1$.

Proposition 4.1. *At the point $q = F(x_0, \bar{x})$, the following expansions hold*

$$\begin{array}{rcl} \bar{g}_{ij} & = & \delta_{ij} + \frac{1}{3}\, \left(\bar{R}(E_i, E_k)\, E_j \cdot E_\ell\right)\, x_k\, x_l + \mathcal{O}(|\bar{x}|^3) \\ \bar{g}_{0i} & = & \mathcal{O}(|\bar{x}|^2) \\ \bar{g}_{00} & = & 1 + \left(\bar{R}(E_0, E_k)\, E_0 \cdot E_\ell\right)\, x_k\, x_\ell + \mathcal{O}(|\bar{x}|^3). \end{array} \tag{4.2}$$

where $i, j, k, \ell = 1, \dots, N-1$ and summation over repeated indices is understood. Here $\bar{R}$ denote the curvature tensor on $(\Sigma, \bar{g})$.

Proof: We compute

$$X_i\, \bar{g}_{ab} = \bar{\nabla}_{X_i} X_a \cdot X_b + X_a \cdot \bar{\nabla}_{X_i} X_b,$$

Using (4.1) we get

$$X_i\, \bar{g}_{ab} = 0.$$

along Γ. This yields the first order Taylor expansion

$$\bar{g}_{ab} = \mathcal{O}(|\bar{x}|^2).$$

To compute the second order terms, it is enough to compute $X_k\, X_k\, \bar g_{ab}$ at a point of Γ and then to polarize (i.e. replace X_k by $X_i + X_j$, $\ldots$). We compute

$$X_k\, X_k\, \bar g_{ab} = \bar\nabla^2_{X_k} X_a \cdot X_b + X_a \cdot \bar\nabla^2_{X_k} X_b + 2\, \bar\nabla_{X_k} X_a \cdot \bar\nabla_{X_k} X_b \tag{4.3}$$

Recall that, since X_a are coordinate vector fields, we have

$$\bar\nabla^2_{X_k} X_a = \bar\nabla_{X_k}\, \bar\nabla_{X_a}\, X_k = \bar\nabla_{X_a}\, \bar\nabla_{X_k} X_k + \bar R(X_k, X_a)\, X_k. \tag{4.4}$$

Therefore, we get

$$\begin{aligned} X_k\, X_k\, \bar g_{ab} &= 2\, \bar R(X_k, X_a) X_k \cdot X_b + 2\, \bar\nabla_{X_k} X_a \cdot \bar\nabla_{X_k} X_b \\ &+ \bar\nabla_{X_a}\, \bar\nabla_{X_k} X_k \cdot X_b + X_a \cdot \bar\nabla_{X_b}\, \bar\nabla_{X_k} X_k \end{aligned} \tag{4.5}$$

Using this, together with (4.1) we get

$$E_k\, E_k\, \bar g_{ij} \quad = \quad 2\, \bar R(E_k, E_i) E_k \cdot E_i + \bar\nabla_{E_i}\, \bar\nabla_{E_k} E_k \cdot E_j + E_i, \bar\nabla_{E_j}\, \bar\nabla_{E_k} E_k \tag{4.6}$$

along Γ. To proceed, first observe that

$$\bar\nabla_X X_{|p} = \bar\nabla^2_X X = 0$$

along Γ, for any $X \in N_p\Gamma$. Indeed, for all $p \in \Gamma$, $X \in N_p\Gamma$ is tangent to the geodesic $s \longrightarrow \exp^\Sigma_p(sX)$, and so $\bar\nabla_X X = \bar\nabla^2_X X = 0$ at p. In particular, taking $X = X_k + \varepsilon\, X_j$, we obtain

$$0 = \bar\nabla_{E_k + \varepsilon\, E_j}\, \bar\nabla_{E_k + \varepsilon\, E_j} (E_k + \varepsilon\, E_j)$$

Equating the coefficient of ε to 0 gives $\bar\nabla_{E_j}\, \bar\nabla_{E_k} E_k = -2\, \bar\nabla_{E_k}\, \bar\nabla_{E_k} E_j$, and hence

$$3\, \bar\nabla^2_{E_k}\, E_j = \bar R(E_k, E_j)\, E_k.$$

So finally, using (4.3) together with (4.6), we get

$$E_k\, E_k\, \bar g_{ij} = \tfrac{2}{3}\, \left(\bar R(E_k, E_i)\, E_k \cdot E_j \right)$$

along Γ. The formula for the second order Taylor coefficient for $\bar g_{ij}$ now follows at once.

Finally, it follows from (4.5) together with (4.1) that

$$E_k\, E_k\, \bar g_{00} \quad = \quad 2\, \bar R(E_k, E_0) E_k \cdot E_0 + 2\, \bar\nabla_{E_0}\, \bar\nabla_{E_k} E_k \cdot E_0$$

along Γ. Since $\bar\nabla_{E_k} E_k = 0$ along K, we also get $\bar\nabla_{E_0}\, \bar\nabla_{E_k} E_k = 0$ along Γ. We conclude that

$$E_k\, E_k\, \bar g_{00} = 2\, \left(\bar R(E_k, E_0) E_k \cdot E_0 \right)$$

along Γ and this gives the formula for the second order Taylor expansion for $\bar g_{00}$. □

Notation In what follows in the paper, we will use the following notation

$$R_{ijlm} = \left(\bar R(E_i, E_j) E_l \cdot E_m \right). \tag{4.7}$$

To parameterize a neighborhood of a point $q \in \Gamma$ in $\bar\Omega$, we consider the system of coordinates $(x_0, x) \in \mathbb{R}^{N+1}$ introduced in (1.8) given by

$$G(x_0, x) = F(x_0, \bar x) - x_N\, \mathbf{n}(F(x_0, \bar x)), \quad x = (\bar x, x_N) \in \mathbb{R}^N$$

where $x \in \mathbb{R}^N$ is close to 0 and $\mathbf{n}$ designates the outward unit normal to Σ.

In these coordinates, the coefficients of the Euclidean metric read

$$g_{NN} = 1, \qquad \text{and} \qquad g_{aN} = g_{Na} = 0, \tag{4.8}$$

for all $a = 0, \ldots, N-1$. Finally, for $a, b = 0, \ldots, N-1$, the coefficients g_{ab} can be expanded, in powers of x_N as

$$g_{ab} = \bar{g}_{ab} + 2\,\bar{h}_{ab}\,x_N + \bar{k}_{ab}\,x_N^2 + O(x_N^3)$$

where $\bar{g}$ is the metric on Σ whose expansion has been given in the last section,

$$\bar{h}_{ab} := -E_a \cdot \nabla_{E_b} \mathbf{n} = -E_b \cdot \nabla_{E_a} \mathbf{n} \tag{4.9}$$

are the coefficients of the second fundamental form $\bar{h}$ of Σ and

$$\bar{k}_{ab} := (\bar{h} \otimes \bar{h})_{ab} = \sum_{c,d} \bar{h}_{ac}\,\bar{g}^{cd}\bar{h}_{db} \tag{4.10}$$

are the coefficients of the square of the second fundamental form. An important remark is that $\bar{h}_{00}$, computed along Γ, is a smooth function of the arclength which represent the normal curvature along the geodesic in the sense that

$$\partial^2_{x_0}\gamma = \nabla_{E_0} E_0 = \bar{h}_{00}\,\mathbf{n} \tag{4.11}$$

along Γ.

Building on the expansion of the metric, which has been obtained above, we give the expansion of the Laplace operator in the above defined coordinates. Recall that the Laplacian is given, in terms of the coefficients of the metric, by

$$\Delta = \frac{1}{\sqrt{|g|}}\partial_{x_\alpha}\left(\sqrt{|g|}\,g^{\alpha\beta}\,\partial_{x_\beta}\right) = g^{pq}\,\partial_{x_\alpha}\,\partial_{x_\beta} + \partial_p\,g^{\alpha\beta}\,\partial_{x_\beta} + \tfrac{1}{2}\,\mathrm{Tr}_g(\partial_{x_\alpha} g)\,g^{\alpha\beta}\,\partial_{x_\beta}$$

where the indices α, β run from 0 to N and where $|g|$ denotes the determinant of the metric. Since (4.8) holds, the above formula simplifies into

$$\Delta = \partial^2_{x_N} + \tfrac{1}{2}\,\mathrm{Tr}_g(\partial_{x_N} g)\,\partial_{x_N} + g^{ab}\partial_{x_a}\partial_{x_b} + \partial_{x_a} g^{ab}\partial_{x_b} + \tfrac{1}{2}\,\mathrm{Tr}_g(\partial_{x_a} g)\,g^{ab}\,\partial_{x_b},$$

where the indices a, b run from 0 to $N-1$.

We have the following decomposition (recall that we agree that the indices $i, j, k, \ell, m, \ldots$ run from 1 to $N-1$) :

$$\begin{array}{rcl} \Delta & = & \partial^2_{x_0} + \sum_j \partial^2_{x_j} + \partial^2_{x_N} + A^{00}\,\partial^2_{x_0} + \sum_j\, A^{0j}\,\partial_{x_0}\,\partial_{x_j} \\ & + & \sum_{i,j}\left(-\frac{1}{3}\sum_{k,l}\,\left(\bar{R}(E_i, E_k)\,E_j \cdot E_\ell\right)\,x_k\,x_\ell - 2\,\bar{h}_{ij}\,x_N + A^{ij}\right)\partial_{x_i}\partial_{x_j} \\ & + & B^0\,\partial_{x_0} + \sum_j\,\left(\sum_k\,\left(\frac{2}{3}\,(\bar{R}(E_i, E_j)\,E_i \cdot E_k) + (\bar{R}(E_0, E_j)E_0 \cdot E_k)\right)\,x_k + B^j\right)\,\partial_{x_j} \\ & + & \left(\mathrm{Tr}_{\bar{g}}\,\bar{h} - \mathrm{Tr}_{\bar{g}}\,\bar{k}\,x_N + B^N\right)\,\partial_{x_N} \end{array} \tag{4.12}$$

where the curvature tensor $\bar{R}$, the metric $\bar{g}$ and the tensors $\bar{h}$ and $\bar{k}$ are computed along Γ, and hence only depend on x_0 while the functions $A^{\alpha\beta}$ and B^α do depend on $x_0, x_1, \ldots, x_N$ and enjoy the following decomposition

$$\begin{aligned}
A^{00} &= A^{00}_N\, x_N + \textstyle\sum_{k,\ell} A^{00}_{k\ell}\, x_k\, x_\ell \\
A^{ij} &= A^{ij}_N\, x_N^2 + \left(\textstyle\sum_k A^{ij}_{Nk}\, x_k\right) x_N + \textstyle\sum_{k,\ell,m} A^{ij}_{k\ell}\, x_k\, x_\ell\, x_m \\
A^{0j} &= A^{0j}_N\, x_N + \textstyle\sum_{k,\ell} A^{0j}_{k\ell}\, x_k\, x_\ell \\
B^{0} &= B^{0}_N\, x_N + \textstyle\sum_{k} B^{0}_{k}\, x_k \\
B^{j} &= B^{j}_N\, x_N + \textstyle\sum_{k,\ell} B^{j}_{k\ell}\, x_k\, x_\ell \\
B^{N} &= B^{N}_N\, x_N^2 + \left(\textstyle\sum_k B^{N}_{k}\, x_k\right) x_N + \textstyle\sum_{j} B^{N}_{j}\, x_j
\end{aligned} \tag{4.13}$$

Here the functions $A^{00}_N, A^{00}_{k\ell}, A^{ij}_N, \ldots$ and the functions $B^0_N, B^0_k, B^j_N, \ldots$ are smooth functions depending on $x_0, \ldots, x_N$ hence they can be further decomposed using Taylor's expansion. More precise expansions can be given in terms of the geometric data defined above but they will not appear in the final result so we have chosen to leave the expansion as it is. For example A^{0j}_N can be further expanded in powers of x_N and we have

$$A^{0j}_N = -4\, \bar h_{0j}\, x_N + \tilde A^{0,j}_N\, x_N^2,$$

where $\tilde A^{0j}_N$ is a smooth function depending on $x_0, \ldots, x_N$.

5. Construction of a first approximation

This section is devoted to the construction of an approximation for a solution to our problem

$$\Delta u + u^{\frac{N+2}{N-2}-\varepsilon} = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega. \tag{5.1}$$

As explained in Section 2, the idea is to build the approximation using the standard bubble ω in $\mathbb{R}^N$, solution of

$$\Delta u + u^{\frac{N+2}{N-2}} = 0 \quad \text{in} \quad \mathbb{R}^N,$$

centered and translated along a curve which is located inside the domain Ω and, at the same time, very close to the geodesic Γ in $\partial\Omega$. We will thus first introduce a precise description of the approximation in a region extremely close to the geodesic, without taking into account the outer region. Since the solution turns out to be very concentrated, this description is accurate enough and a gluing procedure we perform in Section 6 is the key instrument to gather together this thin region close to the geodesic with the outer region.

Let $(x_0, x) \in \mathbb{R}^{N+1}$ be the local coordinates along the geodesic introduced in (1.8). We perform the change of variables introduced in Section 2, formula (2.7),

$$u(G(x_0, x)) = \mu_\varepsilon^{-\frac{N-2}{2}}\, v(\rho^{-1} x_0, \mu_\varepsilon^{-1}(x - d_\varepsilon)), \quad v = v(y_0, y), \quad \rho = \varepsilon^{\frac{N-1}{N-2}},$$

where

$$\mu_\varepsilon(x_0) = \rho \tilde\mu_\varepsilon(x_0), \quad d_\varepsilon(x_0) = \varepsilon \tilde d_\varepsilon(x_0) \tag{5.2}$$

are function of the arclength $x_0 \in (-\ell, \ell)$, see (2.6). We now need to be more precise in the description of μ_ε and d_ε. We assume that

$$\tilde{\mu}_\varepsilon(x_0) = \mu^0_\varepsilon(x_0) + \varepsilon\mu(x_0), \quad \tilde{d}_{\varepsilon N}(x_0) = d^0_{\varepsilon N}(x_0) + \varepsilon d_N(x_0), \tag{5.3}$$

and

$$\tilde{d}_{\varepsilon j}(x_0) = \varepsilon d_j(x_0) \quad \text{ for all } j = 1, \dots, N-1. \tag{5.4}$$

In (5.3), μ^0_ε and $d_{\varepsilon N}(x_0)$ are explicit smooth functions of x_0 of the form

$$\mu^0_\varepsilon = \mu_0(x_0) + \varepsilon^{\frac{1}{N-2}}\mu_1(x_0), \quad d_{\varepsilon N}(x_0) = d_{0N}(x_0) + \varepsilon^{\frac{1}{N-2}} d_{1N}(x_0), \tag{5.5}$$

with

$$\mu_0(x_0) = \frac{\alpha}{\bar{h}_{00}(x_0)}, \quad d_{0N}(x_0) = \frac{\beta}{\bar{h}_{00}(x_0)}, \tag{5.6}$$

where α and β are positive constants depending only on the dimension N and $\bar{h}_{00}$ is the normal curvature along the geodesic Γ, which is assumed to be smooth and strictly positive, see (4.11). The functions μ_1, d_{1N} in (5.5) are smooth functions of x_0, uniformly bounded in ε together with their derivatives, whose precise definition we give later in Section 5, (5.37).

Finally in (5.3) and (5.4), we assume that μ, $d = (d_1, \dots, d_{N-1}, d_N)$ are parameter functions defined in $(-\ell, \ell)$ to be adjusted only in the final finite dimensional reduction. For now, we assume they are smooth functions of x_0 and that they have the following norms bounded

$$\|\mu\|_a = \|\varepsilon^{\frac{N}{N-2}}\ddot{\mu}\|_\infty + \|\varepsilon^{\frac{N}{2(N-2)}}\dot{\mu}\|_\infty + \|\mu\|_\infty \tag{5.7}$$

and

$$\|d\|_d = \|d_N\|_b + \sum_{j=1}^{N-1} \|d_j\|_c, \tag{5.8}$$

where

$$\|d_N\|_b = \|\varepsilon \ddot{d}_N\|_\infty + \|\varepsilon^{\frac{1}{2}}\dot{d}_N\|_\infty + \|d_N\|_\infty, \tag{5.9}$$

for $j = 1, \dots, N-1$,

$$\|d_j\|_c = \|\ddot{d}_j\|_\infty + \|\dot{d}_j\|_\infty + \|d_j\|_\infty. \tag{5.10}$$

In the previous expressions and in the rest of the paper, with the notation $\dot{}$ we denote the derivative with respect to x_0.

