CRITICAL EXPONENTS FOR UNIFORMLY ELLIPTIC EXTREMAL OPERATORS

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1 Introduction

A cornerstone in the study of nonlinear elliptic partial differential equations is

$$\Delta u + u^p = 0, \quad u > 0 \quad \text{in} \quad I\!R^N, \tag{1.1}$$

for which a complete description of the solutions depending on the exponent p is known. The main result is the existence of a number

$$p_N^* = (N+2)/(N-2),$$

known as critical exponent, such that when $1 < p < p_N^*$ no solution to equation (1.1) exists, when $p = p_N^*$ then, up to scaling, equation (1.1) possesses exactly one solution whose behavior at infinity is like $|x|^{-(N-2)}$ and when $p > p_N^*$ then equation (1.1) admits radial solutions with behavior at infinity like $|x|^{-\alpha}$, for $\alpha = 2/(p-1)$.

In the proof of these basic results various important tools has been developed, such as the celebrated Pohozaev identity, energy integrals, the moving planes technique based on the maximum principle, the Kelvin transform and Harnack inequalities. In this respect the work by Pohozaev [29], Serrin [33], Gidas, Ni and Nirenberg [18], Caffarelli, Gidas and Spruck in [5], Gidas and Spruck [20], Chen and Li [6] and, Serrin and Zou [34] have been fundamental.

In the recent article [15], the authors considered a similar equation but replacing the Laplacian by a Pucci's extremal operator

$$\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) + u^p = 0, \quad u > 0 \quad \text{in} \quad I\!R^N, \tag{1.2}$$

with parameters $0 < \lambda \leq \Lambda$ and $p > 1$. While the Pucci's extremal operators share many properties with the Laplacian, like homogeneity, maximum principle and others, they divert from it in a fundamental manner. In fact the Pucci's extremal operators do not have divergence form and they do not possesses the equivalent of the Kelvin transform, preventing the use of many crucial tools in the analysis of the equation. Using different techniques in [15] the authors considered the problem restricted to the radially symmetric case and proved the existence of a critical exponent p^* playing the role of the critical exponent p_N^* for the Laplacian.

In the case of the operator $\mathcal{M}^+_{\lambda,\Lambda}$, the dimension like number

$$\tilde{N}_+ = \frac{\lambda}{\Lambda}(N-1) + 1 \tag{1.3}$$

plays an important role. In fact, the authors prove in [15] that the critical exponent p^* satisfies

$$\max\{\frac{\tilde{N}_+}{\tilde{N}_+ - 2}, p_N^*\} < p^* < \frac{\tilde{N}_+ + 2}{\tilde{N}_+ - 2}.$$

The Pucci's extremal operators represent somehow the simplest version of a fully non-linear, autonomous uniformly elliptic operators, and the techniques developed in [15] are seemingly devised to treat this very particular operator.

It is the purpose of this article to present the analysis of critical exponents, in the case of radially symmetric solutions, for a large class of extremal operators, extending and deepening the understanding started in [15]. More precisely, let $D \subset \mathbb{R}^2_+$ be a nonempty, compact and convex set, then we define the operator $\mathcal{M}$ acting on C^2 radially symmetric functions as

$$\mathcal{M}(D^2u) = \sup_{(a_1,a_2)\in D} \frac{(N-1)}{r}u'a_1 + u''a_2. \tag{1.4}$$

We consider then the study of radially symmetric solutions of the nonlinear equation

$$\mathcal{M}(D^2u) + u^p = 0, \quad u > 0 \quad \text{in} \quad \mathbb{R}^N. \tag{1.5}$$

We prove the existence of a critical exponent p^* that determines the range of $p > 1$ for which we have existence or non-existence of radial solution to (1.5).

Given $N \geq 2$ and D as above, in Section 3 we define two dimension-like numbers N_∞ and N_0, that depend explicitly on N and D. These numbers satisfy $0 < N_\infty \leq N_0$ and play a crucial role in estimating the critical exponent. Now our main theorem can be stated

Theorem 1.1 *If we assume $N \geq 2$ and $N_\infty > 2$ then there is a critical exponent p^* with*

$$\max\{\frac{N_\infty}{N_\infty - 2}, p_0\} \leq p^* \leq p_\infty, \tag{1.6}$$

where

$$p_0 = \frac{N_0+2}{N_0-2}, \qquad p_\infty = \frac{N_\infty+2}{N_\infty-2}, \tag{1.7}$$

and the following statements hold:

i) If $1 < p < p^*$ then there is no radial solution to (1.5).

ii) If $p = p^*$ then there is a unique radial solution of (1.5) whose behavior at infinity is like $r^{-(N_\infty-2)}$.

iii) If $p^* < p$ then there is a unique radial solution to (1.5) whose behavior at infinity is like $r^{-\alpha}$.

In ii) and iii) uniqueness is meant up to scaling.

