AROUND VISCOSITY SOLUTIONS FOR A CLASS OF SUPERLINEAR SECOND ORDER ELLIPTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. In this article we review some recent results of the existence of solution for nonlinear elliptic equations in $\mathbb{R}^N$ having the form

$$-F(D^2u, \nabla u) + |u|^{s-1}u = f(x) \qquad \text{in} \quad I\!R^N,$$

where F is a positively homogeneous function of degree $d > 0$, $s > d$ and f is a given function. In the analysis of this problem one rises some questions about which is the most suitable notion of solutions for its study. Various types of solutions are discussed and open questions are stated.

1. INTRODUCTION

In this article we review some recent results of the existence of solution for nonlinear elliptic equations in $\mathbb{R}^N$ having the form

$$-F(D^2u, \nabla u) + |u|^{s-1}u = f(x) \qquad \text{in} \quad I\!R^N, \tag{1.1}$$

where F is a positively homogeneous function of degree d, $s > d$ and f is a given function. In the analysis of this problem one rises some questions about which is the most suitable notion of solutions for its study. Having as a starting point the notion of viscosity solutions, we think that this problem suggests various directions of research that could lead to new notions of solutions or understanding the relations between known ones.

When F is the Laplace operator the problem was studied by Brezis in [9]. He showed that one can find a (unique) solution to (1.1) assuming only local integrability of f. This very weak assumption is enough when the nonlinearity is increasing and super-linear, as in the case of $|u|^{s-1}u$ with $s > 1$. This result was extended to the case of a general quasilinear operator, including the p-Laplace operator by Boccardo, Gallouet and Vázquez in [8]. See also the work by Leoni in [24] where more general nonlinearities are considered.

In the general case, when F has degree of homogeneity $d = 1$, an important class of operators considered is this discussion is the Pucci operator $\mathcal{M}^+_{\lambda,\Lambda}$, with $0<\lambda\le\Lambda$, which is defined by

$$\mathcal{M}^+_{\lambda,\Lambda}(X)=\Lambda\sum_{e_i>0}e_i+\lambda\sum_{e_i<0}e_i,$$

where $e_i=e_i(X)$ are the eigenvalues of the symmetric matrix X. Since no confusion arises we simply write $\mathcal{M}^+=\mathcal{M}^+_{\lambda,\Lambda}$. The first problem we present is a radially symmetric version of the equation

$$-\mathcal{M}^+(D^2u)+|u|^{s-1}u=f(x)\qquad \text{in}\quad I\!R^N, \tag{1.2}$$

studied by Esteban and the authors in [19].

Theorem 1.1. *Assume $s>1$ and f is a radially symmetric function satisfying*

$$\int_0^R r^{N_+-1}|f(r)|dr<\infty, \tag{1.3}$$

for all $R>0$. Here $N_+:=\frac{\lambda}{\Lambda}(N-1)+1$, with λ and Λ being the parameters defining the Pucci operator. Then equation (1.2) has a unique weak radially symmetric solution and if f is nonnegative then u is also nonnegative.

In this theorem the notion of weak radially symmetric solution, given in Definition 2.1 is used. In defining this notion, advantage is taken from certain variational formulation of the problem which is possible when we restrict the study to radially symmetric functions.

In the case a general function f and considering the notion of L^N-viscosity solutions introduced by Caffarelli, Crandall, Kocan & Świech in [13] (see Definition 4.1), the following theorem was proved in [19].

Theorem 1.2. *Assume that $s>1$. For every function $f\in L^N_{loc}(I\!R^N)$, the equation (1.2) possesses a unique solution in the L^N-viscosity sense and if $f\ge 0$ a.e. then $u(x)\ge 0$ for all $x\in I\!R^N$.*

In a series of papers, Birindelli and Demengel initiated the study of a class of fully nonlinear operators which correspond to a fully nonlinear version of the p-Laplacian, see [3], [4], [5], [6] and [7]. As a representative of this class of equations we may consider the following

$$-|\nabla u|^\alpha\mathcal{M}^+(D^2u)+u|u|^{\alpha-1}=f(x)\ \text{ in }\quad \mathbb{R}^N, \tag{1.4}$$

where $\alpha\in(-1,+\infty)$, corresponding to an operator of degree $d=\alpha+1$. Since this problem is singular or degenerate, depending on the value of α, a special notion of solution is needed. This notion is called S-viscosity solution and was introduced in [3] for continuous function f, extending the usual notion of C-viscosity solution as defined in [14], see Definitions 5.1 and 4.2, respectively.

In a recent paper, Dávila and the authors in [18] obtained a Harnack inequality for a class of singular fully nonlinear equations, see Theorem 5.1. When this results is applied to (1.4) it says precisely that for $\alpha \in (-1, 0)$, if $u \in C(\mathbb{R}^N)$ is a non-negative S-viscosity solution of (1.4) then for every compact Ω'

$$\sup_{\Omega'} u \leq C \left\{ \inf_{\Omega'} u + \|f\|_{N,\Omega'}^{\frac{1}{\alpha+1}} \right\}, \tag{1.5}$$

where the constant C depends on $\lambda, \Lambda, N, \Omega'$ and α. Based on this result, and taking advantage of some ideas used in proving this inequality, the superlinear problem studied above was undertaken for this class of singular operators.

Theorem 1.3. *Under the conditions stated above, for every $f \in C(\mathbb{R}^n)$, equation (1.4) possesses at least one S-viscosity solution.*

If we examine Harnack Inequality (1.5), we observe that the right hand side contains the L^N-norm of the function f. This fact can be used to extend the notion of solutions to equation (1.4) when $f \in L^N_{loc}(\Omega)$, defining good solutions, see Definition 6.1. Then we can prove the following theorem

Theorem 1.4. *Assume $\alpha \in (-1, 0)$ and $s > \alpha + 1$. Then for every $f \in L^N_{loc}(\mathbb{R}^N)$, equation (1.4) possesses at least one good solution.*

The main portion of this paper is devoted to review and discuss the theorems just stated, leaving Section §7 to set up some open questions and discuss some lines of research that are interesting to undertake.