The (y_0, y) variable belong to the set $\mathcal{D}$ defined in (2.9). We recall the definition of $\mathcal{D}$

$$\mathcal{D} = \{(y_0, \bar{y}, y_N) \ / \ -\frac{d_{\varepsilon N}}{\mu_\varepsilon}(\rho y_0) < y_N < \frac{\hat{\delta}}{\rho}, \quad |\bar{y}| < \frac{\hat{\delta}}{\rho}\},$$

for some fixed positive number $\hat{\delta}$ we will chose later. The domain $\mathcal{D}$ is expanding, as $\varepsilon \to 0$ to the whole space $\mathbb{R}^N$. Observe that, with our choice of μ_ε and $d_{\varepsilon N}$ in (5.3)-(5.5), we have

$$-\frac{d_{\varepsilon N}}{\mu_\varepsilon} = -\varepsilon^{-\frac{1}{N-2}}\left[\gamma + \varepsilon^{\frac{1}{N-2}}O(1)\right], \tag{5.11}$$

where γ is a positive constant, depending only on N, and where $O(1)$ denotes a smooth function of x_0, which is uniformly bounded in ε, together with its derivative, for μ and d with $\|\mu\|_a + \|d\|_d \le c$

(see (5.7)-(5.8)). In particular, the function $\frac{d_{\varepsilon N}}{\mu_\varepsilon}$ satisfies assumption (3.8). Not only this. We have that

$$\|\partial_0 \left(\frac{d_{\varepsilon N}}{\mu_\varepsilon}\right)\|_\infty \leq c\rho\varepsilon^{-\frac{1}{N-2}} \left(\varepsilon\|\dot\mu\|_\infty + \varepsilon\|\dot d_N\|_\infty\right) \leq c\varepsilon^{\frac{3}{2}},$$

and

$$\rho^{-1}\|\partial_{00} \left(\frac{d_{\varepsilon N}}{\mu_\varepsilon}\right)\|_\infty \leq c\rho\varepsilon^{-\frac{1}{N-2}} \left(\varepsilon\|\ddot\mu\|_\infty + \varepsilon\|\ddot d_N\|_\infty\right) \leq c\varepsilon^{\frac{3N-8}{2(N-2)}}.$$

Thus the function $\frac{d_{\varepsilon N}}{\mu_\varepsilon}$ satisfies (3.7).

We also define

$$\mathcal{D}_{y_0} = \{y \ / \ (y_0, y) \in \mathcal{D}\}, \tag{5.12}$$

As we rigorously prove in Lemma 5.1 below, the Laplace operator whose expansion is described in (4.12), after the change of variable (2.7) gets transformed by the following relation

$$\mu_\varepsilon^{\frac{N+2}{2}} \Delta u = \mathcal{A}(v) \tag{5.13}$$

where, in the region $\mathcal{D}$, the differential operator $\mathcal{A}$ can be written in the following compact form

$$\mathcal{A}v = a_0\partial_0^2 v + \Delta_y v + \tilde{\mathcal{A}}v. \tag{5.14}$$

In (5.14), a_0 is given by

$$a_0 = \left(\mu_0 + \varepsilon^{\frac{1}{N-2}}\mu_1 + \varepsilon\mu\right)^2,$$

see (2.10). Observe that

$$\rho^{-1}\|\partial_0 a_0\|_\infty \leq c\varepsilon\|\dot\mu\|_\infty \leq c\varepsilon^{\frac{N-4}{2(N-2)}},$$

thus the function a_0 satisfies (3.7).

Furthermore, in $\mathcal{D}$ the differential operator $\tilde{\mathcal{A}}$ can be described as follows

$$\tilde{\mathcal{A}}v = \sum_{(\alpha,\beta)} a_{\alpha,\beta}\partial_{\alpha,\beta}v + \sum_\alpha b_\alpha \partial_\alpha v + cv, \tag{5.15}$$

where $a_{\alpha,\beta}$, d_α and c are functions of the variable $(\rho y_0, y)$, depending in an algebraic way on the parameter functions μ_ε and d_ε. More precisely, given the choice in (5.2), (5.3) and (5.4), one has, in the region under consideration,

$$a_{\alpha,\beta} = O(\varepsilon + \rho^2|y|^2) \quad \text{if} \quad \alpha \neq 0,\, \beta \neq 0, \quad a_{0,\beta} = O(\varepsilon), \quad \text{and} \quad a_{0,0} = 0,$$

while

$$b_\alpha = \rho O(\varepsilon + \rho|y|) \quad \text{and} \quad c = \rho^2 O(1).$$

Condition (3.7) is thus satisfied by the differential operator $\mathcal{A}$. This fact, together with the estimates on $\frac{d_{\varepsilon N}}{\mu_\varepsilon}$ in the definition of $\mathcal{D}$ in (2.9), gives that the linear theory developed in Section 3 for the linear operator $\mathcal{A} + p\omega^{p-1}$ in the domain $\mathcal{D}$ can be applied.

Next Lemma gives the detailed computation of the differential operator $\mathcal{A}$ in terms of the geometry of the problem.

Lemma 5.1. *After the change of variable (2.7), the following holds true*

$$\mu_\varepsilon^{\frac{N+2}{2}} \Delta u = \mathcal{A}(v) := a_0 \partial_0^2 v + \Delta_y v + \sum_{k=0}^{5} \mathcal{A}_k v + B(v), \tag{5.16}$$

where a_0 is defined in (2.10). In the previous expression $\mathcal{A}_k$ denotes the following differential operators

$$\begin{aligned}
\mathcal{A}_0 v \;&=\; \dot\mu_\varepsilon^2 \left[D_{yy} v\,[y]^2 + 2(1+\gamma) D_y v[y] + \gamma(1+\gamma) v\right] \\
&+\; \dot\mu_\varepsilon \left[D_{yy} v[y] + \gamma D_y v\right]\, [\dot d_\varepsilon] + D_{yy} v\,[\dot d_\varepsilon]^2 \\
&-\; 2\mu_\varepsilon \left[\varepsilon^{-\frac{N-1}{N-2}} D_y(\partial_0 v)[\dot\mu_\varepsilon y + \dot d_\varepsilon] + \gamma \dot\mu_\varepsilon \varepsilon^{-\frac{N-1}{N-2}} \partial_0 v\right] \\
&-\; \mu_\varepsilon D_y\, v\,[\ddot d_\varepsilon] - \mu_\varepsilon \ddot\mu_\varepsilon \left(\gamma v + D_y v\,[y]\right),
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
\mathcal{A}_1\, v \;&=\; \textstyle\sum_{i,j} \left[-\frac{1}{3} R_{ikjl} (\mu_\varepsilon y_k + d_{\varepsilon k})(\mu_\varepsilon y_l + d_{\varepsilon l}) - 2\bar h_{ij}(\mu_\varepsilon y_N + d_{\varepsilon N})\right. \\
&+\; \left.\textstyle\sum_k a_{Nk}^{ij} (\mu_\varepsilon y_k + d_{\varepsilon k})(\mu_\varepsilon y_N + d_{\varepsilon N})\right] \partial_{ij} v,
\end{aligned} \tag{5.18}$$

where R_{ikjl} is defined in (4.7), $\bar h_{ij}$ is given in (4.9) and the functions $a_{Nk}^{ij} = a_{Nk}^{ij}(\varepsilon^{\frac{N-1}{N-2}} y_0)$ are given by

$$A_{Nk}^{ij} = a_{Nk}^{ij} x_N + O(x_N^2),$$

with A_{Nk}^{ij} defined in (4.13). Furthermore,

$$\begin{aligned}
\mathcal{A}_2 v \;&=\; \textstyle\sum_j \left[-4\bar h_{0j}(\mu_\varepsilon y_N + d_{\varepsilon N}) \times\right. \\
&\left.\left(-D_y(\partial_j v)\,[d] + \mu_\varepsilon \varepsilon^{-\frac{N-1}{N-2}} \partial_{0j} v - (\gamma \partial_j v + D_y(\partial_j v)\,[y])\dot\mu_\varepsilon\right)\right]
\end{aligned} \tag{5.19}$$

and

$$\begin{aligned}
\mathcal{A}_3 v \;&=\; \left(\textstyle\sum_k b_k^0 [\mu_\varepsilon y_k + d_{\varepsilon k}] + b_N^0 (\mu_\varepsilon y_N + d_{\varepsilon N})\right) \times \\
&\left\{\mu_\varepsilon \left[-D_y v\,[\dot d_\varepsilon] + \mu_\varepsilon \varepsilon^{-\frac{N-1}{N-2}} \partial_0 v - \dot\mu_\varepsilon(\gamma v + D_y v\,[y])\right]\right\}
\end{aligned} \tag{5.20}$$

where b_k^0 are smooth functions of $\varepsilon^{\frac{N-1}{N-2}} y_0$ given by

$$B_k^0 = b_k^0 x_N + O(x_N^2)$$

(see (4.13) for B_k^0). Finally,

$$\mathcal{A}_4 v \;=\; \sum_j \left[\sum_k \left(\frac{2}{3} R_{ijik} + R_{0j0k}\right)(\mu_\varepsilon y_k + d_{\varepsilon k}) + b_N^j (\mu_\varepsilon y_N + d_{\varepsilon N})\right] \mu_\varepsilon \partial_j v, \tag{5.21}$$

where b_N^j are smooth functions of $\varepsilon^{\frac{N-1}{N-2}} y_0$ given by

$$B_N^j = b_N^j x_N + O(x_N^2)$$

(see (4.13) for B_N^j), and

$$\mathcal{A}_5 v \;=\; \left(Tr_{\bar g}\bar h - Tr_{\bar g}\bar k(\mu_\varepsilon y_N + d_{\varepsilon N})\right) \mu_\varepsilon \partial_N v, \tag{5.22}$$

where $\bar h$ is given by (4.9) and $\bar k$ by (4.10). The operator $B(v)$ can be described as follows

$$\begin{aligned}
B(v) \;&=\; O\left(|\mu_\varepsilon \bar y + \bar d_\varepsilon|^2 + (\mu_\varepsilon y_N + d_{\varepsilon N}) + (\mu_\varepsilon y_N + d_{\varepsilon N})(\mu_\varepsilon \bar y + \bar d_\varepsilon)\right) \mathcal{A}_0(v)\\
&+\; O\left(|\mu_\varepsilon \bar y + \bar d_\varepsilon|^3 + (\mu_\varepsilon y_N + d_{\varepsilon N})|\mu_\varepsilon \bar y + \bar d_\varepsilon|^2 + (\mu_\varepsilon y_N + d_{\varepsilon N})^2\right) \partial_{ij} v\\
&+\; O\left(|\mu_\varepsilon \bar y + \bar d_\varepsilon|^2 + (\mu_\varepsilon y_N + d_{\varepsilon N})|\mu_\varepsilon \bar y + \bar d_\varepsilon| + (\mu_\varepsilon y_N + d_{\varepsilon N})^2\right)\\
&\times\; \Big[\mu_\varepsilon \varepsilon^{-\frac{N-1}{N-2}} \partial_{0j} v + \mu_\varepsilon \varepsilon^{-\frac{N-1}{N-2}} \partial_0 v - D_y(\partial_j v)\,[d_\varepsilon]\\
&-\; (\gamma \partial_j v + D_y(\partial_j v)\,[y])\dot\mu_\varepsilon - D_y v \dot d_\varepsilon - \dot\mu_\varepsilon(\gamma v + D_y v[y]) + \mu_\varepsilon \partial_j v\Big]\\
&+\; O\left((\mu_\varepsilon \bar y + \bar d_\varepsilon)^2 + (\mu_\varepsilon \bar y + \bar d_\varepsilon)(\mu_\varepsilon y_N + d_{\varepsilon N}) + (\mu_\varepsilon y_N + d_{\varepsilon N})^2\right) \mu_\varepsilon \partial_N v.
\end{aligned}$$

Proof. We will show first that

$$\mu_\varepsilon^{\gamma+2} \partial_0^2 u(x_0, x) = \rho^{-2} \mu_\varepsilon^2 \partial_0^2 v(y_0, y) + \mathcal{A}_0(v(y_0, y)). \tag{5.23}$$

If $v = v(y_0, y)$, we define

$$\tilde v(z_0, z, \mu_\varepsilon) := \mu_\varepsilon^{-\gamma} v(z_0, \mu_\varepsilon^{-1} z).$$

We have $u(x_0, x) = \tilde v(\rho^{-1} x_0, x - d, \mu_\varepsilon)$. Then we compute

$$\partial_0 u = D_z \tilde v\,[-\dot d_\varepsilon] + \rho^{-1} \partial_0 \tilde v + \dot\mu_\varepsilon \partial_{\mu_\varepsilon} \tilde v,$$

and

$$\begin{aligned}
\partial_0^2 u = D_{zz} \tilde v [\dot d_\varepsilon]^2 + \rho^{-2} \partial_0^2 \tilde v + \dot\mu_\varepsilon^2 \partial_{\mu_\varepsilon}^2 \tilde v - 2\rho^{-1} D_z(\partial_0 \tilde v)[\dot d_\varepsilon] + 2\rho^{-1} \dot\mu_\varepsilon \partial_{0\mu_\varepsilon} \tilde v - 2\dot\mu_\varepsilon D_z(\partial_{\mu_\varepsilon} \tilde v)[\dot d_\varepsilon]\\
- D_z \tilde v\,[\ddot d_\varepsilon] - \ddot\mu_\varepsilon \partial_{\mu_\varepsilon} \tilde v.
\end{aligned}$$

Thus formula (5.23) follows expressing the previous computations in terms of v. To get the rest of (5.16), one argues in a similar way. □

With respect to the local coordinates along the geodesic Γ previously introduced and after scaling the variables as in (2.7), the original equation reduces locally close to the geodesic to

$$\mathcal{A} v + \mu_\varepsilon^{-\frac{N-2}{2}\varepsilon} v^{p-\varepsilon} = 0, \tag{5.24}$$

where $\mathcal{A}$ is defined in (5.14) and $p = \frac{N+2}{N-2}$. We denote by S_ε the operator given by (5.24), namely

$$S_\varepsilon(v) := \mathcal{A} v + \mu_\varepsilon^{-\frac{N-2}{2}\varepsilon} v^{p-\varepsilon} \tag{5.25}$$

In the rest of this section we study equation (5.24) in the set $(y_0, y) \in \mathcal{D}$ and we build an approximate solution to (5.24) which furthermore satisfies zero Dirichlet boundary condition in the region $y_N = -\frac{d_{\varepsilon N}}{\mu_\varepsilon}$. Indeed, our approximation close to the geodesic is

$$\mathtt{W} = \mathtt{w} + \Pi. \tag{5.26}$$

We start with the description of $\mathtt{w}$. The definition of Π will be given at the end of this section.

We define $\mathtt{w}$ to be given by

$$\mathtt{w} = \tilde{\omega} + e_\varepsilon(\rho y_0)\chi_\varepsilon(y) Z_0. \tag{5.27}$$

The first term in (5.27) is $\tilde{\omega}$ defined as follows

$$\tilde{\omega}(y) := (1+\alpha_\varepsilon)\left(\omega(y) - \bar{\omega}(y)\right), \tag{5.28}$$

with ω given in (2.4), $\alpha_\varepsilon := \mu_\varepsilon^{\frac{(N-2)^2}{8}\varepsilon} - 1$ and $\bar{\omega}$

$$\bar{\omega}(y) = \omega(\bar{y}, y_N + 2\frac{d_{\varepsilon N}}{\mu_\varepsilon}).$$

Observe that

$$\Delta\left((1+\alpha_\varepsilon)\omega\right) + \mu_\varepsilon^{-\frac{N-2}{2}\mu_\varepsilon}\left((1+\alpha_\varepsilon)\omega\right)^p = 0 \quad \text{in} \quad \mathbb{R}^N.$$

In the second term in (5.27), Z_0 denotes the first eigenfunction in $L^2(\mathbb{R}^N)$ of the problem

$$\Delta\phi + pw(x)^{p-1}\phi = \lambda\phi \quad \text{in} \quad \mathbb{R}^N, \quad \lambda_1 > 0$$

with $\int Z_0^2 = 1$ and χ_ε is a cut off function defined as follows. Let $\chi = \chi(s)$, for $s \in \mathbb{R}$, with $\chi(s) = 1$ if $s < \hat{\delta}$, $\chi(s) = 0$ if $s > 2\hat{\delta}$, for some fixed $0 < \hat{\delta}$ chosen in such a way that $\chi_\varepsilon(\bar{y}, -\frac{d_{\varepsilon N}}{\mu_\varepsilon}) = 0$, where $\chi_\varepsilon(y) = \chi(\varepsilon^{\frac{1}{N-2}}|y|)$. Observe that the function $\mathtt{w}$ satisfies the Dirichlet boundary condition for $y_N = -\frac{d_{\varepsilon N}}{\mu_\varepsilon}$.