Remark 1.1 We prove this theorem in Section 6. Actually we will further classify the solutions behaving like $r^{-\alpha}$ in some cases and give some examples.

Remark 1.2 The theorem we just stated deals with maximal operators. A completely analogous result can be obtained for minimal operators. See Section 6.

Remark 1.3 Our theorem answer partially a conjecture raised in [13]. If was conjectured that for any uniformly elliptic operator F, with ellipticity constants between $0 < \lambda < \Lambda$ such that

$$\mathcal{M}^-_{\lambda,\Lambda}(M) \le F(M) \le \mathcal{M}^+_{\lambda,\Lambda}(M)$$

for all symmetric matrices, there exists a unique critical exponent p_F such that

$$\frac{\tilde{N}_- + 2}{\tilde{N}_- - 2} \le p_F \le \frac{\tilde{N}_+ + 2}{\tilde{N}_+ - 2}$$

where $\tilde{N}_+$ was defined above and $\tilde{N}_- = (\Lambda/\lambda)(N-1)+1$. In this article we prove the conjecture is valid for all maximal and minimal operators as defined above, in the radial case, with $D \subset [\lambda,\Lambda]^2$. We notice that $\tilde{N}_+ \le N_\infty \le N_0 \le \tilde{N}_-$.

At this point we would like to put our class of radially symmetric operators in perspective. Let $0 < \lambda \le \Lambda$ and $\Delta \subset [\lambda,\Lambda]^N$ be a symmetric set, that is a set satisfying $a = (a_1, a_2, ..., a_N) \in \Delta$ if and only if $a_\pi = (a_{\pi(1)}, a_{\pi(2)}, ..., a_{\pi(N)}) \in \Delta$, for all permutation π. If u is a C^2 function, we let $d = (d_1, ..., d_n)$ be the eigenvalues of the Hessian matrix $D^2u(x)$ and define the maximal operator

$$\mathcal{M}_\Delta(D^2u) = \sup_{a\in\Delta} \; a \cdot d. \tag{1.8}$$

This class of operators includes the Pucci's operators. In fact, if we consider $\Delta = [\lambda,\Lambda]^N$ we recover the Pucci's operators $\mathcal{M}^+_{\lambda,\Lambda}$. Alternatively, if we take $c \in [\lambda,\Lambda]$ and $\Delta_c = \{a \in [\lambda,\Lambda]^N \;/\; \sum_{i=1}^N a_i = cN\}$, we obtain the Pucci's operators $P^+ = \mathcal{M}_{\Delta_c}$, see [27] and [28]. In any case, if $\lambda = \Lambda$ we recover the Laplacian. Thus, the operators $\mathcal{M}_\Delta$ define a large class of uniformly elliptic, nonlinear, autonomous operators.

A major open problem is to understand the structure of the solutions to the nonlinear equation

$$\mathcal{M}_\Delta(D^2u) + u^p = 0, \quad u > 0 \quad \text{in} \quad I\!R^N, \tag{1.9}$$

without further assumptions on u. At this point this problem seems extremely hard, so we specialize to radial solutions. When u is radially symmetric, the Hessian matrix D^2u has u'' and u'/r as eigenvalues, the first is simple and the second has multiplicity $N-1$. Thus, if we define $D(\Delta) = \{(\frac{1}{N-1}\sum_{i=1}^{N-1} a_i, a_N) \in I\!R^2 \,/\, a \in \Delta\}$, we are back to the radial maximal operators, to which Theorem 1.1 applies.

Remark 1.4 It can be easily proved that any convex nonlinear elliptic operator F depending only on D^2u, with ellipticity constants $0 < \lambda < \Lambda$ and which is homogeneous of degree one, can be represented as a maximal operator M_Δ for an appropriate Δ.

Further discussion on the meaning of the non-existence results we are proving comes into place. It is interesting to mention that the nonexistence of solutions to (1.1) when $1 < p < p_N^*$ holds even if we do not assume a given behavior at infinity. This result is known as Liouville type theorem and it was proved by Gidas and Spruck [20] and by Chen and Li [6]. When $1 < p \leq N/(N-2)$ then a Liouville type theorem is known for super-solutions of (1.1), that is solutions of the inequality

$$\Delta u + u^p \leq 0, \quad u \geq 0 \quad \text{in} \quad I\!R^N \tag{1.10}$$

Moreover, it is known that this exponent is optimal, in the sense that solutions to (1.1) exist if $p > N/(N-2)$. This number is called sometimes the second critical exponent for (1.1). See [17].

In this direction we have the paper [10], where Cutri and Leoni extended this result for the Pucci's extremal operators. They consider the inequality

$$\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) + u^p \leq 0, \quad u \geq 0 \quad \text{in} \quad I\!R^N \tag{1.11}$$

and then they prove that for $1 < p \leq \tilde{N}_+/(\tilde{N}_+ - 2)$ equation (1.11), with the operator $\mathcal{M}^+_{\lambda,\Lambda}$, has no non-trivial solution. A similar statement holds for $\mathcal{M}^-_{\lambda,\Lambda}$.