In Section §2 we define the notion of weak radial solution for fully nonlinear equations involving the Pucci operator. Then in Section §3 we sketch the proof of the existence part of Theorem 1.1. In Section §4 we discuss the proof of Theorem 1.2, in the existence part. Section §5 is devoted to introduce the study of non-uniformly fully nonlinear elliptic operators modelled on the p-laplacian, dicussing the notion of solution and presenting Harnack inequality for the singular case. Then in Section §6 we see how to prove Theorem 1.3 presenting a brief sketch. We also introduce here the notion of good solution and we see how to prove Theorem 1.4. Finally, in Section §7 we set up some open problems, research lines and we make some concluding remarks.

2. Radial Solutions for some Fully Nonlinear equations

In this section we define a notion of weak solution for equation (1.2) in the radially symmetric case, taking advantage of a variational formulation of the problem.

When u is a radially symmetric function then the Pucci operator has a much simpler form. In fact, since the eigenvalues of D^2u are u'' and u'/r with multiplicity 1 and $N-1$, respectively, defining $\theta(s) = \Lambda$ if $s \geq 0$ and $\theta(s) = \lambda$ if $s < 0$, we easily see that

$$\mathcal{M}^+(D^2u)(r) = \theta(u''(r))u''(r) + \theta(u'(r))(N-1)\frac{u'(r)}{r}.$$

Then, for a radial function f, equation (1.2) becomes

$$-\theta(u''(r))u'' - \theta(u'(r))(N-1)\frac{u'}{r} + |u|^{s-1}u = f(r). \tag{2.1}$$

In order to write this equation in a variational form, we make some definitions. First we observe that for solutions of (2.1) we have

$$\theta(u''(r)) = \theta\{-\theta(u'(r))(N-1)\frac{u'}{r} + |u|^{s-1}u - f(r)\},$$

which is more convenient as we will see. We define

$$\Theta(r, u(r), u'(r)) = \theta\{-\theta(u'(r))(N-1)\frac{u'}{r} + |u|^{s-1}u - f(r))\},$$

the variable dimension

$$N(r, u(r), u'(r)) = \frac{\theta(u'(r))}{\Theta(r, u(r), u'(r))}(N-1) + 1$$

and the weights

$$\rho(r, u(r), u'(r)) = \exp\left(\int_1^r \frac{N(\tau, u(\tau), u'(\tau)) - 1}{\tau} d\tau\right)$$

and

$$\tilde{\rho}(r, u(r), u'(r)) = \frac{\rho(r, u(r), u'(r))}{\Theta(r, u(r), u'(r))}.$$

We observe that $\lambda \leq \rho/\tilde{\rho} \leq \Lambda$. If we define the dimension numbers $N_+ = \lambda(N-1)/\Lambda + 1$ and $N_- = \Lambda(N-1)/\lambda + 1$, we see that $N_+ \leq N(r, u(r), u'(r)) \leq N_-$ and also,

$$r^{N_- - 1} \leq \rho(r, u(r), u'(r)) \leq r^{N_+ - 1} \text{ if } 0 \leq r \leq 1.$$

With these definitions we find that (2.1) is equivalent to

$$-(\rho u')' + \tilde{\rho}|u|^{s-1}u = \tilde{\rho}f(r). \tag{2.2}$$

When no confusion arises we omit the arguments in the functions ρ and $\tilde{\rho}$, in particular when we write $\rho v'$ we mean $\rho(r, v(r), v'(r))v'(r)$ and so on. What is interesting about equation (2.2) is that it allows to define a notion of weak solution which extends the L^N-viscosity sense to more general f.

We consider the set of test functions defined as

$$H = \{\varphi : [0,\infty) \to I\!R \;/\; \exists\, \phi \in W_0^{1,\infty}(I\!R^N) \text{ such that } \phi(x) = \varphi(|x|)\},$$

where $W_0^{1,\infty}(I\!R^N)$ denotes the space of functions in $W^{1,\infty}(I\!R^N)$ with compact support.

Definition 2.1. *We say that $u : [0,R] \to I\!R$ is a weak radially symmetric solution of (2.2) with Dirichlet boundary condition at $r = R$, if u is absolutely continuous in $(0,R]$, $u(R) = 0$,*

$$\int_0^R \rho |u|^s dr < \infty, \quad \int_0^R \rho |u'| dr < \infty \tag{2.3}$$

and

$$\int_0^R \rho u'\varphi' + \tilde{\rho} |u|^{s-1} u \varphi dr = \int_0^R \tilde{\rho} f \varphi dr \quad \forall \varphi \in H. \tag{2.4}$$

This definition is weaker than the L^N-viscosity one since it permits less regularity on the right hand side and it provides a good framework to study existence of solutions of nonlinear equations in the radial case. It poses the question about the possibility of defining a weaker version of viscosity solution as we discuss in Question 1, Section §7.

3. Application to radial solutions to superlinear equation in $\mathbb{R}^N$ without growth on the data

The goal in this section is to sketch the proof of the existence part of Theorem 1.1. First we state a theorem, which is a more complete version of Theorem 1.1.

Theorem 3.1. *Assume $s > 1$ and f is a radial function satisfying (1.3), for all $R > 0$. Then equation (2.2) has a unique weak radially symmetric solution u and if f is nonnegative then u is also nonnegative. Additionally, for any $1 < q < 2s/(s+1)$*

$$\int_0^r \rho |u'|^q dr < \infty \quad \text{for all} \quad R > 0. \tag{3.1}$$

Moreover, $\rho u'$ is differentiable a.e. in $(0,\infty)$ and it satisfies

$$\lim_{r\to 0} (\rho u')(r) = 0, \quad \lim_{r \to 0} \int_0^r \rho |u'| dr = 0. \tag{3.2}$$

If we consider a continuous function f in $I\!R^N \setminus \{x_i \,/\, i = 1, ..., k\}$, such that near each singularity

$$f(x) \sim \frac{c_i}{|x - x_i|^{\alpha_i}}, \quad x \sim x_i, \; i = 1, ..., k.$$

In order to have $f \in L^N_{loc}(\mathbb{R}^N)$ we need $\alpha_i < 1$ for all $i = 1, ..., k$. In contrast, assuming that f is radially symmetric with a singularity at the origin of the form

$$f(r) \sim \frac{c}{r^\alpha}, \quad r \sim 0, r > 0,$$

in order to have condition (1.3) we only need $\alpha < N_+$. In particular, we see that if $\lambda/\Lambda \to 0$ then $N_+ \to 1$, while if $\lambda/\Lambda = 1$ then $N_+ = N$. When we have a radial function f being in $L^p_{loc}(I\!R^N)$ with $p > N/N_+$ then f satisfies our hypothesis (1.3) and we may apply Theorem 3.1. This is particularly interesting if N and N_+ are close to each other.

