Multiplicity results for extremal operators through bifurcation

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Abstract. We study nonproper uniformly elliptic fully nonlinear equations involving extremal operators of Pucci type. We prove the existence of all radial spectrum for this type of operators and establish a multiplicity results through global bifurcation.

1 Introduction

In the sobability of fully nonlinear elliptic equations of the type

$$F(D^2u, Du, u, x) = 0,$$

have been extensively studied in the framework of classical, strong and viscosity solutions, see for example, Lions [24], Evans [15], Gilbarg, Trundinger [22], Crandall Ishii, Lions [11] and Caffarelli, Cabre [6] and Caffarelli, Crandall, Kocan, Swiech [7], Crandall, Kocan, Lions, Swiech [9]. Most of these works concern proper or coercive operators, that is, F nonincreasing u.

In this paper we want to solve, through global bifurcation, non proper equations of the type

$$\left\{ \begin{array}{rcll} \mathcal{M}^{\pm}_{\mathcal{C}}(D^2u) + g(u) & = & 0, & \text{in} \quad \Omega, \\ \\ u & = & 0 & \text{on} \quad \partial\Omega, \end{array} \right. \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ and $\mathcal{M}^{\pm}_{\mathcal{C}}$ are general Hamilton-Jacobi-Bellman (HJB) operators. Specifically theses operators are defined as:

$$\mathcal{M}^{+}_{\mathcal{C}}(M) = \sup_{\sigma(A)\in\mathcal{C}} \operatorname{tr}(AM) \quad \text{and} \quad \mathcal{M}^{-}_{\mathcal{C}}(M) = \inf_{\sigma(A)\in\mathcal{C}} \operatorname{tr}(AM), \tag{1.2}$$

where $\mathcal{C}$ is any subset of the cube $[\lambda, \Lambda]^N$ invariant with respect to permutations of coordinates and $\sigma(A)$ is the set of eigenvalues of A. These operators reduce to classical Pucci type operators when $\mathcal{C} = [\lambda, \Lambda]^N$, and to the Laplacian when $\lambda = \Lambda = 1$. When $\lambda \le 1/N$, $\Lambda = 1 - \lambda(N-1)$ and $\mathcal{C} = \{a \in [\lambda, \Lambda]^N \mid \sum_{i=1}^N a_i = 1\}$ the operator corresponds to Pucci's operators, see [27], [28].

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This general class of extremal operators were introduced, in this form, by Felmer and Quaas in [19] and [20]. Observe that the operators in the class are positively homogeneous of degree one and so, the associated eigenvalue problem is

$$\begin{cases} \mathcal{M}_{\mathcal{C}}^{+}(D^2u) &= \mu u, \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.3}$$

Therefore in the case $\lambda = \Lambda = 1$ the classical Rabinowitz bifurcation theory [34], [33], [35] gives an answer to existence of solution to (1.1).

Concerning the first half eigenvalue problem, recent results have been established by Quaas Sirakov in [32], see also [31] in the general setting of convex (concave) operators. For the case of Pucci type of operator with $\mathcal{C} = [\lambda, \Lambda]^N$, some of the results were established earlier by Felmer Quaas [18], Quaas in [29] and Busca, Esteban and Quaas [5]. Other previous results of first half eigenvalue are due to Lions [25] and Pucci [27].

The aim of this paper is to extend the results for $\mathcal{C} = [\lambda, \Lambda]^N$ in [5] of local bifurcation from the first two half eigenvalues and from the radial spectrum to this class of HJB operator. Moreover, we establish existence results for equation of (1.1) through global bifurcation. These results are new even in the case $\mathcal{C} = [\lambda, \Lambda]^N$.

We start recalling the results of [32] that we need.

Theorem 1.1 *Let Ω be a regular domain. There exist two positive half-eigenvalues μ_1^+, μ_1^-, and two functions φ_1^+, $\varphi_1^- \in C^2(\Omega) \cap C(\bar{\Omega})$ such that (μ_1^+, φ_1^+), (μ_1^-, φ_1^-) are solutions to (1.3) and $\varphi_1^+ > 0, \varphi_1^- < 0$ in Ω. Moreover, these two half-eigenvalues are simple, that is, all positive solutions to (1.3) are of the form $(\mu_1^+, \alpha\varphi_1^+)$, with $\alpha > 0$. The same holds for the negative solution. Additionally, the half-eigenvalues are strict monotone with respect to the domain.*

Remark 1.1 *Principal eigenvalue for other type of fully nonlinear operators have been recently studied by Birindelli, Demengel [3], [4] and Juutienn [23].*

Now we give our first result which deals with the existence of the radial spectrum for this class of operators. Notice that nothing is known about higher eigenvalue for general domains. We think that the result below gives light on the fact that infinitely many eigenvalues exist. For the radially symmetric situation we can apply ODE techniques to get the results.

Theorem 1.2 *Let $\Omega = B_1$ the unit ball. The set of all the scalars μ such that (1.3) admits a nontrivial radial solution, consists of two unbounded increasing sequences*

$$0 < \mu_1^+ < \mu_2^+ < \cdots < \mu_k^+ < \cdots,$$

$$0 < \mu_1^- < \mu_2^- < \cdots < \mu_k^- < \cdots.$$

Moreover, the set of radial solutions of (1.3) for $\mu = \mu_k^+$ is positively spanned by a function φ_k^+, which is positive at the origin and has exactly k-1 zeros in $(0,1)$, all these zeros being simple. The same holds for $\mu = \mu_k^-$, but considering φ_k^- negative at the origin.

This result is related with the radial Fucik spectrum studied by Arias and Campos in [1], see [5] for more details and other related references.

Finally, we want to adress our original motivation, that is, we want to prove existence results for an equation of the type (1.1). For this purpose we consider the nonlinear bifurcation problem associated, that is:

$$\begin{aligned} -\mathcal{M}_{\mathcal{C}}^+(D^2u) &= \mu u + f(u,\mu) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.4}$$

where f is continuous, $f(s,\mu) = o(|s|)$ near $s = 0$, uniformly for $\mu \in \mathbb{R}$ and Ω is a general regular bounded domain. Concerning this problem we have the following Theorem.

Theorem 1.3 *The pair $(\mu_1^+, 0)$ (resp. $(\mu_1^-, 0)$) is a bifurcation point of positive (resp. negative) solutions to (1.4). Moreover, the set of nontrivial solutions of (1.4) whose closure contains $(\mu_1^+, 0)$ (resp. $(\mu_1^-, 0)$), is either unbounded or contains a pair $(\bar{\mu}, 0)$ for some $\bar{\mu}$, eigenvalue of (1.3) with $\bar{\mu} \neq \mu_1^+$ (resp. $\bar{\mu} \neq \mu_1^-$).*

The first bifurcation results in the context of HJB equation are due to Lions [25] where very particular nonlinearities are considered. In that paper the author also gave the existence of half eigenvalue and its probabilistic interpretation in terms of stochastic control. It is interesting to understand how such a bifurcation result can be extended to the context of [3] or [23].