Liouville type theorems are the basis of existence results in bounded domains via degree theory. Actually, the success of this approach depends on a priori bounds for the positive solutions of the equation, and this a priori bounds are obtained by a blow-up technique from the Liouville type theorems.

Regarding existence results in the line discussed above we have the following result

Theorem 1.2 *Assume $N \geq 2$ and $N_\infty > 2$. Let $R > 0$ and $B = B(0, R) \subset I\!R^N$ be the ball of radius R centered at the origin, then the equation*

$$\begin{aligned} \mathcal{M}(D^2u) + u^p = 0 \quad in \quad B, \\ u(x) = 0 \ \ x \in \partial B, \qquad u > 0 \quad in \quad B, \end{aligned} \tag{1.12}$$

has a radial solution if and only if $1 < p < p^*$. Moreover, if $p > N_0/(N_0 - 2)$, then the solution is unique.

This theorem is a model for more general existence results that can be proven once the Liouville type theorem, that is the critical exponent, is understood. We do not pursue this line, but we mention existence results obtained in the case of the Pucci's extremal operators by Felmer and Quaas [16] and Quaas [30]. It is of special interest the recent results by Quaas and Sirakov [31] on the existence of first eigenvalues for a large class of elliptic operators, which could be combined with the Liouville type theorem in order to obtain existence results for a larger class of non-linearities.

Extremal operators appear in the context of stochastic control when the diffusion coefficient is a control variable, see the book of Bensoussan and J.L. Lions [1] or the papers of P.L. Lions [23], [24], [25] for the relation between a general Hamilton-Jacobi-Bellman and stochastic control.

In the study of equation (1.1) a crucial role is played by the Pohozaev identity. Since the extremal operators do not have a divergence form, this kind of identity is no longer available, posing a special difficulty to the problem. However, since we consider only radial solutions, these operators take a simpler form where we can still take advantage of some techniques developed for equations with operators in divergence form.

Our approach consists in a combination of the Emden-Fowler phase plane analysis with the Coffman-Kolodner technique. We start considering the classical Emden-Fowler transformation that allows us to view the problem in the phase plane. Here is where the main differences with the work in [15] appears. In fact here we cannot use any form of energy argument and the initial value problem poses non-trivial difficulties. We rely more on linearized stability, the Dulac principle and the Poincaré-Bendixon theorem to understand the asymptotic behavior. A phase plane analysis has been used in related problems by Kajikiya [21] and Erbe and Tang [12] among many others.

We continue with the use the Coffman-Kolodner technique. Originally introduced by Kolodner [8] and later used by Coffman [7], this technique consists in the study of the solution of an associated initial value problem, obtained differentiating the solution with respect to the initial value. The function so obtained possesses valuable information on the problem. This idea has been used by several authors in dealing with uniqueness questions. We cite in particular the work of Kwong [22], Kwong and Zhang [26] and Erbe and Tang [12]. In our case though we do not differentiate with respect to the initial value, which is kept fixed, but with respect to the power p. Thus the variation function satisfies a non-homogeneous equation, in contrast with the situations treated earlier.

Our article is organized as follows. In Section 2. we discuss preliminary properties of the operator $\mathcal{M}$ in terms of the set D and its analytical description. In Section 3. we prove an existence and uniqueness result for the initial value problem associated to (1.5). In order to prove the uniqueness we use the Emden-Fowler transformation and a form of the stable-unstable manifold theorem. In Section 4. we further study the dynamical system obtained through the Emden-Fowler transformation. We understand the asymptotic behavior of the solutions, especially when p is outside the range defined by p_0 and p_∞. In section 5. we analyze the system from the point of view of the Coffman-Kolodner technique. We study the variation of the solution with respect to the exponent p. This is a crucial step in obtaining the *uniqueness* of the critical exponent. Here we use ideas coming from [26] and [12]. Finally in Section 6. we prove Theorem 1.1 and Theorem 1.2. We also provide some extensions and give examples.

Notation: $\mathbb{R}_- = (-\infty, 0)$, $\mathbb{R}_+ = (0, \infty)$.

2 Preliminaries

In this section we consider the class of radial maximal operators defined in the introduction in terms of the set D. In order to describe the set D in a more convenient way, and to avoid trivialization, we make a further assumption.

D) The set $D \subset \mathbb{R}^2_+$ is compact, convex and its projection onto the y-axis is not a singleton.

Without mentioning, we assume this condition D) along the paper up to Section 5.

Remark 2.1 Assuming D) we exclude the case when the projection of D onto the y-axis is a singleton, which is equivalent to $D = \{(a_1, a_2)\}$. This particular case can be analyzed as the radial Laplacian, just considering a constant dimension $\tilde{N} = (N-1)a_1/a_2 + 1$. On the other hand, when D satisfies D), we may think that the dimension varies along the solution in a nonlinear way, and it is this situation what make the problem interesting.