Finally, in (5.27) the function $e_\varepsilon(\rho y_0)$ is defined as follows

$$e_\varepsilon = \varepsilon\tilde{e}_\varepsilon, \quad \text{with} \quad \tilde{e}_\varepsilon = e_\varepsilon^0 + \varepsilon e, \quad \text{and} \quad e_\varepsilon^0 = e_0 + \varepsilon^{\frac{1}{N-2}} e_1, \tag{5.29}$$

where e_1 is an explicit smooth function, uniformly bounded in ε, whose expression we give in Section 5, (5.37) and

$$e_0 = \frac{2\int_{\mathbb{R}^N} \partial_{ii}\omega Z_0}{\lambda_1}\left(\mathrm{Tr}_{\bar{g}}\bar{h} - \bar{h}_{00}\right) d_{0N}, \tag{5.30}$$

Finally, in (5.29), the function e is unknown and, for now, it plays the role of a parameter. This function e will be chosen later on, together with μ, $d_1, \ldots, d_N$ in (5.3) and (5.4), to be solution of a system of $(N+2)$ ordinary differential equations. For the moment, we assume that e is a smooth function with the following norm

$$\|e\|_e = \|\varepsilon^{2+\frac{2}{N-2}}\ddot{e}\|_\infty + \|\varepsilon^{1+\frac{1}{N-2}}\dot{e}\|_\infty + \|e\|_\infty \tag{5.31}$$

uniformly bounded by a positive constant independent of ε.

The error one commits in considering $\mathtt{w}$ a real solution to (5.24) is given by the size of $S_\varepsilon(\mathtt{w})$, which is itself a function of the parameter function μ, d and e. Assume that the parameter functions μ, d and e, defined respectively in (5.3), (5.4) and (5.29) satisfy the following assumption

$$\|(\mu, d, e)\| := \|\mu\|_a + \|d\|_d + \|e\|_e \leq c \tag{5.32}$$

for some constant $c > 0$, independent of ε.

Then for all ε small enough and $(y_0, y) \in \mathcal{D}$, we have the validity of the following expansion

$$
\begin{aligned}
S_\varepsilon(\mathtt{w}) \quad &= \quad -p\omega^{p-1}\bar{\omega} - \varepsilon\omega^p \log\omega + \varepsilon\left[-2\bar{h}_{ij} d^0_{\varepsilon N}\partial_{ij}\omega + \lambda_1 e^0_\varepsilon Z_0\right] \\
&+ \quad \varepsilon^{1+\frac{1}{N-2}}\,\mu^0_\varepsilon\,\left[-2\bar{h}_{ij} y_N \partial_{ij}\omega + Tr_{\bar{g}}\bar{h}\partial_N\omega\right] \\
&+ \quad \varepsilon^2\left[\left(\rho^2 a_0 \ddot{e} + \lambda_1 e\right) Z_0 - 2\bar{h}_{ij} d_N \partial_{ij}\omega\right.
\end{aligned}
\tag{5.33}
$$

$$
\begin{aligned}
&+ \quad \sum_{ij}(\dot{d}_i \dot{d}_j - \tfrac{1}{3} R_{ijkl} d_k d_l + a^{ij}_{Nk} d_k d^0_{\varepsilon N} + 4\bar{h}_{0j} d_i d^0_{\varepsilon N})\partial_{ij}\omega + \Upsilon_\varepsilon\Big] \\
&+ \quad \varepsilon^{2+\frac{1}{N-2}}\,\mu^0_\varepsilon\left[-\sum_j \partial_j\omega\cdot\ddot{d}_j + (-\sum_{ij}\tfrac{1}{3} R_{ijkl} y_k d_l \partial_{ij}\omega + 2a^{ij}_{Nk} y_k d^0_{\varepsilon N}\partial_{ij}\omega)\right. \\
&+ \quad \left.(\tfrac{2}{3} R_{ijik} + R_{0j0k}) d_k \partial_j\omega + 4\bar{h}_{0j}\dot{d}_i y_N \partial_{ij}\omega\right] \\
&+ \quad \varepsilon^{3+\frac{1}{N-2}}\,\left[-\mu^0_\varepsilon\partial_N\omega\cdot\ddot{d}_N - \tfrac{\mu^0_\varepsilon}{3} R_{ijkl} y_k d_l \partial_{ij}\omega + \mu(\tfrac{2}{3} R_{ijik} + R_{0j0k}) d_k \partial_j\omega\right. \\
&+ \quad (\mu^0_\varepsilon d_N + \mu d^0_{\varepsilon N})\left(2a^{ij}_{Nk} y_k \partial_{ij}\omega + b^j_N \partial_j\omega - \mathrm{Tr}_{\bar{g}}\bar{h}\partial_N\omega\right) \\
&+ \quad \left.(\mu^0_\varepsilon e + \mu e^0_\varepsilon)\left(-2\bar{h}_{ij} y_N \partial_{ij} Z_0 + \mathrm{Tr}_{\bar{g}}\bar{h}\partial_N Z_0\right)\right] \\
&+ \quad \varepsilon^{3+\frac{2}{N-2}}\,\left[-\ddot{\mu}\mu Z_{N+1}\right. \\
&+ \quad \left.2\mu\mu^0_\varepsilon(-\tfrac{1}{3} R_{ikjl}\, y_k y_l\,\partial_{ij}\omega + (\tfrac{2}{3} R_{ijik} \;+\; R_{0j0k}) y_k \partial_j\omega + b^j_N y_N \partial_j\omega - Tr_{\bar{g}}\bar{k} y_N \partial_N\omega)\right] \\
&+ \quad \varepsilon^4(\log\varepsilon) r
\end{aligned}
$$

where

$$
\Upsilon_\varepsilon = \Upsilon_0 + \varepsilon^{\frac{1}{N-2}}\Upsilon^1_\varepsilon, \tag{5.34}
$$

with

$$
\Upsilon_0 = -2\bar{h}_{ij} d_{0N} e_0 \partial_{ij} Z_0 + p(p-1) e_0^2 \omega^{p-2} Z_0^2 + p e_0 \omega^{p-1}\log\omega Z_0,
$$

Υ^1_ε a sum of functions of the form

$$
f_1(\varepsilon^{1+\frac{1}{N-2}} y_0) f_2(\mu, d, e) f_3(y)
$$

with f_1 a smooth explicit function of the variable $\varepsilon^{1+\frac{1}{N-2}} y_0$, uniformly bounded in ε, f_2 a smooth function of μ, d and e and uniformly bounded in ε for μ, d and e satisfying (5.32), and f_3 a smooth function of the variable y, with $\sup(1 + |y|^{N-2})|f_3(y)| < +\infty$.

In the previous expansion, $\bar{h}$ is the second fundamental form on Σ defined in (4.9), $\bar{k}$ is the square of the second fundamental form defined in (4.10), R_{ijkl} are the components of the curvature tensor $\bar{R}$ on $(\Sigma, \bar{g})$ as defined in (4.7). Here indexes i, j, k, l are understood to run from 1 to $N - 1$ and summation is understood under repeated indexes. Finally a^{ij}_{Nk} is defined as $A^{ij}_{Nk} = a^{ij}_{Nk} x_N + O(x_N^2)$, see (4.13).

Finally the term r the expansion (5.33) is a sum of functions of the form

$$
h_0(\varepsilon^{1+\frac{1}{N-2}} y_0)\left[f_1(\mu, d, \dot{\mu}, \dot{d}) + o(1) f_2(\mu, d, e, \dot{\mu}, \dot{d}, \dot{e}, \ddot{\mu}, \ddot{d}, \ddot{e})\right] f_3(y)
$$

with h_0 a smooth function uniformly bounded in ε, f_1 and f_2 are smooth functions of their arguments, uniformly bounded in ε as μ, d and e satisfy (5.32). An important remark is that the function f_2 depends linearly on the argument $(\ddot\mu, \ddot d, \ddot e)$. Concerning f_3, we have

$$\sup(1+|y|^{N-2})|f_3(y)| < +\infty.$$

We postpone the proof of the expansion (5.33) to the Appendix, Section 9 and we continue the description of $\mathtt{w}$ in (5.27).

We now use formula (5.33) to compute, for each y_0, the $L^2(\mathcal{D}_{y_0})$ projection of the error $S_\varepsilon(\mathtt{w})$ (see (5.25) and (5.27)) along the functions Z_i, $i = 0, 1, \dots, N+1$ (see (2.11) and (2.12)). Here $\mathcal{D}_{y_0}$ denotes the y_0 section of the domain $\mathcal{D}$, defined in (5.12),

$$\mathcal{D}_{y_0} = \{y \,:\, (y_0, y) \in \mathcal{D}\}.$$

Denote

$$C_1 := \int_{\mathbb{R}^N} Z_i^2, \quad C_2 := \int_{\mathbb{R}^N} Z_{N+1}^2, \quad C_3 := \int_{\mathbb{R}^N} Z_0^2$$

We start with the projections in the tangential directions Z_i, for $i = 1, \dots, N-1$. Assume μ, d and e satisfy (5.32). Then for ε small enough, and for any $k = 1, \dots, N-1$, we have

$$\begin{aligned} \textstyle\int_{\mathcal{D}_{y_0}} S_\varepsilon(\mathtt{w}) Z_k &= \varepsilon^{2+\frac{1}{N-2}}\, C_1 \left[\mu_0\,(-\ddot d_k + R_{0j0k} d_j) \; + \alpha_k(\rho y_0) + \varepsilon \beta_k(\rho y_0; \mu, d, e) \right] \\ &+ \varepsilon^3 r. \end{aligned} \tag{5.35}$$

In (5.35), R_{0j0k} are the components as defined in (4.7) of the curvature tensor $\bar R$ on $(\Sigma, \bar g)$ as in Proposition 4.1, the functions α_k are explicit, smooth and uniformly bounded in ε. The functions β_k are smooth functions of their arguments, they are bounded in ε as μ, d and e satisfy (5.32) and they do not depend of the derivatives of μ, d and e. Finally the term r denotes a sum of functions of the form

$$h_0(\rho y_0)[\, h_1(\mu, d, e, \dot\mu, \dot d, \dot e) + o(1) h_2(\mu, d, e, \dot\mu, \dot d, \dot e, \ddot\mu, \ddot d, \ddot e)\,] \tag{5.36}$$

where h_0 is a smooth function uniformly bounded in ε, h_1 and h_2 are smooth functions of their arguments, uniformly bounded in ε as μ, d and e satisfy (5.32), $o(1) \to 0$ as $\varepsilon \to 0$. An important remark is that h_2 depends linearly on the argument $(\ddot\mu, \ddot d, \ddot e)$. We postpone the proof of (5.35) to the Appendix, Section 9.

Concerning the projection os $S_\varepsilon(\mathtt{w})$ in the remaining directions Z_{N+1}, Z_N and Z_0, they turn out to be much bigger in size to the projections along Z_i, for $i = 1, \dots, N-1$. Indeed, roughly speaking, they are at main order of size ε. To reduce this size, we make an expansion of $\tilde\mu_\varepsilon$, $\tilde d_{\varepsilon N}$ and $\tilde e_\varepsilon$ through the functions μ_0, d_{0N}, μ_1, d_{1N} in (5.5) and of e_0, e_1 in (5.30).

Indeed, if we assume μ, d and e satisfy (5.32), then we can prove that there exist a constant $\varpi > 0$ depending on N and smooth functions

$$\mu_0, \quad d_{0N}, \quad e_0, \quad \mu_1, \quad d_{1N}, \quad e_1 : (-\ell, \ell) \to \mathbb{R}, \tag{5.37}$$

in the definitions (5.5), (5.29), (5.30) such that, as $\varepsilon \to 0$, for all $y_0 \in (-\rho^{-1}\ell, \rho^{-1}\ell)$, we have

$$\begin{aligned} \int_{\mathcal{D}_{y_0}} S_\varepsilon(\mathtt{w}) Z_{N+1} &= \varepsilon^2 \left[A\bar{h}_{00}\mu + B\bar{h}_{00} d_N + \alpha_{N+1}(\rho y_0) + \varepsilon \beta_{N+1}(\rho y_0; \mu, d, e) \right] \\ &+ \varepsilon^{3+\frac{2}{N-2}} \left[-C_2 \mu_0 \ddot{\mu} \right] + \varepsilon^4 r \end{aligned} \tag{5.38}$$

and

$$\begin{aligned} \varpi \int_{\mathcal{D}_{y_0}} S_\varepsilon(\mathtt{w}) Z_N &= \varepsilon^{2+\frac{1}{N-2}} \left[B\bar{h}_{00}\mu + C\bar{h}_{00} d_N + \alpha_N(\rho y_0) + \varepsilon \beta_N(\rho y_0; \mu, d, e) \right] \\ &+ \varepsilon^{3+\frac{1}{N-2}} \left[-C_1 \mu_0 \ddot{d}_N \right] + \varepsilon^4 r. \end{aligned} \tag{5.39}$$

In (5.38) and (5.39), A, B and C are explicit constants which depend only on the dimension N, with A, $C > 0$ and $AC - B^2 > 0$. The function $\bar{h}_{00}$ is the curvature of the geodesic Γ on the boundary Σ as defined in (4.11). The functions α_{N+1}, α_N are explicit, smooth and uniformly bounded in ε. The functions β_{N+1}, β_N are smooth functions of their arguments, they are bounded in ε as μ, d and e satisfy (5.32) and they do not depend of the derivatives of μ, d and e.

Finally,

$$\begin{aligned} \int_{\mathcal{D}_{y_0}} S_\varepsilon(\mathtt{w}) Z_0 &= \varepsilon^2 C_3 \left[\rho^2 a_0 \ddot{e} + \lambda_1 e - 2(Tr_{\bar{g}} \bar{h} - \bar{h}_{00}) \left(\textstyle\int \partial_{ii} \omega Z_0\right) d_N + \alpha_0(\rho y_0) \right. \\ &+ \textstyle\sum_i (\dot{d}_i^2 - \frac{1}{3} R_{ikil}\, d_k d_l + a_{Nk}^{ii}\, d_k\, d_{0N} + 4\bar{h}_{0j} d_j d_{0N}) \left(\int \partial_{ii} w Z_0\right) \\ &+ \left. \varepsilon^2 \beta_0(\rho y_0; \mu, d, e) \right] \\ &+ \varepsilon^4 r. \end{aligned} \tag{5.40}$$

In (5.40), a_0 is the function defined in (2.10) and $\bar{h}$ is the second fundamental form of Σ as defined in (4.9). Again α_0 denotes an explicit smooth function, uniformly bounded in ε and β_0 is a smooth function of its arguments, which is bounded in ε as μ, d and e satisfy (5.32) and it does not depend of the derivatives of μ, d and e.

In (5.38), (5.39) and (5.40), the term r denotes a sum of functions of the form (5.36).

We postpone the proof of (5.38), (5.39) and (5.40) to the Appendix, Section 9.

Thanks to the choice of the parameters performed in (5.37), from the expansion given in (5.33) we conclude that the error $S_\varepsilon(\mathtt{w})$, computed in (5.33), reduces to

$$S_\varepsilon(\mathtt{w}) = \varepsilon S_0 + \varepsilon \left[\rho^2 a_0 \ddot{e} + \lambda_1 e \right] \chi_\varepsilon Z_0 + \varepsilon^2 S_1, \tag{5.41}$$

where S_0 is a smooth function of ρy_0, uniformly bounded in ε. Observe that S_0 does not depend on μ, d and e. Furthermore, S_0 satisfies, for all $i = 0, 1, \ldots, N+1$,

$$\int_{\mathcal{D}_{y_0}} S_0 Z_i \, dy = 0, \quad \text{for all } y_0,$$

and

$$\|S_0\|_{**} \leq c,$$

for some positive constant c independent of ε. In (5.41), a_0 is the function defined in (2.10), Z_0 is given by (2.12), e is the parameter function which enters in the definition (5.29) and whose $\|\cdot\|_e$ norm is bounded uniformly in ε (see (5.31)). On the other hand, S_1 depends on μ, d and e.