In the radially symmetric case we obtain a more complete result.

Theorem 1.4 *Let $\Omega = B_1$. For each $k \in \mathbb{N}$, $k \geq 1$ there are two connected components $H_k^\pm \subset S_k^\pm$ of nontrivial solutions to (1.4), whose closures contain $(\mu_k^\pm, 0)$. Moreover, $H_k^\pm$ are unbounded and $(\mu, u) \in S_k^\pm$ implies that u possesses exactly $k-1$ zeros in $(0,1)$.*

Remark 1.2 *1) S_k^+ (resp. S_k^-) denotes the set of solutions that are positive (resp. negative) at the origin, with exactly $k-1$ zeros in $(0,1)$.*

Now we are in position to state our first existence result for (1.1) with $\Omega = B_1$, where $g : \mathbb{R} \to \mathbb{R}$ is a continuos function and $g(0) = 0$.

Theorem 1.5 *Let $\Omega = B_1$, assume*

$$\sup_{s \in \mathbb{R}} \left| \frac{g(s)}{s} \right| < \infty,$$

and that for some positive natural numbers k y n, with $k \leq n$

$$\lim_{s \to 0} \frac{g(s)}{s} < \mu_k^+ \leq \mu_n^+ < \liminf_{|s| \to \infty} \frac{g(s)}{s}.$$

Then (1.1) possesses at least $n-k+1$ nontrivial radial solution. More precisely, for each j such that $k \leq j \leq n$, there is a radial solution of the problem (1.1) positive in the origin with exactly $j-1$ zeros in $(0,1)$.

An analogous result can be established when $\mu_{k,n}^+$ is replaced by $\mu_{k,n}^-$ but the solutions are negative at the origin.

Remark 1.3 *i) Similar results can be obtained for in general domains when $k = n = 1$ using Theorem 1.3 , see [30] in the case of $\mathcal{C} = [\lambda, \Lambda]^N$ with a different proof.*
ii) Another multiplicity result can be obtained in a ball through global bifurcation assuming sub-critical nonlinearity with critical exponent p_* found by Felmer and Quaas in [19] for this type of operators. The sub-critical behavior of the nonlinearity will give the desired bound on the branch though blow-up method, see the proof of Theorem 1.5 and Gidas Spruck [21] .

If g is an odd function, then we have more solutions since we can distinguish the positive and minus the negative solution at the origin. Observe that this result is only valid for nonlinear operators.

Corollary 1.1 *Let $\Omega = B_1$, assume g is odd,*

$$\sup_{s \in \mathbb{R}} |\frac{g(s)}{s}| < \infty,$$

and that for some positive natural numbers k y n, with $k \leq n$

$$\lim_{s \to 0} \frac{g(s)}{s} < \min\{\mu_k^+, \mu_k^-\} \leq max\{\mu_n^+, \mu_n^-\} < \liminf_{|s| \to \infty} \frac{g(s)}{s}.$$

Then (1.1) possesses at least $2(n-k+1)$ nontrivial radial solutions.

Finally, notice that all above results can be rewritten in the case of $\mathcal{M}_{\mathcal{C}}^-$ using that $\mathcal{M}_{\mathcal{C}}^-(M) = -\mathcal{M}_{\mathcal{C}}^+(-M)$ for any $M \in S_N$ the set of all N by N symmetric matrices.

The paper is organized as follows. In the section 2 we define the homotopy of the HJB operator and the Laplacian that is used to compute the degree. Then we give a sketch of the proof Theorem 1.3. In section 3 we study the radial case. First, we prove the existence of the radial spectrum, then the local bifurcation and finally we establish our main existence results.

2 Homotopy and General Domain

We start this section by recalling some results of [32] that we need.

Theorem 2.1 *i) There exists $\varepsilon_0 > 0$ depending on $N, \Omega, \mathcal{C}$ such that the problem (1.3) , has no nontrivial solution, for $\mu \in (-\infty, \mu_1^- + \varepsilon_0) \setminus \{\mu_1^+, \mu_1^-\}$*

ii) For any $f \in L^p(\Omega)$, $p \geq N$ there exists a unique solution $u \in W^{2,p}_{loc}(\Omega) \cup C(\bar{\Omega})$ of

$$\begin{cases} \mathcal{M}^+_{\mathcal{C}}(D^2 u) + \mu u &= f, \quad in \quad \Omega, \\ u &= 0 \quad on \quad \partial\Omega, \end{cases} \tag{2.5}$$

for $\mu < \mu_1^+$. Moreover, if $\mu_1^+ \leq \mu < \mu_1^-$ the the problem (2.5) does not have a solution, provided $f \leq 0$, $f \neq 0$ in Ω.

We need some preliminaries in order to compute the Leray-Schauder degree of our equivalent problem.

We start by constructing an homotopic deformation of $\mathcal{C}$ to a point. More precisely, let $\hat{\mathcal{C}}_* : [0,1] \mapsto [\lambda, \Lambda]^N$ a multivalued function such that

$$\hat{\mathcal{C}}_\alpha = \{a \in [\lambda, \Lambda]^N \,|\, a = \bar{a} + \alpha(a - \bar{a}),\, a \in \mathcal{C}\},$$

where $\bar{a} \in \mathcal{C}$ satisfies $\bar{a}_i = \bar{a}_1$ for all $i = 2, ..., N$.

Notice that $\hat{\mathcal{C}}_\alpha$ is convex for all $\alpha \in [0,1]$ and $\hat{\mathcal{C}}_0 = \{\bar{a}\}$ and $\bar{\mathcal{C}}_1 = \mathcal{C}$. In a natural way we define the extremal operator related to $\hat{\mathcal{C}}_\alpha$ and define

$$\mathcal{M}_\alpha := \mathcal{M}^+_{\hat{\mathcal{C}}_\alpha}.$$

Therefore, we have $\mathcal{M}_0 := \bar{a}\Delta$ and $\mathcal{M}_1 := \mathcal{M}^+_C$.