Under the assumption D) we may describe ∂D by means of functions. Let $0 < \theta_- < \theta_+$ be defined as $\theta_- = \min\{\theta \,/\, (x, \theta) \in D\}$ and $\theta_+ = \max\{\theta \,/\, (x, \theta) \in D\}$, and define the functions $S, \tilde{S} : [\theta_-, \theta_+] \to \mathbb{R}_+$ as

$$S(\theta) = \min\{x \,/\, (x, \theta) \in D\} \quad \text{and} \quad \tilde{S}(\theta) = \max\{x \,/\, (x, \theta) \in D\}.$$

With these definitions we see that S is convex, $\tilde{S}$ is concave and

$$D = \{(x, \theta) \,/\, \theta \in [\theta_-, \theta_+],\ S(\theta) \le x \le \tilde{S}(\theta)\}.$$

In the study of positive solutions to (1.5) for the case of maximal operators, as we will see later, we only need to look at the left hand side of D, which is described by S. Being S convex, it has one-sided derivatives $S'_-(\theta)$ and $S'_+(\theta)$ and consequently it is locally Lipschitz continuous in (θ_-, θ_+). The sub-differential of S is then defined as $\partial S(\theta) = [S'_-(\theta), S'_+(\theta)]$, for $\theta \in (\theta_-, \theta_+)$, and we see that S is differentiable at θ if and only if $\partial S(\theta)$ is a singleton. The cases $\theta = \theta_-$ and $\theta = \theta_+$ are special. At θ_+ we have two possibilities, either $S'_-(\theta_+)$ exists, and then we define $\partial S(\theta_+) = [S'_-(\theta_+), +\infty)$, or

$$\lim_{t \to 0_+^-} \frac{S(\theta_+ + t) - S(\theta_+)}{t} = +\infty.$$

An analogous situation occurs at θ_-. We observe that with these definitions, for every $Q < 0$ there is at least one solution $\theta \in [\theta_-, \theta_+]$ of the equation

$$\partial S(\theta)\theta - S(\theta) \ni Q. \tag{2.1}$$

The case when this equation has multiple solutions is very important for our analysis and occurs when the function S coincide locally with an affine function. We let $\mathcal{I}$ be a set of indices so that for every $i \in \mathcal{I}$, the function S is affine in the maximal interval $[\theta_i^-, \theta_i^+]$, with $\theta_i^- < \theta_i^+$. In each of these intervals we may write

$$S(\theta) = d_i\theta - Q_i \quad \text{for all} \quad \theta \in [\theta_i^-, \theta_i^+], \tag{2.2}$$

for numbers d_i and Q_i. We notice that, depending on the shape of D, the set $\mathcal{I}$ may be empty, finite or countable.

Given equation (2.1) we define the function $d : I\!R_- \to I\!R$ as $d(Q) \in \partial S(\theta)$ such that

$$d(Q)\theta - S(\theta) = Q. \tag{2.3}$$

We would also like to define θ as a function of Q, but we cannot do it in a unique way because of the possible multiplicity of solutions to (2.1). We make a choice considering $\Theta : I\!R_- \to I\!R$ as $\Theta(Q_i) = \theta_i^+$ for $i \in \mathcal{I}$ and as the unique solution of (2.1) otherwise. In the next lemma we state some basic properties of d and Θ

Lemma 2.1 *The function Θ is strictly monotone increasing and it satisfies*

$$\lim_{Q \to Q_i^{\pm}} \Theta(Q) = \theta_i^{\pm}, \quad \textit{for every } i \in \mathcal{I}.$$

The function d is Lipschitz continuous, strictly monotone increasing, with derivative

$$d'(Q) = \frac{1}{\theta} \qquad \textit{for} \quad Q \notin \{Q_i \,/\, i \in \mathcal{I}\},$$

and one-sided derivatives

$$d'_+(Q_i) = \frac{1}{\theta_i^+} \quad \textit{and} \quad d'_-(Q_i) = \frac{1}{\theta_i^-} \qquad \textit{for} \quad i \in \mathcal{I}. \tag{2.4}$$

Proof. We first observe that d and Θ are strictly monotone functions because $\partial S(\theta)\theta - S(\theta)$ is increasing (as a graph), by the convexity of S.