Now we introduce a further correction Π to $\mathtt{w}$, to get the final approximation $\mathtt{W} = \mathtt{w} + \Pi$ (5.26). The correction Π is chosen to Define Π to reduce the size of the error (5.41), eliminating the term εS_0, as the unique solution of the following linear problem

$$a_0 \partial_0^2 \Pi + \Delta_y \Pi + \tilde{\mathcal{A}} \Pi + p\omega^{p-1}\Pi = -\varepsilon S_0 + \sum \mathtt{c}_i Z_i \quad \text{in} \quad \mathcal{D} \tag{5.42}$$

$$\int_{\mathcal{D}_{y_0}} \Pi(y_0, y) Z_i \, dy = 0 \quad \forall y_0, \quad \forall i = 0, \dots, N+1 \tag{5.43}$$

and

$$\Pi(y_0, \bar{y}, y_N)_{|\partial \mathcal{D}_{y_0}} = 0 \quad \text{ for all } y_0. \tag{5.44}$$

In (5.42) a_0 is defined as in (2.10), $\tilde{\mathcal{A}}$ in (5.15). Taking into account the description of the linear operator (5.14) carried out at the beginning of this Section, the assumptions of Proposition 3.1 are satisfied and the linear theory developed in Section 3 can be applied and it gives the validity of the following estimate

$$\|\Pi\|_* \leq c\varepsilon, \tag{5.45}$$

for some given positive constant c. The linear operator in (5.42) depends on μ and d (but not on e). This implies that Π itself depends on μ and d. A direct analysis of (5.42), together with (5.14), shows that

$$\|\Pi_{\mu_1, d_1} - \Pi_{\mu_2, d_2}\|_* \leq c\varepsilon^2 \|(\mu_1 - \mu_2, d_1 - d_2)\|. \tag{5.46}$$

We next compute the size of $\mathtt{c}_i = \mathtt{c}_i(\rho y_0)$. Multiplying equation (5.42) against Z_i, integrating on the section $\mathcal{D}_{y_0}$, we obtain, for all y_0,

$$\mathtt{c}_i \int_{\mathcal{D}_{y_0}} Z_i^2 = a_0 \int_{\mathcal{D}_{y_0}} \partial_0^2 \Pi Z_i + \int_{\mathcal{D}_{y_0}} (\Delta_y \Pi + p\omega^{p-1}\Pi) Z_i + \int_{\mathcal{D}_{y_0}} \tilde{\mathcal{A}}(\Pi) Z_i. \tag{5.47}$$

Taking into account (5.43) and (5.32), we have

$$\Big| \int_{\mathcal{D}_{y_0}} \partial_0 \Pi Z_i \Big| \leq o(1)\varepsilon^3, \quad \Big| \int_{\mathcal{D}_{y_0}} \partial_0^2 \Pi Z_i \Big| \leq o(1)\varepsilon^3,$$

where $o(1)$ denotes a small function of y_0. Furthermore, integrating by parts and using (5.43), we have

$$\Big| \int_{\mathcal{D}_{y_0}} (\Delta_y \Pi + p\omega^{p-1}\Pi) Z_i \Big| \leq o(1)\varepsilon^3.$$

Finally, from (5.14) we obtain

$$\Big| \int_{\mathcal{D}_{y_0}} \tilde{\mathcal{A}}(\Pi) Z_i \Big| \leq o(1)\varepsilon^3.$$

Thus we conclude that

$$\sup |\mathtt{c}_i| \leq o(1)\varepsilon^3. \tag{5.48}$$

Directly from (5.47) and (5.46) we get that $\mathtt{c}_i = \mathtt{c}_i[\mu, d]$ depends smoothly on μ, d and their derivatives. Indeed, we have

$$\|\mathtt{c}_i[\mu_1, d_1] - \mathtt{c}_i[\mu_2, d_2]\|_\infty \leq c\varepsilon^2 \|(\mu_1 - \mu_2, d_1 - d_2)\|. \tag{5.49}$$

Let $\psi := \partial_0 \Pi$. We have

$$a_0 \partial_0^2 \psi + \Delta_y \psi + \tilde{\mathcal{A}} \psi + p\omega^{p-1}\psi + \rho \dot{a}_0 \partial_0 \psi = h + \sum \partial_0 \mathtt{c}_i Z_i \quad \text{in} \quad \mathcal{D} \tag{5.50}$$

with

$$h = -\varepsilon \rho \partial_o S_0 - \partial_0 \tilde{\mathcal{A}}(\Pi)$$

$$\int_{\mathcal{D}_{y_0}} \psi(y_0, y) Z_i \, dy = o(1)\varepsilon \quad \forall y_0, \quad \forall i = 0, \ldots, N+1 \tag{5.51}$$

and

$$\psi(y_0, \bar{y}, y_N)_{|\partial \mathcal{D}_{y_0}} - \partial_0 (\frac{d_{\varepsilon N}}{\mu_\varepsilon}) \partial_N \Pi(y_0, \bar{y}, y_N)_{|\partial \mathcal{D}_{y_0}} = 0 \quad \text{for all } y_0. \tag{5.52}$$

Direct computations show that

$$\|h\|_{**} \leq C\varepsilon\rho$$

and condition (5.52) reduces to

$$\psi(y_0, \bar{y}, y_N)_{|\partial \mathcal{D}_{y_0}} = O(1)\varepsilon^{3 - \frac{1}{N-2}},$$

where $O(1)$ denotes a smooth function of y_0, uniformly bounded in ε, for μ, d and e satisfying (5.32). We thus conclude that

$$\|\partial_0 \Pi\|_* \leq c\rho\varepsilon.$$

With this choice of Π we have that

$$S_\varepsilon(\mathtt{W}) = \varepsilon^2 S_1 + \varepsilon \left[\rho^2 a_0 \ddot{e} + \lambda_1 e\right] \chi_\varepsilon Z_0 + N_1(\Pi) + \sum \mathtt{c}_i Z_i, \tag{5.53}$$

(see (5.41)), where

$$N_1(\Pi) = \mu_\varepsilon^{-\frac{N-2}{2}\varepsilon} \left[(\mathtt{w} + \Pi)^{p-\varepsilon} - \mathtt{w}^{p-\varepsilon}\right] - p\omega^{p-1}\Pi. \tag{5.54}$$

Observe that S_1 depends smoothly on the parameter μ, d and e and

$$\|S_1(\mu_1, d_1, e_1) - S_1(\mu_2, d_2, e_2)\|_{**} \leq c\|(\mu_1 - \mu_2, d_1 - d_2, e_1 - e_2)\|. \tag{5.55}$$

We next estimate $\|N_1(\Pi)\|_{**}$. If $|y| \leq \delta\varepsilon^{-\frac{1}{2}}$, we have

$$|N_1(\Pi)| \leq c|\omega^{p-2}\Pi^2|.$$

Thus in this region, we have that

$$\sup_{|y| < \delta\varepsilon^{-\frac{1}{2}}} |(1+|y|)^{N-2} N_1(\Pi)| \leq c\varepsilon^2.$$

If now $|y| > \delta\varepsilon^{-\frac{1}{2}}$, then

$$|N_1(\Pi)| \leq c|\Pi^p|,$$

so that

$$\begin{aligned} \sup_{|y| > \delta\varepsilon^{-\frac{1}{2}}} |(1+|y|)^{N-2} N_1(\Pi)| &\leq c\varepsilon^p \sup_{|y| > \delta\varepsilon^{-\frac{1}{2}}} |(1+|y|)^{-2+\frac{8}{N-2}}| \\ &\leq c\varepsilon^{2+\frac{8}{N-2}}. \end{aligned}$$

We conclude that

$$\|N_1(\Pi)\|_{**} \leq c\|\omega^{p-2}\Pi^2\|_{**} \leq c\varepsilon^2. \tag{5.56}$$

This concludes the construction of our approximation $\mathtt{W}$ (5.26) and the analysis of the error $S_\varepsilon(\mathtt{W})$ (5.53).

6. The gluing procedure

This section is devoted to perform a gluing procedure that reduces the full problem (2.1) A first observation is that replacing u by $\rho^{\frac{N-2}{2}}u(\rho z)$ the problem becomes equivalent to

$$\begin{cases} \Delta u + \rho^{-\frac{N-2}{2}\varepsilon}u^{p-\varepsilon} = 0 & \text{in } \Omega_\varepsilon, \\ u > 0 & \text{in } \Omega_\varepsilon, \\ u = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{6.1}$$

where $\Omega_\varepsilon = \rho^{-1}\Omega$.

The function $\mathtt{W}(y_0, y)$ built in the previous section in (5.26) defines an approximation W to a solution of (2.1) near the geodesic through the natural change of variables (5.3)-(5.2). More generally, let us denote by $z \in \mathbb{R}^{N+1}$ the original variable in Ω_ε. Then for a function $f(z)$ defined on a small neighborhood of Γ we use in this section the notation

$$f(z) = \tilde\mu_\varepsilon^{\frac{N-2}{2}}(\rho y_0)\tilde f(y_0, y), \quad \text{for } z = \rho^{-1}G(\rho y_0, \rho\tilde\mu_\varepsilon(\rho y_0)y + \varepsilon\tilde d_\varepsilon(\rho y_0))$$

or

$$\tilde f(y_0, y) = \tilde\mu_\varepsilon^{\frac{N-2}{2}}(\rho y_0)f(\rho^{-1}G(\rho y_0, \rho\tilde\mu_\varepsilon(\rho y_0)y + \varepsilon\tilde d_\varepsilon(\rho y_0)\,))$$

so that in particular W and $\mathtt{W}$ are linked as $\mathtt{W} = \tilde W$. In fact we recall that near Γ_ε, setting in this language $v := \tilde u$, the equation in (6.1) becomes

$$S_\varepsilon(v) := \mathcal{A}v + \mu_\varepsilon^{-\frac{N-2}{2}\varepsilon}v^{p-\varepsilon} = 0, \tag{6.2}$$

where $\mathcal{A}$ is the operator defined in (5.14).

Let $\delta > 0$, be a fixed number, with $4\delta < \hat\delta$, where $\hat\delta$ was chosen in (2.9). We consider a smooth cut-off function $\xi_\delta(s)$, such that $\xi_\delta(s) = 1$ if $0 < s < \delta$, and $= 0$ if $s > 2\delta$. Let us consider the cut-off function

$$\zeta_\delta^\varepsilon(y_0, y) = \zeta_\delta(|G(\rho y_0, \tilde\mu_\varepsilon(\rho y_0)\rho y + \varepsilon\tilde d_\varepsilon(\rho y_0)\,)|),$$

and its pull-back to Ω_ε, supported near $\rho^{-1}\Gamma$, defined as

$$\eta_\delta^\varepsilon(z) = \zeta_\delta^\varepsilon(y_0, y) \quad \text{for } z = \rho^{-1}G(\rho y_0, \tilde\mu_\varepsilon(\rho y_0)\rho y + \varepsilon\tilde d_\varepsilon(\rho y_0)\,).$$

We also denote We observe that with this definition $\eta_\delta^\varepsilon(z)$ does not longer carry dependence on the parameter functions and it is well-defined in entire Ω_ε, by just extending it by zero outside the range of the variables (y_0, y). We define our global first approximation $\mathbf{w}(z)$ to a solution of (2.1) to be simply

$$\mathbf{w}(z) = \eta_\delta^\varepsilon(z)\tilde{\mathbf{w}}(z), \tag{6.3}$$

We look for solution to Problem (6.1) of the form $u = \mathbf{w} + \Phi$, namely

$$\begin{cases} \Delta\Phi + p\mathbf{w}^{p-1}\Phi + N(\Phi) + E = 0 & \text{in } \Omega_\varepsilon, \\ \Phi = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{6.4}$$

where
$$N(\Phi) = \rho^{-\frac{N-2}{2}\varepsilon}(\mathbf{w}+\Phi)^{p-\varepsilon} - \mathtt{w}^{p-\varepsilon} - p\mathbf{w}^{p-1}\Phi, \quad E = \Delta \mathbf{w} + \mathtt{w}^{p-\varepsilon}.$$
According to (6.2), near the geodesic $v = \tilde u + \tilde\Phi$ must then satisfy
$$\mathcal{A}\tilde\Phi + p\tilde{\mathbf{w}}^{p-1}\tilde\Phi + \mathtt{N}(\tilde\Phi) + S_\varepsilon(\tilde{\mathbf{w}}) = 0 \tag{6.5}$$
where now
$$\mathtt{N}(\tilde\Phi) = \tilde\mu_\varepsilon^{-\frac{N-2}{2}\varepsilon}(\tilde{\mathbf{w}}+\tilde\Phi)^{p-\varepsilon} - \tilde{\mathtt{w}}^{p-\varepsilon} - p\tilde{\mathbf{w}}^{p-1}\tilde\Phi, \quad S_\varepsilon(\tilde{\mathbf{w}}) = \mathcal{A}\tilde{\mathbf{w}} + \tilde{\mathbf{w}}^{p-\varepsilon}.$$

We look for Φ, solution of (6.4) in the following form:
$$\Phi = \eta_{2\delta}\phi + \psi,$$
where the function ϕ is such that $\tilde\phi$ is in principle defined only in $\mathcal{D}$. It is immediate to check that Φ of this form will satisfy the above problem if the pair (ψ,ϕ) satisfies the following nonlinear coupled system.
$$\mathcal{A}\tilde\phi + p\tilde{\mathbf{w}}^{p-1}\tilde\phi = -\mathtt{N}(\zeta_{2\delta}^\varepsilon\tilde\phi + \tilde\psi) - \mathtt{E} - p\tilde{\mathbf{w}}^{p-1}\tilde\psi \quad \text{in} \quad \mathcal{D}, \tag{6.6}$$
$$\tilde\phi = 0 \quad \text{on} \quad \partial\mathcal{D}. \tag{6.7}$$

$$\begin{aligned}\Delta\psi + (1-\eta_{2\delta}^\varepsilon)p\,\mathbf{w}^{p-1}\psi &= -2\nabla\phi\nabla\eta_{2\delta}^\varepsilon - \phi\Delta\eta_{2\delta}^\varepsilon\\ &\quad -(1-\eta_{2\delta}^\varepsilon)\,N(\eta_{2\delta}^\varepsilon\phi + \psi) \quad \text{in} \quad \Omega_\varepsilon\\ \psi &= 0 \quad \text{on} \quad \partial\Omega_\varepsilon.\end{aligned} \tag{6.8}$$
Given ϕ such that in $\mathcal{D}$ $\tilde\phi$ has a sufficiently small $\|\cdot\|_*$-norm, we first solve problem (6.8) for ψ.