Define now $\mathcal{L}_\alpha$ as the inverse of $-\mathcal{M}_\alpha$. It is well known that $\mathcal{L}_\alpha$ is well defined in $\mathcal{S} := \{u \in C(\bar{\Omega}) \,|\, u = 0 \quad \text{on} \quad \partial\Omega\}$ (see for example [8], also Theorem 3.1 in [36]) and $\mathcal{L}^+_\alpha$ is compact (see [6], also Proposition 4.2 in [9] and Teorem 3.8 [7]). Let observe that in [8] $C^{2,\alpha}$ estimates holds up to the boundary if Ω is smooth, this give compactness in C^2.

The aim is to compute the degree $deg_{\mathcal{S}}(I - \mu\mathcal{L}_\alpha, B(0,r), 0)$ for some values of μ where it is well define..

Proposition 2.1 *Let $r > 0$, $\bar{\alpha} > 0$, $\mu \in \mathbb{R}$. Then*

$$deg_{\mathcal{S}}(I - \mu\mathcal{L}_{\bar{\alpha}}, B(0,r), 0) = \begin{cases} 1 & if \quad \mu < \mu_1^+(\bar{\alpha}), \\ 0 & if \quad \mu_1^+(\bar{\alpha}) < \mu < \mu_1^-(\bar{\alpha}), \\ -1 & if \quad \mu_1^-(\bar{\alpha}) < \mu < \mu_2(\bar{\alpha}). \end{cases}$$

Remark 2.1 *1) Since $\mathcal{L}_\alpha$ is compact, the degree is well defined if $0 \notin (I - \mu\mathcal{L}_{\bar{\alpha}})(\partial B(0,r))$.*
2) $\mu_2(\bar{\alpha})$ could be $+\infty$ since is define as a infimum of solution to (1.3) with $\mu > \mu_1^-(\bar{\alpha})$.

This Proposition and then Theorem 1.3 can be proven without changes form the form the proof in [5] the only different is that we need to use the above homotopy to prove Proposition 2.1 (to get degree -1) and the non-existence result of Theorem 2.1 ii) (to get degree 0) and i) of Theorem 2.1 (degree 1). For more details see the proof of proposition 3.1 below.

3 Spectrum in the Radial Case and existence results

We start this section by studying the operator acting on radial function, most of the this is done in [19] the only point here is that we need to define the operator for negative and positive right hand side simultaneously and where the derivative vanishes.

In the case of radially symmetric solutions then we can define the operator $\mathcal{M}_{\mathcal{C}}^+$ acting on C^2 radially symmetric functions as

$$\mathcal{M}_{\mathcal{C}}(D^2u) = \sup_{(a_1,a_2)\in\tilde{\mathcal{C}}} a_1u'' + \frac{(N-1)a_2u'}{r}, \tag{3.6}$$

where $\tilde{\mathcal{C}} := \{(a_1, \frac{1}{N-1}\sum_{i=2}^N a_i) \in \mathbb{R}^2 \,/\, a \in \mathcal{C}\}$. In the rest of the paper we will write $\mathcal{C}$ for $\tilde{\mathcal{C}}$ to simplify the notation. In order to describe the set $\mathcal{C}$ in a more convenient way, and to avoid trivialization, we make a further assumption.
(D) The set $\mathcal{C} \subset \mathbb{R}^2_+$ is compact, convex and its projection onto the y-axis is not a singleton.

Assuming **(D)** we exclude the case when the projection of **(D)** onto the y-axis is a singleton, which is equivalent to $\mathcal{C} = \{(a_1, a_1)\}$. This particular case can be analyzed as the radial Laplacian. Observe that $\mathcal{C}$ is a symmetric set.
Under the assumption **(D)** we may describe $\partial\mathcal{C}$ by means of two functions. Let $0 < \theta_{min} < \theta_{max}$ be defined as $\theta_{min} = min\{\theta \mid (x,\theta) \in \mathcal{C}\}$ and $\theta_{max} = max\{\theta \mid (x,\theta) \in \mathcal{C}\}$, and define the functions $S, \tilde{S} : [\theta_{min}, \theta_{max}] \to \mathbb{R}_+$ as

$$S(\theta) = min\{x \mid (x,\theta) \in \mathcal{C}\}, \qquad \tilde{S}(\theta) = max\{x \mid (x,\theta) \in \mathcal{C}\}.$$

With these definitions we see that S is convex, $\tilde{S}$ is concave and

$$\mathcal{C} = \{(x,\theta) \mid \theta \in [\theta_{min}, \theta_{max}],\ S(\theta) \le x \le \tilde{S}(\theta)\}.$$

Being S convex, it has one-sided derivatives $S'_-(\theta)$ and $S'_+(\theta)$, consequently it is locally Lipschitz continuous in $(\theta_{min}, \theta_{max})$. The sub-differential of S is then defined as $\partial S(\theta) = [S'_-(\theta), S'_+(\theta)]$, for $\theta \in (\theta_{min}, \theta_{max})$. The cases $\theta = \theta_{min}$ and $\theta = \theta_{max}$ are special. At θ_{max} we have two possibilities, either $S'_-(\theta_{max})$ exists, and then we define $\partial S(\theta_{max}) = [S'_-(\theta_{max}), +\infty)$, or

$$\lim_{t\to 0^-} \frac{S(\theta_{max}+t) - S(\theta_{max})}{t} = +\infty.$$

An analogous situation occurs at θ_{min} . We observe that with these definitions, for every $Q \in \mathbb{R}$ there is at least one solution $\theta \in [\theta_{min}, \theta_{max}]$ of the equation

$$\partial S(\theta)\theta - S(\theta) \ni Q. \tag{3.7}$$

The case when this equation has multiple solutions is very important for our analysis and occurs when the function S coincide locally with an affine function. In each of this maximal intervals $[\theta_i^-, \theta_i^+]$, with $\theta_i^- < \theta_i^+$ where the function S is affine, we may write for

$$S(\theta) = d_i\theta - Q_i \qquad \forall\ \theta \in [\theta_i^-, \theta_i^+],$$

for numbers d_i and Q_i.
We define the function $d : \mathbb{R} \to \mathbb{R}$ as $d(Q) \in \partial S(\theta)$ such that

$$d(Q)\theta - S(\theta) = Q.$$

All the above holds for $\tilde{S}$ with natural modification since $\tilde{S}$ is concave and $\partial\tilde{S}$ is the super-differential of $\tilde{S}$. We would also like to define θ as a function of Q, but we cannot do it in a unique way because of the possible multiplicity of solutions of the equation above. We make a choice considering $\Theta : \mathbb{R} \to \mathbb{R}$ as $\Theta(Q_i) = \theta_i^+$ in each interval where S or $\tilde{S}$ are affine functions.