In order to prove Theorem 3.1 we use an approximation procedure. Let $\{f_n\}$ be a sequence of smooth functions such that for all $0 < R$

$$\lim_{n\to\infty} \int_0^R r^{N_+-1} |f_n(r) - f(r)| dr = 0. \tag{3.3}$$

We may assume that there exists a function $g : (0, \infty) \to I\!R$ such that $|f_n(r)| \le g(r)$ for all $r > 0$ and $\int_0^R r^{N_+-1} |g(r)| dr < +\infty$, for all $R > 0$. Using a general existence theorem together with a symmetry result we can prove the following existence lemma for the approximate problems. See Lemma 2.1 in [19] and [16].

Lemma 3.1. *For every n there is solution u_n in $C^2[0, n]$ satisfying $u_n(n) = 0$, (2.3) with $R = n$ and*

$$\int_0^n \rho_n u_n' \varphi' + \tilde\rho_n |u_n|^{s-1} u_n \varphi = \int_0^n \rho_n f_n \varphi, \tag{3.4}$$

for all $\varphi \in H$, where $\rho_n(r) = \rho(r, u_n(r), u_n'(r))$ (similarly for $\tilde\rho_n$).

Now we present some estimates that will allow us to pass to the limit. These estimates are obtained following the ideas of Boccardo, Gallouet and Vazquez in [8] and use the divergence form of the equation.

Lemma 3.2. *Let $\{u_n\}$ be the sequence of solutions found in Lemma 3.1. Then, for all $0 < R$ and $m \in (0, s - 1)$ there is a constant C depending on R, m, s, N, λ and Λ, such that for all $n \in \mathbb{N}$ we have*

$$\int_0^R \rho_n |u_n|^s ds \le C(1 + \int_0^{2R} r^{N_+-1} |f| dr) \tag{3.5}$$

and for all $q \in (1, 2s/(s + 1))$

$$\int_0^R \rho_n |u_n'|^q dr \le C(1 + \int_0^{2R} r^{N_+-1} |f| dr). \tag{3.6}$$

Proof of Theorem 3.1 (Sketch of the existence part). Let $\{u_n\}$ be the solutions found in 3.1. From Lemma 3.2, we see that the function $\rho_n u_n'$ has weak derivatives in any interval of the form (r_0, R_0) with

$0 < r_0 < R_0$. Since the function ρ_n is differentiable a.e., we obtain then that u_n is twice differentiable a.e. and u_n'' is in $L^1(r_0, R_0)$. From here we conclude that u_n' and u_n are uniformly bounded in (r_0, R_0). By the Arzela-Ascoli Theorem there exists a differentiable function u in the interval (r_0, R_0) such that, up to a subsequence, u_n and u_n' converges to u and u' respectively, in a uniform way in the interval (r_0, R_0).

Repeating the argument in any interval and using a diagonal procedure we can prove that $\{u_n\}$ and $\{u_n'\}$ converge point-wise to a differentiable function $u : (0, \infty) \to I\!R$. Notice that $\{\rho_n\}$ converges point-wise to $\rho(r) = \rho(r, u(r), u'(r))$.

Next we use the estimate (3.6), to prove that the sequence $\{\rho_n u_n'\}$ is equi-integrable in $[0, R]$ and then it converges in $L^1[0, R]$ to $\rho u'$, for all $R > 0$. It is only left to prove that $\{\tilde{\rho}_n |u_n|^s\}$ converges in $L^1[0, R]$. For this purpose we introduce, as in [8], a new function ϕ in $I\!R$ defined as $\phi(\nu) = \min\{\nu - t, 1\}$ if $\nu \geq 0$ and extended as an odd function to all $I\!R$, for a parameter $t > 0$. Testing the equation with cut-off function $\phi(u_n)\theta$ we get

$$\int_{E_n^{t+1} \cap (0,R)} \tilde{\rho}_n |u_n|^s dr \leq \int_{E_n^t \cap (0,2R)} \tilde{\rho}_n |f_n| dr + C \int_{E_n^t \cap (0,2R)} \rho_n |u_n'| dr,$$

where $E_n^t = \{r > 0 \,/\, |u_n(r)| > t\}$. From (3.5) and (3.6) it follows that the second integral approaches zero if $t \to \infty$. From here the equi-integrability of $\rho_n |u_n|^s$ follows and we conclude.

Finally (3.2) is consequence of the integrability properties just proved for u_n that also hold for u. This finishes the proof. □

We do not discuss in detail the the proof of uniqueness and nonnegativity of weak solutions in Theorem 3.1. One needs to use a comparison argument which is bit delicate in this case. In a natural way we may define the notion of weak subsolutions (supersolution) by writing $\leq$ and use only nonnegative test functions in (2.4). It happen that, if u is a weak subsolution and v is a weak supersolution, we cannot be sure that $w = u - v$ is a weak subsolution, since we do not have good control of $\rho w'$ at the origin.

Remark 3.1. *Given a f a function in $I\!R^N$ we define*

$$g(r) = \max\{|f(x)| \,/\, |x| = r\}$$

and assume that g satisfies (1.3). Then we may construct a solution of (2.2). This solution would be a 'candidate' for a supersolution of equation (1.2), however this is not so since the two notion of solutions are not compatible.

4. General superlinear equation in $\mathbb{R}^N$ without growth assumption on the data

In this section we study the solvability of the differential equation (1.2) or the more general version

$$-F(D^2u) + |u|^{s-1}u = f(x) \qquad \text{in} \quad I\!R^N, \tag{4.1}$$

when F is a fully nonlinear, uniformly elliptic operator, $s > 1$ and for f having only local integrability properties, but without assuming any growth condition at infinity.