Let us assume first that Ω is bounded. Since $\Omega_\varepsilon = \rho^{-1}\Omega$, the problem
$$-\Delta\psi = h \quad \text{in} \quad \Omega_\varepsilon, \quad \psi = 0 \quad \text{on} \quad \partial\Omega_\varepsilon, \tag{6.9}$$
has a unique solution $\psi := (-\Delta)^{-1}(h)$ for each given $h \in L^\infty(\Omega_\varepsilon)$. Besides
$$\|\psi\|_\infty \le C\left(\frac{N-1}{N-2}\right)^{-2}\|h\|_\infty.$$

Let us observe that, for instance,
$$\|\Delta\eta_{2\delta}^\varepsilon\phi\|_\infty \le C\rho^2\|\tilde\phi\|_{L^\infty(|y|>\delta\rho^{-1})} \le C\rho^{N-2}\|\tilde\phi\|_*$$
We obtain similarly
$$\|\nabla\eta_{2\delta}^\varepsilon\nabla\phi\|_\infty \le C\rho^{N-2}\|\tilde\phi\|_*.$$
Let us assume now $\|\psi\|_\infty \le R\rho^{N-4}\|\tilde\phi\|_*$ and consider in this ball the operator
$$M(\psi) := (1-\eta_{2\delta}^\varepsilon)\,N(\eta_{2\delta}^\varepsilon\phi + \psi) = (1-\eta_{2\delta}^\varepsilon)\,(\eta_{2\delta}^\varepsilon\phi + \psi)^p$$
we have that
$$\|M(\psi_1) - M(\psi_2)\|_\infty \le C(\|\tilde\phi\|_{L^\infty(|y|>\delta\rho^{-1}} + R\rho^{N-4}\|\tilde\phi\|_*)^{p-1}\|\psi_1-\psi_2\|_\infty \le$$
$$C(1+R)^{p-1}\rho^{\frac{4(N-4)}{N-2}}\|\phi\|_*^{p-1}\|\psi_1-\psi_2\|_\infty.$$

Observe that, also,

$$\| (1 - \eta_{2\delta}^{\varepsilon}) p\, \mathbf{w}^{p-1}\, \psi \|_{\infty} \leq C\rho^4 \|\psi\|_{\infty}.$$

By adjusting R suitable large but fixed, we see directly from an application of contraction mapping principle that the fixed point problem, equivalent to (6.8),

$$\psi = (-\Delta)^{-1}(M(\psi) + (1-\eta_{2\delta}^{\varepsilon}) p\, \mathbf{w}^{p-1}\, \psi + 2\nabla\phi\nabla\eta_{2\delta}^{\varepsilon} + \phi\Delta\eta_{2\delta}^{\varepsilon})$$

has a unique solution $\psi = \psi(\phi)$ with $\|\psi\|_{\infty} \leq R\rho^{N-4}\|\tilde{\phi}\|_*$, whenever $\|\tilde{\phi}\|_*$ is sufficiently small, independently of ε. Note that $\rho^{N-4} = \varepsilon^{N-3-\frac{2}{N-2}}$. In addition, the nonlinear operator ψ satisfies a Lipschitz condition of the form

$$\|\psi(\phi_1) - \psi(\phi_2)\|_{\infty} \leq C\varepsilon^{N-3-\frac{2}{N-2}} \|\phi_1 - \phi_2\|_*. \tag{6.10}$$

Let us consider now the case $\Omega = \mathbb{R}^N \setminus \Lambda$ with Λ bounded. In this case, exactly the same arguments go through. Indeed, let us pull back the equation for ψ to Ω in the following way: Associated to $f(z)$ defined in Ω_ε let us write $\hat{f}(z) := f(z/\varepsilon)$. Equation (6.8) then becomes

$$\begin{gathered} \Delta\hat{\psi} + \rho^{-2}(1 - \hat{\eta}_{2\delta}^{\varepsilon}) p\, \hat{\mathbf{w}}^{p-1}\psi \;= \\ -2\rho^{-2}\widehat{\nabla\phi}\widehat{\nabla\eta_{2\delta}^{\varepsilon}} - \hat{\phi}\rho^{-2}\widehat{\Delta\eta_{2\delta}^{\varepsilon}} - \rho^{-2}(1-\hat{\eta}_{2\delta}^{\varepsilon})\,(\hat{\eta}_{2\delta}^{\varepsilon}\hat{\phi} + \hat{\psi}\,)^p \quad \text{in} \quad \Omega \\ \hat{\psi} = 0 \quad \text{on} \quad \partial\Omega, \end{gathered}$$

or

$$\Delta\hat{\psi} + O(\rho^2)\chi\psi \;= -2O(\rho^{N-6})\|\tilde{\phi}\|_*\chi - \rho^{-2}(\,O(\rho^{N-4})\|\tilde{\phi}\|_*\chi + \hat{\psi}\,)^p \quad \text{in} \quad \Omega$$

where χ is just a function with bounded support. In the case of the exterior domain, after a Kelvin transform we see that the problem (in $\mathbb{R}^{N+1}$),

$$-\Delta\hat{\psi} \;= h \quad \text{in} \quad \Omega, \quad \hat{\psi} = 0 \quad \text{on} \quad \partial\Omega, \tag{6.11}$$

has a solution $\hat{\psi} := (-\Delta)^{-1}(h)$ with

$$\|(1+|z|^{N-1})\hat{\psi}(z)\|_{\infty} \leq C\|(1+|z|^{N+3})h(z)\|_{\infty} < +\infty.$$

We can do a Fixed point scheme similar to that before in this setting, the reason being that if

$$\|(1+|z|^{N-1})\hat{\psi}(z)\|_{\infty} \leq C\rho^{N-6}\|\tilde{\phi}\|_*,$$

then

$$|\hat{\psi}(z)|^p \leq \rho^{-2+(N-6)p}\|\tilde{\phi}\|_*^p(1+|z|)^{-p(N-1)}$$

and we also have $p(N-1) = (N+2)(N-1)/(N-2) > N-3$. Thus (6.8) can be solved in the same way as before, and the conclusion remains unchanged. It is worthwhile observing that the energy of ψ in Ω_ε is small with ε indeed small in any case, provided that $\|\tilde{\phi}\|_*$ is bounded by some small fixed constant.

As a conclusion, substituting $\tilde{\psi} = \tilde{\psi}(\tilde{\phi})$ in equation (6.6), we have reduced the full problem (2.1) h to solving the following (nonlocal) problem in $\mathcal{D}$.

$$\begin{aligned} \mathcal{A}\tilde{\phi} + p\tilde{\mathbf{w}}^{p-1}\tilde{\phi} \;&=\; -\mathtt{N}(\zeta_{2\delta}^{\varepsilon}\tilde{\phi} + \tilde{\psi}(\tilde{\phi})) - S_\varepsilon(\tilde{\mathbf{w}}) - p\tilde{\mathbf{w}}^{p-1}\tilde{\psi}(\phi) \quad \text{in} \quad \mathcal{D}, \\ \tilde{\phi} \;&=\; 0 \quad \text{on} \quad \partial\mathcal{D}. \end{aligned} \tag{6.12}$$

We will solve a projected version of this problem in the next section, and in Section 8 we will solve it in full.

7. The nonlinear projected problem

This section is devoted to solve a projected problem associated to (6.12). We shall relieve the notation in (6.12) dropping the ˜ symbol and write it as

$$\begin{aligned} L(\phi) &= S_\varepsilon(\mathbf{w}) + N(\phi) \quad \text{in} \quad \mathcal{D} \\ \phi(y_0 + \rho^{-1}\ell, y) &= \phi(y_0, y), \quad \text{for all } y_0, y \\ \phi &= 0 \quad \text{on} \quad \partial\mathcal{D}, \end{aligned}$$

where $L(\phi) = \mathcal{A}\phi + p\omega^{p-1}\phi$, with $\mathcal{A}$ defined in (5.14) and ω in (2.4), and $N(\phi)$ is given by

$$N(\phi) = p(\omega^{p-1} - \mathtt{w}^{p-1})\phi - \mathtt{N}(\zeta_{2\delta}^\varepsilon \phi + \psi(\phi)) + \zeta_{2\delta}^\varepsilon p\mathbf{w}^{p-1}\psi(\phi), \tag{7.1}$$

with

$$\mathtt{N}(\phi) = \tilde\mu_\varepsilon^{-\frac{N-2}{2}\varepsilon}(\mathbf{w}+\phi)^{p-\varepsilon} - \mathtt{w}^{p-\varepsilon} - p\mathbf{w}^{p-1}\tilde\phi.$$

Let us observe that $S_\varepsilon(\mathtt{W})$ can be decomposed in the following way.

$$S_\varepsilon(\mathtt{W}) = E + \left\{\varepsilon\left[\rho^2 a_0 \ddot e(\rho y_0) + \lambda_1 e(\rho y_0)\right]\right\}\chi_\varepsilon Z_0, \tag{7.2}$$

(see (5.53)). The projected version of the problem is as follows: Given μ, d and e satisfying (5.32), the projected problem we want to solve is: find functions ϕ, $c_i(y_0)$, for $i = 0, \ldots, N+1$, so that

$$L(\phi) = E + N(\phi) + \sum_i c_i Z_i \quad \text{in} \quad \mathcal{D} \tag{7.3}$$

$$\phi(y_0 + \rho^{-1}\ell, y) = \phi(y_0, y), \quad \text{for all } y_0, y \tag{7.4}$$

$$\phi = 0 \quad \text{on} \quad \partial\mathcal{D}, \tag{7.5}$$

$$\int_{\mathcal{D}_{y_0}} \phi Z_i = 0, \text{ for all } i = 0, \ldots, N+1, \quad \text{for all } y_0. \tag{7.6}$$

Observe that the last term in (7.2) have been absorbed in $c_0 Z_0$.

For further reference, it is useful to point out the Lipschitz dependence of the term of error S_1 on the parameters μ, d and e for the norms defined in (5.7)-(5.31). We have the validity of the estimate

$$\|E(\mu_1, d_1, e_1) - E(\mu_1, d_1, e_1)\|_\infty \le c\varepsilon^2 \|(\mu_1 - \mu_2, d_1 - d_2, e_1 - e_2)\| \tag{7.7}$$

This is consequence of (5.53), (5.49), (5.46), (5.55). As already observed, we can apply the linear theory developed in Section 3. Given Proposition 3.1, solving (7.3)–(7.6) reduces to solve a fix point problem, namely

$$\phi = T(E + N(\phi)) := A(\phi), \tag{7.8}$$

where T is the operator defined in Proposition 3.1.

Consider the set

$$\mathcal{M} := \{\phi \,:\, \|\phi\|_* \le c\varepsilon^2\}$$

for a certain positive constant c.

We first show that A maps $\mathcal{M}$ in itself. Assume $\|\phi\|_* \le c\varepsilon^2$. Then

$$\|A(\phi)\|_* \le C\|E + N(\phi)\|_{**}.$$

We first estimate $\|E\|_{**}$. Given the definition (5.53) for S_1, we get that

$$\|\chi_\varepsilon E\|_{**} \le C\varepsilon^2. \tag{7.9}$$

Next we estimate $\|N(\phi)\|_{**}$. We have

$$\|N(\phi)\|_{**} \le C\Bigg[\|(\omega^{p-1} - \mathtt{w}^{p-1})\phi\|_{**} + \|\eta^\varepsilon_{3\delta}\mathtt{N}(\eta^\varepsilon_{3\delta}\phi + \psi(\phi))\|_{**} + \|\eta^\varepsilon_{3\delta}\mathbf{w}^{p-1}\psi(\phi)\|_{**}\Bigg].$$

We get

$$\begin{aligned} \|(\omega^{p-1} - \mathtt{w}^{p-1})\phi\|_{**} &\le C\|\left[(\omega + \varepsilon e Z_0 + \Pi)^{p-1} - \omega^{p-1}\right]\phi\|_{**} \\ &\le C\|\omega^{p-2}(\varepsilon e Z_0 + \Pi)\phi\|_{**} \\ &\le C\varepsilon\|\phi\|_*; \end{aligned}$$

furthermore

$$\begin{aligned} \|\zeta^\varepsilon_{3\delta}\mathtt{N}(\zeta^\varepsilon_{3\delta}\phi + \psi(\phi))\|_{**} &\le C\sup\nolimits_{|y|\le c\varepsilon^{-\frac12}} |(1+|y|)^{N-2}\omega^{p-2}(\phi+\psi)^2| \\ &+ \sup\nolimits_{|y|\ge c\varepsilon^{-\frac12}} |(1+|y|)^{N-2}(|\phi|^p + |\psi|^p)| \\ &\le C\varepsilon^4 \end{aligned}$$

and

$$\|\zeta^\varepsilon_{3\delta}\mathbf{w}^{p-1}\psi(\phi)\|_{**} \le C\varepsilon^{N-3-\frac{2}{N-2}} \sup_{|y|\le c\varepsilon^{-\frac{N-1}{N-2}}} (1+|y|)^{N-6}\,\|\phi\|_* \le C\varepsilon^{2+\frac{2}{N-2}}\|\phi\|_*.$$

Thus we get

$$\|N(\phi)\|_{**} \le C\varepsilon^3$$

for all $\|\phi\|_* \le c\varepsilon^2$. Given (7.9), we conclude that $A(\phi) \in \mathcal{M}$ for any $\phi \in \mathcal{M}$, provided c in the definition of $\mathcal{M}$ is chosen large enough.

We next prove that A is a contraction mapping, so that the fixed point problem (7.8) can be uniquely be solved in $\mathcal{M}$. This fact is a direct consequence of (6.10). Indeed, arguing as in the estimates above

$$\|A(\phi_1) - A(\phi_2)\|_* \le C\|N(\phi_1) - N(\phi_2)\|_{**} \le C\varepsilon\|\phi_1 - \phi_2\|_*.$$

Emphasizing the dependence on μ, d, e what we find for the Linear operator T is the Lipschitz dependence

$$\|T_{\mu_1, d_1, e_1} - T_{\mu_2, d_2, e_2}\| \le C\varepsilon\|(\mu_1 - \mu_2,\, d_1 - d_2,\, e_1 - e_2)\|.$$

We recall that we have the Lipschitz dependence (7.7). Moreover, the operator N also has Lipschitz dependence on (μ, d, e). It is easily checked that for $\|\phi\|_* \le C\varepsilon^2$ we have, with obvious notation,

$$\|N_{(\mu_1, d_1, e_1)}(\phi) - N_{(\mu_2, d_2, e_2)}(\phi)\|_{**} \le C\varepsilon^3\|(\mu_1 - \mu_2,\, d_1 - d_2,\, e_1 - e_2)\|.$$

Hence from the fixed point characterization we then see that

$$\|\phi_{(\mu_1,\,d_1,\,e_1)} - \phi_{(\mu_2,\,d_2,\,e_2)}\|_* \le C\varepsilon^4 \;\|(\mu_1-\mu_2,\, d_1-d_2,\, e_1-e_2)\|. \tag{7.10}$$

We have thus proved the following

Proposition 7.1. *There is a number $c>0$ such that for all sufficiently small ε and all μ, d, e satisfying respectively (5.32), problem (7.3)-(7.6) has a unique solution $\phi = \phi(\mu, d, e)$ and $c_i = c_i(\mu, d, e)$ which satisfies*

$$\|\phi\|_* \le c\varepsilon^2. \tag{7.11}$$

Besides ϕ depends Lipschitz-continuously on μ, d and e in the sense of estimate (7.10).

8. The final adjustment of parameters: conclusion of the proof

In this section we will find the equations relating μ, d and e to get all the coefficients c_i in (7.3) identically equal to zero. To get this, we multiply equation (7.3) against Z_i, for all $i = 0, \dots, N+1$, (see (2.11) and (2.12)) and we integrate in y. Thus, the system

$$c_i(\rho y_0) = 0 \quad \text{ for all } i = 0, \dots, N+1$$

is equivalent to

$$\int_{\mathcal{D}_{y_0}} S_\varepsilon(\mathtt{W}) Z_i dy + \int_{\mathcal{D}_{y_0}} \left(N(\phi) - \mathcal{A}\phi - \omega^{p-1}\phi\right) Z_i = 0, \quad \text{ for all } i, \quad \forall y_0$$

where $S_\varepsilon(\mathtt{W})$ is defined in (5.53), $N(\phi)$ in (7.1), $\mathcal{A}$ in (5.14), ω in (2.4)

Taking into account Section 7 and the result of Proposition 7.1, we get that

$$\int_{\mathcal{D}_{y_0}} \left(N(\phi) - \mathcal{A}\phi - \omega^{p-1}\phi\right) Z_i = \varepsilon^3 r,$$

where r is the sum of functions of the form

$$h_0(\rho y_0)[\, h_1(\mu, d, e, \dot\mu, \dot d, \dot e) + o(1) h_2(\mu, d, e, \dot\mu, \dot d, \dot e, \ddot\mu, \ddot d, \ddot e)\,]$$

where h_0 is a smooth function uniformly bounded in ε, h_1 depends smoothly on μ, d, e and their first derivative, it is bounded in the sense that

$$\|h_1\|_\infty \le c\|(\mu, d, e)\|$$

and it is compact, as a direct application of Ascoli Arzelá Theorem shows. The function h_2 depends on (μ, d, e), together with their first and second derivatives. An important remark is that h_2 depends linearly on $\ddot\mu$, $\ddot d$ and $\ddot e$. Furthermore it is Lipschtz, with

$$\|h_2(\mu_1, d_1, e_1) - h_2(\mu_2, d_2, e_2)\|_\infty \le o(1)\|(\mu_1-\mu_2, d_1-d_2, e_1-e_2)\|.$$

We next study $\int S_\varepsilon(\mathtt{W}) Z_i dy$, with $S_\varepsilon(\mathtt{W})$ given by (5.53). First we have that

$$\int_{\mathcal{D}_{y_0}} \left[N_1(\Pi) + \sum \mathtt{c}_i Z_i\right] Z_j = \varepsilon^2 h_0(\rho y_0) + o(1)\varepsilon^3 r$$

where $h_0(\rho y_0)$ is a smooth function of ρy_0, which does not depend on μ, d, e, and r as before.