Now we are interested in the study of the problem

$$v''(r) + (N_d - 1)\frac{v'(r)}{r} + \frac{\mu}{\theta}v(r) = 0 \quad in \quad (0, 1), \tag{3.8}$$

$$v'(0) = 0, \quad v(1) = 0, \tag{3.9}$$

where $\theta = \Theta(v/(N-1)v')$ when $v' \neq 0$ and

$$N_d = \begin{cases} \frac{S(\theta)}{\theta}(N-1) & \text{if } v' < 0, \\ \frac{\tilde{S}(\theta)}{\theta}(N-1) & \text{if } v' > 0. \end{cases} \tag{3.10}$$

When $v' = 0$, then $\theta := \theta_{min}$ if $v > 0$ and $\theta := \theta_{max}$ if $v < 0$. Notice that the functions $\theta(r)$ y $N_d(r)$ are mesurable functions, having discontinuities whenever r is so that $v(r)r/(N-1)v'(r) = Q_i$ and $S, \tilde{S}$ are affine functions. Moreover, both $\theta(r)$ and $N_d(r)$ are bounded and bounded away from 0.

Next we briefly study the existence, uniqueness, global existence, and oscillation of the solutions to the related initial value problem

$$u''(r) + (N_d - 1)\frac{u'(r)}{r} + \frac{1}{\theta}u(r) = 0 \quad in \quad (0, \infty), \tag{3.11}$$

$$u'(0) = 0, \quad u(0) = 1. \tag{3.12}$$

Then we will come back to (3.8), (3.9) and to the proof of Theorem 1.2. In the rest of the paper we will use the following two important remarks.

Remark 3.1 *By the definition of the maximal operator, for a solution u of equation (3.11) we have that*

$$u''(r) + \frac{S(\hat{\theta})}{\hat{\theta}}(N-1)\frac{u'(r)}{r} + \frac{u(r)}{\hat{\theta}} \leq 0,$$

for all $\hat{\theta} \in [\theta_{min}, \theta_{max}]$ and the same is valid for $\tilde{S}$.

Remark 3.2 *We have the following estimates for the dimension number $N_d(r)$*

$$N^+ \leq N_d(r) \leq N^-,$$

where $N^+ - 1 := \frac{S(\theta^+)}{\theta^+}(N-1)$ and $N^- - 1 := \frac{\tilde{S}(\theta^-)}{\theta^-}(N-1)$ and $\theta^\pm$ are solutions of the equations

$$\partial S(\theta^+) \ni \frac{S(\theta^+)}{\theta^+}, \qquad \partial \tilde{S}(\theta^-) \ni \frac{\tilde{S}(\theta^-)}{\theta^-},$$

for $\theta^+, \theta^- \in [\theta_{min}, \theta_{max}]$.

First using Theorem 1.1 and the symmetry result of [12] we can conclude, after a rescaling if necessary, the existence of a unique $u \in C^2$ solution to

$$\{u'(r)\exp(\int_{r_0}^{r} \frac{N_d(\tau)-1}{\tau} d\tau)\}' = -\frac{1}{\theta}\exp(\int_{r_0}^{r} \frac{N_d(\tau)-1}{\tau} d\tau) u(r), \tag{3.13}$$

$$u'(0) = 0, \quad u(0) = 1.$$

Observe that the existence can be obtain from Schauder fixed point (see [19] for similar argument) but the uniqueness is not trivial to obtain and is related to the simplicity result in Theorem 1.1 .

Now for some $\delta > 0$ small, u satisfies

$$u'' = -(N_d - 1)\frac{u'}{r} - \frac{u}{\theta}, \quad in \quad (0, \delta].$$

Next we consider (3.11) with initial values $u(\delta) = 0$ and $u'(\delta)$ at $r = \delta$. From the standard theory of ordinary differential equations we find a unique C^2 solution of this problem for $r \in [\delta, a)$, for $a > \delta$. Using Gronwall's inequality we can extend the local solution to $[0, \infty)$.
In the following Lemma we will show that the solution u is oscillatory.

Lemma 3.1 *The unique solution u to (3.11)-(3.12) is oscillatory, that is, given any $r > 0$, there is a $\tau > r$ such that $u(\tau) = 0$.*

Proof. Suppose that u is not oscillatory, that is, for some r_0, u does not vanish on (r_0,∞). Assume first that $u>0$ in (r_0,∞). Let ϕ be a solution to (3.11) and (3.12) for fix $\tilde\theta$ and $S(\theta)=\theta=\tilde\theta$, then it is known, since it corresponds to the Laplace operator, that ϕ is oscillatory. So, we can take $r_0<r_1,r_2$ such that $\phi(r)>0$ if $r\in(r_1,r_2)$ and $\phi(r_1)=\phi(r_2)=0$. From the definition and the optimality of the operator, u and ϕ satisfy

$$u''+(N-1)\frac{u'}{r}+\frac{u}{\tilde\theta}\le 0,$$
$$\phi''+(N-1)\frac{\phi'}{r}+\frac{\phi}{\tilde\theta}=0,$$

for $\tilde\theta\in[\theta_{min},\theta_{max}]$. If we multiply the first equation by ϕ and the second by u, subtract them and then integrate, we get

$$r_1^{N-1}u(r_1)\phi'(r_1)-r_2^{N-1}u(r_2)\phi'(r_2)\le 0,$$

getting a contradiction.
Suppose now that $u<0$ in (r_0,∞). In that case, from the the equation (3.13), we claim that $u'>0$ in (r_0,∞), taking if necessary a larger r_0. If there exists a r^* such that $u'(r^*)=0$, then using the equation we have that $u'>0$ in (r^*,∞). So we only need to discard the case $u'<0$ in (r_0,∞). In that case u satisfies

$$\left\{u'\exp\Big(\int_{r_0}^{r}\frac{N_d(\tau)-1}{\tau}d\tau\Big)\right\}'=-\frac{u}{\theta}\exp\Big(\int_{r_0}^{r}\frac{N_d(\tau)-1}{\tau}d\tau\Big).$$

Let denote by $g(r)=u'\exp\Big(\int_{r_0}^{r}\frac{N_d(\tau)-1}{\tau}d\tau\Big)$, we have that $g(r)$ is monotone, then there exists a finite $c_1<0$ such as $\lim_{r\to\infty}g(r)=c_1$. On the other hand, since $u'<0$, there exists $c_2\in[-\infty,0)$ such that $\lim_{r\to\infty}u(r)=c_2$, then from the equation, we get that

$$\lim_{r\to\infty}g'(r)=-\lim_{r\to\infty}\frac{u(r)}{\theta}\exp\Big(\int_{r_0}^{r}\frac{N_d(\tau)-1}{\tau}d\tau\Big)=+\infty.$$