The limit properties of Θ are straight forward, so we only prove the differentiability properties of d. We let $Q > \bar{Q}$ and we consider the equations

$$d(Q)\theta - S(\theta) = Q \quad \text{and} \quad d(\bar{Q})\bar{\theta}^{+} - S(\bar{\theta}^{+}) = \bar{Q}, \tag{2.5}$$

satisfied by $\theta \in [\theta^-, \theta^+]$ and $\bar{\theta} \in [\bar{\theta}^-, \bar{\theta}^+]$. We notice that as Q approaches $\bar{Q}$ we have that θ approaches $\bar{\theta}^+$, for all $\theta \in [\theta^-, \theta^+]$. Next we use (2.5) to find

$$-\bar{\theta}^{+}\left[\frac{d(Q)-d(\bar{Q})}{Q-\bar{Q}} - \frac{1}{\bar{\theta}^{+}}\right] = (\theta - \bar{\theta}^{+})/\left[\theta - \bar{\theta}^{+} + \bar{\theta}^{+}\{\frac{d(Q)-d(\bar{Q})}{d(Q) - \frac{S(\bar{\theta}^{+})-S(\theta)}{\bar{\theta}^{+}-\theta}}\}\right].$$

But, since S is convex we have

$$d(\bar{Q}) \leq \frac{S(\bar{\theta}^{+}) - S(\theta)}{\bar{\theta}^{+} - \theta}$$

then

$$|\frac{d(Q)-d(\bar{Q})}{Q-\bar{Q}} - \frac{1}{\bar{\theta}^{+}}| \leq \frac{\theta - \bar{\theta}^{+}}{\theta\bar{\theta}^{+}},$$

proving that d has a derivative from the right and $d'_{+}(\bar{Q}) = \frac{1}{\bar{\theta}^{+}}$. A similar argument can be used to prove that d has a derivative from the left, and that when $\bar{\theta}^{+} = \bar{\theta}^{-}$ the function d is differentiable at $\bar{Q}$. This completes the proof. $\square$

Now we define the function $g : I\!R \times I\!R_{-} \to I\!R$ as

$$g(u, w) = -w(N-1)\, d(\frac{u^p}{(N-1)w}), \tag{2.6}$$

which is locally Lipschitz continuous. This function can be written making θ explicit as

$$g(u, w) = -\frac{S(\theta)}{\theta}(N-1)w - \frac{u^p}{\theta}, \tag{2.7}$$

where θ is a any solution of (2.1), with $Q = u^p/(N-1)w$. In particular we may use $\theta = \Theta(u^p/(N-1)w)$, and in this case we write

$$\tilde{N} = (N-1)S(\theta)/\theta + 1. \tag{2.8}$$

The regularity properties of g are determined by those of d, so we have the following

Proposition 2.1 *The function g is Lipschitz continuous in $I\!R \times I\!R_{-}$. Outside the curves*

$$Q_i = \frac{u^p}{(N-1)w}, \quad i \in I, \tag{2.9}$$

the function g is differentiable and its partial derivatives are given by

$$\frac{\partial g}{\partial w} = -\frac{S(\theta)}{\theta}(N-1) \quad \textit{and} \quad \frac{\partial g}{\partial u} = -\frac{pu^{p-1}}{\theta}.$$

At points on the curve (2.9), in transversal directions, directional derivatives always exists. If $\vec{d}$ points into the region above the curve (2.9) then the directional derivative is

$$D(g, \vec{d}) = -(\frac{S(\theta_i^+)}{\theta_i^+}(N-1), \frac{pu^{p-1}}{\theta_i^+}) \cdot \vec{d}.$$

If $\vec{d}$ points into the region below the curve (2.9) we replace θ_i^+ by θ_i^-.

Proof. Direct from previous lemma. □

3 Emden-Fowler Analysis: the initial value problem

We devote this section to study the the initial value problem

$$\begin{aligned} \mathcal{M}(D^2 u) + u^p &= 0, \quad r > 0, && (3.1) \\ u(0) = \gamma, u'(0) &= 0, && (3.2) \end{aligned}$$

for $\gamma > 0$, where $\mathcal{M}$ is a radial maximal operator associated to a set D satisfying condition D). We want to understand the existence and uniqueness theory for this equation. While existence of a solution can be obtained by standard arguments, surprisingly uniqueness property requires extra arguments. We relay on the stable-unstable manifold theorem, after we transform the problem into the phase space using the Emden-Fowler transformation.

We start writing the equation in terms of the function S, solving the maximization problem. Let us assume we have a C^2 radially symmetric solution u of the equation (3.1). Then by definition of $\mathcal{M}$ and the description of D in terms of the function S, while $u'(r) < 0$, we have that

$$\mathcal{M}(D^2 u(r)) = u''(r)\theta + (N-1)\frac{u'(r)}{r}S(\theta),$$

where $\theta \in [\theta_-, \theta_+]$ is characterized by

$$\partial S(\theta)\theta - S(\theta) \ni \frac{u^p}{(N-1)(u'/r)}. \qquad (3.3)$$

We call this equation the optimality condition. Thus equation (3.1) can be written as

$$u''(r) = g(u(r), \frac{u'(r)}{r}),$$

where g was defined in (2.6), or alternatively as

$$u''(r)+\frac{S(\theta)}{\theta}(N-1)\frac{u'(r)}{r}+\frac{u^p(r)}{\theta}=0. \tag{3.4}$$