On the operator F we assume uniform ellipticity, that is:

$$M^-_{\lambda,\Lambda}(M - N) \leq F(M) - F(N) \leq \mathcal{M}^+_{\lambda,\Lambda}(M - N) \text{ for all } N, M \in S_N,$$

and $F(0) = 0$. Here S_N denotes the set of $N \times N$ symmetric matrices. In order to find a solution to (4.1), we have to work in the viscosity solution framework so we cannot use test functions and integration by parts to derive *a priori* estimates. In the case of the Laplacian, solution exists asumming that $f \in L^1_{loc}(R^N)$, however in this more general case we need to assume that $f \in L^N_{loc}(I\!R^N)$, because we only known local estimates in this case and because there is no $L^1_{loc}(R^N)$ theory of viscosity solutions.

We start recalling the notion of solution suitable when the right hand side in (4.1) is in $L^p_{loc}(I\!R^N)$. Following the work by Caffarelli, Crandall, Kocan and Swiech [13], we notice that the framework requires $p > N - \varepsilon_0$, where $\varepsilon_0 > 0$ depends on the ellipticity constants λ and Λ. Thus the case $p = N$, which will be our framework, is covered by the theory. According to [13] we have the following definition:

Definition 4.1. *Assume that $f \in L^p_{loc}(I\!R^N)$, then we say that a continuous function $u : I\!R^N \to I\!R$ is an L^p-viscosity subsolution (supersolution) of the equation (4.1) in $I\!R^N$ if for all $\varphi \in W^{2,p}_{loc}(I\!R^N)$ and $\hat{x} \in I\!R^N$ at which $u - \varphi$ has a local maximum (respectively, minimum) one has*

$$\operatorname{ess}\liminf_{x\to\hat{x}}(-F(D^2\varphi(x)) + |\varphi(x)|^{s-1}\varphi(x) - f(x)) \leq 0 \tag{4.2}$$

$$(\operatorname{ess}\limsup_{x\to\hat{x}}(-F(D^2\varphi(x)) + |\varphi(x)|^{s-1}\varphi(x) - f(x)) \geq 0). \tag{4.3}$$

Moreover, u is an L^p-viscosity solution of (4.1) if it is both and L^p-viscosity subsolution and an L^p-viscosity supersolution.

This notion extends the notion of C-viscosity (sub or super) solution of (4.1) replacing the tests function space $C^2(I\!R^N)$ by $\varphi \in W^{2,p}_{loc}(I\!R^N)$:

Definition 4.2. *Assume that $f \in C(I\!R^N)$, then we say that a continuous function $u : I\!R^N \to I\!R$ is a C-viscosity solution subsolution* *(supersolution) of the equation (4.1) in* $I\!R^N$ *if for all* $\varphi \in C^2(I\!R^N)$ *and* $\hat{x} \in I\!R^N$ *at which* $u - \varphi$ *has a local maximum (respectively, minimum) one has*

$$-F(D^2\varphi(\hat{x})) + |\varphi(\hat{x})|^{s-1}\varphi(\hat{x}) - f(\hat{x}) \leq 0 \tag{4.4}$$

$$-F(D^2\varphi(\hat{x})) + |\varphi(\hat{x})|^{s-1}\varphi(\hat{x}) - f(\hat{x}) \geq 0. \tag{4.5}$$

The proof of Theorem 1.2 is done by an approximation procedure together with a local estimate based on a appropriate cut-off of the solution and the application of the Alexandroff-Bakelman-Pucci inequality. Given $f \in L^N_{loc}(I\!R^N)$ we assume $\{f_n\}$ is a sequence of $C^\infty(I\!R^N)$ functions so that for every bounded set Ω

$$\lim_{n\to\infty} \int_\Omega |f_n - f|^N dx = 0. \tag{4.6}$$

The following is a basic existence and regularity result we need in our construction of a solution to (4.1).

Lemma 4.1. *For every* $n \in \mathbb{N}$ *there is a solution* $u_n \in C^1(B_n)$ *of the equation*

$$-F(D^2u_n) + |u_n|^{s-1}u_n \quad = \quad f_n(x) \quad in \quad B_n \tag{4.7}$$

where $B_n = B(0,n)$ *is the ball centered at* 0 *and with radius* n*.*

Now we consider a crucial local estimate for solutions of (4.7). This result was proved by Brezis [9] in the context of the Laplacian, see also [8], and its proof is based on the use suitable test functions and integration by parts. This cannot be done here since the differential operator does not have divergence form. For this result the fact that $s > 1$ is essential.

Lemma 4.2. *Let* $s > 1$ *and* g *continuous in* $I\!R^N$*. Suppose that* $g \geq 0$ *and* u *is a* C^1 *nonnegative* C*-viscosity solution of*

$$-\mathcal{M}^+(D^2u) + |u|^{s-1}u \leq g \qquad in \quad \Omega,$$

then for all $R > 0$ *and* $R' > R$

$$\sup_{B_R} u \leq C(1 + \|g\|_{L^N(B_{R'})}), \tag{4.8}$$

where $C = C(s, R, R', N, \lambda, \Lambda)$ *does not depend on* g *nor* n*.*

Sketch of the Proof. Consider the functions $\xi(x) = (R')^2 - |x|^2$, $\beta = 2/(s-1)$ and $v = \xi^\beta u$. Suppose that $v - \varphi$ has a local maximum, $v(\hat{x}) = \varphi(\hat{x})$, $Dv(\hat{x}) = D\varphi(\hat{x})$ and $\varphi \in C^2$. Then $u - \xi^{-\beta}\varphi$ has a local maximum at $\hat{x}$. Therefore $\xi^{-\beta}\varphi$ is a test function for u and so

$$-\mathcal{M}^+(D^2\varphi) + \xi^{-2}|\varphi|^{s-1}\varphi \leq \xi^\beta g + I + II + III,$$

$$\begin{aligned} I &:= -\beta\xi^{-1}v\mathcal{M}^-(D^2\xi) \\ II &:= \beta(\beta+1)\xi^{-2}v\mathcal{M}^+(D\xi\otimes D\xi) \\ III &:= -\beta\xi^{-1}\mathcal{M}^-(D\xi\otimes D\varphi + D\varphi\otimes D\xi) \end{aligned}$$

So v satisfies the equation

$$-\mathcal{M}^+(D^2v) + v|v|^{s-1} \le \xi^\beta g + I + II + III \tag{4.9}$$

in $B(R')$ in the C-viscosity sense. Here in I, II and III we replace $D\varphi$ by Dv. Consider the contact set for the function v, which is defined as

$$\Gamma_v^+ = \{x \in B_{R'} \,/\, \exists\, p \in \mathbb{R}^N \text{ with } v(y) \le v(x) + \langle p, y-x\rangle,\ \forall y \in B_{R'}\}.$$