That is a contradiction with $\lim_{r\to\infty}g(r)=c_1$, proving the claim.
Using $u'>0$, Remark 3.2 and the definition of the maximal operator we get

$$u''(r)+(N^--1)\frac{u'(r)}{r}\ge-\frac{u(r)}{\theta_{max}},$$

now if we multiply this equation by $r^{N^--1}/u(r)<0$ we have that

$$b'(r)+\frac{b^2(r)}{r^{N^--1}}+\frac{r^{N^--1}}{\theta_{max}}\le 0, \tag{3.14}$$

where

$$b(r)=r^{N^--1}\frac{u'(r)}{u(r)},\quad r\in(r_0,\infty).$$

Integrating (3.14) from r_0 to $t > r_0$ we get

$$b(t) - b(r_0) + \tilde{C}_1\{t^{N^-} - r_0^{N^-}\} + \int_{r_0}^{t} \frac{b^2(r)}{r^{N^- - 1}} dr \leq 0. \tag{3.15}$$

In particular we have

$$-b(t) \geq Ct^{N^-}, \quad \text{for some} \quad C > 0 \quad \text{and } t \text{ large.}$$

Define now

$$k(t) = \int_{r_0}^{t} \frac{b^2(r)}{r^{N^- - 1}} dr,$$

and notice from the previous fact that

$$k(t) \geq \tilde{c}t^{N^+ + 2} \quad \text{for some} \quad c > 0 \quad \text{and } t \text{ large.} \tag{3.16}$$

By (3.15) $k(t) \leq -b(t)$, and so

$$k'(t) \geq t^{1-N^-} k^2(t), \text{for t large.}$$

From this last inequality follows that

$$C_1\Big(\frac{1}{k(t)} - \frac{1}{k(s)}\Big) \geq \Big(\frac{1}{t^{N^- - 2}} - \frac{1}{s^{N^- - 2}}\Big),$$

for some C_1 and t, s large. Noting that $N^- > N > 2$ and taking $s \to \infty$ we find that $k(t)$ satisfies

$$k(t) \leq C_1 t^{N^- - 2} \tag{3.17}$$

However (3.16) and (3.17) are not compatible, hence u must be oscillatory. □

Notice that the same proof holds when the initial conditions are $u(0) = -1$, $u'(0) = 0$ in (3.11).

Next Lemma is a principal step in proving that the branches conserve the number of zeros and that the zeros are isolated .

Lemma 3.2 *Consider $a \in L^\infty(0, \alpha)$ and $u \in C^2[0, \alpha]$ with satisfying*

$$\Big\{\rho_u(r)u'(r)\Big\}' + a(r)\rho_u(r)u(r) = 0 \qquad a.e \quad (0, \alpha),$$

where $\rho_u(r) := exp(\int_{r_0}^{r}(N_d(\tau) - 1/\tau)d\tau)$, denote an integral factor of the equation and

$$u(r_0) = 0, \qquad u'(r_0) = 0,$$

for some $r_0 \in (0, \alpha]$. Then $u \equiv 0$.

Moreover, if $u(0) \neq 0$ then first zero of u is not arbitrarily close to 0.

Remark 3.3 *It can also be proven that all pairs of zeros of the above equation can not be arbitrarily close.*

Proof. Observe that u satisfies the equation

$$u(r) = \int_{r_0}^{r} \frac{1}{\rho_u(s)} \int_{r_0}^{s} \rho_u(\tau)a(\tau)u(\tau)d\tau ds,$$

now taken $r_0 \in (0, \alpha)$ we have that

$$|u(r)| \leq C\delta \|a\|_{L^\infty} \sup_{\tau\in[r_0-\delta,r_0+\delta]} |u(\tau)| \qquad r \in (r_0 - \delta, r_0 + \delta),$$

with C a positive constant. Then $u \equiv 0$ in $(\rho_0 - \delta, r_0 + \delta)$ for δ small enough. So, $u \equiv 0$ in $[0, \alpha]$.

Let r^* the first zero of u, then by ABP estimate in B_{r^*} since u satisfies $M^+_{\mathcal{C}}(D^2u(|x|)) + a(|x|)u(|x|) = 0$ in the L^N viscosity sense , see for example [7], we have

$$\sup_{[0,r^*)} |u| \leq r^* \|a\|_{L^\infty} \sup_{[0,r^*)} |u|,$$

getting a contradiction if r^* is sufficiently small.

Proof of Theorem 1.2. Let denote u^ν the above solutions of (3.11) with initial conditions $u^\nu = \pm 1$ (here and in the rest of the proof $\nu \in \{+, -\}$). From Lemma 3.1, u^ν has infinitely many zeros:

$$0 < \beta_1^\nu < \beta_2^\nu < \ldots < \beta_k^\nu < \ldots \tag{3.18}$$

Using the previous Lemma all the zeros of the equation are simples.
Now if we take $r = \beta_k^\nu \tilde{r}$ where $\tilde{r} \in [0, 1]$ and define $\mu_k^\nu = (\beta_k^\nu)^2$, then μ_k^ν is an eigenvalue for $\mathcal{M}^+_{\mathcal{C}}$ in B_1, with $u(\beta_k^\nu \tilde{r})$ and $\tilde{r} \in [0, 1]$ the corresponding eigenfunction with $k - 1$ zeros.

Let now μ be an eigenvalue of $\mathcal{M}^+_{\mathcal{C}}$ in B_1 with a radial eigenfunction $z(r)$ such that $z(0) > 0$. Notice that $\mu > 0$ and by uniqueness $z(r) = z(0)u^+(\mu^{1/2}r)$. But $z(1) = 0$, then $\mu = (\beta_k^+)^2$ for some $k \in \mathbb{N}$ and $z = z(0)u^+$. Similarly if $z(0) < 0$. Therefore, there are not others radial eigenvalues different form $\mu_k^\nu s$. From here we obtain Theorem 1.2. □

Now we will show some properties about the distibution of the eigenvalues.