In this last form θ may be chosen as any solution of (3.3), however for our future analysis it is convenient to have θ as a function of r, so we define

$$\theta(r)=\Theta(\frac{u^p(r)r}{(N-1)u'(r)}) \tag{3.5}$$

and then the variable dimension numbers $\tilde{N}(r)$ replacing $\theta(r)$ in (2.8). The functions $\theta(r)$ and $\tilde{N}(r)$ are measurable functions, having discontinuities whenever r is so that $u^p(r)r/(N-1)u'(r)=Q_i$, with $i\in\mathcal{I}$. Moreover, both $\theta(r)$ and $\tilde{N}(r)$ are bounded and bounded away from 0.

Remark 3.1 By the definition of the maximal operator, for a solution u of equation (3.4) we have that

$$u''(r)+\frac{S(\tilde{\theta})}{\tilde{\theta}}(N-1)\frac{u'(r)}{r}+\frac{u^p(r)}{\tilde{\theta}}\leq 0, \tag{3.6}$$

for all $\tilde{\theta}\in[\theta_-,\theta_+]$.

Remark 3.2 Assume $u=u(r,p,\gamma)$ a solution to (3.1)-(3.2) then by rescaling we obtain solutions for other initial values. In fact, we have the following relation that can be proved by a straight forward computation.

$$u(r,p,\gamma_0\gamma)=\gamma u(\gamma^{1/\alpha}r,p,\gamma_0),$$

for all $\gamma_0,\gamma>0$.

Our next goal is to obtain an existence result for the initial value problem (3.1)-(3.2). The difficulty to solve this equation comes from the singularity at $r=0$ and the lack of control of the Lipschitz constant when $u'(r)/r$ is small.

Lemma 3.1 *There is an $\varepsilon>0$, so that the initial value problem (3.1)-(3.2) possesses a C^2 solution $u:[0,\varepsilon]\to\mathbb{R}$.*

Proof. We consider the space B of continuous functions $u,v:[0,\varepsilon]\to\mathbb{R}$ so that $u(0)=\gamma$ and $v(0)=0$, satisfying additionally that

$$C_1\leq-\frac{v(r)}{r}\leq C_2,\quad\forall r\in(0,\varepsilon], \tag{3.7}$$

with constants C_1 and C_2 to be made precise later. Next we define the operator $T : B \to B$ as

$$T_u(u,v)(r) = \gamma + \int_0^r v(s)ds, \tag{3.8}$$

$$T_v(u,v)(r) = -\int_0^r \frac{u^p(s)}{\theta(s)} \exp(-\int_s^r \frac{\tilde{N}(\tau)-1}{\tau} d\tau)ds, \tag{3.9}$$

where $\theta(r)$ is defined in (3.5) and $\tilde{N}(r)$ obtained from (2.8), replacing θ by $\theta(r)$. We see that given $(u,v) \in B$ we obtain the following estimates, for all $r \in (0,\varepsilon]$,

$$\gamma - C_1\frac{r^2}{2} \le T_u(u,v) \le \gamma + C_2\frac{r^2}{2},$$

and

$$\frac{(\gamma - C_1 r^2/2)^p}{\theta_+ n_+} \le -\frac{T_v(u,v)}{r} \le \frac{(\gamma + C_2 r^2/2)^p}{\theta_- n_-},$$

where n_+ and n_- are such that $n_+ \le \tilde{N}(r) \le n_-$. From here we choose $\varepsilon > 0$ small enough and C_1 and C_2 so that (3.7) holds. Thus, we have that T is well defined. Since T is compact, the Schauder fixed point theorem guarantees the existence of a fixed point (u,v) of T. To complete the proof we see that by differentiating $v = T_v(u,v)$ we find that

$$v' = g(u(r), \frac{v(r)}{r}), \qquad \text{a. e. in } (0,\varepsilon],$$

but actually the right hand side is continuous, so that u is C^2. $\square$

We notice that given a solution of the equation of (3.1)-(3.2) in $[0,\varepsilon]$ we can uniquely extend it beyond ε. Since $T_v(u,v)$ is negative, when u is positive, we have

Lemma 3.2 *The solutions of (3.1)-(3.2) are decreasing while they remain positive.*

As we already mention, in order to prove that the solution we just found is unique we should further argue through the Emden-Fowler transformation. We will also use this transformation to obtain asymptotic behavior of the solutions in the next section.

