Multispike solutions for a nonlinear elliptic problem involving critical Sobolev exponent.

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Abstract

The main purpose of this paper is to construct families of positive solutions for the equation

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} + \varepsilon u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

which blow-up and concentrate in $k \geq 1$ different points of Ω as ε goes to 0. We exhibit some examples of contractible domains where a large number of solutions exists.

Keywords: critical Sobolev exponent, blowing-up solution, Robin's function.

AMS subject classification: 35J20, 35J60.

0 Introduction

In this paper we are concerned with the problem

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} + \varepsilon u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$ and $\varepsilon > 0$ is a positive parameter.

In [6] (see also [2]) Brezis and Nirenberg showed that if $N \geq 4$ problem (0.1) has a solution for any $\varepsilon \in (0, \lambda_1)$ where λ_1 denotes the first eigenvalue of $-\Delta$ with Dirichlet boundary condition on Ω. When $N = 3$ the problem is much more delicate and a complete answer can be given only when Ω is a ball. In that case problem (0.1) has a solution if and only if $\varepsilon \in (\frac{1}{4}\lambda_1, \lambda_1)$.

In [15] Rey showed that if u_ε are solutions of (0.1) which concentrate around a point x_0 as ε goes to 0 then x_0 is a critical point of the Robin's function τ_Ω (see 0.4). Conversely he proved that if $N \geq 5$ any nondegenerate critical point x_0 of τ_Ω generates a family of solutions of (0.1) concentrating around x_0 as ε goes to 0. Successively in [16] the author proved that for ε small enough (0.1) has at least as many solutions as cat Ω, i.e. the Ljusternik-Schnirelmann category of Ω. In [14] Passaseo showed that the number of solutions of (0.1) is not related to the topology of Ω but to the topology of a domain Ω' which differs from Ω by a set of small capacity. For instance if Ω is obtained from Ω' by cutting off a set with small capacity, then problem (0.1) has at least cat $\Omega' + 1$ distinct solutions even if the domain Ω is contractible in itself.

In this paper we still consider the case $N \geq 5$ and we study existence of solutions which concentrate in one or more than one point of Ω in the sense of the following definition.

Definition 0.1 *Let u_ε be a family of solutions for (0.1). We say that u_ε blow-up and concentrate at k points $x_1, \ldots, x_k$ if there exist rates of concentration $\mu_{1\varepsilon}, \ldots, \mu_{k\varepsilon} > 0$, and points $x_{1\varepsilon}, \ldots, x_{k\varepsilon} \in \Omega$ with $\lim_{\varepsilon \to 0} \mu_{i\varepsilon} = 0$ and $\lim_{\varepsilon \to 0} x_{i\varepsilon} = x_{i0}$, $x_{i0} \neq x_{j0}$ for $i, j = 1, \ldots, k$, $i \neq j$, such that*

$$u_\varepsilon - \sum_{i=1}^{k} i_\Omega^* \left(U_{\mu_{i_\varepsilon}, x_{i_\varepsilon}}^{\frac{N+2}{N-2}} \right) \longrightarrow 0 \qquad in \quad \mathrm{H}_0^1(\Omega) \qquad as \quad \varepsilon \to 0$$

where i_Ω^* is the adjoint operator of the embedding $i_\Omega : H_0^1(\Omega) \to L^{\frac{2N}{N-2}}(\Omega)$ (see Definition 1.1).

Here (see [1], [7] and [17])

$$U_{\lambda,y}(x) = C_N \frac{\lambda^{\frac{N-2}{2}}}{(\lambda^2 + |x-y|^2)^{\frac{N-2}{2}}}, \qquad x \in \mathbb{R}^N, \ y \in \mathbb{R}^N, \ \lambda > 0,$$

with $C_N = [N(N-2)]^{(N-2)/4}$, are all the solutions of the equation

$$-\Delta u = u^{\frac{N+2}{N-2}} \qquad \text{in } \mathbb{R}^N.$$

Before stating our results it is useful to introduce some notation.

Let us denote by $\Gamma_x(y) = \frac{\gamma_N}{|x-y|^{N-2}}$, for every $x, y \in \mathbb{R}^N$, the fundamental solution for the negative Laplacian. For every point $x \in \Omega \cup \partial\Omega$, let us define the regular part of the Green's function, $H_\Omega(x, \cdot)$, as the solution of the following Dirichlet problem

$$\begin{cases} \Delta_y H_\Omega(x,y) = 0 & \text{in } \Omega, \\ H_\Omega(x,y) = \Gamma_x(y) & \text{on } \partial\Omega. \end{cases} \tag{0.2}$$

The Green's function of the Dirichlet problem for the Laplacian is then defined by $G_x(y) = \Gamma_x(y) - H_\Omega(x, y)$ and it satisfies

$$\begin{cases} -\Delta_y G_x(y) = \delta_x(y) & \text{in } \Omega, \\ \\ G_x(y) = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.3}$$

For every $x \in \Omega$ the leading term of the regular part of the Green's function

$$\tau_\Omega(x) := H_\Omega(x, x) \tag{0.4}$$

is called *Robin function of* Ω *at the point* x.

In this paper we study the existence of solutions which blow-up and concentrate at $k \geq 1$ different points of Ω. Let us introduce the function $\Psi_k : (\mathbb{R}^+)^k \times \Omega^k \longrightarrow \mathbb{R}$ defined by

$$\Psi_k(\lambda, x) = \frac{1}{2} A^2 \left(M(x) \lambda^{\frac{N-2}{2}}, \lambda^{\frac{N-2}{2}} \right) - \frac{1}{2} B \sum_{i=1}^{k} \lambda_i^2, \tag{0.5}$$

where $\lambda^{\frac{N-2}{2}} = (\lambda_1^{\frac{N-2}{2}}, \dots, \lambda_k^{\frac{N-2}{2}})^T$ and $M(x) = \left(m_{ij}(x)\right)_{1 \leq i,j \leq k}$ is the matrix defined by

$$m_{ii}(x) = \tau(x_i), \quad m_{ij}(x) = G(x_i, x_j) \quad \text{if} \quad i \neq j. \tag{0.6}$$

The constants A, B are given in (2.4). We prove the following result.

Theorem 0.2 *Let* (λ_0, x_0) *be a stable critical point of* Ψ_k *(see Definition (2.4)). Then there exists a family of solution of (0.1) which blow-up and concentrate at the points* $x_0^1, \dots, x_0^k$ *with rates of concentration* $\mu_{1\varepsilon}, \dots, \mu_{k\varepsilon}$ *such that* $\mu_\varepsilon \varepsilon^{\frac{1}{N-4}} \longrightarrow \lambda_0$ *as* $\varepsilon \to 0$.

In particular, as far as it concerns the existence of solutions which blow-up and concentrate at one point, i.e $k = 1$, we improve the results of Rey (see [15] and [16]).

Theorem 0.3 *If* x_0 *is a stable critical point of* τ_Ω *(see Definition (2.4)), then there exists a family of solutions of (0.1) which blow-up and concentrate at* x_0.

The problem of existence of a family of solution of (0.1) which blow-up and concentrate at k points of Ω, becomes a purely geometric problem.

Firstly we find many solutions of (0.1) which blow-up and concentrate at one point of Ω, by constructing a domain Ω for which τ_Ω has many stable critical points, which are local minimum points. In order to do this we follow the idea of perturbing domains. We start with a domain Ω such that τ_Ω has many stable critical points (for example Ω is the union of many disjoint domains) and we perturb Ω adding a set of small capacity (for example we add some very thin handles). It is easy to prove that the Robin's function of the perturbed domain converges in the C^1-topology to the Robin's function of Ω. Therefore the Robin's function of the perturbed domain has a large number of stable critical points, even if the perturbed domain is contractible in itself.

More precisely we can prove the following result.

Theorem 0.4 *For any $h \geq 2$ there exists a contractible domain Ω for which problem (0.1) has at least h different families of solutions which blow-up and concentrate at a point x_i in Ω, $i = 1, \ldots, h$.*

Secondly we find a family of solutions of (0.1) which blow-up and concentrate in k points of Ω, by constructing a domain Ω for which the function Ψ_k has a stable critical point. Again we follow the idea of perturbing domains. We start with a domain Ω such that Ψ_k has a stable critical point, which is a local minimum point, (for example Ω is the union of many disjoint domains) and we perturb Ω adding a set of small capacity (for example we add some very thin handles). It is easy to prove that the function Ψ_k of the perturbed domain converges in the C^1-topology to the function Ψ_k of Ω. Therefore the function Ψ_k of the perturbed domain has one stable critical point, even when the perturbed domain is contractible in itself. More precisely we prove the following result.

Theorem 0.5 *For any $k \geq 2$ there exists a contractible domain Ω for which problem (0.1) has a family of solutions which blow-up and concentrate at different k points.*

Moreover, using the results of [3], we can prove that Theorem 0.4 and Theorem 0.5 hold also for the slightly subcritical problem (4.1) (see Section 4).

We would like to point out that in [8] Dancer already emphasized that the number of positive solutions of critical problems, like (0.1) or (4.1), is strongly affected by the geometry of the domain and not just by its topology. In [8] he considered a large class of problems with subcritical growth, he constructed domains as connected approximations to a finite number of disjoint or touching balls and he proved that the number of positive solutions which are not "large" grows with the number of these balls.

The proof of our results is based on a Ljapunov-Schmidt procedure as developed in [2], [9] and [10]. The paper is organized as follows. In Section 1 we reduce the problem to a finite dimensional one. In Section 2 we study the reduced problem. In Section 3 we prove our main results. In Section 4 we briefly treat the slightly subcritical problem. The proof of Theorem 0.2 requires some technical computations which are given in Appendix A and Appendix B.

1 The finite-dimensional reduction

Let α be a fixed positive number which will be choosen later. Let us set

$$\Omega_\varepsilon := \Omega/\varepsilon^\alpha = \{x/\varepsilon^\alpha \mid x \in \Omega\}$$

and let us introduce the following problem

$$\begin{cases} -\Delta u = u^p + \varepsilon^{2\alpha+1} u & \text{in } \Omega_\varepsilon \\ u > 0 & \text{in } \Omega_\varepsilon \\ u = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{1.1}$$

Here $p = \frac{N+2}{N-2}$. By a rescaling argument one sees that $u(x)$ is a solution of (0.1) if and only if $w(x) = \varepsilon^{\alpha\frac{N-2}{2}} u\big(\varepsilon^\alpha x\big)$ is a solution of (1.1).

Now let $\mathrm{H}^1_0(\Omega_\varepsilon)$ be the Hilbert space equipped with the usual inner product

$$(u,v) = \int\limits_{\Omega_\varepsilon} \nabla u \nabla v, \quad \text{which induces the norm} \quad \|u\| = \left(\int\limits_{\Omega_\varepsilon} |\nabla u|^2 \right)^{1/2}.$$

It will be useful to rewrite problem (1.1) in a different setting. To this end let us introduce the following operator.

Definition 1.1 *Let* $i^*_\varepsilon : \mathrm{L}^{\frac{2N}{N+2}}(\Omega_\varepsilon) \longrightarrow \mathrm{H}^1_0(\Omega_\varepsilon)$ *be the adjoint operator of the immersion* $i_\varepsilon : \mathrm{H}^1_0(\Omega_\varepsilon) \hookrightarrow \mathrm{L}^{\frac{2N}{N-2}}(\Omega_\varepsilon)$, *i.e.*

$$i^*_\varepsilon(u) = v \quad \Longleftrightarrow \quad (v,\varphi) = \int\limits_{\Omega_\varepsilon} u(x)\varphi(x)dx \quad \forall\ \varphi \in \mathrm{H}^1_0(\Omega_\varepsilon).$$

Lemma 1.2 $i^*_\varepsilon : \mathrm{L}^{\frac{2N}{N+2}}(\Omega_\varepsilon) \longrightarrow \mathrm{H}^1_0(\Omega_\varepsilon)$ *is a continuous function, i.e. there exists a constant* $c > 0$ *such that*

$$\|i^*_\varepsilon(u)\| \le c\|u\|_{\frac{2N}{N+2}} \qquad \forall\ u \in \mathrm{L}^{\frac{2N}{N+2}}(\Omega_\varepsilon), \quad \forall\ \varepsilon > 0.$$

Proof. It follows from the fact that the costant of the Sobolev embedding $\mathrm{H}^1_0(\Omega_\varepsilon) \hookrightarrow \mathrm{L}^{\frac{2N}{N-2}}(\Omega_\varepsilon)$ does not depend on the domain. □

Now by scaling argument and by using the i^*_ε operator, we introduce the equivalent problem

$$\begin{cases} u = i^*_\varepsilon[f(u) + \varepsilon^{2\alpha+1}u] \\ \\ u \in \mathrm{H}^1_0(\Omega_\varepsilon). \end{cases} \tag{1.2}$$

where $f(s) = (s^+)^p$ and $p = \frac{N+2}{N-2}$.

Let now fix an integer $k \ge 1$.

Definition 1.3 *For any* $\delta > 0$ *set*

$$\begin{aligned} \mathcal{O}_\delta = \quad & \Big\{(\lambda,x) \in (\mathrm{I\!R}^+)^k \times \Omega^k \mid \mathrm{dist}\ (x_i,\partial\Omega) \ge \delta,\ \delta < \lambda_i < 1/\delta, \\ & |x_i - x_l| \ge \delta, \quad i = 1\dots,k,\ i \ne l\Big\}. \end{aligned}$$

Let us fix some notation.

If $(\lambda,x) \in \mathcal{O}_\delta$, let $y_i = x_i/\varepsilon^\alpha$ for $i = 1,\dots,k$ and set $y := x/\varepsilon^\alpha \in \Omega^k_\varepsilon$. Set

$$U_i := U_{\lambda_i,y_i} \qquad \text{and} \qquad P_\varepsilon U_i := i^*_\varepsilon\left(U^p_{\lambda_i,y_i}\right).$$

and for $j = 1,\dots,n$ and $i = 1,\dots,k$

$$\psi^0_i := \frac{\partial U_{\lambda_i,y_i}}{\partial \lambda_i}, \qquad \psi^j_i := \frac{\partial U_{\lambda_i,y_i}}{\partial y^j_i} \qquad \text{and} \qquad P_\varepsilon \psi^j_i := i^*_\varepsilon\left(pU^{p-1}_{\lambda_i,y_i}\psi^j_i\right).$$

Definition 1.4 *For any* $\varepsilon > 0$, $\lambda \in (\mathbb{R}^+)^k$ *and* $y \in \Omega_\varepsilon^k$ *set*

$$K_{\lambda,y}^\varepsilon = \Big\{ u \in \mathrm{H}_0^1(\Omega_\varepsilon) \mid \big(u, P_\varepsilon \psi_i^j\big)_{\mathrm{H}_0^1(\Omega_\varepsilon)} = 0, \qquad i = 1, \dots, k, \ j = 0, 1, \dots, n \Big\}.$$

Lemma 1.5 *Let* $\Pi_{\lambda,y}^\varepsilon : \mathrm{H}_0^1(\Omega_\varepsilon) \longrightarrow K_{\lambda,y}^\varepsilon$ *be the projection, i.e.*

$$\Pi_{\lambda,y}^\varepsilon(u) = u - \sum_{\substack{i=1,\dots,k \\ j=0,1,\dots,N}} \big(u, P_\varepsilon \psi_i^j\big)_{\mathrm{H}_0^1(\Omega_\varepsilon)} P_\varepsilon \psi_i^j.$$

Then $\Pi_{\lambda,y}^\varepsilon$ *is a continuous map, i.e. there exists* $c > 0$ *such that for any* $\varepsilon > 0$ *and for any* $(\lambda, y) \in (\mathbb{R}^+)^k \times \Omega_\varepsilon^k$ *it holds*

$$\|\Pi_{\lambda,y}^\varepsilon(u)\| \le c\|u\| \qquad \forall\ u \in \mathrm{H}_0^1(\Omega_\varepsilon).$$