Lemma 3.3 *For $k \in \mathbb{N}$, $k > 1$ we have $\mu_k^- < \mu_{k+1}^+$ and $\mu_k^+ < \mu_{k+1}^-$.*

Proof. We will prove the Lemma in terms of the functions defined above.
We claim that if u^+ has to change sign between two consecutive zeros of u^-, if u^+ has the same sign of u^-.
Suppose first by contradiction that $u^-(r_1) = u^-(r_2) = 0$, $u^-(r) > 0$ for all $r \in (r_1, r_2)$ and $u^+ > 0$ for all $r \in [r_1, r_2]$. Let $r_3 < r_1 < r_2 < r_4$ be the next zeros

of u^+, that is, $u^+(r_3) = u^+(r_4) = 0$, $u^+ > 0$ for all $r \in (r_3, r_4)$. Then, the first half-eigenvalue in $A_1 := \{r_1 < |x| < r_2\}$ is $\mu^+(A_1) = 1$ and first half-eigenvalue in $A_2 := \{r_3 < |x| < r_4\}$ is $\mu^+(A_2) = 1$. Define now $u(r) = u^+(\beta r)$, with $\beta > 1$ such that $r_4/\beta > r_2$ and $r_3/\beta < r_1$. So, u is a positive eigenfunction in $A_3 := \{r_3/\beta < |x| < r_4/\beta\}$ with eigenvalue $\mu^+(A_3) = \beta^2$. But $A_1 \subset A_3$, therefore $\mu^+(A_1) = 1 \geq \mu^+(A_3) = \beta^2$ getting a contradiction with the monotonicity respect the domain of the eigenvalues see Theorem 1.1 . The same kind of argument can be used in the case when u^- negative in (r_1, r_2) and u^+ negative in! $[r_1, r_2]$. Hence, the claim follows. By inverting the roles of u^-, if u^+ we get the conclusion of the Lemma. $\square$

Remark 3.4 *The above Lemma implies that in the case $\beta_k^+ < \beta_k^-$, $u^+u^- > 0$ for all $r \in (\beta_k^-, \beta_k^+)$. The same holds true in the case $\beta_k^+ > \beta_k^-$.*

Next, we prove some preliminary non existence result to prepare the proof of Theorem 1.4, the proof use some ideas that can be found for example in the book by [13].

Lemma 3.4 *Assume that $\mu_k^+ \neq \mu_k^-$ and that there exists $r_0 \in (0,1)$ such that $u^\pm(r) > 0$ for all $r \in (r_0, 1]$. Then, there exists a continuous function g such that there is no solution to the problem*

$$u'' = -\frac{(N-1)s(\theta)}{\theta}\frac{u'}{r} - \frac{\mu u}{\theta} + g \qquad \text{in } [0, r_0], \tag{3.19}$$

and

$$u'' \geq -\frac{(N-1)s(\theta)}{\theta}\frac{u'}{r} - \frac{\mu u}{\theta} + g \qquad \text{in } (r_0, 1], \tag{3.20}$$

$$u'(0) = 0, \quad u(1) = 0, \tag{3.21}$$

for μ between μ_k^+ and μ_k^- and s,θ given by the optimal condition

$$\frac{-u''r}{(N-1)u'} \in \partial S(\theta),$$

in the case $u' > 0$, $s = S$. Similar in the other cases, so that we have the extremal operator.

Remark 3.5 *1) Let us denote by u^+ and u^- the solutions of (3.19) with $r_0 = 1$ and $g = 0$ and respective initial conditions $u(0) = 1$, $u'(0) = 0$ and $u(0) = -1$, $u'(0) = 0$. Let us suppose that μ is between μ_k^+ and μ_k^-, then by previos remark we deduce that $u^+(1)u^-(1) > 0$.*
2) There is a similar non-existence result in the case when there exists $r_0 \in (0,1)$ such that $u^\pm < 0$ for all $r \in (r_0, 1]$, replacing (3.20) by

$$u'' \leq -\frac{(N-1)s(\theta)}{\theta}\frac{u'}{r} - \frac{\mu u}{\theta} + g \qquad \text{in } (r_0, 1]. \tag{3.22}$$

Proof. Consider then the particular case

$$u_{\pm}(r) > 0, \quad u'_{\pm}(r) \leq 0 \quad \text{for all } r \in (r_0, 1],$$

where we denote $u_{\pm} := u^{\pm}$ for notation convinence in this proof. All other cases can be treated similarly.
Let $g : [0, 1] \to \mathbb{R}$ be a continuous function such that $g(r) = 0$ for all $r \in [0, r_0]$ and $g(r) > 0$ for all $r \in (r_0, 1]$. For $\gamma \in \mathbb{R}$ let ϕ_γ be the solution to (3.19), (3.20) and (3.21) with, $\phi_\gamma(0) = \gamma$. Then, we have

$$\phi_\gamma = \gamma u_+ \qquad \forall\, r \in [0, r_0],$$

since uniqueness holds when g = 0. Put $r_1 = inf\{r \in (r_0, 1) : \phi_\gamma(r) = 0\}$. The interval (r_0, r_1) contains a point τ_1 such that

$$\left[\frac{\phi_\gamma}{u_+}\right](\tau_1) < 0.$$

If this is not the case, then

$$\frac{\phi_\gamma(\tau)}{u_+(\tau)} \geq \frac{\phi_\gamma(r_0)}{u_+(r_0)} = \gamma > 0 \qquad \tau \in (r_0, r_1),$$

which is impossible. So, we obtain

$$(\phi'_\gamma u_+ - \phi_\gamma u'_+)(\tau_1) < 0.$$

Define

$$G(r) = \exp\Big(\int_{r_0}^{r} \frac{N_d(t, \phi_\gamma) - 1}{t} dt\Big)(\phi'_\gamma u_+ - \phi_\gamma u'_+),$$

where $N_d(t, \phi_\gamma)$ is the dimension number for ϕ_γ denote $\rho := \int_{r_0}^{r} \frac{N_d(t,\phi_\gamma)-1}{t} dt.$

Now we claim that there exists a τ_2, $r_0 \leq \tau_2 < \tau_1$ such that

$$\phi'_\gamma(r) < 0 \quad \text{for all } r \in (\tau_2, \tau_1) \;\; y \;\; G(\tau_2) \geq 0.$$

If $\phi'_\gamma(r) < 0$ for all $r \in (r_0, \tau_1)$, since $G(r_0) = 0$, we conclude taken $\tau_2 = r_0$. If not, define $\tau_2 = \sup\{\tau \in [r_0, \tau_1)/\phi'_\gamma(\tau) = 0\}$.
Notice that $\tau_2 < \tau_1$ and $\phi'_\gamma(\tau_1) < 0$, so $\phi'_\gamma(r) < 0$ for all $r \in (\tau_2, \tau_1)$.
From the definition of τ_2 we have that $\phi'_\gamma(\tau_2) = 0$. Thus, $G(\tau_2) > 0$ and the claim follows. By equation satisfied by u_+ we get using the optimal condition

$$\Big\{\rho u'_+\Big\}' \leq -\frac{\mu u_+}{\theta}\rho \;\; in \; (\tau_2, \tau_1), \tag{3.23}$$

here θ is give by ϕ_α Since ϕ_γ is positive in (τ_2, τ_1), we obtain

$$G'(r) \geq \frac{\rho}{\theta} u_+(r) g(r) > 0$$

for all $r \in (\tau_2, \tau_1)$, so we get a contradiction.
The case when $\gamma \leq 0$ are quite analogous. All the above shows that there is no solution for (3.19), (3.20) y (3.21).