Taking into account the previous computation and the results of Section 5, (5.35), (5.38), (5.39), (5.40), we conclude that the equations

$$c_i = 0$$

are equivalent to solve the following limit system of $N+2$ non linear ordinary differential equations in the unknowns μ, $d_1, \ldots, d_N$, e,

$$\left\{\begin{array}{rcl} L_{N+1}(\mu) & := & -C_2 \varepsilon^{1+\frac{2}{N-2}} \mu_0 \ddot{\mu} + A\mu + B d_N = \alpha_{N+1} + \varepsilon M_{N+1} \\ L_N(d_N) & := & -C_1 \varpi \varepsilon \mu_0 \ddot{d}_N + B\mu + C d_N = \alpha_N + \varepsilon M_N \\ L_k(d_k) & := & -\ddot{d}_k + \sum_{j=1}^{N-1} R_{0j0k} d_j = \alpha_k + \varepsilon M_k \\ & & k = 1, \ldots, N-1 \\ L_0(e) & := & \rho^2 a_0 \ddot{e}(\rho y_0) + \lambda_1 e(\rho y_0) + \gamma_0 d_N = \alpha_0 + \varepsilon Q_0 + \varepsilon^2 M_0 \end{array}\right. \tag{8.1}$$

where μ, $d_1, \ldots, d_N$ and e satisfy periodic boundary conditions in $[-\ell, \ell]$. In (8.1), we have $A > 0$, $C > 0$ and $AC - B^2 > 0$. The functions α_i are explicit functions of x_0, smooth and uniformly bounded in ε. The function γ_0 is given by $\gamma_0 = 2(Tr_{\bar{g}} \bar{h} - \bar{h}_{00}) (\int \partial_{ii} \omega Z_0)$. The operators $M_i = M_i(\mu, d, e)$ can be decomposed in the following form:

$$M_i(f, e) = A_i(\mu, d, e) + K_i(\mu, d, e)$$

where K_i is uniformly bounded in $L^\infty(-\ell, \ell)$ for (μ, d, e) satisfying constraints (5.32) and is also compact. The operator A_i depends on (μ, d, e) and their first and second derivatives and it is Lipschitz in this region, namely

$$\|A_i(\mu_1, d_1, e_1) - A_i(\mu_2, d_2, e_2)\|_\infty \le C o(1) \|(\mu_1 - \mu_2, d_1 - d_2, e_1 - e_2)\|.$$

We remark that the dependence on $\ddot{\mu}$, $\ddot{d}$ and $\ddot{e}$ is linear. Finally, the operator Q_0 is quadratic in d and it is uniformly bounded in $L^\infty(-\ell, \ell)$ for (μ, d, e) satisfying constraints (5.32).

Our goal is now to solve (8.1) in μ, d and e. To do so, we first analyze the invertibility of the linear operators L_i.

We start with a linear theory in L^∞ setting for the problem of finding 2ℓ-periodic solutions of the problem

$$L_{N+1}(\mu) = h_1, \quad L_N(d) = h_2, \tag{8.2}$$

with h_1 and h_2 bounded. This is the content of next Lemma.

Lemma 8.1. *Assume that $A > 0$, $C > 0$ and $AC - B^2 > 0$ and that $\|h_1\|_\infty + \|h_2\|_\infty$ is bounded. Then there exist (μ, d) 2ℓ-periodic solution to the above system and a constant c such that*

$$\|\mu\|_\infty + \|d\|_\infty + \varepsilon^{\frac{1}{2}+\frac{1}{N-2}} \|\dot{\mu}\|_\infty + \varepsilon^{\frac{1}{2}} \|\dot{d}\|_\infty \le$$

$$c\,[\, \|h_1\|_\infty + \|h_2\|_\infty \,]\ .$$

Proof. System (8.2) has a variational structure. The associated energy functional on the class of 2ℓ-periodic functions is positive, bounded from below away from zero and convex. Existence of solution thus follows.

In order to get the a-priori estimate, we will argue by contradiction. Assuming the opposite we have the existence of a sequence (h_{1n}, h_{2n}) with

$$\|h_{1n}\|_\infty + \|h_{2n}\|_\infty \to 0 \ ,$$

and a sequence of solutions (μ_n, d_n) with

$$\|\mu_n\|_\infty + \|d_n\|_\infty + \varepsilon^{\frac{1}{2}+\frac{1}{N-2}} \|\dot\mu_n\|_\infty + \varepsilon^{\frac{1}{2}} \|\dot d_n\|_\infty = 1.$$

Since $A > 0$ and $C > 0$, applying the maximum principle to each equation in the system, we see that $\|\mu_n\|_\infty \leq c\|d_n\|_\infty$ and $\|d_n\|_\infty \leq c\|\mu_n\|_\infty$. Hence we can assume $d_n(m_n) = \|d_n\|_\infty > \delta$ and $m_n \to m$. Scaling the system with $y = \frac{x-m}{\varepsilon}$, we obtain that the scaled functions, which we denote by $\hat\mu_n$ and $\hat d_n$ solve

$$\begin{aligned} -\varepsilon^{\frac{1}{N-2}} C_2 \hat\mu \ddot{\hat\mu}_n + A\hat\mu_n &= -B\hat d_n + o(1) \\ -C_1 \frac{A_2}{A_1} \hat\mu \ddot{\hat d}_n + C\hat d_n &= -B\hat\mu_n + o(1). \end{aligned} \tag{8.3}$$

From the second equation we read that $\|\hat d_n\|_\infty + \|\dot{\hat d}_n\|_\infty \leq c$ and a direct application of Ascoli-Arzelá theorem implies that

$$\hat d_n \to \hat d$$

uniformly on compact sets.

We state that

$$A\hat\mu_n \to -B\hat d. \tag{8.4}$$

Assume by contradiction that this is not true. There exists a compact interval I and a sequence of points $x_n \in I$ such that

$$\left| A\hat\mu_n(x_n) + B\hat d(x_n) \right| > a \tag{8.5}$$

for a certain fixed positive constant a. Up to subsequence, that we still denote x_n, we have $x_n \to x_0$. We now scale with $z = \frac{y-x_0}{\varepsilon^{\frac{1}{N-2}}}$, so that the scaled functions $\bar\mu_n$ and $\bar d_n$ satisfy

$$-C_2 \hat\mu \ddot{\bar\mu}_n + A\bar\mu_n = -B\bar d_n + o(1).$$

In this scale, we get $\|\dot{\bar d}_n\|_\infty \leq c\varepsilon^{\frac{1}{2(N-2)}} \to 0$. This implies that $\bar d_n$ converges uniformly over compact sets to a constant and this constant has to be $\hat d(x_0)$. Hence $A\bar\mu_n + B\bar d_n$ converges to 0 locally over compacts. This is in contradiction with (8.5). This proves (8.4).

We now fo back to (8.3), which reduces to say that $\hat d$ solves

$$-C_1 \hat\mu \ddot{\hat d} + \left(C - \frac{B^2}{A} \right) \hat d = 0.$$

Since $C - \frac{B^2}{A} > 0$, we conclude that $\hat d = 0$. A contradiction. $\square$

Concerning the invertibility of the operator L_0, we have the validity of the following Lemma.

Lemma 8.2. *Assume that condition (1.7) holds. If $f \in C(-\ell,\ell) \cap L^\infty(-\ell,\ell)$ then there is a unique solution e of $L_0(e) = f$ which is 2ℓ-periodic and satisfies*

$$\rho^2 \|\ddot{e}\|_\infty + \rho\|\dot{e}\|_\infty + \|e\|_\infty \le C\rho^{-1}\|f\|_\infty.$$

Moreover, if f is in $C^2(-\ell,\ell)$, then

$$\rho^2 \|\ddot{e}\|_\infty + \rho\|\dot{e}\|_\infty + \|e\|_\infty \le C[\|\ddot{f}\|_\infty + \|\dot{f}\|_\infty + \|f\|_\infty].$$

Proof. Consider the following transformation

$$l = \int_{-\ell}^{\ell} \frac{1}{\sqrt{a_0(s)}}\, ds, \quad t = \frac{\int_{-\ell}^{s} (\sqrt{a_0(\theta)})^{-1}\, d\theta}{l}, \quad \tilde{\lambda}_1 = \frac{l^2}{\pi^2}\lambda_1$$

and

$$y(t) = \tilde{e}(s).$$

Then problem

$$L_0(\tilde{e}) = f, \quad \tilde{e}(-\ell) = \tilde{e}(\ell), \quad \dot{\tilde{e}}(-\ell) = \dot{\tilde{e}}(\ell)$$

reduces to

$$\rho^2 \ddot{y} + \tilde{\lambda}_1 \ddot{y} = \tilde{f}, \quad y(0) = y(\pi), \quad \dot{y}(0) = \dot{y}(\pi). \tag{8.6}$$

Thus (8.6) is solvable if and only if $\rho^2\tilde{\lambda}_1 \ne \lambda_k$, for all $k \ge 0$, where λ_k in an infinite sequence of eigenvalues for (8.6), with $\tilde{f} = 0$, where $y_k(t)$ is an orthonormal basis on $L^2(0,\pi)$ constituted by the eigenfunctions

$$\ddot{y}_k + 4k^2\ddot{y} = 0, \quad y_k(0) = y_k(\pi), \quad \dot{y}_k(0) = \dot{y}_k(\pi)$$

Furthermore,

$$\sqrt{\lambda_k} = 2k + O(\frac{1}{k^3}).$$

When solvable, the solution to (8.6) is given by

$$y(t) = \sum_{k=0}^{\infty} \frac{\tilde{f}_k}{\tilde{\lambda}_1 - 4k^2\rho^2} y_k(t), \tag{8.7}$$

and $\|\tilde{f}\|_{L^2} = \left(\int_0^\pi \tilde{f}_k^2\right)^{\frac{1}{2}}$. Choose

$$|\rho^2 4k^2 - \tilde{\lambda}_1| \;\ge\; c\,\rho \tag{8.8}$$

for all k, where c is small. This corresponds precisely to the condition (1.7) in the statement of the theorem with

$$\kappa = \frac{\pi}{2}\sqrt{\lambda_1}\int_{-\ell}^{\ell} \frac{1}{\sqrt{a_0(s)}}\, ds\ . \tag{8.9}$$

From (8.8) we then find that $|\tilde{\lambda}_1 - \lambda_k\rho^2| \ge \frac{c}{2}\rho$ if ρ is also sufficiently small. It follows directly from expression (8.7) that $\|y\|_{L^\infty(0,\pi)} \le C\rho^{-1}\|\tilde{f}\|_{L^\infty(0,\pi)}$. Observe also that

$$\|y'\|^2_{L^\infty(0,\pi)} \le \sum_{k=0}^{\infty} |\tilde{f}_k|^2 \frac{1 + |\lambda_k|^2}{(\tilde{\lambda}_1 - \lambda_k\rho^2)^2} \le C\sum_{k=0}^{\infty}(1 + k^4)|\tilde{f}_k|^2.$$

Hence
$$\rho\|y'\|_{L^\infty(0,\pi)} + \|y\|_{L^\infty(0,\pi)} \leq C\rho^{-1}\|\tilde f\|_{L^\infty(0,\pi)}.$$
Besides, if $\tilde f$ is in $C^2(0,\pi)$ with $f(0)=f(\pi)$, $f'(0)=f'(\pi)$, then the sum $\sum_k k^4 \tilde f_k^2$ is finite and bounded by the C^2-norm of $\tilde f$. This automatically implies
$$\rho^2\|y''\|_{L^\infty(0,\pi)} + \|y'\|_{L^\infty(0,\pi)} + \|y\|_{L^\infty(0,\pi)} \leq C\|\tilde f\|_{c^2(0,\pi)},$$
and the proof is complete. □

We now conclude with

Proof of Theorem 1.1. Since the geodesic Γ is non degenerate, the linear operator L_k is invertible in the set of 2ℓ-periodic functions. More precisely, for any $f \in L^\infty(-\ell,\ell)$, there exist a 2ℓ periodic function d_k and a positive constant C such that $L_k(d_k)=f$ and
$$\|\ddot d_k\|_\infty + \|\dot d_k\|_\infty + \|d_k\|_\infty \leq C\|f\|_\infty,$$
Define $\tilde\mu_0$, $\tilde d_{0N}$ and d_{0k} to be solution of
$$L_{N+1}(\tilde\mu_0) = \alpha_{N+1}, \quad L_N(\tilde d_{0N}) = \alpha_N$$
and
$$L_k(\tilde d_{0k}) = \alpha_k \quad \text{ for all } k=1,\dots,N-1.$$
Thus we have
$$\varepsilon\|\ddot{\tilde d}_{0N}\|_\infty + \varepsilon^{\frac12}\|\dot{\tilde d}_{0N}\|_\infty + \|\tilde d_{0N}\|_\infty \leq c, \quad \|\ddot{\tilde d}_{0k}\|_\infty + \|\dot{\tilde d}_{0k}\|_\infty + \|\tilde d_{0k}\|_\infty \leq c$$
and
$$\varepsilon^{1+\frac{1}{N-2}}\|\ddot{\tilde\mu}_0\|_\infty + \varepsilon^{\frac12+\frac{1}{N-2}}\|\dot{\tilde\mu}_0\|_\infty + \|\tilde\mu_0\|_\infty \leq c.$$
We now solve $L_0(\tilde E_0) = -\,2(\,Tr_{\bar g}\bar h - \bar h_{00})\,(\int \partial_{ii}\omega Z_0)\,\tilde d_{0N} + \alpha_0 + \varepsilon Q_0(\tilde d_0)$, where $\tilde d_0 = (\tilde d_{01},\dots,\tilde d_{0N})$. Since the right hand side is regular, by Lemma 8.2 we have
$$\varepsilon^{2+\frac{2}{N-2}}\|\ddot e_0\|_\infty + \|E_0\|_\infty \leq c.$$
We have
$$\|(\tilde\mu_0, \tilde d_0, \tilde E_0)\| \leq c.$$
Define
$$\mu = \tilde\mu_0 + \tilde\mu_1, \quad d = \tilde d_0 + \tilde d_1, \quad e = \tilde E_0 + \tilde e_1.$$
The system (8.1) reduces to
$$\left\{ \begin{aligned} L_{N+1}(\tilde\mu_1) = \varepsilon M_{N+1}, \quad L_N(\tilde d_{1N}) = \varepsilon M_N \\ L_k(\tilde d_{1k}) = \varepsilon M_k \quad k=1,\dots,N-1 \\ L_0(\tilde e_1) = -\,2(\,Tr_{\bar g}\bar h - \bar h_{00})\,(\int \partial_{ii}\omega Z_0)\,\tilde d_{1N} + \varepsilon^2 M_0 \end{aligned} \right. \tag{8.10}$$
Let us observe now that the linear operator
$$\mathcal{L}(\mu_1, d_1, e_1) = \big(\,L_{N+1}(\mu_1), L_N(d_{1N}), L_{N-1}(d_{1(N-1)}), \dots, L_1(d_{11}), L_0(e_1)\,\big)$$

is invertible with bounds for $L(\mu_1, d_1, e_1) = (f, g, h)$ given by

$$\|(\mu_1, d_1, e_1)\| \leq C\left[\|f\|_\infty + \|g\|_\infty + \varepsilon^{-\frac{N-1}{N-2}}\|h\|_\infty\right].$$

It then follows from contraction mapping principle that, given $\sigma > 0$, the problem

$$[\mathcal{L} + (\varepsilon M_{N+1}, \varepsilon M_N, \varepsilon M_{N-1}, \ldots, \varepsilon M_1, \varepsilon^2 M_0)](\mu_1, d_1, e_1) = (f, g, h)$$

is uniquely solvable for $\|(\mu_1, d_1, e_1)\| \leq c\varepsilon^\sigma$ if $\|f\|_\infty < \varepsilon^{\sigma+\rho}$, $\|g\|_\infty < \varepsilon^{\sigma+\rho}$ $\|h\|_2 < \varepsilon^{\sigma+\rho-\frac{N-1}{N-2}}$, for some $\rho > 0$. The desired result for the full problem (8.10) then follows directly from Schauder's fixed point theorem. In fact we get $\|(\tilde{\mu}_1, \tilde{d}_1, \tilde{e}_1)\| = O(\varepsilon^{\frac{N-3}{N-2}})$ for the solution. □