Proof. It follows by Remark 5.2 and Lemma 1.2. □

Definition 1.6 *Let* $L_{\lambda,y}^\varepsilon : K_{\lambda,y}^\varepsilon \longrightarrow K_{\lambda,y}^\varepsilon$ *be defined by*

$$L_{\lambda,y}^\varepsilon(\phi) = \Pi_{\lambda,y}^\varepsilon \Big\{ \phi - i_\varepsilon^* \big[f'(\sum_{i=1}^k P_\varepsilon U_i)\phi + \varepsilon^{2\alpha+1}\phi \big] \Big\}.$$

Lemma 1.7 *For any* $\delta > 0$ *there exist* $\varepsilon_0 > 0$ *and* $c > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *and for any* $(\lambda, x) \in \mathcal{O}_\delta$ *if* $y = x/\varepsilon^\alpha$ *it holds*

$$\Big\| L_{\lambda,y}^\varepsilon(\phi) \Big\| \ge C\|\phi\| \qquad \forall \phi \in K_{\lambda,y}^\varepsilon.$$

Proof. We argue by contradiction. Assume there exist $\delta > 0$ and sequences $\varepsilon_n > 0$, $(\lambda_n, x_n) \in \mathcal{O}_\delta$, $\phi_n \in \mathrm{H}_0^1(\Omega_{\varepsilon_n})$ such that $\lim_n \varepsilon_n = 0$, $\lim_n \lambda_{in} = \lambda_i > 0$, $\lim_n x_{in} = x_i$,

$$\phi_n \in K_{\lambda_n, y_n}^{\varepsilon_n} \quad \text{and} \quad \|\phi_n\|_{\mathrm{H}_0^1(\Omega_{\varepsilon_n})} = 1 \tag{1.3}$$

and

$$L_{\lambda_n, y_n}^{\varepsilon_n}(\phi_n) = h_n \quad \text{with} \quad \|h_n\|_{\mathrm{H}_0^1(\Omega_{\varepsilon_n})} \longrightarrow 0. \tag{1.4}$$

Set $\Omega_n = \Omega_{\varepsilon_n}$, $P_n U_{in} = P_{\varepsilon_n} U_{\lambda_{in}, y_{in}}$ and $P_n \psi_{i\,n}^j = P_{\varepsilon_n} \psi_{\lambda_{in}, y_{in}}^j$. Therefore we have

$$\phi_n - i_{\varepsilon_n}^* [f'(\sum_{i=1}^k P_n U_{in})\phi_n] = h_n + w_n \qquad \text{in } \Omega_n, \tag{1.5}$$

where $w_n = \sum_{l,j} c_{l,j}^n P_n \psi_{l\,n}^j$ for certain coefficients $c_{l,j}^n$.

Step1. It holds

$$\lim_n \|w_n\|_{\mathrm{H}_0^1(\Omega_{\varepsilon_n})} = 0. \tag{1.6}$$

By (1.5) we deduce

$$
\begin{aligned}
&\|w_n\|^2_{\mathrm{H}^1_0(\Omega_{\varepsilon_n})} = (\phi_n, w_n) - \int\limits_{\Omega_n} f'\big(\sum_i P_n U_{in}\big)\phi_n w_n - (h_n, w_n) \\
&\le \int\limits_{\Omega_n} \left| f'\big(\sum_i P_n U_{in}\big) - \sum_i f'\big(U_{in}\big) \right| \, |\phi_n| \, |w_n| \\
&+ \int\limits_{\Omega_n} \left| \sum_i f'\big(U_{in}\big) \right| \, |\phi_n| \, \sum_{l,j} |c^n_{l,j}| \, |P_n \psi^j_{l\,n} - \psi^j_{l\,n}| \\
&+ \|h_n\|_{\mathrm{H}^1_0(\Omega_{\varepsilon_n})} \, \|w_n\|_{\mathrm{H}^1_0(\Omega_{\varepsilon_n})} \\
&\le \|f'\big(\sum_i P_n U_{in}\big) - \sum_i f'\big(U_{in}\big)\|_{\frac{N}{2}} \, \|\phi_n\|_{\frac{2N}{N-2}} \, \|w_n\|_{\frac{2N}{N-2}} \\
&+ \sum_i \|f'\big(U_{in}\big)\|_{\frac{N}{2}} \, \|\phi_n\|_{\frac{2N}{N-2}} \sum_{l,j} |c^n_{l,j}| \|P_n \psi^j_{l\,n} - \psi^j_{l\,n}\|_{\frac{2N}{N-2}} \\
&+ \|h_n\|_{\mathrm{H}^1_0(\Omega_{\varepsilon_n})} \, \|w_n\|_{\mathrm{H}^1_0(\Omega_{\varepsilon_n})}
\end{aligned}
\tag{1.7}
$$

since

$$
(\phi_n, w_n) = \sum_{l,j} c^n_{l,j} \int\limits_{\Omega_n} f'(U_{in}) \phi_n \psi^j_{l\,n} = 0.
$$

Using (1.3), (1.7), Lemma 5.3, Lemma 6.4 and the fact that

$$
\|w_n\|^2_{\mathrm{H}^1_0(\Omega_{\varepsilon_n})} = \sum_{l,j} c^n_{l,j} c^n_{r,s} \big(P_n \psi^j_{l\,n}, P_n \psi^r_s\big) = \sum_{l,j} c^n_{l,j} c^n_{r,s} \big[\delta_{j,r}\delta_{l,s} + o(1)\big]
$$

the claim follows.

Step 2. Let $\chi : \mathbb{R} \longrightarrow [0,1]$ be a smooth cut-off function such that $\chi(x) = 1$ if $|x| \le \delta$ and $\chi(x) = 0$ if $|x| \ge 2\delta$.

For any $h = 1, \ldots, k$ set

$$
\phi^h_n(x) = \phi_n\left(x + y_{h\,n}\right) \chi_n(x), \qquad x \in \Omega_n - y_{h\,n}, \tag{1.8}
$$

where $\chi_n(x) = \chi(\varepsilon^\alpha_n x)$.

It holds

$$
\lim_n \phi^h_n = 0 \quad \textit{weakly in } \mathcal{D}^{1,2}(\mathbb{R}^N), \quad h = 1, \ldots, k. \tag{1.9}
$$

Here $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the space obtained by taking the completion of $C^\infty_0(\mathbb{R}^N)$ with the norm $\|u\| = (\int\limits_{\mathbb{R}^N} |\nabla u|^2 dx)^{1/2}$.

First of all by (1.3) and the smoothness of χ it follows that $\|\phi^h_n\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}$ is bounded. So, up to a subsequence, we can assume that

$$
\lim_n \phi^h_n = \phi^h_\infty \quad \textit{weakly in } \mathcal{D}^{1,2}(\mathbb{R}^N).
$$

By (1.5) we deduce that for any $\psi \in C_0^\infty(\mathbb{R}^N)$

$$
\begin{aligned}
&\int\limits_{\Omega_n - y_{h_n}} \nabla\phi_n^i \nabla\psi \\
&= \int\limits_{\Omega_n - y_{h_n}} \nabla\phi_n \nabla(\chi_n\psi) + \int\limits_{\Omega_n - y_{h_n}} \nabla\chi_n(\phi_n\nabla\psi - \psi\nabla\psi_n) \\
&\int\limits_{\Omega_n - y_{h_n}} f'\big(\sum_i P_n U_{in}(x + y_{hn})\big)\phi_n(x + y_{in})\chi_n\psi dx \\
&+ \int\limits_{\Omega_n - y_{h_n}} \nabla h_n(x + y_{hn})\nabla(\chi_n\psi)dx \\
&+ \int\limits_{\Omega_n - y_{h_n}} \nabla w_n(x + y_{hn})\nabla(\chi_n\psi)dx \\
&+ \int\limits_{\Omega_n - y_{h_n}} \nabla\chi_n(\phi_n\nabla\psi - \psi\nabla\psi_n).
\end{aligned}
\tag{1.10}
$$

By (1.4), (1.6) and (1.8) we get

$$
\begin{aligned}
&\int\limits_{\Omega_n - y_{h_n}} \nabla h_n(x + y_{hn})(\chi_n\psi)dx \\
&+ \int\limits_{\Omega_n - y_{h_n}} \nabla w_n(x + y_{hn})(\chi_n\psi)dx \\
&+ \int\limits_{\Omega_n - y_{h_n}} \nabla\chi_n(\phi_n\nabla\psi - \psi\nabla\psi_n) = o(1).
\end{aligned}
\tag{1.11}
$$

Finally

$$
\begin{aligned}
&\int\limits_{\Omega_n - y_{h_n}} f'\big(\sum_i P_n U_{in}(x + y_{hn})\big)\phi_n(x + y_{hn})\chi_n(x)\psi(x)dx \\
&= \int\limits_{\Omega_n} f'\big(\sum_i P_n U_{in}(x)\big)\phi_n(x)\chi_n(x - y_{hn})\psi(x - y_{hn})dx \\
&= \varepsilon^{-\alpha(N-2)} \int\limits_{|x - x_h| \le 2\delta} f'\big(\sum_i PU_{\lambda_{in}\varepsilon_n^\alpha, x_{in}}(x)\big)\phi_n(x/\varepsilon_n^\alpha)\chi_n\Big(\frac{x - x_{hn}}{\varepsilon_n^\alpha}\Big)\psi\Big(\frac{x - x_{hn}}{\varepsilon_n^\alpha}\Big)dx \\
&= \varepsilon^{-\alpha(N-2)} \int\limits_{|x - x_h| \le 2\delta} f'\big(\sum_i U_{\lambda_{in}\varepsilon_n^\alpha, x_{in}}(x)\big)\phi_n(x/\varepsilon_n^\alpha)\chi_n\Big(\frac{x - x_{hn}}{\varepsilon_n^\alpha}\Big)\psi\Big(\frac{x - x_{hn}}{\varepsilon_n^\alpha}\Big)dx + o(1) \\
&= \int\limits_{\mathbb{R}^N} f'(U_{\lambda_h,0})\phi_\infty^h\psi.
\end{aligned}
\tag{1.12}
$$

Hence, from (1.10), (1.11) and (1.12) we deduce that $\phi^h_\infty \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ is a weak solution of

$$-\Delta\phi^h_\infty = f'(U_{\lambda,0})\phi^h_\infty \qquad \text{in} \quad \mathcal{D}^{1,2}(\mathbb{R}^N). \tag{1.13}$$

Moreover the function ϕ^h_∞ satisfies the condition

$$\int_{\mathbb{R}^N} \nabla\phi^h_\infty(x)\nabla\psi^j_{\lambda_h,0}(x)dx = 0 \qquad j = 0,1,\dots,N. \tag{1.14}$$

In fact

$$\begin{aligned}
&\left|\int_{\Omega_n - y_{h_n}} \phi^h_n(x) f'\big(U_{\lambda_{h_n},0}(x)\big)\psi^j_{\lambda_{h_n},0}(x)dx\right| \\
&= \left|\int_{\Omega_n} \phi_n(y)\chi_n(y-y_{hn}) f'\big(U_n(y)\big)\psi^j_{hn}(y)dy\right| \\
&= \left|\int_{\Omega_n} \phi_n(y)\big[\chi_n(y-y_{hn})-1\big] f'\big(U_n(y)\big)\psi^j_{hn}(y)dy\right| \\
&\le \left|\int_{|y-y_{h_n}|\ge 2\delta/\varepsilon_n^\alpha} \phi_n(y) f'\big(U_n(y)\big)\psi^j_{hn}(y)dy\right| \\
&\le \|\phi_n\|_{\frac{2N}{N-2}} \left[\int_{|y-y_{h_n}|\ge 2\delta/\varepsilon_n^\alpha} \big(U_n(y)\big)^{\frac{2N}{N-2}}dy\right]^{\frac{2}{N}} \left[\int_{|y-y_{h_n}|\ge 2\delta/\varepsilon_n^\alpha} \big(\psi^j_{hn}(y)\big)^{\frac{2N}{N-2}}\right]^{\frac{N-2}{2N}} \\
&= o(1).
\end{aligned} \tag{1.15}$$

¿From [4] and using (1.13) and (1.14), we deduce (1.9).

Step 3. A contradiction arises!

First of all we want to show that

$$\lim_n \int_{\Omega_n} f'\big(\sum_i P_nU_{in}\big)\phi_n^2 = 0. \tag{1.16}$$

Using the definition of ϕ^h_n we deduce that

$$\begin{aligned}
\int_{\Omega_n} f'\big(\sum_i P_nU_{in}\big)\phi_n^2 &= \sum_{h=1}^k \int_{\Omega_n} f'\big(\sum_i P_nU_{in}\big)(y)\phi_n(y)\phi^h_n(y)dy \\
&+ \int_{\Omega_n \setminus \bigcup_{h=1}^k B(y_{h_n},\delta\varepsilon_n^\alpha)} f'\big(\sum_i P_nU_{in}\big)(y)\phi_n^2(y)dy.
\end{aligned} \tag{1.17}$$

By (1.9) we deduce that

$$\int_{\Omega_n} f'\big(\sum_i P_n U_{in}\big)(y)\phi_n(y)\phi_n^h(y)dy \longrightarrow 0 \qquad \forall\ h=1,\dots,k. \tag{1.18}$$

Moreover we have

$$\begin{aligned}
&\int_{\Omega_n\setminus \overset{k}{\underset{h=1}{\cup}} B(y_{h\,n},\delta\varepsilon_n^\alpha)} f'\big(\sum_i P_n U_{in}\big)(y)\phi_n^2(y)dy \\
&\le C\sum_i \int_{\Omega_n\setminus \overset{k}{\underset{h=1}{\cup}} B(y_{h\,n},\delta\varepsilon_n^\alpha)} U^{p-1}_{\lambda_{i\,n},y_{i\,n}}(y)\phi_n^2(y)dy \\
&\le C\varepsilon_n^{4\alpha}\|\phi_n\|^2_{\mathrm{L}^2(\Omega_{\varepsilon_n})}.
\end{aligned} \tag{1.19}$$

Therefore (1.16) follows by (1.17), (1.18) and (1.19).

Finally by (1.5) we deduce that

$$\int_{\Omega_n} |\nabla\phi_n|^2 = \int_{\Omega_n} f'\big(\sum_i P_n U_{in}\big)\phi_n^2 + \int_{\Omega_n}(\nabla h_n+\nabla w_n)\nabla\phi_n. \tag{1.20}$$

¿From (1.4), (1.6), (1.7) and (1.16) it follows that $\lim_n \|\phi_n\|_{\mathrm{H}^1_0(\Omega_{\varepsilon_n})} = 0$ and (1.3) gives a contradiction.

Proposition 1.8 *Let* $\alpha = \frac{1}{N-4}$. *For any* $\delta > 0$ *there exist* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0,\varepsilon_0)$ *and for any* $(\lambda, x) \in \mathcal{O}_\delta$, *if* $y = x/\varepsilon^\alpha$, *there exists a unique* $\phi^\varepsilon_{\lambda,y} \in K^\varepsilon_{\lambda,y}$ *such that*

$$\Pi^\varepsilon_{\lambda,y}\Big\{\sum_{i=1}^k P_\varepsilon U_i + \phi - i^*_\varepsilon\big[f(\sum_{i=1}^k P_\varepsilon U_i+\phi) + \varepsilon^{2\alpha+1}(\sum_{i=1}^k P_\varepsilon U_i+\phi)\big]\Big\} = 0 \tag{1.21}$$

and

$$\|\phi\| \le \varepsilon^\mu \tag{1.22}$$

with

$$\mu = \begin{cases} \frac{1}{2}+2\alpha = \frac{N}{2(N-4)} & \textit{if } N\ge 6 \\[2ex] \frac{1}{4}+2\alpha = \frac{9}{4} & \textit{if } N=5. \end{cases} \tag{1.23}$$

Proof.