We observe that $\Gamma_v^+ \subset \Omega^+ \cap B_{R'}$ and that if $\bar v$ is the concave envelope of v in $\overline{B}_{R'}$ then for $x \in B_{R'}$ we have $v(x) = \bar v(x)$ if and only if $x \in \Gamma_v^+$. Here $\Omega^+ = \{x \in \Omega \,/\, v(x) > 0\}$. The function $\bar v$, being concave, satisfies

$$|Dv(x)| \le \frac{v(x)}{R' - |x|}, \qquad \text{for all} \quad x \in \Gamma_v^+. \tag{4.10}$$

Then we prove that the function v satisfies

$$-\mathcal{M}^+(D^2v) + \xi^{-2}v(|v|^{s-1} - C) \le \xi^\beta g \qquad \text{for all} \quad x \in \Gamma_v^+.$$

Now we define $w = \max\{v - C^{1/(p-1)}, 0\}$ in $B_{R'}$ and we observe that $\Gamma_w^+ \subset \Gamma_v^+$ and $\Gamma_w^+ \subset \{x \in B_{R'} \,/\, w > 0\}$. Consequently

$$-\mathcal{M}^+(D^2w) \le \xi^\beta g, \qquad \text{a.e in} \quad \Gamma_w^+.$$

Thus, from Alexandroff-Bakelman-Pucci inequality

$$\sup_{B_{R'}} w \le C\|\xi^\beta g\|_{L^N(B_{R'})},$$

but then

$$c\sup_{B_R} u \le \sup_{B_{R'}} v \le \sup_{B_{R'}} w + C^{1/(p-1)} \le C(1 + \|g\|_{L^N(B_{R'})}),$$

where c and C represent generic constants depending on the desired quantities. $\square$

Proof of Theorem 1.2 (Existence) We use Lemma 4.1 to construct a sequence of solutions $\{u_n\}$ of equation (4.7). We can prove that u_n satisfies the hypothesis of Lemma 4.2, so that for every $0 < R < R' < n$ we have

$$\sup_{B_R} |u_n| \le C(1 + \|f\|_{L^N(B'_R)}),$$

where C does not depend on f nor in n. With this inequality in hand we look at equation (4.7) and use Proposition 4.10 in [14] to obtain, for every bounded open set Ω,

$$\|u_n\|_{C^\alpha(\Omega)} \le C,$$

where C does not depend on n, but only on f, Ω and the other parameters. Then we obtain a subsequence satisfying

$$-F(D^2u_n) + |u_n|^{s-1}u_n = f_n,$$

and converging uniformly over every bounded subset of $I\!R^N$. Then using Theorem 3.8 in [13] we conclude that u is an L^N-viscosity solution of (4.1), completing the proof of existence in Theorem 1.2. □

5. Singular operators and Harnack inequality

In this section we discuss Harnack inequality for positive solutions a class of singular fully nonlinear elliptic equations of the form

$$-F\left(\nabla u, D^2u\right) + u\,|u|^{\alpha} = f \quad \text{in} \quad \Omega, \tag{5.1}$$

where $\alpha \in (-1, 0)$ and $F : \mathbb{R}^N \setminus \{0\} \times S_N \to \mathbb{R}$ is a continuous function that satifies

- $(H1)$ $F\,(tp, \mu X) = t^{\alpha}\mu F\,(p, X)\,, \quad \forall t, \mu \in \mathbb{R}^+.$
- $(H2)$ For all $p \neq 0$ and $M, N \in S_N, N \geq 0$
  $\lambda\,|p|^{\alpha}\,tr\,(N) \leq F\,(p, M + N) - F\,(p, N) \leq \Lambda\,|p|^{\alpha}\,tr\,(N)\,,$
  where $\Lambda \geq \lambda > 0$.

In equation (5.1) we consider that Ω is a domain in $\mathbb{R}^N$ and $c, f : \Omega \to \mathbb{R}$ are continuous functions.

The study of equation (5.1) was initiated in a series of papers by Birindelli and Demengel [3], [4], [5], [6] and [7], where a notion of solution is introduced and an existence theory for the Dirichlet boundary value problem based on Perron's method is developed. They also studied first eigenvalue theory and existence for some non-proper operators. One of the difficulties in the study of these equations is that the associated differential operator does not have divergence form and simultaneously it is not sublinear in any reasonable sense.

In [18] Dávila with the authors continued the study of equation (5.1) proving the Harnack inequality in the singular case, that is when $\alpha \in (-1, 0)$. This inequality is usually obtained as a consequence of the Alexandroff-Bakelman-Pucci (ABP) inequality in the fully nonlinear case, using a sublinearity property of the operator. Here however, even though a version of the ABP inequality is known [17], this is not possible.

The Harnack inequality was derived originally by Harnack for two dimensional harmonic functions, it was later extended by Moser [25], Serrin [27] and Trudinger [29] for divergence form operators. The proof of these results uses in an esential way the divergence structure, integrating by parts against appropriate tests functions and an iteration procedure. For linear elliptic operators in non-divergence form with general coefficients, the Harnack inequality was proved by Krylov and Safonov [22], opening the way to the general theory of fully nonlinear elliptic operators, case studied by Caffarelli in [12].

Before stating the Harnack inequality in precise terms we recall the notion of solution introduced in [3]-[6], which is a notion of viscosity solution adapted to (5.1), since we cannot test functions with vanishing gradient. (5.1).