9. Appendix

Proof of (5.33). We write

$$\begin{aligned} S_\varepsilon(\mathtt{w}) &= S_\varepsilon(\tilde{\omega}) + \left\{\rho^2 a_0 \ddot{e}_\varepsilon(\rho y_0) + \lambda_1 e_\varepsilon(\rho y_0)\right\}\chi_\varepsilon Z_0 + \tilde{\mathcal{A}}(e_\varepsilon \chi_\varepsilon Z_0) \\ &+ 2e_\varepsilon \nabla\chi_\varepsilon \nabla Z_0 + N_0(e_\varepsilon \chi_\varepsilon Z_0) \end{aligned} \tag{9.1}$$

where

$$N_0(e_\varepsilon\chi_\varepsilon Z_0) = \mu_\varepsilon^{-\frac{N-2}{2}\varepsilon}\left[(\tilde{\omega} + e_\varepsilon\chi_\varepsilon Z_0)^{p-\varepsilon} - \tilde{\omega}^{p-\varepsilon}\right] - pe_\varepsilon\omega^{p-1}\chi_\varepsilon Z_0. \tag{9.2}$$

We start analyzing $S_\varepsilon(\tilde{\omega})$. Expanding $S_\varepsilon(\tilde{\omega})$ in ε and taking into account that

$$\Delta\left[(1+\alpha_\varepsilon)\omega\right] + \mu_\varepsilon^{-\frac{N-2}{2}\varepsilon}\left[(1+\alpha_\varepsilon)\omega\right]^p = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{9.3}$$

we have

$$\begin{aligned} S_\varepsilon(\tilde{\omega}) &= \textstyle\sum_{k=0}^5 \mathcal{A}_k\omega - p\omega^{p-1}\bar{\omega} - \varepsilon\omega^p\log\omega \\ &+ B(\omega) - \mathcal{A}(\bar{\omega}) + \alpha_\varepsilon\mathcal{A}(\omega - \bar{\omega}) + a_0\partial_0^2\left[\alpha_\varepsilon(\omega - \bar{\omega})\right] \\ &+ b(\rho y_0, y; \mu, d)\varepsilon^2\omega^p, \end{aligned} \tag{9.4}$$

where the operators $\mathcal{A}_k$ and $\mathcal{A}$ are defined in Lemma 5.1 and 5.14, the operator B is given by (5.16) and b is a sum of functions of the form

$$b_0(\rho y_0)b_1(\mu, d)$$

with b_0 a smooth of ρy_0, uniformly bounded in ε together with its derivatives, and b_1 a smooth function of its arguments, uniformly bounded in ε. A remark to be made is that b_1 does not depend on the derivatives of its arguments.

The main part in (9.4) is

$$e_0 := \sum_{k=0}^5 \mathcal{A}_k\omega - p\omega^{p-1}\bar{\omega} - \varepsilon\omega^p\log\omega. \tag{9.5}$$

Indeed, $B(\omega)$ is of lower order with respect to $\sum_{k=0}^5 \mathcal{A}_k\omega$ as shown by Lemma 5.1, so is the term given by $\mathcal{A}(\bar{\omega})$ since $\bar{\omega} = O(\varepsilon)\omega$ and also the term $\alpha_\varepsilon\mathcal{A}(\omega - \bar{\omega})$ since $\alpha_\varepsilon = O(\varepsilon|\log\varepsilon|)$ as $\varepsilon \to 0$.

Observe furthermore that $\partial_0^2 \alpha_\varepsilon = \rho^2 O(\alpha_\varepsilon)$, so that $a_0 \partial_0^2 \left[\alpha_\varepsilon(\omega - \bar\omega)\right] = o(1)\rho^2\omega$. Summarizing, we can write

$$S_\varepsilon(\tilde\omega) = e_0 + \varepsilon^2 b(\rho y_0; \mu, d)\omega^p + \varepsilon^3 r, \tag{9.6}$$

where r is a sum of functions of the form

$$h_0(\rho y_0) f_1(\mu, d, \dot\mu, \dot d) f_2(y)$$

with h_0 a smooth function uniformly bounded in ε, f_1 a smooth function of its arguments, homogeneous of degree 3, uniformly bounded in ε and

$$\sup(1 + |y|^{N-2})|f_2(y)| < +\infty.$$

By means of Lemma 5.1 and taking into account notation (5.2), we can expand in power of ε the first term in (9.5)

$$\begin{aligned}
\textstyle\sum_{k=0}^5 \mathcal{A}_k(\omega) \quad = \quad & \varepsilon \left[-2\bar h_{ij}\tilde d_N \partial_{ij}\omega\right] + \varepsilon^{1+\frac{1}{N-2}}\, \tilde\mu \left[-2\bar h_{ij} y_N \partial_{ij}\omega + Tr_{\bar g}\bar h \partial_N \omega\right] \\
+ \quad & \varepsilon^2 \left[\textstyle\sum_{ij}(\ \dot{\tilde d}_i \dot{\tilde d}_j - \frac13 R_{ijkl}\tilde d_k \tilde d_l + a_{Nk}^{ij}\tilde d_k \tilde d_N + 4\bar h_{0j}\tilde d_i \tilde d_N)\partial_{ij}\omega\right] \\
+ \quad & \varepsilon^{2+\frac{1}{N-2}} \Big[-\tilde\mu D_y\omega \cdot \ddot{\tilde d} - \tfrac{\tilde\mu}{3} R_{ijkl} y_k \tilde d_l \partial_{ij}\omega + 2\tilde\mu a_{Nk}^{ij} y_k \tilde d_N \partial_{ij}\omega \\
+ \quad & \tilde\mu(\tfrac23 R_{ijik} + R_{0j0k})\tilde d_k \partial_j\omega + 4\bar h_{0j}(\tilde\mu y_N D_y(\partial_j\omega)\dot\delta + \dot{\tilde\mu}\tilde d_N(\gamma\partial_j\omega + D_y(\partial_j\omega)y)) \\
+ \quad & b_N^j \tilde\mu\, \tilde d_N \partial_j\omega \ - \ Tr_{\bar g}\bar k \tilde\mu\, \tilde d_N \partial_N\omega - 2\dot{\tilde\mu} D_y Z_{N+1} \cdot \ddot{\tilde d}\Big] \\
+ \quad & \varepsilon^{2+\frac{2}{N-2}} \ \big[-\ddot{\tilde\mu}\tilde\mu Z_{N+1} \\
+ \quad & \tilde\mu^2(-\tfrac13 R_{ikjl}\, y_k y_l\, \partial_{ij}\omega + (\tfrac23 R_{ijik} \ + \ R_{0j0k}) y_k \partial_j\omega + b_N^j y_N \partial_j\omega - Tr_{\bar g}\bar k y_N \partial_N\omega) \\
+ \quad & 4\bar h_{0j}\tilde{\tilde\mu}\dot{\tilde\mu} y_N(\gamma\partial_j\omega + D_y(\partial_j\omega)\cdot y) \\
+ \quad & (\dot{\tilde\mu})^2(D_{yy}\omega[y]^2 + 2(1+\gamma)D_y\omega\cdot y + \gamma(1+\gamma)\omega)\big] \\
+ \quad & \varepsilon^3\, r
\end{aligned} \tag{9.7}$$

where r denotes the sum of functions of the form

$$h_0(\rho y_0)[f_1(\nu, d, \dot\mu, \dot d) + o(1) f_2(\mu, d, \dot\mu, \dot d, \ddot\mu, \ddot d)] f_3(y)$$

with h_0 a smooth function of ρy_0 uniformly bounded in ε, f_1, f_2 smooth functions of their arguments, f_1 homogeneous of degree 3, f_2 linear in the variables $(\ddot\mu, \ddot d)$, and

$$\sup(1 + |y|^{N-2})|f_3(y)| < +\infty.$$

The previous expansion, together with (9.5), (9.6) and the notation (5.2), give precise description of the first term $S_\varepsilon(\tilde\omega)$ in (9.1). Let us now consider the term $\mathcal{A}(e_\varepsilon \chi_\varepsilon Z_0)$. Arguing as before, we have that

$$\mathcal{A}(e_\varepsilon \chi_\varepsilon Z_0) = \sum_{k=0}^5 \mathcal{A}_k(e_\varepsilon Z_0) + \varepsilon^3 r,$$

where r is the sum of functions of the form

$$h_0(\rho y_0)[f_1(\nu, d, e, \dot\mu, \dot d, \dot e) + o(1) f_2(\mu, d, e, \dot\mu, \dot d, \dot e, \ddot\mu, \ddot d, \ddot e)] f_3(y)$$

with h_0 a smooth function of ρy_0 uniformly bounded in ε, f_1, f_2 smooth functions of their arguments, f_1 homogeneous of degree 3, f_2 linear in the variables $(\ddot\mu, \ddot d, \ddot e)$, and

$$\sup(1+|y|^{N-2})|f_3(y)| < +\infty.$$

Let us then consider the term $\sum_{k=0}^{5} \mathcal{A}_k(e_\varepsilon Z_0)$. Directly from Lemma 5.1 and taking into account (5.30), we obtain

$$\sum_{k=0}^{5} \mathcal{A}_k(e_\varepsilon Z_0) = \varepsilon \tilde e A + \varepsilon^{2+\frac{1}{N-2}} \dot{\tilde e} B$$

where

$$\begin{aligned}
A \;=\;& \varepsilon \left[-2\bar h_{ij} \tilde d_N \partial_{ij} Z_0\right] + \varepsilon^{1+\frac{1}{N-2}}\, \tilde\mu \left[-2\bar h_{ij} y_N \partial_{ij} Z_0 + Tr_{\bar g} \bar h \partial_N Z_0\right] \\
+\;& \varepsilon^2 \left[\textstyle\sum_{ij} (\dot{\tilde d}_i \dot{\tilde d}_j - \frac{1}{3} R_{ijkl}\, \tilde d_k \tilde d_l \;+\; a_{Nk}^{ij} \tilde d_k \tilde d_N + 4\bar h_{0j} \tilde d_i \tilde d_N) \partial_{ij} Z_0\right] \\
+\;& \varepsilon^{2+\frac{1}{N-2}} \left[-\tilde\mu D_y Z_0 \cdot \ddot{\tilde d} - \tfrac{\tilde\mu}{3} R_{ijkl}\, y_k \tilde d_l \partial_{ij} Z_0 + 2\tilde\mu a_{Nk}^{ij} y_k \tilde d_N \partial_{ij} Z_0 + \tilde\mu (\tfrac{2}{3} R_{ijik} \;+\; R_{0j0k})\, \tilde d_k \partial_j Z_0 \right. \\
+\;& 4\bar h_{0j} (\tilde\mu y_N D_y (\partial_j Z_0) \dot\delta + \dot{\tilde\mu} \tilde d_N (\gamma \partial_j Z_0 + D_y(\partial_j Z_0) y)) \\
+\;& \left. b_N^j \tilde\mu \tilde d_N \partial_j Z_0 - Tr_{\bar g} \bar k \tilde\mu \tilde d_N \partial_N Z_0 - 2\dot{\tilde\mu} (\gamma D_y Z_0 + D_{yy} Z_0[y]) \cdot \dot{\tilde d}\right] \\
+\;& \varepsilon^{2+\frac{2}{N-2}} \left[-\ddot{\tilde\mu} \tilde\mu Z_{N+1} \right. \\
+\;& \tilde\mu^2 (-\tfrac{1}{3} R_{ikjl}\, y_k y_l \partial_{ij} Z_0 \;+\; (\tfrac{2}{3} R_{ijik} \;+\; R_{0j0k})\, y_k \partial_j Z_0 + b_N^j y_N \partial_j Z_0 - Tr_{\bar g} \bar k y_N \partial_N Z_0) \\
+\;& \bar h_{0j} \tilde\mu \dot{\tilde\mu} y_N (\gamma \partial_j Z_0 + D_y(\partial_j Z_0) \cdot y) \\
+\;& \left. (\dot{\tilde\mu})^2 (D_{yy} Z_0[y]^2 + 2(1+\gamma) D_y Z_0 \cdot y + \gamma(1+\gamma) Z_0)\right] \\
+\;& \varepsilon^3\, r
\end{aligned}$$

and r is as before. On the other hand,

$$\begin{aligned}
B \;=\;& \varepsilon \left[-2\tilde\mu D_y Z_0 \cdot \dot\delta - 4\bar h_{0j} \tilde\mu \tilde d_N \partial_j Z_0\right] \\
+\;& \varepsilon^{1+\frac{1}{N-2}} \left[-2\tilde\mu \dot{\tilde\mu} D_y Z_0 \cdot y - 2\gamma \tilde\mu \dot{\tilde\mu} Z_0 - 4(\tilde\mu)^2 \bar h_{0j} y_N \partial_j Z_0\right] \\
+\;& \varepsilon^2 r,
\end{aligned}$$

with r as before.

Expanding in ε the term $N_0(e_\varepsilon \chi_\varepsilon Z_0)$ defined in (9.2), we get

$$\begin{aligned}
N_0(e_\varepsilon \chi_\varepsilon Z_0) \;=\;& \varepsilon^2 \left[p(p-1) E_0^2 \omega^{p-2} Z_0^2 + p E_0 \omega^{p-1} \log\omega Z_0\right] \\
+\;& \varepsilon^3 |\log\varepsilon| r
\end{aligned} \tag{9.8}$$

where r is the sum of functions of the form

$$h_0(\rho y_0)h_1(\mu, d, e)h_2(y)$$

with h_0 a smooth function, uniformly bounded in ε, h_1 a smooth function of its arguments and $\sup(1+|y|)^{N+2}|h_2|(y) \le C$. Summing up all the computation, we obtain the proof of (5.33). □

Proof of (5.35), (5.38), (5.39), (5.40). The proof consists of two steps. In the first step we compute the expansion in ε of the projections assuming that

$$\mu_\varepsilon = \varepsilon^{\frac{N-1}{N-2}}\tilde\mu, \quad d_{\varepsilon N} = \varepsilon \tilde d_N, \quad d_{\varepsilon j} = \varepsilon d_j \quad \text{and} \quad e_\varepsilon = \varepsilon \tilde e.$$

In the second part, we will chose μ_1, d_{N1} and e_1 to get the above expansion when μ, d and e are defined as in (5.3), (5.2), (5.5), (5.29) and (5.30).

Step 1. We start with the projection of the non linear part

$$h = -p\omega^{p-1}\bar\omega - \varepsilon\omega^p \log\omega.$$

We have the validity of the following facts: as $\varepsilon \to 0$,

$$\int_{\mathcal{D}_{y_0}} hZ_{N+1}\,dy = \varepsilon\left[A_2\left(\frac{\tilde\mu}{\tilde d_N}\right)^{N-2} - A_3 + \varepsilon^{\frac{1}{N-2}}\left(\frac{\tilde\mu}{\tilde d_N}\right)^{N-1} g_{N+1}(\frac{\tilde\mu}{\tilde d_N})\right] \tag{9.9}$$

$$\int_{\mathcal{D}_{y_0}} hZ_N\,dy = \varepsilon^{1+\frac{1}{N-2}}\left[-A_1\left(\frac{\tilde\mu}{\tilde d_N}\right)^{N-1} + \varepsilon^{\frac{1}{N-2}}\left(\frac{\tilde\mu}{\tilde d_N}\right)^{N} g_N(\frac{\tilde\mu}{\tilde d_N})\right], \tag{9.10}$$

and

$$\int_{\mathcal{D}_{y_0}} hZ_k\,dy = \varepsilon^{2+\frac{3}{N-2}} g_k(\frac{\tilde\mu}{\tilde d_N}) \tag{9.11}$$

for $k = 1, \dots, N-1$, and

$$\int_{\mathcal{D}_{y_0}} hZ_0\,dy = \varepsilon\left[-A_4\left(\frac{\tilde\mu}{\tilde d_N}\right)^{N-2} - A_5 + \varepsilon^{\frac{1}{N-2}}\left(\frac{\tilde\mu}{\tilde d_N}\right)^{N-1} g_0(\frac{\tilde\mu}{\tilde d_N})\right]. \tag{9.12}$$

In the previous formula, the functions g_i are smooth function with $g_i(0) \ne 0$ and A_i are positive constants.