First of all we point out that ϕ solves equation (1.21) if and only if ϕ is a fixed point of the operator $T^\varepsilon_{\lambda,y} : K^\varepsilon_{\lambda,y} \longrightarrow K^\varepsilon_{\lambda,y}$ defined by

$$\begin{aligned}
T^\varepsilon_{\lambda,y}(\phi) \quad &= \Big[\big(L^\varepsilon_{\lambda,y}\big)^{-1}\circ \Pi^\varepsilon_{\lambda,y}\circ i^*_\varepsilon\Big] \\
&\Bigg[f(\sum_{i=1}^k P_\varepsilon U_i+\phi) - \sum_{i=1}^k f(U_i) - f'(\sum_{i=1}^k P_\varepsilon U_i)\phi + \varepsilon^{2\alpha+1}\sum_{i=1}^k P_\varepsilon U_i\Bigg].
\end{aligned}$$

Step 1: there exist $\varepsilon_0 > 0$ and $\mu > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ we have

$$\|\phi\| \le \varepsilon^\mu \quad \Longrightarrow \quad \|T^\varepsilon_{\lambda,y}(\phi)\| \le \varepsilon^\mu. \tag{1.24}$$

¿From Lemma 1.7, Lemma 1.5 and Lemma 1.2 we deduce that

$$\|T^\varepsilon_{\lambda,y}(\phi)\| \le c\Big[\|f(\sum_{i=1}^k P_\varepsilon U_i + \phi) - f(\sum_{i=1}^k P_\varepsilon U_i) - f'(\sum_{i=1}^k P_\varepsilon U_i)\phi\|_{\frac{2N}{N+2}}$$
$$+\|f(\sum_{i=1}^k P_\varepsilon U_i) - \sum_{i=1}^k f(U_i)\|_{\frac{2N}{N+2}} + \varepsilon^{2\alpha+1}\|\sum_{i=1}^k P_\varepsilon U_i\|_{\frac{2N}{N+2}}\Big]. \tag{1.25}$$

Now it is easy to see that

$$\|f(\sum_{i=1}^k P_\varepsilon U_i + \phi) - f(\sum_{i=1}^k P_\varepsilon U_i) - f'(\sum_{i=1}^k P_\varepsilon U_i)\phi\|_{\frac{2N}{N+2}} \le c\|\phi\|^{p\wedge 2} \tag{1.26}$$

By Lemma 5.3 we deduce that

$$\|f(\sum_{i=1}^k P_\varepsilon U_i) - \sum_{i=1}^k f(U_i)\|_{\frac{2N}{N+2}} \le \begin{cases} c\varepsilon^{\frac{N+2}{2(N-4)}} & \text{if } N \ge 7, \\ c\varepsilon^2|\log\varepsilon| & \text{if } N = 6, \\ c\varepsilon^3 & \text{if } N = 5. \end{cases} \tag{1.27}$$

Moreover Remark 5.2 implies

$$\varepsilon^{2\alpha+1}\|\sum_{i=1}^k P_\varepsilon U_i\|_{\frac{2N}{N+2}} \le \begin{cases} c\varepsilon^{\frac{N-2}{N-4}} & \text{if } N \ge 7, \\ c\varepsilon^{2\frac{r-1}{r}},\ r > 0 & \text{if } N = 6, \\ c\varepsilon^{\frac{6r-7}{2r}},\ r \in (0,7) & \text{if } N = 5. \end{cases} \tag{1.28}$$

Finally from (1.25), (1.26), (1.28) and (1.27) the claim (1.24) easily follows.
Step 2: there exist $\varepsilon_0 > 0$ and $\mu > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$

$$T^\varepsilon_{\lambda,y} : \{\|\phi\| \le \varepsilon^\mu\} \longrightarrow \{\|\phi\| \le \varepsilon^\mu\} \textit{ is a contraction mapping.} \tag{1.29}$$

In fact arguing as in the previous step we can prove that if $\|\phi_1\|, \|\phi_2\| \le \varepsilon^\mu$ then

$$\begin{aligned}&\|T^\varepsilon_{\lambda,y}(\phi_1) - T^\varepsilon_{\lambda,y}(\phi_2)\| \\ &\le c\Big[\|f(\sum_{i=1}^k P_\varepsilon U_i + \phi_1) - f(\sum_{i=1}^k P_\varepsilon U_i + \phi_2) - f'(\sum_{i=1}^k P_\varepsilon U_i + \phi_2)\phi_1\|_{\frac{2N}{N+2}} \\ &+\|\big[f'(\sum_{i=1}^k P_\varepsilon U_i + \phi_2) - f'(\sum_{i=1}^k P_\varepsilon U_i)\big](\phi_1 - \phi_2)\|_{\frac{2N}{N+2}}\Big] \\ &\le c\big(\|\phi_1 - \phi_2\|^p + \|\phi_2\|^{p-1}\|\phi_1 - \phi_2\|\big) \le L\|\phi_1 - \phi_2\|,\end{aligned}$$

for some $L \in (0, 1)$. The claim (1.29) follows.

2 The reduced problem

¿From Proposition 1.8 we deduce that the function $u_\varepsilon = \sum_{i=1}^{k} P_\varepsilon U_{\lambda_i^\varepsilon, y_i^\varepsilon} + \phi^\varepsilon_{\lambda_\varepsilon, y_\varepsilon}$ is a solution of (1.2) if and only if the parameters λ_ε and the points y_ε are such that for any $i = 1, \ldots, k$ and $j = 0, 1, \ldots, n$

$$\left(\sum_{i=1}^{k} P_\varepsilon U_i + \phi, P_\varepsilon \psi^j_{\lambda_i^\varepsilon, y_i^\varepsilon} \right)_{\mathrm{H}^1_0(\Omega_\varepsilon)} - \left(i^*_\varepsilon \left[f(\sum_{i=1}^{k} P_\varepsilon U_i + \phi) + \varepsilon^{2\alpha+1} (\sum_{i=1}^{k} P_\varepsilon U_i + \phi) \right], P_\varepsilon \psi^j_{\lambda_i^\varepsilon, y_i^\varepsilon} \right)_{\mathrm{H}^1_0(\Omega_\varepsilon)} = 0. \tag{2.1}$$

Now we establish the asymptotic expansion of the left-hand side of the previous expression using the crucial estimates in Appendix B.

Proposition 2.1 *Let $\alpha = \frac{1}{N-4}$. If $j = 1, \ldots, N$ and $h = 1, \ldots, k$ then*

$$\begin{aligned} &\left(\sum_{i} P_\varepsilon U_i + \phi, P_\varepsilon \psi^j_h \right)_{\mathrm{H}^1_0(\Omega_\varepsilon)} \\ &- \left(i^*_\varepsilon \left[f(\sum_{i} P_\varepsilon U_i + \phi) + \varepsilon^{2\alpha+1} (\sum_{i} P_\varepsilon U_{i+}\phi) \right], P_\varepsilon \psi^j_h \right)_{\mathrm{H}^1_0(\Omega_\varepsilon)} \\ &= A^2 \left[\frac{\partial H}{\partial x^j_h}(x_h, x_h) \lambda_h^{N-2} - \sum_{\substack{l=1 \\ l \neq h}}^{k} \frac{\partial G}{\partial x^j_h}(x_h, x_l)(\lambda_h \lambda_l)^{\frac{N-2}{2}} \right] \varepsilon^{\frac{N-1}{N-4}} \\ &+ o\left(\varepsilon^{\frac{N-1}{N-4}} \right) \end{aligned} \tag{2.2}$$

as $\varepsilon \to 0$ uniformly with respect to $(\lambda, x) \in \mathcal{O}_\delta$.

Moreover if $j = 0$ and $h = 1, \ldots, k$ then

$$\begin{aligned} &\left(\sum_{i} P_\varepsilon U_i + \phi, P_\varepsilon \psi^j_h \right)_{\mathrm{H}^1_0(\Omega_\varepsilon)} \\ &- \left(i^*_\varepsilon \left[f(\sum_{i} P_\varepsilon U_i + \phi) + \varepsilon^{2\alpha+1} (\sum_{i} P_\varepsilon U_{i+}\phi) \right], P_\varepsilon \psi^j_h \right)_{\mathrm{H}^1_0(\Omega_\varepsilon)} \\ &= \left\{ \frac{N-2}{2} A^2 \left[H(x_h, x_h) \lambda_h^{N-3} - \sum_{\substack{l=1 \\ l \neq h}}^{k} G(x_h, x_l) \lambda_h^{\frac{N}{2}-2} \lambda_l^{\frac{N-2}{2}} \right] + B \lambda_h \right\} \varepsilon^{\frac{N-2}{N-4}} \\ &+ o\left(\varepsilon^{\frac{N-2}{N-4}} \right) \end{aligned} \tag{2.3}$$

as $\varepsilon \to 0$ *uniformly with respect to* $(\lambda, x) \in \mathcal{O}_\delta$.

Here the constants A *and* B *are given by*

$$A = \int\limits_{\mathbb{R}^N} U^p(x)\,dx \qquad \text{and} \qquad B = \int\limits_{\mathbb{R}^N} U^2(x)\,dx. \tag{2.4}$$

Proof. We have

$$\begin{aligned}
&\left(\sum_i P_\varepsilon U_i + \phi, P_\varepsilon \psi_h^j\right)_{\mathrm{H}_0^1(\Omega_\varepsilon)} \\
&-\left(i_\varepsilon^*\big[f(\sum_i P_\varepsilon U_i + \phi) + \varepsilon^{2\alpha+1}(\sum_i P_\varepsilon U_{i+}\phi)\big], P_\varepsilon \psi_h^j\right)_{\mathrm{H}_0^1(\Omega_\varepsilon)} \\
&= \int\limits_{\Omega_\varepsilon} \sum_i f(U_i) P_\varepsilon \psi_h^j - \int\limits_{\Omega_\varepsilon} f(\sum_i P_\varepsilon U_i + \phi) P_\varepsilon \psi_h^j \\
&-\varepsilon^{2\alpha+1} \sum_i \int\limits_{\Omega_\varepsilon} P_\varepsilon U_i P_\varepsilon \psi_h^j - \varepsilon^{2\alpha+1} \int\limits_{\Omega_\varepsilon} \phi P_\varepsilon \psi_h^j \\
&= \int\limits_{\Omega_\varepsilon} \big[\sum_i f(U_i) - f(\sum_i P_\varepsilon U_i)\big] P_\varepsilon \psi_h^j - \varepsilon^{2\alpha+1} \sum_i \int\limits_{\Omega_\varepsilon} P_\varepsilon U_i P_\varepsilon \psi_h^j \\
&- \int\limits_{\Omega_\varepsilon} \big[f(\sum_i P_\varepsilon U_i + \phi) - f(\sum_i P_\varepsilon U_i) - f'(\sum_i P_\varepsilon U_i)\phi\big] P_\varepsilon \psi_h^j \\
&- \int\limits_{\Omega_\varepsilon} \big[f'(\sum_i P_\varepsilon U_i) - \sum_i f'(U_i)\big] \phi P_\varepsilon \psi_h^j \\
&- \sum_i \int\limits_{\Omega_\varepsilon} f'(U_i) \phi P_\varepsilon \psi_h^j \\
&- \varepsilon^{2\alpha+1} \int\limits_{\Omega_\varepsilon} \phi P_\varepsilon \psi_h^j.
\end{aligned} \tag{2.5}$$

We will estimate first the terms involving the function ϕ taking in account (1.22) of Proposition 1.8. We get firstly

$$\begin{aligned}
&\left|\int\limits_{\Omega_\varepsilon} \big[f(\sum_i P_\varepsilon U_i + \phi) - f(\sum_i P_\varepsilon U_i) - f'(\sum_i P_\varepsilon U_i)\phi\big] P_\varepsilon \psi_h^j\right| \\
&\le c\|\phi\|^2 \le c\varepsilon^{2\mu}.
\end{aligned} \tag{2.6}$$

Secondly by (5.9) of Lemma 5.4 we get

$$\left|\int\limits_{\Omega_\varepsilon} \big[f'(\sum_i P_\varepsilon U_i) - \sum_i f'(U_i)\big] \phi P_\varepsilon \psi_h^j\right|$$

$$\leq c\|\phi\|_{\frac{2N}{N-2}}\|P_\varepsilon\psi_h^j(f'(\sum_i P_\varepsilon U_i)-\sum_i f'(U_i))\|_{\frac{2N}{N+2}}$$
$$\leq c\|\phi\|\varepsilon^{\alpha\frac{N+2}{2}}\leq c\varepsilon^{\mu+\alpha\frac{N+2}{2}}. \tag{2.7}$$

Moreover by Lemma 6.4 we get

$$\left|\int_{\Omega_\varepsilon} f'(\sum_i U_i)\phi P_\varepsilon\psi_h^j\right| = \left|\int_{\Omega_\varepsilon} f'(\sum_i U_i)\phi(P_\varepsilon\psi_h^j-\psi_h^j)\right|$$
$$\leq c\|\phi\|_{\frac{2N}{N-2}}\|f'(\sum_i U_i)\|_{\frac{N}{2}}\|P_\varepsilon\psi_h^j-\psi_h^j\|_{\frac{2N}{N-2}}$$
$$\leq \begin{cases} c\varepsilon^{\alpha\frac{N}{2}+\mu} & \text{if } j\neq 0,\\ c\varepsilon^{\alpha\frac{N-2}{2}+\mu} & \text{if } j=0,\end{cases} \tag{2.8}$$

and finally

$$\varepsilon^{2\alpha+1}\left|\int_{\Omega_\varepsilon}\phi P_\varepsilon\psi_h^j\right|\leq\varepsilon^{2\alpha+1}\|\phi\|_{\frac{2N}{N-2}}\|P_\varepsilon\psi_h^j\|_{\frac{2N}{N+2}}\leq\begin{cases} c\varepsilon^{2\alpha+1+\mu} & \text{if } j\geq 0,\ N\geq 7,\\ c\varepsilon^{2\alpha+1+\mu} & \text{if } j\geq 1,\ N=5,6\\ c\varepsilon^{\alpha+1+\mu} & \text{if } j=0,\ N=5,6.\end{cases} \tag{2.9}$$

Taking into account (1.23) the claim follows from Lemma 6.5 in Appendix B. □

Finally we can prove the following crucial expansions.

Proposition 2.2 *Let Ψ_k be the function defined by (0.5). If $j=1,\dots,N$ and $h=1,\dots,k$ then*

$$\left(\sum_i P_\varepsilon U_i+\phi, P_\varepsilon\psi_h^j\right)_{\mathrm{H}_0^1(\Omega_\varepsilon)}$$
$$-\left(i_\varepsilon^*\left[f(\sum_i P_\varepsilon U_i+\phi)+\varepsilon^{2\alpha+1}(\sum_i P_\varepsilon U_{i+}\phi)\right], P_\varepsilon\psi_h^j\right)_{\mathrm{H}_0^1(\Omega_\varepsilon)}$$
$$=\varepsilon^{\frac{N-1}{N-4}}\left[\frac{\partial\Psi_k}{\partial x_h^j}(\lambda,x)+o(1)\right] \tag{2.10}$$

as $\varepsilon\to 0$ uniformly with respect to $(\lambda,x)\in\mathcal{O}_\delta$.

Moreover if $j=0$ and $h=1,\dots,k$ then

$$\left(\sum_i P_\varepsilon U_i+\phi, P_\varepsilon\psi_h^j\right)_{\mathrm{H}_0^1(\Omega_\varepsilon)}$$
$$-\left(i_\varepsilon^*\left[f(\sum_i P_\varepsilon U_i+\phi)+\varepsilon^{2\alpha+1}(\sum_i P_\varepsilon U_{i+}\phi)\right], P_\varepsilon\psi_h^j\right)_{\mathrm{H}_0^1(\Omega_\varepsilon)}$$
$$=\varepsilon^{\frac{N-2}{N-4}}\left[\frac{\partial\Psi_k}{\partial\lambda_h}(\lambda,x)+o(1)\right] \tag{2.11}$$

as $\varepsilon\to 0$ uniformly with respect to $(\lambda,x)\in\mathcal{O}_\delta$.