Definition 5.1. *Let $\Omega \subseteq \mathbb{R}^N$ an open bounded set, $\alpha \in (-1, \infty)$ and $u \in C(\Omega)$. We say that u is a viscosity supersolution of*

$$-F\left(D^2 u, \nabla u\right) = g(x, u),$$

in Ω if for every $x_0 \in \Omega$ we have

(i) *Either for all $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum at x_0 and $\nabla\varphi(x_0) \neq 0$ then*

$$-F\left(D^2\varphi(x_0), \nabla\varphi(x_0)\right) \geq g(x_0, u(x_0)). \tag{5.2}$$

(ii) *Or there is an open ball $B(x_0, \delta) \subset \Omega$, $\delta > 0$ where u is constant, $u = C$ and*

$$0 \geq g(x, C) \quad \forall x \in B(x_0, \delta). \tag{5.3}$$

Analogously, we say that u is a viscosity subsolution of

$$-F\left(D^2 u, \nabla u\right) = g(x, u),$$

in Ω if for every $x_0 \in \Omega$ we have

(i) *Either for all $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum at x_0 and $\nabla\varphi(x_0) \neq 0$ then*

$$-F\left(D^2\varphi(x_0), \nabla\varphi(x_0)\right) \leq g(x_0, u(x_0)). \tag{5.4}$$

(ii) *Or there is an open ball $B(x_0, \delta) \subset \Omega$, $\delta > 0$ inside which u is constant, $u = C$ and*

$$0 \leq g(x, C) \quad \forall x \in B(x_0, \delta). \tag{5.5}$$

Now we state the main theorem in [18] on Harnack inequality for equation (5.1), which may be seen as a fully nonlinear version of the p-laplacian operator in case $\alpha \in (-1, 0)$, corresponding to $1 < p < 2$, that is the singular case.

Theorem 5.1. *Assume $\alpha \in (-1,0)$ and F satisfies $(H1)$ and $(H2)$. If $u \in C(\Omega)$ is a non-negative viscosity solution of (5.1), with a, b and f continuous functions in Ω, then for every $\Omega' \subset\subset \Omega$ we have*

$$\sup_{\Omega'} u \leq C \left\{ \inf_{\Omega'} u + \|f\|_{N,\Omega'}^{\frac{1}{\alpha+1}} \right\}. \tag{5.6}$$

where the constant C depends on $\lambda, \Lambda, N, \Omega', \Omega$ and α.

Here and in what follows we denote by $\|\cdot\|_{N,A}$ the norm in $L^N(A)$. A proof of this theorem is given in [18], following the presentation in Gilbarg and Trudinger [20], which is based on Krylov, Safonov [22] original approach. By taking advantage of the fact that $\alpha \in (-1,0)$ we can reduce the application of ABP to inequalities having the Pucci operator as the main term. We think that inequality (5.6) is also true for solutions of the equation (5.1), but with $\alpha > 0$, but at this point we are not able to prove it.

It is interesting (and useful) to see that in case the differential operator is continuous, that is, $\alpha \geq 0$, then Definition 5.1 is equivalent with the definition of C-viscosity solution as given in Definition 4.2 (with the adequate changes). Next lemma is the precise statement of this remark and it is proved in [18].

Lemma 5.1. *Let $F : \mathbb{R}^N \times S_N \to \mathbb{R}$ be a continuous function satisfying $F(p,0) = 0$ for all $p \in \mathbb{R}^N$. Then u is a S-viscosity supersolution (subsolution) of*

$$-F\left(D^2 u, \nabla u\right) = g(x,u) \quad \text{in } \Omega, \tag{5.7}$$

if and only if u is a C-viscosity supersolution (subsolution) of (5.7).

This lemma is useful even when the operator is not continuous, when $\alpha \in (-1,0)$, since one can prove that solutions of equation (5.1) also satisfy an equation with a continuous operator. We discuss this in detail in [18].

6. Singular superlinear equation in $\mathbb{R}^N$ without growth on the data: Continuous case and good solutions case

In section §5 we considered equation (5.1) with F satisfying (H1)-(H2), $\alpha \in (-1,0)$, $s > 1+\alpha$ and f continuous. This problem was discussed in section §4 in the case of a fully nonlinear operator. When the operator is singular, satisfying (H1)-(H2) with $\alpha \in (-1,0)$ we obtain existence of a solution generalizing the result of Esteban, Felmer and Quaas in [19], however we assume f is continuous. We proved in [18] the following theorem, which is more general that Theorem 1.3.

Theorem 6.1. *Assume that $\alpha \in (-1,0)$ and F satisfies $(H1)$ and $(H2)$. If $s > 1+\alpha$, then for every $f \in C^0\left(\mathbb{R}^N\right)$, equation (5.1) possesses at least one solution.*

In the proof of this theorem we use a compactness property derived from the C^β regularity obtained from the Harnack as proved in [18], together with a local estimate. This estimate is similar to (4.8) and it is proved with ideas from the proof of Theorem 5.1. In the earlier works in [9] and [19] uniqueness and positivity of solutions is also proved. However here we cannot do it since we lack of sublinearity of the differential operator, the use of the Osserman function as in [9] and the standard uniqueness argument is not possible.

In what follows we sketch the proof of Theorem 6.1. We start with a sequence of approximate problems in bounded domains B_n. The next lemma summarizes what we know about existence and regularity in bounded domains for the approximate problems and its proof follows from the existence theory developed in [6].

Lemma 6.1. *For every $n \in \mathbb{N}$ there exists an S-viscosity solution $u_n \in C^\gamma(B_n)$ of equation*

$$\left\{ \begin{array}{rcl} -F\left(\nabla u_n, D^2 u_n\right) + |u_n|^{s-1} u_n &=& f(x) \quad in\ B_n, \\ u_n &=& 0 \quad in\ \partial B_n \end{array} \right. , \tag{6.1}$$

where B_n is the ball centered at 0 and with radius n and $\gamma \in (0,1)$.

In the study of the sequence $\{u_n\}$ of solutions of (6.1), the next lemma us crucial, since it allows for local uniform estimates.

Lemma 6.2. *Let $s > 1+\alpha$ and g continuous in $\mathbb{R}^N$. Assume u is an S-viscosity solution of*

$$-F\left(\nabla u, D^2 u\right) + |u|^{s-1} u \le g \ \ in\ \Omega,$$

where Ω is a subset of $\mathbb{R}^N$. Then for every $R > 0$ and $R' > R$ such that $B_{R'} \subset \Omega$ we have

$$\sup_{B_R} u \le C\left(1 + \|g\|_{L^N(B_{R'})}^{\frac{1}{1+\alpha}}\right), \tag{6.2}$$

where $C = C\left(s, R, R', N, \lambda, \Lambda\right)$ does not depend on g.