We consider the change of variables $x(t) = r^\alpha u(r), r = e^t$. With this transformation, if u is a solution of (3.1) then x satisfies

$$\ddot{x}(t) = f(x(t), \dot{x}(t)), \tag{3.10}$$

where the field f is given by

$$f(x,y) = (2\alpha+1)y - \alpha(\alpha+1)x + g(x, y - \alpha x).$$

Displaying g as in (2.7), we may write f as

$$f(x,y) = -\tilde{a}y + \tilde{b}x - \frac{x^p}{\theta},$$

with $\tilde{a} = \tilde{N} - 2 - 2\alpha$, $\tilde{b} = \alpha(\tilde{N} - 2 - \alpha)$ and

$$\theta = \Theta(\frac{x^p}{(N-1)(y-\alpha x)}),$$

where $\tilde{N}$ is as in (2.8). If we have a solution of (3.10) we may think $\tilde{a}$, $\tilde{b}$, θ and $\tilde{N}$ as functions of t.

Thus, given a solution u of the initial value problem (3.1) we have the function $x(t)$ that satisfies (3.10) and associated to it we have the corresponding orbit $(x(t), \dot{x}(t))$ in the phase plane. Moreover, if u satisfies the initial condition (3.2) then

$$\lim_{t \to -\infty} (x(t), \dot{x}(t)) = (0,0). \tag{3.11}$$

In order to analyze the possible values of the dimension like numbers $\tilde{N}(t)$ and to understand the asymptotic behavior of the solutions, at infinity and minus infinity we define the extreme dimensions N_0 and N_∞. We consider the equation

$$\partial S(\theta_0) \ni -\frac{1}{N-1}.$$

If this equation has more than one solution, we take θ_0 as the right extreme of the interval of solutions. Then we define

$$N_0 = (N-1)\frac{S(\theta_0)}{\theta_0} + 1 \quad \text{and} \quad Q_0 = -\frac{\theta_0 N_0}{(N-1)}.$$

We also consider the equation

$$\partial S(\theta_\infty) \ni \frac{S(\theta_\infty)}{\theta_\infty}$$

and if this equation has more than one solution, we take θ_∞ as the left extreme of the interval of solutions. We define

$$N_\infty = (N-1)\frac{S(\theta_\infty)}{\theta_\infty} + 1.$$

We observe that by hypotheses on the set D we have that $N_0 \geq N_\infty$. The following preliminary lemma gives us the possible range of values for $\tilde{N}(t)$.

Lemma 3.3 *Given a solution x of (3.10) with (3.11). Then, while $x(t) > 0$, we have that*

$$N_0 \geq \tilde{N}(t) \geq N_\infty.$$

Proof. While the solution is positive, it is clear that

$$Q(t) = \frac{x^p(t)}{(N-1)(\dot{x}(t) - \alpha x(t))} \tag{3.12}$$

is negative and it never vanishes, so that $\tilde{N}(t) \leq N_\infty$. To prove the other inequality it is enough to prove that any solution stays below the curve

$$y = \alpha x + \frac{x^p}{(N-1)Q_0}.$$

To do this we consider the maximality of the differential operator to find that

$$(r^{N_0-1}u')' \leq -r^{N_0-1}\frac{u^p}{\theta_0},$$

from where, integrating and using the fact that u is decreasing, we find

$$u' \leq -\frac{ru^p}{N_0\theta_0}.$$

The claim follows from here, doing the Emden-Fowler transformation. □

Remark 3.3 Depending on the dimension N and on the set D it may happen that $N_0 = N_\infty$. In this case the operator $\mathcal{M}$ becomes a constant coefficient operator, and the study of critical exponents trivialize, see Remark 2.1.

A more precise behavior of $\tilde{N}(t)$ for t approaching $-\infty$ is given in the following

Proposition 3.1 *Given a solution u of the initial value problem (3.1)-(3.2), that is a solution x of (3.10) with (3.11) we have*

$$\lim_{t\to-\infty} \tilde{N}(t) = N_0 \quad \textit{and} \quad \lim_{t\to-\infty} Q(t) = Q_0.$$

In order to prove this proposition we will first prove a transversality lemma for all solutions of the initial value problem. This lemma is interesting in itself.

Lemma 3.4 *Let x be a solution of (3.10). Then for every $Q \in (Q_0, 0)$, if the orbit $(x(t), \dot{x}(t))$ touches the curve*

$$y = \alpha x + \frac{x^p}{(N-1)Q}, \tag{3.13}$$

then either:

a) it crosses the curve transversally or

b) it stays below the curve in a neighborhood of the touching point, if the crossing occurs in the first quadrant or

c) it stays above the curve in a neighborhood of the touching point, if the crossing occurs in the fourth quadrant.