Now we can compute the Leray-Schauder degree in the radial case and get.

Proposition 3.1 *Let $r > 0$, $\overline{\alpha}$,$\mu \in \mathbb{R}$. Then*

$$deg_C(I - \mu\mathcal{L}_{\overline{\alpha}}, B(0,r), 0) = \begin{cases} 1 & \text{if } \mu < \mu_1^+(\overline{\alpha}), \\ 0 & \text{if } \mu_k^+(\overline{\alpha}) < \mu < \mu_k^-(\overline{\alpha}) \\ & \quad \text{or if } \mu_k^-(\overline{\alpha}) < \mu < \mu_k^+(\overline{\alpha}), \\ (-1)^k & \text{if } \mu_k^+(\overline{\alpha}) < \mu < \mu_{k+1}^-(\overline{\alpha}) \\ & \quad \text{or } \mu_k^-(\overline{\alpha}) < \mu < \mu_{k+1}^+(\overline{\alpha}). \end{cases} \tag{3.24}$$

where $C := \{u \in C([0,1]) | \ u(0) = 1, \ u'(0) = 0\}$.

Remark 3.6 *1) For $k \in \mathbb{N}$, $k \geq 1$, we do not expect in general*

$$\mu_k^+ \leq \mu_k^-.$$

2) If $\mu_k^+ = \mu_k^-$, the case $deg_{\mathcal{C}}(I - \mu\mathcal{L}_\alpha^+, B(0,r), 0)$ is not present in this proposition.

Proof Assume first that $\mu_k^+(\overline{\alpha}) < \mu < \mu_{k+1}^-(\overline{\alpha})$ or $\mu_k^-(\overline{\alpha}) < \mu < \mu_{k+1}^+(\overline{\alpha})$. Since $\mu_j^\pm(\alpha)$ are continuous functions of α we find a continuous function $\bar{\mu} : \ [0,1] \to \mathbb{R}$ such that $\max\{\mu_k^+(\alpha), \mu_k^-(\alpha)\} < \bar{\mu}(\alpha) < \min\{\mu_{k+1}^+(\alpha), \mu_{k+1}^-(\alpha)\}$ and $\bar{\mu}(\overline{\alpha}) = \mu$. The invariance of the Leray-Schauder's degree under compact homotopies implies

$$d(\lambda) = (I - \bar{\mu}(\alpha)\mathcal{L}_{\overline{\alpha}}^+, B(0,r), 0) = constant,$$

for $\alpha \in [0,1]$. In particular $d(\overline{\alpha}) = d(0) = (-1)^k$ and the result follows.
The case $\mu < \mu_1^+(\overline{\alpha})$ is direct homotopy with the identity map. In the case $\mu_k^+(\overline{\alpha}) < \mu < \mu_k^-(\overline{\alpha})$ or $\mu_k^-(\overline{\alpha}) < \mu < \mu_k^+(\overline{\alpha})$, we will prove that $(I - \mu\mathcal{L}_{\overline{\alpha}}^+)(B(r,0))$ is not a neighborhood of zero.
Suppose by contradiction that $(I - \mu\mathcal{L}_{\overline{\alpha}}^+)(B(r,0))$ is a neighborhood of zero. Then, for any smooth h with $\|h\|_{C([0,1])}$ small, there exist a solution u for

$$u - \mu\mathcal{L}_{\overline{\alpha}}^+ u = h \tag{3.25}$$

In particular we can take h being the solution to

$$\mathcal{M}_{\mathcal{C}}^+(D^2 h) = \psi \quad in \quad \Omega \quad and \quad h = 0 \quad on \quad \partial\Omega,$$

where $\|\psi\|_{C([0,1])} > 0$ is small enought. Now from the properties of the maximal operator and the definition of $\mathcal{L}_{\overline{\alpha}}^+$, we obtain

$$\mathcal{M}_{\mathcal{C}}^+(D^2(u)) + \mu(u) \leq \psi,$$

Taking $\psi = g$, where g is a function of the type used in Lemma 3.4, we will wet that u satisfies (3.19), (3.22)) and (3.21). Then, we wet a contradiction with Lemma 3.4, see also remark below it. So, $deg_C(I - \mu\mathcal{L}^{+}_{\alpha}, B(r,0), 0) = 0$ in this case, and the proof is finished. □

Proof of Theorem 1.4. Using the same argument of Rabinowitz (change of degree produce solution, see [5] for this setting), we obtain the existence of a half-component $H_k^+ \subset \mathbb{R} \times C([0,1])$ of radially symmetric solutions of (1.4), whose clausure $\overline{H}_k^+$ contains $(\mu_k^+, 0)$, and is either unbounded or contains a point $(\mu_j^{\pm}, 0)$, with $j \neq k$ in the case of μ_j^+.
Let us first prove that if $H_k^+ \subset S_k^+$, By the convergence to the eigenfunction we find a neighborhood N of $(\mu_k^+, 0)$ such that $N \cup H_k^+ \subset S_k^+$.
Moreover, from Lemma 3.2 we can extend the previous local properties of H_k^+ to all of it. Hence, H_k^+ must be unbounded. □

In the rest of the paper we will apply Theorem 1.4 to prove existence of the problem (1.1) and prove Theorem 1.5 and its corollary.
For the prove we will need the following Sturm comparison Lemma.

Lemma 3.5 *Let $a, b \in L^\infty(0,1)$ with $a \geq b$ in $(0,1)$. Assume that $u, v \in C^2[0,1] \setminus \{0\}$, $u'(0) = v'(0) = 0$, and respectively satisfy*

$$
\begin{aligned}
-\Big\{\rho_u(r)u'(r)\Big\}' &= \tilde{\rho}_u(r)u(r)a(r) \quad a.e. \quad (0,1),\\
-\Big\{\rho_v(r)v'(r)\Big\}' &= \tilde{\rho}_v(r)v(r)b(r) \quad a.e. \quad (0,1),
\end{aligned}
$$

where $\rho_u(r)$ denote the integral factor of the equation , $\tilde{\rho}_u(r) := \rho_u(r)/\theta$, ρ_u and θ are characterized by the the optimal condition.
Then

i) If v has a zero in $(0,1)$, then u does too. The first zero of u is less than or equal to the first zero of v.

ii) If $(r_0, r_1) \in [0,1]$, $v(r_0) = v(r_1) = 0$, $u(r) \neq 0$, $r \in (r_0, r_1)$ and $a \geq b$ in some subset of (r_0, r_1), then u has at least one zero in (r_0, r_1).