Idea of the Proof. We get the desired bound using the fact that u satisfies

$$-\left|\nabla u\right|^{\alpha} \mathcal{M}^{+}\left(D^2 u\right) + |u|^{s-1} u \le g \ \text{ in } \Omega.$$

We assume, without lose of generality, that u is non-trivial, $B = B_1(0)$ and $\lambda = 1$. We may also assume that $u \in C^2(\Omega) \cap C(\bar{\Omega})$, since we can use an appropriate approximation procedure. For $\beta \geq 1$, let us define the function η as follows: $\eta(x) = (1-|x|^2)^\beta$ if $x \in B$ and $\eta(x) = 0$ if $x \notin B$ and consider the function $v = \eta u$. We have

$$\mathcal{M}^+\left(D^2 v\right) \;\geq\; \eta\mathcal{M}^+\left(D^2 u\right) + \mathcal{M}^-\left(\overline{\nabla u \otimes \nabla \eta} + uD^2\eta\right)$$

and then

$$\mathcal{M}^+\left(D^2 v\right) \geq C_1\eta\left\{\frac{-|g| + |u|^{s-1}u}{\left(\eta^{-1/\beta}u\right)^\alpha}\right\} - C_2 v\eta^{\frac{-2}{\beta}} \quad \text{in } \Gamma^+(v),$$

where $\Gamma^+(v)$ is the contact set of v and C_1, C_2 are constants. For details see [18]. Now, given $u > 0$ we have that in $\Gamma^+(v)$

$$\eta\frac{-|g| + |u|^{s-1}u}{\left(\eta^{-1/\beta}u\right)^\alpha} \;=\; -\eta^{1+\alpha/\beta+\alpha}|g|\,v^{-\alpha} + \eta^{\alpha/\beta+1+\alpha-s}v^{s-\alpha}.$$

Thus we have

$$\begin{aligned} -\mathcal{M}^+\left(D^2 v\right) \;&+\; C_1\eta^{\alpha/\beta+1+\alpha-s}v^{s-\alpha} - C_2\eta^{-2/\beta}v \\ &\leq\; C_1\eta^{1+\alpha/\beta+\alpha}|g|\,v^{-\alpha}. \end{aligned} \tag{6.3}$$

Taking $\beta = \max\{-\alpha/(1+\alpha), (\alpha+2)/(s-1-\alpha)\}$, we obtain that $\eta^{\alpha/\beta+1+\alpha-s} \geq \eta^{-2/\beta}$ and then, from (6.3), we obtain

$$-\mathcal{M}^+\left(D^2 v\right) + \eta^{-2/\beta}v\left(v^{s-\alpha-1} - C\right) \leq C_1\eta^{1+\alpha/\beta+\alpha}|g|\,v^{-\alpha}.$$

As in [19], we define $w = \max\left\{v - C^{1/(s-\alpha-1)}, 0\right\}$ and notice that $\Gamma^+(w) \subset \Gamma^+(v)$ and $\Gamma^+(w) \subset \{x \in \Omega \,|\, w(x) \geq 0\}$. Thus, w satisfies

$$-\mathcal{M}^+\left(D^2 w\right) \leq C\eta^{1+\alpha/\beta+\alpha}|g|\,v^{-\alpha} \quad \text{in } \Gamma^+(w).$$

Here we apply ABP inequality to get

$$\sup_B w \leq C\left\|\eta^{1+\alpha/\beta+\alpha}gv^{-\alpha}\right\|_{L^N(B)}$$

and then

$$\sup_B v \leq \sup_B w + C^{1/(s-\alpha-1)} \leq C(1 + \left\|\eta^{1+\alpha/\beta+\alpha}gv^{-\alpha}\right\|_{L^N(B)}).$$

Finally we notice that

$$\left\|\eta^{1+\alpha/\beta+\alpha}gv^{-\alpha}\right\|_{L^N(B)} \leq \left(\sup_B v\right)^{-\alpha}\|g\|_{L^N(B)} \leq \sup_B v\varepsilon^{-\frac{1}{\alpha}} + \|g\|_{L^N(B)}^{\frac{1}{1+\alpha}}$$

from where we conclude

$$\sup_B v \leq C\left(1 + \|g\|_{L^N(B)}\right) \square$$

In order to prove Theorem 6.1 we use Lemma 6.2 to get compactness of the sequence of approximate solutions and a limit function u. Then we prove that u is an actual solution by carefully arguing taking the different possibilities in the definition of S-viscosity solution. These type of arguments were used in [5], see also [18].

It is interesting to see that in case $f \in L^N_{loc}(\mathbb{R}^N)$ we may always find a sequence of smooth functions f_n such that

$$\lim_{n\to\infty} \int_\Omega |f_n - f|^N \, dx = 0 \tag{6.4}$$

for any bounded domain Ω. Then we can use (6.1) with right hand side f_n instead of f to find a sequence of approximate solutions $\{u_n\}$ and we can apply Lemma 6.2 to get (6.2) with u_n on the left and the norm of f on the right. Together with Harnack inequality, we the obtain a function u which is the locally uniform limit of a subsequence of u_n. According to the definition below, u is a good solution, so the proof of Theorem 1.4 is complete.

To end, let us see the precise definition

Definition 6.1. *Given $f \in L^N_{loc}(\mathbb{R}^N)$ we say that a continuous function $u \in C(\mathbb{R}^N)$ is a good solution of (5.1) if there is a sequence of functions $\{f_n\} \subset C^\infty(\mathbb{R}^N)$ satisfying (6.4) and a sequence of domains Ω_n such that $B_n \subset \Omega_n$ for all n, such that u is the locally uniform limit of the sequence u_n of solutions of*

$$\left\{ \begin{array}{rcl} -F\left(\nabla u_n, D^2 u_n\right) + |u_n|^{s-1} u_n &=& f_n(x) \quad \text{in } \Omega_n, \\ u_n &=& 0 \quad \text{in } \partial\Omega_n. \end{array} \right. \tag{6.5}$$

7. Open Questions

We conclude this article with some open questions that are motivated by the study of the various superlinear problems that we have presented in the previous sections. Most of these question are related with the notion of solution we can consider for the superlinear equations for right hand side with less regularity. Notice that in the case of divergence form operators it is possible to have a solution if the right hand side is in $L^1_{loc}(\mathbb{R}^N)$.