We first prove (9.10). Expanding in Taylor we have

$$\begin{aligned} -p\int_{\mathcal{D}_{y_0}} \bar\omega\omega^{p-1}Z_N &= pc_N^{\frac{N+2}{2}} \int_{\mathcal{D}_{y_0}} \frac{N-2}{(1+|\bar y|^2+|y_N+2\varepsilon^{-\frac{1}{N-2}}\frac{\tilde d_N}{\tilde\mu}|^2)^{\frac{N-2}{2}}} \frac{y_N}{(1+|y|^2)^{\frac{N+4}{2}}}\,dy \\ &= \varepsilon^{1+\frac{1}{N-2}}\left[-A_1\left(\frac{\tilde\mu}{\tilde d_N}\right)^{N-1} + \varepsilon^{\frac{1}{N-2}}\left(\frac{\tilde\mu}{\tilde d_N}\right)^{N} g_N(\frac{\tilde\mu}{\tilde d_N})\right]. \end{aligned}$$

The constant A_1 which appears in (9.9) is precisely given by

$$A_1 = \frac{pc_N^{\frac{N+2}{2}}(N-2)^2}{2^{N-1}}(\int \frac{y_N^2}{(1+|y|^2)^{\frac{N+4}{2}}}).$$

Furthermore, we have

$$-\varepsilon\int_{\mathcal{D}_{y_0}} \omega^p \log\omega Z_N = \varepsilon^{2+\frac{2}{N-2}} O((\frac{\tilde\mu}{\tilde d_N})^N).$$

This proves (9.10). Concerning the projection along Z_{N+1}, arguing as before we get

$$-p\int_{\mathcal{D}_{y_0}} \bar\omega\omega^{p-1}Z_{N+1} = \varepsilon\left[A_2\left(\frac{\tilde\mu}{\tilde d_N}\right)^{N-2} + \varepsilon^{\frac{1}{N-2}}(\frac{\tilde\mu}{\tilde d_N})^{N-1}g_N(\frac{\tilde\mu}{\tilde d_N})\right],$$

for a positive constant A_2 which can be computed explicitly.

Finally, we get

$$-\varepsilon\int_{\mathcal{D}_{y_0}} \omega^p\log\omega Z_{N+1} = -\varepsilon A_3 + \varepsilon^{2+\frac{2}{N-2}}O((\frac{\tilde\mu}{\tilde d_N})^N)$$

where A_3 is the positive constant given by

$$A_3 = \int \omega^p\log\omega Z_{N+1} = \frac{N-2}{2}\int\omega^{p+1}\log\omega + \int\log\omega\nabla(\frac{\omega^{p+1}}{p+1})\cdot y$$
$$= -\frac{1}{p+1}\int\omega^p\nabla\omega\cdot y = \frac{N}{(p+1)^2}\int\omega^{p+1}.$$

This proves (9.9). Estimate (9.12) follows in a similar way. We conclude with (9.11) which follows from the observation that

$$p\int\omega^{p-1}\bar\omega Z_k = \int\omega^p\log\omega Z_k = 0 \quad \text{for all } k=1,\dots,N-1,$$

due to symmetry. This gives (9.11).

We continue with the projections of $S := S_\varepsilon(\mathtt{w}) - h$. We have

$$\begin{aligned}
\textstyle\int_{\mathcal{D}_{y_0}} S\,Z_{N+1} \;&=\; \varepsilon^2\left[\textstyle\int \Upsilon_\varepsilon Z_{N+1}\,(1+o(\varepsilon))\right]\\
&+\; \varepsilon^{2+\frac{2}{N-2}}\left[-C_2\tilde\mu\ddot{\tilde\mu} + (\dot{\tilde\mu})^2\textstyle\int[D_{yy}\omega y^2 + 2(1+\gamma)D_y\omega y + \gamma(1+\gamma)\omega]Z_{N+1}\right.\\
&-\; \left.(\tilde\mu)^2[Tr_{\bar g}\bar k\textstyle\int y_N\partial_N\omega Z_{N+1} + \frac{1}{3}R_{ikjl}\int y_ky_l\partial_{ij}\omega Z_{N+1}]\right]\\
&+\; \varepsilon^3 r
\end{aligned}\tag{9.13}$$

where r is a sum of functions of the form (5.36).

Concerning the projection along Z_N, we get at main order

$$\begin{aligned}
\textstyle\int_{\mathcal{D}_{y_0}} SZ_N \;&=\; \varepsilon^{1+\frac{1}{N-2}}\tilde\mu\left[-2\bar h_{ii}\textstyle\int y_N\partial_N\omega\partial_{ii}\omega + Tr_{\bar g}\bar h\int(\partial_N\omega)^2\right]\\
&+\; \varepsilon^{2+\frac{1}{N-2}}\left[-C_1\tilde\mu\ddot{\tilde d}_N - 2\dot{\tilde\mu}\textstyle\int D_yZ_{N+1}[\dot{\tilde d}]Z_N\right.\\
&+\; 4\bar h_{0j}\left(\tilde\mu d_j\textstyle\int y_N\partial_{jj}\omega Z_N + \dot{\tilde\mu}d_N\int\partial_N\partial_j\omega y_N\partial_N\omega\right) - C_1\tilde\mu d_N Tr_{\bar g}\bar k\\
&-\; \left.\tilde A_1\left(\tfrac{\tilde\mu}{\tilde d_N}\right)^{N-1}\tilde e - 2\bar h_{00}\tilde e\tilde d_N\textstyle\int y_N\omega^{p-1}Z_0Z_N\right]\\
&+\; \varepsilon^{3+\frac{2}{N-2}}r\\
&=\; \varepsilon^{1+\frac{1}{N-2}}C_1\tilde\mu\bar h_{00} + \varepsilon^{2+\frac{1}{N-2}}C_1\left[-\tilde\mu\ddot{\tilde d}_N - Tr_{\bar g}\bar k\tilde\mu\tilde d_N + 2\bar h_{0j}\tilde\mu d_j\right.\\
&-\; \left.A_1\left(\tfrac{\tilde\mu}{\tilde d_N}\right)^{N-1}\tilde e - 2\bar h_{00}\tilde e\tilde d_N\textstyle\int y_N\omega^{p-1}Z_0Z_N\right] + \varepsilon^{3+\frac{2}{N-2}}r,
\end{aligned}\tag{9.14}$$

where we use the following computations

$$\int y_N \partial_{jj}\omega \partial_N \omega = \frac{1}{2} C_1, \quad \int \partial_j Z_{N+1} \partial_N = 0, \quad \text{for all } j.$$

We now see the projection along Z_k, for $k = 1, \ldots, N-1$. First we write

$$\begin{aligned} \int_{\mathcal{D}_{y_0}} S Z_k &= \varepsilon^{2+\frac{1}{N-2}} \tilde{\mu} \Big[-C_1 \ddot{d}_k + \big(-\tfrac{2}{3} R_{iljm} \int y_m \partial_{ij}\omega Z_k + C_1 (\tfrac{2}{3} R_{ijil} + R_{0j0l}) \big) d_l \Big] \\ &+ \tilde{d}_N \Big(2 a_{Nl}^{ij} \int y_l \partial_{ij}\omega Z_k + b_N^j C_1 \Big) + \dot{\tilde{d}}_N \left(4 \bar{h}_{0k} \int y_N \partial_{Nk}\omega Z_k \right) \Big] \\ &+ \varepsilon^{3+\frac{2}{N-2}} r \\ &= \varepsilon^{2+\frac{1}{N-2}} \tilde{\mu} C_1 \Big[-\ddot{d}_k + R_{0j0l} d_l + \gamma_{0k} \tilde{d}_N + \gamma_{1k} \dot{\tilde{d}}_N \Big] + \varepsilon^{3+\frac{2}{N-2}} r \end{aligned} \tag{9.15}$$

since we have the validity of the following fact

$$\begin{aligned} -\tfrac{2}{3} R_{iljm} d_l \int_{D_N} y_m \partial_{ij}\omega Z_k &= -\tfrac{2}{3} \Big[R_{ilik} \int_{D_N} y_k \partial_{ii}\omega Z_k + R_{ilki} \int_{D_N} y_i \partial_{ik}\omega Z_k \\ &+ R_{kljj} \int_{D_N} y_j \partial_{kj}\omega Z_k \Big] d_l \\ &= -\tfrac{1}{3} C_1 \left[R_{ilik} - R_{ilki} \right] d_l = -\tfrac{2}{3} C_1 R_{ilik} d_l. \end{aligned}$$

In (9.15), γ_{0k} and γ_{1k} denote smooth explicit functions of ρy_0.

Finally, using the orthogonality in L^2 of Z_0 with respect to Z_i, for $i = 1, \ldots, N+1$, direct computations show

$$\begin{aligned} \int_{\mathcal{D}_{y_0}} S Z_0 &= \varepsilon C_3 \Big[-2 (Tr_{\bar{g}} \bar{h} - \bar{h}_{00}) \tilde{d}_N \Big] \\ &+ \varepsilon^2 C_3 \Big[\rho^2 a_0 \ddot{\tilde{e}} + \lambda_1 \tilde{e} + \dot{d}_i^2 - \tfrac{1}{3} R_{ikil} d_k d_l + a_{Nk}^{ii} d_k \tilde{d}_N + 4 \bar{h}_{0j} d_j \tilde{d}_N + \int \Upsilon_\varepsilon Z_0 \Big] \\ &+ \varepsilon^{2+\frac{2}{N-2}} \left[(\dot{\tilde{\mu}})^2 + f_1(\rho y_0) \tilde{\mu}^2 + f_2(\rho y_0) \tilde{\mu} \dot{\tilde{\mu}} \right] + \varepsilon^3 r \end{aligned} \tag{9.16}$$

where f_i are explicit smooth functions, uniformly bounded in ε, and r is as before.

Summing up the previous calculations, we conclude that at main order

$$\begin{aligned} \int_{\mathcal{D}_{y_0}} S_\varepsilon(\mathtt{w}) Z_{N+1} \, dy &= \varepsilon \left[A_2 \left(\tfrac{\tilde{\mu}}{\tilde{d}_N} \right)^{N-2} - A_3 + \varepsilon^{\frac{1}{N-2}} \left(\tfrac{\tilde{\mu}}{\tilde{d}_N} \right)^{N-1} g_{N+1} (\tfrac{\tilde{\mu}}{\tilde{d}_N}) \right] (1 + o(1)) \\ \varpi \int_{\mathcal{D}_{y_0}} S_\varepsilon(\mathtt{w}) Z_N \, dy &= \varepsilon^{1+\frac{1}{N-2}} \left[C_1 \tfrac{A_2}{A_1} \bar{h}_{00} \tilde{\mu} - A_1 \left(\tfrac{\tilde{\mu}}{\tilde{d}_N} \right)^{N-1} + \varepsilon^{\frac{1}{N-2}} \left(\tfrac{\tilde{\mu}}{\tilde{d}_N} \right)^{N} \tilde{g}_N \left(\tfrac{\tilde{\mu}}{\tilde{d}_N} \right) \right] (1 + o(1)) , \end{aligned}$$

and

$$\begin{aligned} \int_{\mathcal{D}_{y_0}} S_\varepsilon(\mathtt{w}) Z_0 \, dy &= \varepsilon \Bigg[\lambda_1 \tilde{e} - 2 (Tr_{\bar{g}} \bar{h} - \bar{h}_{00}) (\textstyle\int \partial_{ii}\omega Z_0) \, \tilde{d}_N - A_4 \left(\tfrac{\tilde{\mu}}{\tilde{d}_N} \right)^{N-2} - A_5 \\ &+ \varepsilon^{\frac{1}{N-2}} \left(\tfrac{\tilde{\mu}}{\tilde{d}_N} \right)^{N-1} \tilde{g}_0 \left(\tfrac{\tilde{\mu}}{\tilde{d}_N} \right) \Bigg] (1 + o(1)) , \end{aligned}$$

Part 2. Let now $(\mu_\varepsilon^0, d_{\varepsilon N}^0, e_\varepsilon^0) \in (0,\infty)\times(0,\infty)\times\mathbb{R}$ be the solution to the following system of nonlinear equations

$$\begin{cases} A_2\left(\frac{\mu}{d_N}\right)^{N-2} - A_3 + \varepsilon^{\frac{1}{N-2}}\left(\frac{\mu}{d_N}\right)^{N-1}\tilde g_{N+1}\left(\frac{\mu}{d_N}\right) \;=\; 0 \\ \\ C_1\frac{A_2}{A_1}\bar h_{00}\mu - A_2\left(\frac{\mu}{d_N}\right)^{N-1} + \varepsilon^{\frac{1}{N-2}}\left(\frac{\mu}{d_N}\right)^{N}\tilde g_N\left(\frac{\mu}{d_N}\right) = 0 \\ \\ \lambda_1 e - 2(Tr_{\bar g}\bar h - \bar h_{00})(\int \partial_{ii}\omega Z_0)\, d_N - \; A_4\left(\frac{\mu}{d_N}\right)^{N-2} - A_5 \\ \\ +\varepsilon^{\frac{1}{N-2}}\left(\frac{\mu}{d_N}\right)^{N-1}\tilde g_0\left(\frac{\mu}{d_N}\right) \;=\; 0\;. \end{cases} \tag{9.17}$$

It is easy to show that the solution $(\mu_\varepsilon^0, d_{\varepsilon N}^0, e_\varepsilon^0)$ has the form

$$\hat\mu = \mu_0 + \varepsilon^{\frac{1}{N-2}}\mu_1,\quad \hat d_N = d_0 + \varepsilon^{\frac{1}{N-2}}d_{1N},\quad \hat e = e_0 + \varepsilon^{\frac{1}{N-2}}e_1,$$

where μ_0, d_{0N} and E_0 is the solution to

$$F(\mu, d_N, e) := \begin{bmatrix} A_2\left(\frac{\mu}{d_N}\right)^{N-2} - A_3 \\ C_1\frac{A_2}{A_1}\bar h_{00}\mu - A_2\left(\frac{\mu}{d_N}\right)^{N-1} \\ \lambda_1\, e - 2(Tr_{\bar g}\bar h - \bar h_{00})(\int \partial_{ii}\omega Z_0)\, d_N - \; A_4\left(\frac{\mu}{d_N}\right)^{N-2} - A_5 \end{bmatrix} . = \; 0$$

Observe that $\mu_0 > 0$ and $d_0 > 0$. Direct computations show that

$$F_0 := \nabla_{\mu, d_N, e}F(\mu_0, d_0, E_0) = \begin{bmatrix} (N-2)A_2\frac{\mu_0^{N-3}}{d_0^{N-2}} & -(N-2)A_2\frac{\mu_0^{N-2}}{d_0^{N-1}} & 0 \\ -(N-2)A_2\frac{\mu_0^{N-2}}{d_0^{N-1}} & (N-1)A_2\frac{\mu_0^{N-1}}{d_0^{N}} & 0 \\ 0 & -2(Tr_{\bar g}\bar h - \bar h_{00})\int \partial_{ii}\omega Z_0 & \lambda_1 \end{bmatrix}.$$

Since

$$det\left(\nabla_{\mu, d_N, e}F(\mu_0, d_0, E_0)\right) = (N-2)A_2C_1\lambda_1\frac{\mu_0^{N-2}}{d_0^{N-1}}\bar h_{00} > 0,$$

solving system (9.17) is equivalent to solve a fixed point problem, which is uniquely solvable in the set

$$\{(\mu_1, d_{1N}, e_1) \,:\, \|\mu_1\|_\infty \le \delta, \|d_{1N}\|_\infty \le \delta, \|e_1\|_\infty \le \delta\}$$

for some proper small $\delta > 0$.

We conclude the validity of the expansions (5.38), (5.39) and (5.40), with

$$A = (N-2)A_2\frac{\mu_0^{N-3}}{d_0^{N-2}} > 0,\quad B = -(N-2)A_2\frac{\mu_0^{N-2}}{d_0^{N-1}},\quad C = N-1)A_2\frac{\mu_0^{N-1}}{d_0^{N}} > 0.$$

An easy computation shows that $AC - B^2 > 0$. Thus this concludes the proof of the Proposition. $\square$

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