Proof. Let us recall that if $\tau(x) = H(x,x)$ then $\frac{\partial \tau}{\partial x_i}(x) = 2\frac{\partial H}{\partial x_i}(x,x)$. Therefore the claim follows by (0.5) and Proposition 2.1. $\square$

At this point we can give the necessary condition.

Theorem 2.3 *Let $u_\varepsilon = \sum_{i=1}^{k} P_\varepsilon U_{\lambda_{i\varepsilon}, y_{i\varepsilon}} + \phi^\varepsilon_{\lambda_\varepsilon, y_\varepsilon}$ be a family of solution of (1.1) such that $\lim_{\varepsilon \to 0} \lambda_\varepsilon = \lambda_0 > 0$ and $\lim_{\varepsilon \to 0} \varepsilon^{\frac{1}{N-4}} y_\varepsilon = x_0$ with $(\lambda_0, x_0) \in \mathcal{O}_\delta$ for some $\delta > 0$. Then (λ_0, x_0) is a critical point of Ψ_k.*

Proof. Set $x_{i\varepsilon} = \varepsilon^\alpha y_{i\varepsilon} \in \Omega$ for $i = 1, \ldots, k$. By Proposition 2.2 we deduce that for $j = 1, \ldots, N$ and $h = 1, \ldots, k$ we have

$$\frac{\partial \Psi_k}{\partial x_h^j}(\lambda_\varepsilon, x_\varepsilon) + o(1) = 0 \qquad \text{and} \qquad \frac{\partial \Psi_k}{\partial \lambda_h}(\lambda_\varepsilon, x_\varepsilon) + o(1) = 0. \tag{2.12}$$

Since estimates (2.10) and (2.11) hold uniformly with respect to (λ, x) in $\mathcal{O}_\delta$, we can pass to the limit as ε goes to zero in (2.12) and hence the claim follows. $\square$

The next result gives a sufficient condition which ensures the existence of a family of solutions which blow-up and concentrate at k given points of Ω according to Definition 0.1.

Firstly we need to recall the following definition (see [13]).

Definition 2.4 *Let $g : D \longrightarrow \mathbb{R}$ be a C^1-function, where $D \subset \mathbb{R}^m$ is an open set. We say that x_0 is a stable critical point of g if $\nabla g(x_0) = 0$ and there exists a neighbourhood U of x_0 such that*

$$\nabla g(x) \neq 0 \quad \forall\ x \in \partial U,$$

$$\nabla g(x) = 0, \quad x \in U \quad \Longrightarrow \quad g(x) = g(x_0)$$

and

$$\deg\ \big(\nabla g, U, 0\big) \neq 0,$$

where deg denotes the Brouwer degree.

It is clear that any nondegenerate critical point of g is a stable critical point in the sense of Definition (2.4). Moreover it easy to see that if x_0 is a minimum point or a maximum point of the function g (not necessarily nondegenerate) then x_0 is a stable critical point of g according to Definition (2.4).

Proof of Theorem 0.2. We will prove that for some $\delta > 0$ there exists $(\lambda_\varepsilon, x_\varepsilon) \in \mathcal{O}_\delta$ with $\lim_{\varepsilon \to 0} \lambda_\varepsilon = \lambda_0$ and $\lim_{\varepsilon \to 0} x_\varepsilon = x_0$ such that if $y_\varepsilon = x_\varepsilon / \varepsilon^\alpha$ then $u_\varepsilon = \sum_{i=1}^{k} P_\varepsilon U_{\lambda_{i\varepsilon}, y_{i\varepsilon}} + \phi^\varepsilon_{\lambda_\varepsilon, y_\varepsilon}$ is a family of solution of (1.1). The claim will follow by scaling such a function and by assuming $\mu_{i_\varepsilon} = \lambda_i \varepsilon^\alpha$ (see Definition 0.1).

By Proposition 2.2 and Definition 2.4 we deduce that for ε small enough there exist $(x_\varepsilon, \lambda_\varepsilon)$ such that $\lim_{\varepsilon \to 0} \lambda_\varepsilon = \lambda_0$ and $\lim_{\varepsilon \to 0} x_\varepsilon = x_0$ such that for $j = 1, \ldots, N$ and $h = 1, \ldots, k$

$$\varepsilon^{-\alpha(N-1)}\left(\sum_i P_\varepsilon U_i+\phi, P_\varepsilon\psi_h^j\right)_{\mathrm{H}_0^1(\Omega_\varepsilon)}$$

$$-\varepsilon^{-\alpha(N-1)}\left(i_\varepsilon^*\big[f(\sum_i P_\varepsilon U_i+\phi)+\varepsilon^{2\alpha+1}(\sum_i P_\varepsilon U_{i+}\phi)\big], P_\varepsilon\psi_h^j\right)_{\mathrm{H}_0^1(\Omega_\varepsilon)}$$

$$=\frac{\partial\Psi_k}{\partial x_h^j}(\lambda_\varepsilon,x_\varepsilon)+o(1)=0 \tag{2.13}$$

and also

$$\varepsilon^{-\alpha(N-2)}\left(\sum_i P_\varepsilon U_i+\phi, P_\varepsilon\psi_h^j\right)_{\mathrm{H}_0^1(\Omega_\varepsilon)}$$

$$-\varepsilon^{-\alpha(N-2)}\left(i_\varepsilon^*\big[f(\sum_i P_\varepsilon U_i+\phi)+\varepsilon^{2\alpha+1}(\sum_i P_\varepsilon U_{i+}\phi)\big], P_\varepsilon\psi_h^j\right)_{\mathrm{H}_0^1(\Omega_\varepsilon)}$$

$$=\frac{\partial\Psi_k}{\partial \lambda_h}(\lambda_\varepsilon,x_\varepsilon)+o(1)=0 \tag{2.14}$$

Hence by (2.13), (2.14) and Proposition 2.2 the claim follows. □

3 Examples

Firstly let us consider the case $k=1$. In this case the function $\Psi_1:\mathbb{R}^+\times\Omega\longrightarrow\mathbb{R}$ reduces to

$$\Psi_1(\lambda,x)=\frac{1}{2}A^2\tau(x)\lambda^{N-2}-\frac{1}{2}B\lambda^2.$$

We have the following result.

Lemma 3.1 *If x_0 is a stable critical point of τ, then $\big(\lambda_0,x_0\big)$ with $\lambda_0=\left[\frac{2B}{(N-2)A^2}\frac{1}{\tau(x_0)}\right]^{\frac{1}{N-4}}$ is a stable critical point of Ψ_1.*

Proof. First of all we have

$$\nabla\Psi_1(\lambda,x)=\big(\frac{N-2}{2}A^2\tau(x)\lambda^{N-3}-B\lambda,\frac{1}{2}A^2\nabla\tau(x)\lambda^{N-2}\big).$$

Let $H:[0,1]\times\mathbb{R}^+\times\Omega\longrightarrow\mathbb{R}^N\times\mathbb{R}$ be the homotopy defined by

$$H(t,\lambda,x)=t\nabla\Psi_1(\lambda,x)+(1-t)\left(h(\lambda),\nabla\tau(x)\right),$$

where $h(\lambda)=\frac{N-2}{2}A^2\tau(x_0)\lambda^{N-3}-B\lambda$. It is easy to check, using Definition (2.4), that for some $\rho>0$

$$H(t,\lambda,x)\neq 0\qquad\forall\ t\in[0,1],\quad\forall\ (\lambda,x)\in\partial\big(U\times V\big),$$

where U and V are neighborhoods of λ_0 and x_0 respectively. By the homotopy invariance of the degree we deduce that

$$\deg\Big(\nabla\Psi_1, U\times V, 0\Big) = \deg\big(h, U, 0\big)\cdot\deg\big(\nabla\tau, V, 0\big)$$

and the claim follows because $\deg\big(h, V, 0\big) = 1$.

□

Proof of Theorem 0.3. It follows by Theorem 0.2 and Lemma 3.1. □

Our next step consists in giving examples of contractible domains on which problem (0.1) has an arbitrary number of family of solutions which blow-up and concentrate at one point or a family of solutions which blow-up and concentrate at an arbitrary number of points.

Let $\Omega_0 = \Omega_1 \cup \Omega_2$, where Ω_1 and Ω_2 are two smooth bounded domains such that $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset$. Assume that

$$\Omega_1 \subset \{(x_1, x') \in \mathbb{R}\times\mathbb{R}^{N-1} \ \mid 0 < a \le x_1 \le b\}$$

and

$$\Omega_2 \subset \{(x_1, x') \in \mathbb{R}\times\mathbb{R}^{N-1} \ \mid \ -b \le x_1 \le -a < 0\}.$$

For any $\delta > 0$ let

$$C_\delta = \{(x_1, x') \in \mathbb{R}\times\mathbb{R}^{N-1} \ \mid x_1 \in (-b, b),\ |x'| \le \delta\}.$$

Let Ω_δ be a smooth connected domain such that

$$\Omega_0 \subset \Omega_\delta \subset \Omega_0 \cup C_\delta. \tag{3.1}$$

Lemma 3.2 *It holds*

$$\lim_{\delta\to 0}\tau_{\Omega_\delta}(x) = \tau_{\Omega_0}(x) \quad C^1\text{−uniformly on compact sets of } \Omega_0 \tag{3.2}$$

and

$$\lim_{\delta\to 0} G_{\Omega_\delta}(x, y) = G_{\Omega_0}(x, y) \quad C^1\text{−uniformly on compact sets of } \Omega_0\times\Omega_0\setminus\{x = y\}. \tag{3.3}$$

Proof. Let us prove (3.2). For any $x \in \Omega_0$ and $y \in \Omega_0$ we have, by a comparison argument, that $H_{\Omega_\delta}(x, y)$ is decreasing with respect to δ and $0 < H_{\Omega_\delta}(x, y) \le H_{\Omega_0}(x, y)$. Then $H_{\Omega_\delta}(x, y)$ converges increasingly as δ decreases to 0. By harmonicity the pointwise limit of $H_{\Omega_\delta}(\cdot,\cdot)$ in $\Omega_0\times\Omega_0$ is therefore uniform on compact sets of $\Omega_0\times\Omega_0$ as δ goes to zero. Moreover for any $x \in \Omega_0$ the resulting limit is an harmonic function with respect to y in Ω_0 which coincides with $\frac{1}{|x-y|^{N-2}}$ on $\partial\Omega_0$, namely the resulting limit is $H_{\Omega_0}(x,\cdot)$. Moreover if K is a compact set of $\Omega_0\times\Omega_0$ we have the following interior derivative estimate (see Theorem (2.10), [11])

$$\begin{aligned}&\max_{(x,y)\in K}|\nabla H_{\Omega_\delta}(x, y) - \nabla H_{\Omega_0}(x, y)|\\ &\le \frac{N}{\mathrm{dist}(K, \partial(\Omega_0\times\Omega_0))}\max_{(x,y)\in K}|H_{\Omega_\delta}(x, y) - H_{\Omega_0}(x, y)|,\end{aligned}$$

which proves our claim.

The proof of (3.3) is similar. □

Lemma 3.3 *It holds*

$$\#\{\textit{stable critical points of } \tau_{\Omega_\delta}\} \geq \#\{\textit{stable critical points of } \tau_{\Omega_1}\} + \#\{\textit{stable critical points of } \tau_{\Omega_2}\} \tag{3.4}$$

Proof. It follows from Definition 2.4 and (3.2) of Lemma 3.2. □

Proof of Theorem 0.4. We point out that in virtue of Theorem 0.3 it is enough to construct a domain Ω so that the Robin's function τ_Ω has at least h different stable critical points.

Firstly we consider the case $h = 2$. Let us fix two smooth disjoint bounded domains Ω_1 and Ω_2, so that the function τ_{Ω_1} has a strict minimum point in Ω_1 and τ_{Ω_2} has a strict minimum point in Ω_2. Let Ω_δ be defined as in (3.1). By (3.2) of Lemma 3.2 we deduce that if δ is small enough τ_{Ω_δ} has two different strict minimum points, which are stable according to Definition 2.4. The claim is proved. The general case can be proved by using Lemma 3.3. □

Proof of Theorem 0.5. We point out that in virtue of Theorem 0.2 it is enough to construct a domain Ω so that the function ${\Psi_k}^{\Omega} : (\mathbb{R}^+)^k \times (\Omega)^k \longrightarrow \mathbb{R}$ defined by

$${\Psi_k}^{\Omega}(\lambda, x) = \frac{1}{2}A^2\left(\sum_{i=1}^{k}\tau_\Omega(x_i)\lambda_i^{N-2} - \sum_{\substack{i,j=1,\ldots,k\\ i\neq j}} G_\Omega(x_i, x_j)\lambda_i^{\frac{N-2}{2}}\lambda_j^{\frac{N-2}{2}}\right) - \frac{1}{2}B\sum_{i=1}^{k}\lambda_i^2$$

has a stable critical point.

Let $\Omega_0 = \Omega_1 \cup \ldots \cup \Omega_k$, where $\Omega_1, \ldots, \Omega_k$ are k smooth bounded domains such that $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ if $i \neq j$. It is easy to check that the function ${\Psi_k}^{\Omega_0}$ has a strict minimum point in the connected component $(\mathbb{R}^+)^k \times \Omega_1 \times \ldots \times \Omega_k$ of the set $(\mathbb{R}^+)^k \times (\Omega_0)^k$.

Assume that

$$\Omega_i \subset \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} \mid a_i \leq x_1 \leq b_i\} \qquad \text{with} \quad b_i < a_{i+1}, \quad i = 1, \ldots, k.$$

For any $\delta > 0$ let

$$C_\delta = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} \mid x_1 \in (a_1, b_k),\ |x'| \leq \delta\}.$$

Let Ω_δ be a smooth connected domain such that $\Omega_0 \subset \Omega_\delta \subset \Omega_0 \cup C_\delta$.

Arguing as in the proof of Lemma 3.2 we can prove that

$$\lim_{\delta\to 0}\tau_{\Omega_\delta}(x) = \tau_{\Omega_0}(x) \quad C^1\text{−uniformly on compact sets of } \Omega_0$$

and

$$\lim_{\delta\to 0} G_{\Omega_\delta}(x, y) = G_{\Omega_0}(x, y) \quad C^1\text{−uniformly on compact sets of } \Omega_0\times\Omega_0\backslash\{x = y\}.$$

Therefore we deduce that ${\Psi_k}^{\Omega_\delta}$ converges C^1−uniformly on compact sets of $(\Omega_0)^k \times (\mathbb{R}^+)^k$. Therefore if δ is small enough the function ${\Psi_k}^{\Omega_\delta}$ has a strict minimum point, which is stable according to Definition 2.4. The claim is proved. □

4 Some remarks on a slightly subcritical problem

Let us consider the problem

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}-\varepsilon} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$ and $\varepsilon > 0$ is a positive parameter.

Let ${\Phi_k}^{\Omega} : (\mathbb{R}^+)^k \times (\Omega)^k \longrightarrow \mathbb{R}$ be defined by

$$\Phi_k(\lambda, x) = \frac{1}{2} A^2 \left(M(x)\lambda^{\frac{N-2}{2}}, \lambda^{\frac{N-2}{2}} \right) - \frac{N-2}{2} \log \left(\lambda_1 \cdot \ldots \cdot \lambda_k \right), \tag{4.2}$$

where the matrix M is defined in (0.6).

Arguing as in Section 1 and Section 2 and using estimates contained in [3], one can prove the following result.

Theorem 4.1 *Let (λ_0, x_0) be a stable critical point of Φ_k. Then there exists a family of solution of (4.1) which blow-up and concentrate at the points $x_0^1, \ldots, x_0^k$, in the sense of Definition (0.1).*

Proof. We argue as in the proof of Theorem 0.2. □

Arguing exactly as in Section 3 we can show the following examples.