Question 1: Define a notion of solution for equation (1.2) such that in the radially symmetric case it coincide with weak radial solutions. This will allow for an extension of Theorem 1.1 to the non radial case. Notice that if $f \in L^p$ then viscosity solution is defined if $p > N/2$, see [13]. However, for weak radial solutions we may consider f with less regularity at the origin.

Question 2: Prove that the solution in Theorem 1.1 is an L^N-viscosity solution if f is radially symmetric and $f \in L^N(B_R)$.

Question 3: Is it possible to define a notion of solution so that equation (1.2) has a (unique) solution for $f \in L^1_{loc}(\mathbb{R}^N)$, even in the radial case .

Question 4: Define a notion of solution for singular equation (1.4) with right hand side in $L^N_{loc}(\mathbb{R}^N)$ such that good solution satisfies the equation in that sense, and the problem is well posed, that means: existence, uniqueness and continuity with respect to the right hand side holds. Does the coercivity ($s > 1 + \alpha$) plays a role here?

Question 6: Is there a notion of solutions to equation (1.4) well posed?

Question 7: Is the notion of S-viscosity solution to equation (1.4) well posed?

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References

[1] A.D. Aleksandrov, *Majorization of solutions of second-order linear equations.* Amer. Mat. Soc. Transl. 68 (1968), 120-143

[2] I. Bakelman, *Theory of quasilinear elliptic equations.* Siberian Math. J. 2 (1961), 179-186.

[3] I. Birindelli, F. Demengel, *Comparison principle and Liouville type results for singular fully nonlinear operators.* Ann. Fac. Sci Toulouse Math, (6)**13** (2004), N.2, 261-287.

[4] I. Birindelli, F. Demengel, *Eigenvalue and Maximun Principle for fully non-linear singular operators.* Advances in Partial Diff. equations **11** n.1 (2006), 91-119.

[5] I. Birindelli, F. Demengel, *Eigenvalue, maximun principle and regularity for fully non linear homogeneous operators.* Commun. Pure Appl. Anal. 6 (2007), pp. 335366.

[6] I. Birindelli, F. Demengel, *The Dirichlet problem for singular fully nonlinear operators.* Discrete and Continuous Dynamical Systems. Supplement 2007, 110-121.

[7] I. Birindelli, F. Demengel, *Eigenvalue and Dirichlet problem for fully-nonlinear operators in non-smooth domains*, J. Math. Anal. Appl. (2009), to appear.

[8] l. Boccardo, T. Gallouet, J.L. Vázquez, *Nonlinear Elliptic Equations in* $\mathbb{R}^n$ *withouth Growth Restrictions on the Data.* Journal of Differential Equations, Volume **105**(2) (1993), 334-363.

[9] H. Brezis, *Semilinear equations in* $\mathbb{R}^n$ *without condition at infinity.* Appl. Math. Optim. **12** (1984), 271-282.

[10] L. Caffarelli, *A priori estimates for fully nonlinear second order elliptic equations, Nonlinear variational problems.* Vol. II, Pitman Research Notes in Math., vol. 193, Longamna, New York, 1989, 99-106.

[11] L. Caffarelli, *Elliptic second order equations.* Rendiconti del Seminario Matematico e Fisico di Milano 57 (1988), 253-284.

[12] L. Caffarelli, *Interior a priori estimates for solutions of fully non-linear equations.* Ann. Math. 130 (1989), 189-213.

[13] L. Caffarelli, M. Crandall, M. Kocan, A. Świech, *On Visosity Solutions of Fully Nonlinear Equations with measurable Ingredients.* Comm. Pure and Applied Mathematics **49**(4) (1996), 365-398.

[14] L. Caffarelli, X. Cabré, *Fully Nonlinear Elliptic Equations.* American Mathematical Society, Colloquium Publication, **43** (1995).

[15] M. Crandall, H. Ishii, P.L. Lions, *User's guide to viscosity solutions of second order partial differential equations.* Bulletin of the AMS, **27**(1) (1992), 1-67.

[16] F. Da Lio, B. Sirakov, Symmetry results for viscosity solutions of fully nonlinear uniformly elliptic equations. J. Eur. Math. Soc. (JEMS) **9**(2) (2007), p. 317-330.

[17] G. Dávila, P. Felmer. A. Quaas, *Alexandroff-Bakelman-Pucci estimate for singular or degenerate fully nonlinear equations.* Submitted.

[18] G. Dávila, P. Felmer. A. Quaas, *Harnack Inequality for Singular Fully Nonlinear Operators and some Existence Results* . Submitted.

[19] M. Esteban, P. Felmer, A. Quaas, *Super-linear elliptic equation for fully nonlinear operators without growth restrictions for the data.* To appear in Proceedings of the Royal Society of Edinburgh.

[20] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order.* 2nd edition, Springer-Verlag 1983.

[21] H. Ishii, *Viscosity solutions of non-linear partial differential equations.* Sugaku Expositions vol 9, (1996).

[22] N.V. Krylov, M.V. Safonov, *An estimate of the probability that a diffusion process hits a set of positive measure.* Dokl. Akad. Nauk. SSSR 245 (1979), 253-255 [Russian]. Engl. Transl. in Soviet Math. Dokl. **20** (1979), 253-255.

[23] N.V. Krylov, M.V. Safonov, *Certain properties of solutions of parabolic equations with measurable coefficients.* Izvestia Akad. Nauk. SSSR **40** (1981), 161-175 [Russian]. Engl. Transl. in Math. USSr Izv. **161** (1981), 151-164.

[24] F. Leoni, Nonlinear elliptic equations in R^N with "absorbing" zero order terms, Adv. Differential Equations, **5** (2000), p. 681-722.

[25] J. Moser, *On Harnack's theorem for elliptic differential equations.* Comm Pure Appl. Math., 14 (1961), 577-591.

[26] C. Pucci, *Limitazioni per soluzioni di equazioni ellittiche.* Ann. Mat. Pura Appl. 74 (1966), 15-30.

[27] J. Serrin, *Local behavior of solution of quasi-linear elliptic equations.* Acta Math. **111** (1964), 247-302.

[28] N.S. Trudinger, *Comparison principles and pointwise estimates for viscosity solutions.* Rev. Mat. Iberoamericana 4 (1988), 453-468.

[29] N.S. Trudinger, *Harnack inequalities for Nonuniformly Elliptic Divergence structure equations* Inventiones math. 64, 517-531, 1981.

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