Proof. First we consider the case when $u, v > 0$ in the corresponding interval for i) and ii). To prove i), suppose that u do not have a zero, then using the definition of the maximal operator we have

$$
\begin{aligned}
-\Big\{\rho_v(r)u'(r)\Big\}' v(r) &\geq \tilde{\rho}_v(r)a(r)u(r)v(r),\\
-\Big\{\rho_v(r)v'(r)\Big\}' u(r) &= \tilde{\rho}_v(r)b(r)u(r)v(r).
\end{aligned}
\tag{3.26}
$$

Now integrating in $[0, r_0]$ where $v(r_0) = 0$, and then sustracting the first equation from the second one, we obtain

$$-\rho_v(r_0)v'(r_0)u(r_0) - \rho_v(0)v'(0)u(0) \ \leq \ \int_0^{r_0} \tilde{\rho}_v(s)u(s)v(s)(b(s) - a(s))ds,$$

getting a contradiction.
To prove ii), supose that u do not have a zero in $[r_0, r_1]$, then integrating (3.26) in $[r_0, r_1]$, then we have that

$$-\rho_v(r_1)v'(r_1)u(r_1) + \rho_v(r_0)v'(r_0)u(r_0) \ \leq \ \int_{r_0}^{r_1} \tilde{\rho}_v(s)u(s)v(s)(b(s) - a(s))ds,$$

getting a contradiction.
Now if we consider the case when $u < 0$ and $v > 0$ in the corresponding interval for i) and ii), to prove i), using the same argument we have that

$$\begin{aligned} -\Big\{\rho_u(r)u'(r)\Big\}'v(r) \ &= \ \tilde{\rho}_u(r)a(r)u(r)v(r), \\ -\Big\{\rho_u(r)v'(r)\Big\}'u(r) \ &\geq \ \tilde{\rho}_u(r)b(r)u(r)v(r). \end{aligned} \tag{3.27}$$

Now we can use a similar Sturm type argument as in the previous case.
The others cases are similar, using in an apropiate form the definition of the maximal operator to get the same integral factor in both inequalities . □

Proof of Theorem 1.5. If we denote by $\underline{\lambda} = lim_{s \to 0} g(s)/s$, the we can write $g(s) = \underline{\lambda}s + f(s)$, with $f(s) = o(|s|)$ near $s = 0$. Consider the bifurcation problem

$$\begin{aligned} -\Big\{\rho_v(r)v'\Big\}' = \tilde{\rho}_v(r)(\lambda v + f(r)) \qquad & r \in (0,1) \\ v'(0) = 0, \qquad & v(1) = 0, \end{aligned} \tag{3.28}$$

and H_j^+ for $k \leq j \leq n$ be the set of nontrivial solutions of (3.28) given by Theorem 1.4.
Let $(\lambda, v) \in H_j^+$ and $C = \mu_j + sup_{s \in \mathbb{R}}|g(s)/s|$. Then we claim that $\lambda \leq C$. Indeed, observe that the pair (λ, v) satisfy

$$\begin{aligned} -\Big\{\rho_v(r)v'\Big\}' = b(r)\tilde{\rho}_v(r)v \qquad & r \in (0,1), \\ v'(0) = 0, \qquad v(1) = 0, & \end{aligned}$$

where $b(r) = \lambda + g(v(r))/v(r)$. Suppose $\lambda > C$. Then $b(r) > \mu_j$ for $r \in (0,1)$. Using Lemma (3.5) we have that v has at least j zeros in $(0,1)$, this is impossible, then $\lambda \leq C$.
Now, if $\lambda \in [\underline{\lambda}, C]$, we claim that exist $M > 0$ such as $(\lambda, u) \in H_j^+$ then $\|u\|_{C[0,1]} \leq M$. Indeed, suppose by contradiction, then exist a sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ in $[\underline{\lambda}, C]$ and

$\{u_n\}_{n\in\mathbb{N}}$ in $C^1[0,1]$ such as $(\lambda_n, u_n) \in H_j^+$, where $\lambda_n \to \lambda_0$ and $\|u\_n|_{L^\infty} \to \infty$. Define now $\hat{u}_n = u_n/\|u_n\|_{L^\infty}$, then $\hat{u}_n$ satisfy the equation

$$-\Big\{\rho_{\hat{u}_n}(r)\hat{u}_n'\Big\}' = \tilde{\rho}_{\hat{u}_n}\hat{u}_n\Big(\lambda_n + \frac{f(u_n)}{u_n}\Big). \tag{3.29}$$

We can assume, up to a subsequence, that $\hat{u}_n \to \hat{u}$ in C^2. Also, from the boundedness of $g(s)/s$, we obtain that the sequence $\{f(u_n)/u_n\} \to h \in L^\infty$. Taking limits in (3.29) we obtain a solution of

$$-\Big\{\rho_{\hat{u}}(r)\hat{u}'\Big\}' = \tilde{\rho}_{\hat{u}}(r)\hat{u}h(r), \qquad a.e \quad (0,1).$$

Observe that if $\hat{u}(r) \neq 0$, then $u_n(r) \to \pm\infty$ when n goes to $+\infty$. If this is not the case, then $\hat{u}_n(r) = u_n(r)/\|u_n\|_{L^\infty} \to 0$, which is a contradiction. Then we have that $u_n \to \pm\infty$ a.e. in $(0,1)$ when n goes to $+\infty$ using Lemma 3.2 for $\hat{u}$.
We claim that $h(r) > \mu_j$ a.e on $(0,1)$. Indeed, since $f(s)/s + \lambda_0 = g(s)/s + \lambda_0 - \underline{\lambda}$, $\lambda_0 - \underline{\lambda} \geq 0$ and $\mu_j < liminf_{|s|\to\infty} g(s)/s$ we obtain the result.
Now using again Lemma 3.5 we get a contradiction comparing the numbers of zeros. □

Proof of Corollary 1.1. We only need to prove that a solution u_1 positive at the origin is different from minus a negative solution at the origin u_2. Suppose by contradiction that $u_1 = -u_2$, then using that g is odd and the equation satisfies by u_1 and u_2 we fine

$$\mathcal{M}_{\mathcal{C}}^-(D^2u_1) = \mathcal{M}_{\mathcal{C}}^+(D^2u_1) \quad in \quad B_1.$$

This is a contradiction since $\mathcal{C}$ and u_1 are both not trivial . □

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