Theorem 4.2 *If x_0 is a stable critical point of τ, then there exists a family of solutions of (4.1) which blow-up and concentrate at x_0.*

Proof. Firstly one has to prove that if x_0 is a stable critical point of τ, then $\big(\lambda_0, x_0\big)$ with $\lambda_0 = \left[\frac{1}{A^2 \tau(x_0)}\right]^{\frac{1}{N-2}}$ is a stable critical point of Φ_1 (see Lemma 3.1). Finally one gets the claim, arguing as in the proof of Theorem 0.3 and using Theorem 4.1. □

Proposition 4.3 *For any $h \geq 2$ there exists a contractible domain Ω for which problem (4.1) has at least h different families of solutions which blow-up and concentrate at a point x_i in Ω, $i = 1, \ldots, h$.*

Proof. We argue as in the proof of Proposition 0.4, using Theorem 4.1. □

Proposition 4.4 *For any $k \geq 2$ there exists a contractible domain Ω for which problem (4.1) has a family of solutions which blow-up and concentrate at different k points.*

Proof. We argue as in the proof of Proposition 0.5, using Theorem 4.1. □

5 Appendix A

Set for $y \in \mathbb{R}^N$ and $\lambda > 0$

$$PU_{\lambda,y}(x) = i^*_\Omega \left(U^p_{\lambda,y} \right)(x), \qquad x \in \Omega$$

and

$$P_\varepsilon U_{\lambda,y}(z) = i^*_{\Omega_\varepsilon} \left(U^p_{\lambda,y} \right)(z), \qquad z \in \Omega_\varepsilon \qquad (\text{ see } (1.1)).$$

In particular it holds

$$PU_{\varepsilon^\alpha \lambda, \varepsilon^\alpha y}(x) = \varepsilon^{-\alpha \frac{N-2}{2}} P_\varepsilon U_{\lambda,y}\left(\frac{x}{\varepsilon^\alpha} \right) \qquad x \in \Omega. \tag{5.1}$$

Lemma 5.1 *Set $\xi = \varepsilon^\alpha y$. We have*

$$PU_{\varepsilon^\alpha \lambda,\xi}(x) = U_{\varepsilon^\alpha \lambda,\xi}(x) - A(\varepsilon^\alpha \lambda)^{\frac{N-2}{2}} H(x,\xi) + o\left(\varepsilon^{\alpha(\frac{N-2}{2})} \right), \quad x \in \Omega$$

and

$$PU_{\varepsilon^\alpha \lambda,\xi}(x) = A(\varepsilon^\alpha \lambda)^{\frac{N-2}{2}} G(x,\xi) + o\left(\varepsilon^{\alpha(\frac{N-2}{2})} \right), \quad x \in \Omega$$

as $\varepsilon \longrightarrow 0$ uniformly on compact sets of $\Omega \setminus \{\xi\}$ where A is given in (2.4).

Proof. See [15]. □

If $(\lambda, x) \in \mathcal{O}_\delta$ (see Definition 1.3) let $y_i = x_i/\varepsilon^\alpha$ for $i = 1, \dots, k$ and set $y := x/\varepsilon^\alpha \in \Omega^k_\varepsilon$. Set

$$U_i := U_{\lambda_i, y_i} \qquad \text{and} \qquad P_\varepsilon U_i := i^*_\varepsilon \left(U^p_{\lambda_i, y_i} \right),$$

and for $j = 1, \dots, n$ and $i = 1, \dots, k$

$$\psi^0_i := \frac{\partial U_{\lambda_i, y_i}}{\partial \lambda_i}, \qquad \psi^j_i := \frac{\partial U_{\lambda_i, y_i}}{\partial y^j_i} \qquad \text{and} \qquad P_\varepsilon \psi^j_i := i^*_\varepsilon \left(p U^{p-1}_{\lambda_i, y_i} \psi^j_i \right).$$

Remark 5.2 *There exists $c > 0$ such that for any $\varepsilon > 0$ and for any $i = 1, \dots, k$ and $j = 0, 1, \dots, n$ it holds*

$$\|P_\varepsilon U_i\| \le c, \quad \|P_\varepsilon U_i\|_{\frac{2N}{N-2}} \le c \qquad \text{and} \qquad \|P_\varepsilon \psi^j_i\|_{\frac{2N}{N-2}} \le c.$$

Moreover

$$\|P_\varepsilon U_i\|_{\frac{2N}{N+2}} \le \begin{cases} c & \text{if } N \ge 7, \\ c\varepsilon^{-\frac{4\alpha}{r}}, \ r > 0 & \text{if } N = 6, \\ c\varepsilon^{-\frac{7\alpha}{2r}}, \ r \in (0,7) & \text{if } N = 5, \end{cases}$$

$$\|P_\varepsilon \psi^j_i\|_{\frac{2N}{N+2}} \le c \qquad if \ j \ne 0,$$

$$\|P_\varepsilon \psi^0_i\|_{\frac{2N}{N+2}} \le \begin{cases} c & \text{if } N \ge 7, \\ c\varepsilon^{-\frac{\alpha}{2}} & \text{if } N = 5, 6. \end{cases}$$

Lemma 5.3 *For any $\delta > 0$ and for any $\varepsilon_0 > 0$ there exists $C > 0$ such that for any $(\lambda, x) \in \mathcal{O}_\delta$ and for any $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\|f(\sum_{i=1}^{k} P_\varepsilon U_i) - \sum_{i=1}^{k} f(U_i)\|_{\frac{2N}{N+2}} \leq \begin{cases} C\varepsilon^{\alpha\frac{N+2}{2}} & \text{if } N \geq 7, \\ C\varepsilon^{4\alpha}|\log\varepsilon| & \text{if } N = 6, \\ C\varepsilon^{3\alpha} & \text{if } N = 5, \end{cases} \tag{5.2}$$

and

$$\|f'\big(\sum_i P_\varepsilon U_i\big) - \sum_i f'\big(U_i\big)\|_{\frac{N}{2}} \leq C\varepsilon^{2\alpha} \tag{5.3}$$

Proof. Let us prove (5.2). The proof of (5.3) is similar. Since $(\lambda, x) \in \mathcal{O}_\delta$ it holds $|x_i - x_j| > \delta$ for any $i \neq j$. We have by using (5.1)

$$\int_{\Omega_\varepsilon} \left| \big(\sum_{i=1}^{k} P_\varepsilon U_i(y)\big)^p - \sum_{i=1}^{k} U_i^p(y) \right|^{\frac{2N}{N+2}} dy \ (\text{set } x = \varepsilon^\alpha y)$$

$$= \int_{\Omega} \left| \big(\sum_{i=1}^{k} PU_{\lambda_i\varepsilon^\alpha, x_i}(x)\big)^p - \sum_{i=1}^{k} U^p_{\lambda_i\varepsilon^\alpha, x_i}(x) \right|^{\frac{2N}{N+2}} dx$$

$$= \sum_{j=1}^{k} \int_{B(x_j,\frac{\delta}{2})} \left| \big(\sum_{i=1}^{k} PU_{\lambda_i\varepsilon^\alpha, x_i}(x)\big)^p - \sum_{i=1}^{k} U^p_{\lambda_i\varepsilon^\alpha, x_i}(x) \right|^{\frac{2N}{N+2}} dx$$

$$+ \int_{\Omega\setminus \bigcup_{j=1}^{k} B(x_j,\frac{\delta}{2})} \left| \big(\sum_{i=1}^{k} PU_{\lambda_i\varepsilon^\alpha, x_i}(x)\big)^p - \sum_{i=1}^{k} U^p_{\lambda_i\varepsilon^\alpha, x_i}(x) \right|^{\frac{2N}{N+2}} dx. \tag{5.4}$$

Firstly

$$\int_{\Omega\setminus \bigcup_{j=1}^{k} B(x_j,\frac{\delta}{2})} \left| \big(\sum_{i=1}^{k} PU_{\lambda_i\varepsilon^\alpha, x_i}(x)\big)^p - \sum_{i=1}^{k} U^p_{\lambda_i\varepsilon^\alpha, x_i}(x) \right|^{\frac{2N}{N+2}} dx$$

$$\leq C \sum_{i=1}^{k} \int_{\Omega\setminus \bigcup_{j=1}^{k} B(x_j,\frac{\delta}{2})} U^{\frac{2N}{N-2}}_{\lambda_i\varepsilon^\alpha, x_i} dx \leq C \sum_{i=1}^{k} (\lambda_i\varepsilon^\alpha)^N \leq C\varepsilon^{\alpha N}. \tag{5.5}$$

Secondly for $j = 1, \ldots, k$

$$\int_{B(x_j,\frac{\delta}{2})} \left| \big(\sum_{i=1}^{k} PU_{\lambda_i\varepsilon^\alpha, x_i}(x)\big)^p - \sum_{i=1}^{k} U^p_{\lambda_i\varepsilon^\alpha, x_i}(x) \right|^{\frac{2N}{N+2}} dx$$

$$\leq \int_{B(x_j,\frac{\delta}{2})} \left| \big(\sum_{i=1}^{k} PU_{\lambda_i\varepsilon^\alpha, x_i}(x)\big)^p - U^p_{\lambda_j\varepsilon^\alpha, x_j}(x) \right|^{\frac{2N}{N+2}} dx$$

$$+\sum_{\substack{i=1\\ i\neq j}}^{k}\int\limits_{B(x_j,\frac{\delta}{2})}\left|U^p_{\lambda_i\varepsilon^\alpha,x_i}(x)\right|^{\frac{2N}{N+2}}dx. \qquad (5.6)$$

It holds

$$\sum_{\substack{i=1\\ i\neq j}}^{k}\int\limits_{B(x_j,\frac{\delta}{2})}\left|U^p_{\lambda_i\varepsilon^\alpha,x_i}(x)\right|^{\frac{2N}{N+2}}dx$$
$$\leq\sum_{\substack{i=1\\ i\neq j}}^{k}\int\limits_{B(x_j,\frac{\delta}{2})}\left(\frac{\lambda_i\varepsilon^\alpha}{(\lambda_i\varepsilon^\alpha)^2+|x-x_i|^2}\right)^N\leq C\varepsilon^{\alpha N}. \qquad (5.7)$$

Finally by Lemma 5.1 using the mean value theorem we get

$$\int\limits_{B(x_j,\frac{\delta}{2})}\left|PU^p_{\lambda_j\varepsilon^\alpha,x_j}(x)-U^p_{\lambda_j\varepsilon^\alpha,x_j}(x)\right|^{\frac{2N}{N+2}}dx$$
$$=p\int\limits_{B(x_j,\frac{\delta}{2})}\left|\Big(U_{\lambda_j\varepsilon^\alpha,x_j}+\theta(x)\big(PU_{\lambda_j\varepsilon^\alpha,x_j}-U_{\lambda_j\varepsilon^\alpha,x_j}\big)(x)\Big)^{p-1}\big(PU_{\lambda_j\varepsilon^\alpha,x_j}-U_{\lambda_j\varepsilon^\alpha,x_j}\big)(x)\right|^{\frac{2N}{N+2}}dx$$
$$\leq C(\varepsilon^\alpha)^N \qquad \text{if } N\geq 7. \qquad (5.8)$$

Therefore if $N\geq 7$ the claim follows by (5.4), (5.5), (5.6), (5.7) and (5.8). If $N=5$ or $N=6$ we need only to give a different estimate of (5.8) in order to get the claim.

In fact, if $N=6$, we have

$$\int\limits_{B(x_j,\frac{\delta}{2})}\left|PU^p_{\lambda_j\varepsilon^\alpha,x_j}(x)-U^p_{\lambda_j\varepsilon^\alpha,x_j}(x)\right|^{\frac{2N}{N+2}}dx=$$

$$C\varepsilon^{\frac{2N}{N+2}\alpha(N-2)}\int_0^{\frac{1}{\varepsilon^\alpha}}\frac{\varrho^{N-1}}{(1+\varrho^2)^{\frac{4N}{N+2}}}\,d\varrho\leq C\varepsilon^{\frac{2N}{N+2}\alpha(N-2)}|\ln\varepsilon|;$$

on the other hand, if $N=5$, using the substitution $x-x_j=\lambda_j\varepsilon^\alpha z$, we get

$$\int\limits_{B(x_j,\frac{\delta}{2})}\left|PU^p_{\lambda_j\varepsilon^\alpha,x_j}(x)-U^p_{\lambda_j\varepsilon^\alpha,x_j}(x)\right|^{\frac{2N}{N+2}}dx\leq C\varepsilon^{\frac{2N}{N+2}\alpha(N-2)}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{4N}{N+2}}}\,dz.$$

□

Lemma 5.4 *For any $\delta>0$ and for any $\varepsilon_0>0$ there exists $C>0$ such that for any $(\lambda,x)\in\mathcal{O}_\delta$ and for any $\varepsilon\in(0,\varepsilon_0)$ we have for $h=1,\dots,k$ and $j=0,1,\dots,N$*

$$\|\Big[f'\big(\sum_i P_\varepsilon U_i\big)-\sum_i f'\big(U_i\big)\Big]P_\varepsilon\psi^j_h\|_{\frac{2N}{N+2}}\leq C\varepsilon^{\alpha\frac{N+2}{2}} \qquad (5.9)$$

Proof. Since $(\lambda, x) \in \mathcal{O}_\delta$ it holds $|x_i - x_j| > \delta$ for any $i \neq j$. First of all by (5.3) and Lemma 6.4 we get

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} \left(\left| f'\big(\sum_i P_\varepsilon U_i\big) - \sum_i f'\big(U_i\big) \right| |P_\varepsilon \psi_h^j| \right)^{\frac{2N}{N+2}} \\
&\leq \int_{\Omega_\varepsilon} \left(\left| f'\big(\sum_i P_\varepsilon U_i\big) - \sum_i f'\big(U_i\big) \right| |P_\varepsilon \psi_h^j - \psi_h^j| \right)^{\frac{2N}{N+2}} \\
&+ \int_{\Omega_\varepsilon} \left(\left| f'\big(\sum_i P_\varepsilon U_i\big) - \sum_i f'\big(U_i\big) \right| |\psi_h^j| \right)^{\frac{2N}{N+2}} \\
&\leq \| f'\big(\sum_i P_\varepsilon U_i\big) - \sum_i f'\big(U_i\big) \|_{\frac{N}{2}}^{\frac{N+2}{2N}} \| P_\varepsilon \psi_h^j - \psi_h^j \|_{\frac{2N}{N-2}}^{\frac{N+2}{2N}} \\
&+ \int_{\Omega_\varepsilon} \left(\left| f'\big(\sum_i P_\varepsilon U_i\big) - \sum_i f'\big(U_i\big) \right| |\psi_h^j| \right)^{\frac{2N}{N+2}}. \qquad (5.10)
\end{aligned}
$$

Now by using (5.1) we have

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} \left(\left| f'\big(\sum_i P_\varepsilon U_i\big) - \sum_i f'\big(U_i\big) \right| |\psi_h^j| \right)^{\frac{2N}{N+2}} \qquad (\text{set } x = \varepsilon^\alpha y) \\
&= \varepsilon^{-\alpha N + \alpha N \frac{N+4}{N+2}} \int_{\Omega} \left(\left| f'\big(\sum_i P U_{\lambda_i \varepsilon^\alpha, x_i}\big) - \sum_i f'\big(U_{\lambda_i \varepsilon^\alpha, x_i}\big) \right| |\psi^j_{\lambda_h \varepsilon^\alpha, x_h}| \right)^{\frac{2N}{N+2}} \\
&\leq \varepsilon^{-\alpha N + \alpha N \frac{N+4}{N+2}} \int_{B(x_h, \frac{\delta}{2})} \left(\left| f'\big(\sum_i P U_{\lambda_i \varepsilon^\alpha, x_i}\big) - \sum_i f'\big(U_{\lambda_i \varepsilon^\alpha, x_i}\big) \right| |\psi^j_{\lambda_h \varepsilon^\alpha, x_h}| \right)^{\frac{2N}{N+2}} \\
&+ \varepsilon^{-\alpha N + \alpha N \frac{N+4}{N+2}} \int_{\Omega \setminus B(x_h, \frac{\delta}{2})} \left(\left| f'\big(\sum_i P U_{\lambda_i \varepsilon^\alpha, x_i}\big) - \sum_i f'\big(U_{\lambda_i \varepsilon^\alpha, x_i}\big) \right| |\psi^j_{\lambda_h \varepsilon^\alpha, x_h}| \right)^{\frac{2N}{N+2}}. \\
&\qquad (5.11)
\end{aligned}
$$

Firstly we have (by using Lemma 5.1)

$$
\begin{aligned}
&\int_{B(x_h, \frac{\delta}{2})} \left(\left| f'\big(\sum_i P U_{\lambda_i \varepsilon^\alpha, x_i}\big) - \sum_i f'\big(U_{\lambda_i \varepsilon^\alpha, x_i}\big) \right| |\psi^j_{\lambda_h \varepsilon^\alpha, x_h}| \right)^{\frac{2N}{N+2}} \\
&\leq C \int_{B(x_h, \frac{\delta}{2})} \left(\left| f'\big(\sum_i P U_{\lambda_i \varepsilon^\alpha, x_i}\big) - f'\big(U_{\lambda_h \varepsilon^\alpha, x_h}\big) \right| |\psi^j_{\lambda_h \varepsilon^\alpha, x_h}| \right)^{\frac{2N}{N+2}}
\end{aligned}
$$

$$+C\sum_{i\neq h}\int_{B(x_h,\frac{\delta}{2})}\left(\left|f'(U_{\lambda_i\varepsilon^\alpha,x_i})\right||\psi^j_{\lambda_h\varepsilon^\alpha,x_h}|\right)^{\frac{2N}{N+2}}$$

$$\leq C\int_{B(x_h,\frac{\delta}{2})}|PU_{\lambda_h\varepsilon^\alpha,x_h}-U_{\lambda_h\varepsilon^\alpha,x_h}|^{\frac{8N}{(N-2)(N+2)}}|\psi^j_{\lambda_h\varepsilon^\alpha,x_h}|^{\frac{2N}{N+2}}$$

$$+C\sum_{i\neq h}\int_{B(x_h,\frac{\delta}{2})}|U_{\lambda_i\varepsilon^\alpha,x_i}|^{\frac{8N}{(N-2)(N+2)}}|\psi^j_{\lambda_h\varepsilon^\alpha,x_h}|^{\frac{2N}{N+2}}$$

$$\leq C\varepsilon^{\alpha\frac{N}{2}\frac{2N}{N+2}}. \tag{5.12}$$

Secondly we have

$$\int_{\Omega\setminus B(x_h,\frac{\delta}{2})}\left(\left|f'\Big(\sum_i P_\varepsilon U_i\Big)-\sum_i f'(U_i)\right||\psi^j_{\lambda_h\varepsilon^\alpha,x_h}|\right)^{\frac{2N}{N+2}}$$

$$\leq\sum_i\int_{\Omega\setminus B(x_h,\frac{\delta}{2})}U_i^{\frac{8N}{(N-2)(N+2)}}|\psi^j_{\lambda_h\varepsilon^\alpha,x_h}|^{\frac{2N}{N+2}}$$

$$\leq C\varepsilon^{\alpha\frac{N}{2}\frac{2N}{N+2}}. \tag{5.13}$$

By (5.10), (5.11), (5.12) and (5.13) the claim follows. □

6 Appendix B

Set for $y\in\mathbb{R}^N$ and $\lambda>0$

$$\psi^0_{\lambda,y}(x)=\frac{\partial U_{\lambda,y}}{\partial\lambda}(x)=C_N\frac{N-2}{2}\lambda^{\frac{N-4}{2}}\frac{|x-y|^2-\lambda^2}{(\lambda^2+|x-y|^2)^{N/2}},\qquad x\in\mathbb{R}^N$$

and for $j=1,\dots,N$

$$\psi^j_{\lambda,y}(x)=\frac{\partial U_{\lambda,y}}{\partial y^j}(x)=-C_N(N-2)\lambda^{\frac{N-2}{2}}\frac{x^j-y^j}{(\lambda^2+|x-y|^2)^{N/2}},\qquad x\in\mathbb{R}^N.$$

This family satisfies the equation

$$-\Delta\psi^j_{\lambda,y}=pU^{p-1}_{\lambda,y}\psi^j_{\lambda,y}\qquad\text{in }\mathbb{R}^N.$$

Set for $y\in\mathbb{R}^N$

$$P\psi^j_{\lambda,y}(x)=i^*_\Omega\left(pU^{p-1}_{\lambda_i,y_i}\psi^j_{\lambda,y}\right)(x),\qquad x\in\Omega$$

and

$$P_\varepsilon\psi^j_{\lambda,y}(z)=i^*_\varepsilon\left(pU^{p-1}_{\lambda_i,y_i}\psi^j_{\lambda,y}\right)(z),\qquad z\in\Omega_\varepsilon$$

For $j=0,1,\dots,N$ and $i=1,\dots,k$ we have

$$P\psi^j_{\varepsilon^\alpha\lambda,\varepsilon^\alpha y}(x)=\varepsilon^{-\alpha\frac{N}{2}}P_\varepsilon\psi^j_{\lambda,y}\left(\frac{x}{\varepsilon^\alpha}\right)\qquad x\in\Omega. \tag{6.1}$$

Lemma 6.1 *Let $\xi \in \Omega$. We have for $j = 1, \dots, N$*

$$P\psi^j_{\varepsilon^\alpha\lambda,\xi}(x) = A(\varepsilon^\alpha\lambda)^{\frac{N-2}{2}}\frac{\partial G}{\partial \xi^j}(x,\xi) + o\left(\varepsilon^{\alpha\frac{N-2}{2}}\right), \quad x \in \Omega$$

and

$$P\psi^0_{\varepsilon^\alpha\lambda,\xi}(x) = A\frac{N-2}{2}(\varepsilon^\alpha\lambda)^{\frac{N}{2}-2}G(x,\xi) + o\left(\varepsilon^{\alpha(\frac{N}{2}-2)}\right), \quad x \in \Omega$$

as $\varepsilon \longrightarrow 0$ uniformly on compact sets of $\Omega \setminus \{\xi\}$, where A is given in (2.4).

Proof. We recall that

$$\begin{cases} -\Delta P\psi^j_{\varepsilon^\alpha\lambda,\xi}(x) = pU^{p-1}_{\varepsilon^\alpha\lambda,\xi}(x)\psi^j_{\varepsilon^\alpha\lambda,\xi}(x) & \text{in } \Omega, \\ P\psi^j_{\varepsilon^\alpha\lambda,\xi} = 0 & \text{on } \partial\Omega, \end{cases}$$

If $j = 1, \dots, N$, we have for $x \in \Omega$

$$P\psi^j_{\varepsilon^\alpha\lambda,\xi}(x) = \int_\Omega pU^{p-1}_{\varepsilon^\alpha\lambda,\xi}(z)\psi^j_{\varepsilon^\alpha\lambda,\xi}(z)G(x,z)dz$$

$$= -pC^p_N(N-2)(\varepsilon^\alpha\lambda)^{\frac{N}{2}+1}\int_\Omega G(x,z)\frac{z^j-\xi^j}{\left((\varepsilon^\alpha\lambda)^2+|z-\xi|^2\right)^{\frac{N}{2}+2}}dz$$

(set $z = \varepsilon^\alpha\lambda w + \xi$)

$$= -pC^p_N(N-2)(\varepsilon^\alpha\lambda)^{\frac{N}{2}-2}\int_{\frac{\Omega-\xi}{\varepsilon^\alpha\lambda}} G(x,\varepsilon^\alpha\lambda w+\xi)\frac{w^j}{\left(1+|w|^2\right)^{\frac{N}{2}+2}}dw$$

$$= -(\varepsilon^\alpha\lambda)^{\frac{N}{2}-2}\int_{\frac{\Omega-\xi}{\varepsilon^\alpha\lambda}} G(x,\varepsilon^\alpha\lambda w+\xi)\frac{\partial}{\partial w^j}\left(U^p(w)\right)dw$$

$$= (\varepsilon^\alpha\lambda)^{\frac{N}{2}-2}\int_{\frac{\Omega-\xi}{\varepsilon^\alpha\lambda}} \frac{\partial}{\partial w^j}\left(G(x,\varepsilon^\alpha\lambda w+\xi)\right)U^p(w)dw$$

$$= (\varepsilon^\alpha\lambda)^{\frac{N}{2}-1}\int_{\frac{\Omega-\xi}{\varepsilon^\alpha\lambda}} \frac{\partial G}{\partial w^j}(x,\varepsilon^\alpha\lambda w+\xi)U^p(w)dw$$

$$= (\varepsilon^\alpha\lambda)^{\frac{N}{2}-1}\frac{\partial G}{\partial y^j}(x,\xi)\left(\int_{\mathbb{R}^N} U^p(w)dw\right) + o\left(\varepsilon^{\alpha\frac{N-2}{2}}\right).$$

Moreover for $x \in \Omega$

$$P\psi^0_{\varepsilon^\alpha\lambda,\xi}(x) = \int_\Omega pU^{p-1}_{\varepsilon^\alpha\lambda,\varepsilon^\alpha y}(z)\psi^0_{\varepsilon^\alpha\lambda,\xi}(z)G(x,z)dz$$

$$= pC^p_N\frac{N-2}{2}(\varepsilon^\alpha\lambda)^{\frac{N}{2}}\int_\Omega G(x,z)\frac{|z-\xi|^2-(\varepsilon^\alpha\lambda)^2}{\left((\varepsilon^\alpha\lambda)^2+|z-\xi|^2\right)^{\frac{N}{2}+2}}dz$$

(set $z=\varepsilon^\alpha\lambda w+\xi$)

$$= pC_N^p\frac{N-2}{2}(\varepsilon^\alpha\lambda)^{\frac{N}{2}-2}\int\limits_{\frac{\Omega-\xi}{\varepsilon^\alpha\lambda}} G(x,\varepsilon^\alpha\lambda w+\xi)\frac{|w|^2-1}{\left(1+|w|^2\right)^{\frac{N}{2}+2}}dw$$

$$= pC_N^p\frac{N-2}{2}(\varepsilon^\alpha\lambda)^{\frac{N}{2}-2}\int\limits_{\frac{\Omega-\xi}{\varepsilon^\alpha\lambda}} G(x,\varepsilon^\alpha\lambda w+\xi)\frac{|w|^2-1}{\left(1+|w|^2\right)^{\frac{N}{2}+2}}dw$$

(because of Remark 6.2)

$$= (\varepsilon^\alpha\lambda)^{\frac{N}{2}-2}G(x,\xi)\left(\int\limits_{\mathbb{R}^N} U^p(w)dw\right)+o\left(\varepsilon^{\alpha(\frac{N}{2}-2)}\right).$$

□

Remark 6.2 *It holds*

$$pC_N^p\frac{N-2}{2}\int\limits_{\mathbb{R}^N}\frac{|w|^2-1}{\left(1+|w|^2\right)^{\frac{N}{2}+2}}dw=\frac{N-2}{2}\int\limits_{\mathbb{R}^N}U^p(w)dw.$$

Proof. Let us remark that

$$pC_N^p\frac{N-2}{2}\int\limits_{\mathbb{R}^N}\frac{|x|^2-1}{(1+|x|^2)^{\frac{N}{2}+2}}dx=p\int\limits_{\mathbb{R}^N}U^{p-1}(z)\left(\frac{\partial U_{\lambda,0}}{\partial\lambda}\right)_{|\lambda=1}(z)dz.$$

Hence we get

$$pC_N^p\frac{N-2}{2}\int\limits_{\mathbb{R}^N}\frac{|x|^2-1}{(1+|x|^2)^{\frac{N}{2}+2}}dx=p\int\limits_{\mathbb{R}^N}\left(U_{\lambda,0}^{p-1}(z)\frac{\partial U_{\lambda,0}}{\partial\lambda}\right)_{|\lambda=1}(z)dz$$

$$=\int\limits_{\mathbb{R}^N}\frac{\partial}{\partial\lambda}\left(U_{\lambda,0}^p\right)_{|\lambda=1}(z)dz=\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\int\limits_{\mathbb{R}^N}U_{\lambda,0}^p(z)dz\right)_{|\lambda=1}$$

$$=\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\lambda^{\frac{N}{2}-1}\int\limits_{\mathbb{R}^N}U^p(z)dz\right)_{|\lambda=1}=\left(\frac{N}{2}-1\right)\int\limits_{\mathbb{R}^N}U^p(z)dz.$$

□

Let us now set

$$R^0_{\varepsilon^\alpha\lambda,\xi}(x)=\frac{\partial U_{\varepsilon^\alpha\lambda,\xi}}{\partial(\varepsilon^\alpha\lambda)}(x)-P\psi^0_{\varepsilon^\alpha\lambda,\xi}(x),\qquad x\in\Omega$$

and for $j=1,\dots,N$

$$R^j_{\varepsilon^\alpha\lambda,\xi}(x)=\frac{\partial U_{\varepsilon^\alpha\lambda,\xi}}{\partial(\xi)}(x)-P\psi^j_{\varepsilon^\alpha\lambda,\xi}(x)\qquad x\in\Omega.$$

Lemma 6.3 *Let $\xi \in \Omega$. We have for $j = 1, \dots, N$*

$$R^j_{\varepsilon^\alpha\lambda,\xi}(x) = A(\varepsilon^\alpha\lambda)^{\frac{N-2}{2}} \frac{\partial H}{\partial \xi^j}(x,\xi) + o\left(\varepsilon^{\alpha\frac{N-2}{2}}\right), \quad x \in \Omega$$

and

$$R^0_{\varepsilon^\alpha\lambda,\xi}(x) = A\frac{N-2}{2}(\varepsilon^\alpha\lambda)^{\frac{N}{2}-2} H(x,\xi) + o\left(\varepsilon^{\alpha(\frac{N}{2}-2)}\right), \quad x \in \Omega$$

as $\varepsilon \longrightarrow 0$ uniformly on compact sets of $\Omega \setminus \{\xi\}$, where A is given in (2.4).

Proof. We argue as in the proof of Lemma 6.1. □

First of all we deduce the following estimate.

Lemma 6.4 *For $i = 1, \dots, k$ we have*

$$\|P_\varepsilon \psi^j_i - \psi^j_i\|_{\frac{2N}{N-2}} \le C\varepsilon^{\alpha\frac{N}{2}} \qquad \text{if } j = 1, \dots, N$$

and

$$\|P_\varepsilon \psi^0_i - \psi^0_i\|_{\frac{2N}{N-2}} \le C\varepsilon^{\alpha\frac{N-2}{2}}.$$

Proof. It follows easily by (6.1) and Lemma 6.3. □

A crucial estimate is needed to get the expansion in Proposition 2.1. We give it here.

Lemma 6.5 *Let $\alpha = \frac{1}{N-4}$. If $j = 1, \dots, N$ and $h = 1, \dots, k$ then*

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \big[\sum_i f(U_i) - f(\sum_i P_\varepsilon U_i)\big] P_\varepsilon \psi^j_h - \varepsilon^{2\alpha+1} \sum_i \int_{\Omega_\varepsilon} P_\varepsilon U_i P_\varepsilon \psi^j_h \\
&= A^2 \left[\frac{\partial H}{\partial x^j_h}(x_h,x_h)\lambda_h^{N-2} - \sum_{\substack{l=1\\ l\neq h}}^k \frac{\partial G}{\partial x^j_h}(x_h,x_l)(\lambda_h\lambda_l)^{\frac{N-2}{2}}\right] \varepsilon^{\frac{N-1}{N-4}} \\
&+o\left(\varepsilon^{\frac{N-1}{N-4}}\right)
\end{aligned} \tag{6.2}$$

as $\varepsilon \to 0$ uniformly with respect to $(\lambda,\xi) \in \mathcal{O}_\delta$.

Moreover if $j = 0$ and $h = 1, \dots, k$ then

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \big[\sum_i f(U_i) - f(\sum_i P_\varepsilon U_i)\big] P_\varepsilon \psi^0_h - \varepsilon^{2\alpha+1} \sum_i \int_{\Omega_\varepsilon} P_\varepsilon U_i P_\varepsilon \psi^0_h \\
&= \left\{\frac{N-2}{2} A^2 \left[H(x_h,x_h)\lambda_h^{N-3} - \sum_{\substack{l=1\\ l\neq h}}^k G(x_h,x_l)\lambda_h^{\frac{N}{2}-2}\lambda_l^{\frac{N-2}{2}}\right] + \lambda_i B\right\} \varepsilon^{\frac{N-2}{N-4}} \\
&+o\left(\varepsilon^{\frac{N-2}{N-4}}\right)
\end{aligned} \tag{6.3}$$

as $\varepsilon \to 0$ uniformly with respect to $(\lambda,\xi) \in \mathcal{O}_\delta$. A and B are given in (2.4).

The proof of the previous Lemma is a consequence of the following three Lemmas.

Lemma 6.6 *If $j = 1, \dots, N$ and $i, h = 1, \dots, k,\ i \neq h$ then*

$$\int\limits_{\Omega_\varepsilon} U_i^p P_\varepsilon \psi_h^j = A^2 (\lambda_i \lambda_h)^{\frac{N-2}{2}} \frac{\partial G}{\partial x_h^j}(x_i, x_h) \varepsilon^{\alpha(N-1)} + o\left(\varepsilon^{\alpha(N-1)}\right) \tag{6.4}$$

and if $i = h$

$$\int\limits_{\Omega_\varepsilon} U_i^p P_\varepsilon \psi_i^j = -A^2 \lambda_i^{N-2} \frac{\partial H}{\partial x_i^j}(x_i, x_i) \varepsilon^{\alpha(N-1)} + o\left(\varepsilon^{\alpha(N-1)}\right) \tag{6.5}$$

as $\varepsilon \to 0$ uniformly with respect to $(\lambda, x) \in \mathcal{O}_\delta$.

Moreover if $j = 0$ and $i, h = 1, \dots, k,\ i \neq h$ then

$$\int\limits_{\Omega_\varepsilon} U_i^p P_\varepsilon \psi_h^0 = \frac{N-2}{2} A^2 \lambda_i^{\frac{N-2}{2}} \lambda_h^{\frac{N}{2}-2} G(x_i, x_h) \varepsilon^{\alpha(N-2)} + o\left(\varepsilon^{\alpha(N-2)}\right) \tag{6.6}$$

and if $i = h$

$$\int\limits_{\Omega_\varepsilon} U_i^p P_\varepsilon \psi_i^0 = -\frac{N-2}{2} A^2 \lambda_i^{N-3} H(x_i, x_i) \varepsilon^{\alpha(N-2)} + o\left(\varepsilon^{\alpha(N-2)}\right) \tag{6.7}$$

as $\varepsilon \to 0$ uniformly with respect to $(\lambda, x) \in \mathcal{O}_\delta$.

Proof. Set

$$\hat{\psi}_i^0 := \frac{\partial U_{\lambda_i \varepsilon^\alpha, x_i}}{\partial(\lambda_i \varepsilon^\alpha)} \qquad \text{and} \qquad \hat{\psi}_i^j := \frac{\partial U_{\lambda_i \varepsilon^\alpha, x_i}}{\partial x_i^j}. \tag{6.8}$$

In the following we will always use estimate (6.1), Lemma 6.1 and Lemma 6.3. Let $j = 1, \dots, N$ and $i \neq h$ then we have

$$\begin{aligned}
&\int\limits_{\Omega_\varepsilon} U_i^p(y) P_\varepsilon \psi_h^j(y) dy = \varepsilon^\alpha \int\limits_{\Omega} U^p_{\varepsilon^\alpha \lambda, x_i}(x) P \hat{\psi}_h^j(x) dx \\
&= \varepsilon^\alpha (\lambda_h \varepsilon^\alpha)^{\frac{N-2}{2}} A \int\limits_{\Omega} \frac{\partial G}{\partial x_h^j}(x, x_h) U^p_{\varepsilon^\alpha \lambda_i, x_i}(x) dx \\
&+ o\left(\varepsilon^{\alpha N/2} \int\limits_{\Omega} \frac{\partial G}{\partial x_h^j}(x, x_h) U^p_{\varepsilon^\alpha \lambda_i, x_i}(x) dx\right) \\
&= A (\lambda_h \lambda_i)^{\frac{N-2}{2}} \varepsilon^{\alpha(N-1)} \int\limits_{\frac{\Omega - x_i}{\lambda_i \varepsilon^\alpha}} \frac{\partial G}{\partial x_h^j}(\lambda_i \varepsilon^\alpha z + x_i, x_h) U^p_{\lambda_i, \frac{x_i}{\varepsilon^\alpha}}(z) dz + o\left(\varepsilon^{\alpha(N-1)}\right) \\
&= A^2 (\lambda_h \lambda_i)^{\frac{N-2}{2}} \varepsilon^{\alpha(N-1)} \frac{\partial G}{\partial x_h^j}(x_i, x_h) + o\left(\varepsilon^{\alpha(N-1)}\right).
\end{aligned}$$

Let $j = 1, \ldots, N$ and $i = h$ then we have

$$\int\limits_{\Omega_\varepsilon} U_i^p(y) P_\varepsilon \psi_i^j(y) dy = \varepsilon^\alpha \int\limits_\Omega U^p_{\varepsilon^\alpha \lambda_i, x_i}(x) P\hat{\psi}_i^j(x) dx$$

$$= \varepsilon^\alpha \int\limits_\Omega U^p_{\varepsilon^\alpha \lambda_i, x_i}(x) \frac{\partial U_{\varepsilon^\alpha \lambda, x_i}}{\partial x_i^j}(x) dx - \varepsilon^\alpha \int\limits_\Omega U^p_{\varepsilon^\alpha \lambda_i, x_i}(x) R_i^j(x) dx$$

$$= \varepsilon^\alpha (\lambda_i \varepsilon^\alpha)^{N + \frac{N}{2} - N - \frac{N+2}{2}} \int\limits_{\frac{\Omega - x_i}{\lambda_i \varepsilon^\alpha}} U^p(z) \frac{\partial U}{\partial z_i^j}(z) dz$$

$$- \varepsilon^\alpha (\lambda_i \varepsilon^\alpha)^{N-2} \frac{\partial H}{\partial x_i^j}(x_i, x_i) A^2 + o\left(\varepsilon^{\alpha(N-1)}\right)$$

$$= \lambda_i \int\limits_{\mathbb{R}^N} U^p(z) \frac{\partial U}{\partial z_i^j}(z) dz + o\left(\varepsilon^{\alpha(N+1)}\right)$$

$$- \varepsilon^{\alpha(N-1)} \lambda_i^{N-2} \frac{\partial H}{\partial x_i^j}(x_i, x_i) A^2 + o\left(\varepsilon^{\alpha(N-1)}\right)$$

$$= - \varepsilon^{\alpha(N-1)} \lambda_i^{N-2} \frac{\partial H}{\partial x_i^j}(x_i, x_i) A^2 + o\left(\varepsilon^{\alpha(N-1)}\right).$$

Let $j = 0$ and $i \neq h$ then we have

$$\int\limits_{\Omega_\varepsilon} U_i^p(y) P_\varepsilon \psi_h^0(y) dy = \varepsilon^\alpha \int\limits_\Omega U^p_{\varepsilon^\alpha \lambda_i, x_i}(x) P\hat{\psi}_h^0(x) dx$$

$$= \varepsilon^\alpha (\lambda_h \varepsilon^\alpha)^{\frac{N}{2} - 2} \frac{N-2}{2} A \int\limits_\Omega U^p_{\varepsilon^\alpha \lambda_i, x_i}(x) G(x, x_h) dx + o\left(\varepsilon^{\alpha(N-2)}\right)$$

$$= \varepsilon^{\alpha(N-2)} \lambda_i^{\frac{N-2}{2}} \lambda_h^{\frac{N}{2} - 2} \frac{N-2}{2} A^2 G(x_i, x_h) + o\left(\varepsilon^{\alpha(N-2)}\right).$$

Let $j = 0$ and $i = h$ then we have

$$\int\limits_{\Omega_\varepsilon} U_i^p(y) P_\varepsilon \psi_i^0(y) dy = \varepsilon^\alpha \int\limits_\Omega U^p_{\varepsilon^\alpha \lambda_i, x_i}(x) P\hat{\psi}_i^0(x) dx$$

$$= \varepsilon^\alpha \int\limits_\Omega U^p_{\varepsilon^\alpha \lambda_i, x_i}(x) \frac{\partial U_{\varepsilon^\alpha \lambda, x_i}}{\partial (\varepsilon^\alpha \lambda)}(x) dx - \varepsilon^\alpha \int\limits_\Omega U^p_{\varepsilon^\alpha \lambda_i, x_i}(x) R_i^0(x) dx$$

$$= \varepsilon^\alpha (\lambda_i \varepsilon^\alpha)^{N - \frac{N+2}{2} + \frac{N-4}{2} + 2 - N} \int\limits_{\frac{\Omega - x_i}{\lambda_i \varepsilon^\alpha}} U^p(z) \left(\frac{\partial U}{\partial \lambda}\right)_{|\lambda=1}(z) dz$$

$$- \varepsilon^\alpha (\lambda_i \varepsilon^\alpha)^{\frac{N}{2} - 2} \frac{N-2}{2} A \int\limits_\Omega U^p_{\lambda_i \varepsilon^\alpha, x_i}(x) H(x, x_i) dx + o\left(\varepsilon^{\alpha(N-2)}\right)$$

$$= \lambda_i^{-1} \int\limits_{\mathbb{R}^N} U^p(z) \left(\frac{\partial U}{\partial \lambda}\right)_{|\lambda=1}(z) dz + o\left(\varepsilon^{\alpha N}\right)$$

$$-\varepsilon^{\alpha}(\lambda_i\varepsilon^{\alpha})^{\frac{N}{2}-2+N-\frac{N+2}{2}}\frac{N-2}{2}A\int\limits_{\frac{\Omega-x_i}{\lambda_i\varepsilon^{\alpha}}} U^p_{\lambda_i,\frac{x_i}{\varepsilon^{\alpha}}}(z)H(\lambda_i\varepsilon^{\alpha}z+x_i,x_i)dz$$

$$+o\left(\varepsilon^{\alpha(N-2)}\right)$$

$$=-\frac{N-2}{2}A^2\varepsilon^{\alpha(N-2)}\lambda_i^{N-3}H(x_i,x_i)+o\left(\varepsilon^{\alpha(N-2)}\right).$$

□

Lemma 6.7 *If $j=1,\dots,N$ and $h=1,\dots,k$ we have*

$$\int\limits_{\Omega_\varepsilon} f(\sum_{i=1}^k P_\varepsilon U_i)P_\varepsilon\psi_h^j$$

$$=2A^2\left[\sum_{\substack{l=1\\ l\neq h}}^k \frac{\partial G}{\partial x_h^j}(x_h,x_l)(\lambda_h\lambda_l)^{\frac{N-2}{2}}-\frac{\partial H}{\partial x_h^j}(x_h,x_h)\lambda_h^{N-2}\right]\varepsilon^{\alpha(N-1)}$$

$$+o\left(\varepsilon^{\alpha(N-1)}\right) \tag{6.9}$$

as $\varepsilon\to 0$ uniformly with respect to $(\lambda,x)\in\mathcal{O}_\delta$.

Moreover $j=0$ and $h=1,\dots,k$ we have

$$\int\limits_{\Omega_\varepsilon} f(\sum_{i=1}^k P_\varepsilon U_i)P_\varepsilon\psi_h^0$$

$$=(N-2)A^2\left[\sum_{\substack{l=1\\ l\neq h}}^k G(x_h,x_l)\lambda_h^{\frac{N}{2}-2}\lambda_l^{\frac{N-2}{2}}-H(x_h,x_h)\lambda_h^{N-3}\right]\varepsilon^{\alpha(N-2)}$$

$$+o\left(\varepsilon^{\alpha(N-2)}\right) \tag{6.10}$$

as $\varepsilon\to 0$ uniformly with respect to $(\lambda,x)\in\mathcal{O}_\delta$.

Proof. In the following we will always use estimate (5.1), Lemma 5.1, estimate (6.1), Lemma 6.1 and Lemma 6.3.

Let $j\neq 0$ and $h=1$. Fix δ such that $|x_i-x_j|>\delta$ for any $i\neq j$. We have, by 6.8,

$$\int\limits_{\Omega_\varepsilon}\left(\sum_{i=1}^k P_\varepsilon U_i(x)\right)^p P_\varepsilon\psi_1^j(x)dx=\varepsilon^{\alpha}\int\limits_{\Omega}\left(\sum_{i=1}^k PU_{\lambda_i\varepsilon^{\alpha},x_i}(z)\right)^p P\hat{\psi}_1^j(z)dz$$

$$=\varepsilon^{\alpha}\int\limits_{B(x_1,\delta)}\left[\left(\sum_{i=1}^k PU_{\lambda_i\varepsilon^{\alpha},x_i}(z)\right)^p-PU^p_{\lambda_1\varepsilon^{\alpha},x_1}(z)\right]P\hat{\psi}_1^j(z)dz$$

$$+\varepsilon^{\alpha}\int\limits_{B(x_1,\delta)} PU^p_{\lambda_1\varepsilon^{\alpha},x_1}(z)P\hat{\psi}_1^j(z)dz$$

$$+\varepsilon^{\alpha}\sum_{l=2}^{k}\int\limits_{B(x_l,\delta)}\left(\sum_{i=1}^{k}PU_{\lambda_i\varepsilon^{\alpha},x_i}(z)\right)^{p}P\hat{\psi}_1^{j}(z)dz$$

$$+\varepsilon^{\alpha}\int\limits_{\Omega\setminus\bigcup_{l=1}^{k}B(x_l,\delta)}\left(\sum_{i=1}^{k}PU_{\lambda_i\varepsilon^{\alpha},x_i}(z)\right)^{p}P\hat{\psi}_1^{j}(z)dz. \tag{6.11}$$

Firstly we have for any $j=1,\dots,k$

$$\left|\int\limits_{\Omega\setminus\bigcup_{l=1}^{k}B(x_l,\delta)}\left(\sum_{i=1}^{k}PU_{\lambda_i\varepsilon^{\alpha},x_i}(z)\right)^{p}P\hat{\psi}_1^{j}(z)dz\right|$$
$$\leq C\sum_{i=1}^{k}\int\limits_{\Omega\setminus\bigcup_{l=1}^{k}B(x_l,\delta)}U_{\lambda_i\varepsilon^{\alpha},x_i}\left|\frac{\partial U_{\lambda_1\varepsilon^{\alpha},x_1}}{\partial x_1^{j}}\right|dz$$
$$\leq C\int\limits_{\Omega}\varepsilon^{\alpha\frac{N+2}{2}}\varepsilon^{\alpha\frac{N-2}{2}}dx\leq C\varepsilon^{\alpha N}. \tag{6.12}$$

because $|x-x_i|>\delta$ for any $i=1,\dots,k$ and $x\in\Omega\setminus\bigcup_{l=1}^{k}B(x_l,\delta)$. Secondly

$$\int\limits_{B(x_1,\delta)}\left[\left(\sum_{l=1}^{k}PU_{\lambda_l\varepsilon^{\alpha},x_l}(x)\right)^{p}-PU^{p}_{\lambda_1\varepsilon^{\alpha},x_1}(x)\right]P\hat{\psi}_1^{j}(x)dx$$
$$=p\int\limits_{B(x_1,\delta)}\left[PU_{\lambda_1\varepsilon^{\alpha},x_1}+t(x)\sum_{l=2}^{k}PU_{\lambda_l\varepsilon^{\alpha},x_l}\right]^{p-1}\sum_{l=2}^{k}PU_{\lambda_l\varepsilon^{\alpha},x_l}P\hat{\psi}_1^{j}dx$$
$$=-\sum_{l=2}^{k}\int\limits_{B(x_1,\delta)}\frac{\partial}{\partial x_1^{j}}\left[(PU_{\lambda_1\varepsilon^{\alpha},x_1}+t(x)PU_{\lambda_l\varepsilon^{\alpha},x_l})^{p}\right]PU_{\lambda_l\varepsilon^{\alpha},x_l}dx$$
$$=-\sum_{l=2}^{k}\int\limits_{B(x_1,\delta)}\frac{\partial}{\partial x_1^{j}}\left[(PU_{\lambda_1\varepsilon^{\alpha},x_1}+t(x)PU_{\lambda_l\varepsilon^{\alpha},x_l})^{p}\right](\lambda_l\varepsilon^{\alpha})^{\frac{N-2}{2}}G(x,x_l)Adx$$
$$+o\left(\varepsilon^{\alpha(N-2)}\right)$$
$$=-A\sum_{l=2}^{k}(\lambda_1\lambda_l)^{\frac{N-2}{2}}\varepsilon^{\alpha(N-2)}$$
$$\int\limits_{B(0,\frac{\delta}{\lambda_1\varepsilon^{\alpha}})}\frac{\partial}{\partial z_1^{j}}\left[(PU_{\lambda_1\varepsilon^{\alpha},x_1}+t(x)PU_{\lambda_l\varepsilon^{\alpha},x_l})^{p}(x_1+\lambda_1\varepsilon^{\alpha}z)\right]G(x_1+\lambda_1\varepsilon^{\alpha}z,x_l)dx$$

$$+o\left(\varepsilon^{\alpha(N-2)}\right)$$

$$= A^2\sum_{l=2}^{k}(\lambda_1\lambda_l)^{\frac{N-2}{2}}\varepsilon^{\alpha(N-2)}\frac{\partial G}{\partial x_1^j}(x_1,x_l)+o\left(\varepsilon^{\alpha(N-2)}\right). \tag{6.13}$$

Moreover for any $l \neq 1$

$$\int_{B(x_l,\delta)}\left(\sum_{l=1}^{k}PU_{\lambda_l\varepsilon^\alpha,x_l}(z)\right)^p P\hat{\psi}_1^j(z)dz$$

$$= (\lambda_1\varepsilon^\alpha)^{\frac{N-2}{2}}A\int_{B(x_l,\delta)}\left(\sum_{l=1}^{k}PU_{\lambda_l\varepsilon^\alpha,x_l}(z)\right)^p\frac{\partial G}{\partial x_1^j}(x,x_1)dx$$

$$+o\left(\varepsilon^{\alpha(N-2)}\right)$$

$$= (\lambda_1\lambda_l)^{\frac{N-2}{2}}\varepsilon^{\alpha(N-2)}A$$

$$\int_{B(0,\frac{\delta}{\lambda_l\varepsilon^\alpha})}\left(\sum_{l=1}^{k}PU_{\lambda_l\varepsilon^\alpha,x_l}(x_l+\lambda_l\varepsilon^\alpha z))\right)^p\frac{\partial G}{\partial x_1^j}(x_l+\lambda_l\varepsilon^\alpha z,x_1)dz$$

$$+o\left(\varepsilon^{\alpha(N-2)}\right)$$

$$= (\lambda_1\lambda_l)^{\frac{N-2}{2}}\varepsilon^{\alpha(N-2)}A^2\frac{\partial G}{\partial x_1^j}(x_l,x_1)dz+o\left(\varepsilon^{\alpha(N-2)}\right). \tag{6.14}$$

Finally we have

$$\int_{B(x_1,\delta)}(PU_{\lambda_1\varepsilon^\alpha,x_1})^p P\hat{\psi}_1^j(x)dz$$

$$= \int_{B(x_1,\delta)}(PU_{\lambda_1\varepsilon^\alpha,x_1})^p\frac{\partial U_{\lambda_1\varepsilon^\alpha,x_1}}{\partial x_1^j}dx-\int_{B(x_1,\delta)}(PU_{\lambda_1\varepsilon^\alpha,x_1})^p R_1^j(x)dx \tag{6.15}$$

Now setting $\phi_{\lambda_1\varepsilon^\alpha,x_1}=U_{\lambda_1\varepsilon^\alpha,x_1}-PU_{\lambda_1\varepsilon^\alpha,x_1}$ we have

$$\int_{B(x_1,\delta)}(PU_{\lambda_1\varepsilon^\alpha,x_1})^p\frac{\partial U_{\lambda_1\varepsilon^\alpha,x_1}}{\partial x_1^j}dx$$

$$= \int_{B(x_1,\delta)}(U_{\lambda_1\varepsilon^\alpha,x_1}-\phi_{\lambda_1\varepsilon^\alpha,x_1})^p\frac{\partial U_{\lambda_1\varepsilon^\alpha,x_1}}{\partial x_1^j}dx$$

$$= \int_{B(x_1,\delta)}\left[(U_{\lambda_1\varepsilon^\alpha,x_1}-\phi_{\lambda_1\varepsilon^\alpha,x_1})^p-U^p_{\lambda_1\varepsilon^\alpha,x_1}\right]\frac{\partial U_{\lambda_1\varepsilon^\alpha,x_1}}{\partial x_1^j}dx$$

$$+\int_{B(x_1,\delta)}U^p_{\lambda_1\varepsilon^\alpha,x_1}\frac{\partial U_{\lambda_1\varepsilon^\alpha,x_1}}{\partial x_1^j}dx$$

$$= p \int\limits_{B(x_1,\delta)} \left(U_{\lambda_1\varepsilon^\alpha,x_1} - t(x)\phi_{\lambda_1\varepsilon^\alpha,x_1}\right)^{p-1} \phi_{\lambda_1\varepsilon^\alpha,x_1} \frac{\partial U_{\lambda_1\varepsilon^\alpha,x_1}}{\partial x_1^j} dx$$

$$+ \int\limits_{\mathbb{R}^N} U^p(z) \frac{\partial U}{\partial z^j}(z) dz + o\left(\varepsilon^{\alpha(N-2)}\right)$$

$$= - \int\limits_{B(x_1,\delta)} \frac{\partial \phi_{\lambda_1\varepsilon^\alpha,x_1}}{\partial x_1^j} U^p_{\lambda_1\varepsilon^\alpha,x_1} dx + o\left(\varepsilon^{\alpha(N-2)}\right)$$

$$= -\lambda_1^{N-2}\varepsilon^{\alpha(N-2)} A^2 \frac{\partial H}{\partial x_1^j}(x_1,x_1) + o\left(\varepsilon^{\alpha(N-2)}\right). \tag{6.16}$$

On the other hand we have

$$\int\limits_{B(x_1,\delta)} (PU_{\lambda_1\varepsilon^\alpha,x_1})^p R_1^j(x) dx$$

$$= \lambda_1^{N-2}\varepsilon^{\alpha(N-2)} A^2 \frac{\partial H}{\partial x_1^j}(x_1,x_1) + o\left(\varepsilon^{\alpha(N-2)}\right). \tag{6.17}$$

By (6.15), (6.16) and (6.17) we get

$$\int\limits_{B(x_1,\delta)} (PU_{\lambda_1\varepsilon^\alpha,x_1})^p P\hat{\psi}_1^j(x) dz$$

$$= -2\lambda_1^{N-2}\varepsilon^{\alpha(N-2)} A^2 \frac{\partial H}{\partial x_1^j}(x_1,x_1) + o\left(\varepsilon^{\alpha(N-2)}\right). \tag{6.18}$$

If $j = 0$ and $h = 1$ we write

$$\int\limits_{\Omega_\varepsilon} \left(\sum_{i=1}^k P_\varepsilon U_i(x)\right)^p P_\varepsilon \psi_1^0(x) dx$$

$$= \varepsilon^\alpha \int\limits_{\Omega} \left(\sum_{i=1}^k PU_{\lambda_i\varepsilon^\alpha,x_i}(z)\right)^p P\hat{\psi}_1^0(z) dz$$

$$= \varepsilon^\alpha \int\limits_{B(x_1,\delta)} \left[\left(\sum_{i=1}^k PU_{\lambda_i\varepsilon^\alpha,x_i}(z)\right)^p - PU^p_{\lambda_1\varepsilon^\alpha,x_1}(z)\right] P\hat{\psi}_1^0(z) dz$$

$$+\varepsilon^\alpha \int\limits_{B(x_1,\delta)} PU^p_{\lambda_1\varepsilon^\alpha,x_1}(z) P\hat{\psi}_1^0(z) dz$$

$$+\varepsilon^\alpha \sum_{l=2}^k \int\limits_{B(x_l,\delta)} \left(\sum_{i=1}^k PU_{\lambda_i\varepsilon^\alpha,x_i}(z)\right)^p P\hat{\psi}_1^0(z) dz$$

$$+\varepsilon^\alpha \int\limits_{\Omega\setminus \bigcup_{l=1}^k B(x_l,\delta)} \left(\sum_{i=1}^k PU_{\lambda_i\varepsilon^\alpha,x_i}(z)\right)^p P\hat{\psi}_1^0(z) dz. \tag{6.19}$$

Firstly arguing as in the proof of (6.12), we have for any $j = 1, \ldots, k$

$$\left| \int\limits_{\Omega \setminus \bigcup_{l=1}^{k} B(x_l,\delta)} \left(\sum_{i=1}^{k} PU_{\lambda_i \varepsilon^\alpha, x_i}(z) \right)^p P\hat{\psi}_1^0(z) dz \right|$$
$$\leq C \int\limits_{\Omega} \varepsilon^{\alpha \frac{N+2}{2}} \varepsilon^{\alpha \frac{N-4}{2}} dx$$
$$\leq C \varepsilon^{\alpha(N-1)}. \tag{6.20}$$

Secondly, arguing as in the proof of (6.13), we get

$$\int\limits_{B(x_1,\delta)} \left[\left(\sum_{l=1}^{k} PU_{\lambda_l \varepsilon^\alpha, x_l}(x) \right)^p - PU^p_{\lambda_1 \varepsilon^\alpha, x_1}(x) \right] P\hat{\psi}_1^0(x) dx$$
$$= A^2 \frac{N-2}{2} \varepsilon^{\alpha(N-3)} \lambda_1^{\frac{N}{2}-2} \sum_{l=2}^{k} \lambda_l^{\frac{N-2}{2}} G(x_1, x_l) + o\left(\varepsilon^{\alpha(N-3)}\right). \tag{6.21}$$

Moreover, arguing as in the proof of (6.14), we get for any $l \neq 1$

$$\int\limits_{B(x_l,\delta)} \left(\sum_{l=1}^{k} PU_{\lambda_l \varepsilon^\alpha, x_l}(z) \right)^p P\hat{\psi}_1^0(z) dz$$
$$= \frac{N-2}{2} A^2 \varepsilon^{\alpha(N-3)} \lambda_1^{\frac{N}{2}-2} \lambda_l^{\frac{N-2}{2}} G(x_l, x_1) + o\left(\varepsilon^{\alpha(N-3)}\right). \tag{6.22}$$

Moreover, arguing as in the proof of (6.18), we get

$$\int\limits_{B(x_1,\delta)} (PU_{\lambda_1 \varepsilon^\alpha, x_1})^p P\hat{\psi}_1^0(x) dz$$
$$= -\frac{N-2}{2} A^2 \lambda_1^{N-3} \varepsilon^{\alpha(N-3)} H(x_1, x_1) + o\left(\varepsilon^{\alpha(N-3)}\right). \tag{6.23}$$

□

Lemma 6.8 *If $j = 1, \ldots, N$ and $i, h = 1, \ldots, k$, then*

$$\int\limits_{\Omega_\varepsilon} P_\varepsilon U_i P_\varepsilon \psi_h^j = o\left(\varepsilon^\alpha\right) \tag{6.24}$$

as $\varepsilon \to 0$ uniformly with respect to $(\lambda, x) \in \mathcal{O}_\delta$.

Moreover if $j = 0$ and $i \neq h$ then

$$\int\limits_{\Omega_\varepsilon} P_\varepsilon U_i P_\varepsilon \psi_h^0 = o\left(1\right) \tag{6.25}$$

and if $i = h$ then

$$\int\limits_{\Omega_\varepsilon} P_\varepsilon U_i P_\varepsilon \psi_i^0 = \lambda_i B + o\,(1) \tag{6.26}$$

as $\varepsilon \to 0$ uniformly with respect to $(\lambda, x) \in \mathcal{O}_\delta$.

Proof. In the following we will always use estimate (5.1), Lemma 5.1, estimate (5.1), Lemma 6.1 and Lemma 6.3.

Let $j = 0$ and $i = h$. We have

$$\begin{aligned}
&\int\limits_{\Omega_\varepsilon} P_\varepsilon U_i(x) P_\varepsilon \psi_i^0(x) dx \quad (\text{set } x = z/\varepsilon^\alpha, \quad \text{use (5.1) and Lemma 6.1}) \\
&= \varepsilon^{-\alpha} \int\limits_{\Omega} PU_{\lambda_i \varepsilon^\alpha, x_i}(z) P\psi^0_{\lambda_i \varepsilon^\alpha, x_i}(z) dz \\
&= \varepsilon^{-\alpha} \left[(\lambda_i \varepsilon^\alpha)^{N-3} \int\limits_{\Omega} \frac{|z - x_i|^2 - (\lambda_i \varepsilon^\alpha)^2}{\left((\lambda_i \varepsilon^\alpha)^2 + |z - x_i|^2\right)^{N-1}} dz + o\left(\varepsilon^{\alpha(N-3)}\right) \right] \\
&\quad (\text{set } z = x_i + \lambda_i \varepsilon^\alpha w) \\
&= \varepsilon^{-\alpha} \left[(\lambda_i \varepsilon^\alpha) \int\limits_{\frac{\Omega - x_i}{\lambda_i \varepsilon^\alpha}} \frac{|w|^2 - 1}{(1 + |w|^2)^{N-1}} dw + o\,(\varepsilon^\alpha) \right] \\
&= \lambda_i B + o\,(1)\,.
\end{aligned}$$

because, arguing exactly as in the proof of Remark 6.2, we can prove that $B = \int\limits_{\mathbb{R}^N} \frac{|w|^2-1}{(1+|w|^2)^{N-1}} dw$.

Let $j = 1, \ldots, N$ and $i = h$. We have

$$\begin{aligned}
&\int\limits_{\Omega_\varepsilon} P_\varepsilon U_i(x) P_\varepsilon \psi_i^j(x) dx \quad (\text{set } x = z/\varepsilon^\alpha, \quad \text{use (5.1) and (5.1)}) \\
&= \varepsilon^{-\alpha} \int\limits_{\Omega} PU_{\lambda_i \varepsilon^\alpha, x_i}(z) P\psi^j_{\lambda_i \varepsilon^\alpha, x_i}(z) dz \\
&= \varepsilon^{-\alpha} \left[(\lambda_i \varepsilon^\alpha)^{N-1} \int\limits_{\Omega} \frac{z_j - x_{ij}}{\left((\lambda_i \varepsilon^\alpha)^2 + |z - x_i|^2\right)^{N-1}} dz + o\left(\varepsilon^{\alpha(N-1)}\right) \right] \\
&\quad (\text{set } z = x_i + \lambda_i \varepsilon^\alpha w) \\
&= \varepsilon^{-\alpha} \left[(\lambda_i \varepsilon^\alpha)^2 \int\limits_{\frac{\Omega - x_i}{\lambda_i \varepsilon^\alpha}} \frac{w_j}{(1 + |w|^2)^{N-1}} dw + o\left(\varepsilon^{2\alpha}\right) \right] \\
&= o\,(\varepsilon^\alpha)\,.
\end{aligned}$$

In a analogous way we can prove that if $i \neq h$

$$\int\limits_{\Omega_\varepsilon} P_\varepsilon U_i P_\varepsilon \psi_h^0 = o\,(1)\,.$$

and if $j \neq 0$

$$\int\limits_{\Omega_\varepsilon} P_\varepsilon U_i P_\varepsilon \psi_h^j = o\,(\varepsilon^\alpha)\,.$$

$\square$

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