Proof. Let us assume that there is t such that $(x(t), \dot{x}(t))$ crosses the curve (3.13). Then, from the equation (3.10) we have

$$\frac{dy}{dx} = (2\alpha + 1 - (N-1)d(Q)) - \alpha(\alpha + 1 - (N-1)d(Q))\frac{x}{y}.$$

On the other hand, the slope of the curve (3.13) at this point is given by

$$m = -2 + \frac{\alpha+2}{\alpha}\frac{y}{x}.$$

It is convenient to write $z = \frac{y}{x}$ and $\rho = 1 - (N-1)d(Q)$, and notice that for all $Q \in (Q_0, 0)$ we have $\rho < 2$. Now we see that the slopes will coincide at the point of intersection if and only if

$$\frac{\alpha+2}{\alpha}z^2 - (2\alpha + 2 + \rho)z + \alpha(\alpha + \rho) = 0.$$

This equation has two roots

$$\lambda_1 = \alpha\frac{\alpha+\rho}{\alpha+2} < \lambda_2 = \alpha.$$

Thus, if the crossing occurs with the same slope then it occurs at a point (x_1, y_1), with

$$x_1^{p-1} = -\alpha Q(N-1)(\frac{2-\rho}{\alpha+2}) \quad \text{and} \quad y_1 = x_1\alpha\frac{\alpha+\rho}{\alpha+2}. \tag{3.14}$$

Now we can reach our first conclusion. Let (x, y) be the point of intersection then: a) If $y > 0$ we have $x < x_1$ if and only if $\frac{dy}{dx} > m$ and b) If $y < 0$ we have $x < x_1$ if and only if $\frac{dy}{dx} < m$. This proves the transversality property.

Next we look at the case when $\frac{dy}{dx} = m$. Assume that $y_1 > 0$, that is that the crossing point is in the first quadrant. Then consider the case when, near (x_1, y_1), the solution stays at one side of the curve, thus we can differentiate the equation at (x_1, y_1) to obtain

$$\frac{d^2y}{dx^2}y + (\frac{dy}{dx})^2 = (2\alpha + \rho)\frac{dy}{dx} - \alpha(\alpha + \rho).$$

Evaluating the derivative at (x_1, y_1) we find that

$$\frac{dy}{dx} = \alpha(1 + p\frac{\rho-2}{\alpha+2}),$$

and then we find that

$$\frac{d^2y}{dx^2} = \frac{p(\rho-2)}{x_1(\alpha+\rho)}(\rho - \alpha p\frac{\rho-2}{\alpha+2}).$$

On the other hand, a direct computation shows that on the curve we have

$$\frac{d^2y_c}{dx^2} = \frac{2p}{x_1}\frac{\rho-2}{\alpha+2}.$$

Then using that $\rho < 2$ and $\alpha + \rho > 0$ we find that

$$\frac{d^2 y}{dx^2} < \frac{d^2 y_c}{dx^2}. \tag{3.15}$$

This implies that near the crossing point the orbit stays below the curve.

Finally we observe that the same argument can be given when $y_1 < 0$, with the only difference that the inequality in (3.15) is reversed, since $\alpha + \rho < 0$. This completes the proof of c). $\Box$

Remark 3.4 A consequence of this result is that given $Q \in (Q_0, 0)$, the orbit cannot enter and leave the region below the curve (3.13). If this occurs, we can decrease the value of Q up to reach one where the orbit stays above the curve and touches it at a point. We saw above that this is impossible.

Proof of Proposition 3.1. In view of Lemma 3.4 we see that $Q(t)$, as defined in (3.12), is monotone in t, as t approaches $-\infty$. Then, both $\tilde{N}(t)$ and $\theta(t)$ are also monotone and consequently there exist $\bar{N}$ and $\bar{\theta}$ such that

$$\lim_{t\to-\infty} \tilde{N}(t) = \bar{N} \quad \text{and} \quad \lim_{t\to-\infty} \tilde{\theta}(t) = \bar{\theta}.$$

Since the function $v = u'(r) = T_v(u, v)$ as in (3.9), we find that

$$\lim_{r\to 0} \frac{u'(r)}{r} = -\frac{\gamma^p}{\bar{N}\bar{\theta}}.$$

Next, using the equation (3.1) satisfied by u, we find that

$$\lim_{r\to 0} u''(r) = (\bar{N} - 1)\frac{\gamma^p}{\bar{N}\bar{\theta}} - \frac{\gamma^p}{\bar{\theta}} = \lim_{r\to 0} \frac{u'(r)}{r}.$$

Then, from the optimality condition (3.3) we finally obtain

$$\partial S(\bar{\theta}) \ni -\frac{1}{N-1}.$$

From here we conclude that $\lim_{t\to\infty} Q(t) = Q_0$ and then $Q(t)$ has to be monotone increasing, $\bar{\theta} = \theta_0$ and $\bar{N} = N_0$. $\Box$

As a consequence of Lemma 3.4 and the above proof we have the following important

Corollary 3.1 *Let u be a solution of the initial value problem (3.1)-(3.2) and x its Endem-Fowler transformation. If the orbit $(x(t), \dot{x}(t))$ crosses the x-axis only once, while $x(t)$ remains positive, then $Q(t)$ is strictly monotone increasing.*

Now we are in a position to prove the uniqueness of solution for the initial value problem (3.1)-(3.2).

Proposition 3.2 *Assuming that $N_0/(N_0-2) < p$, then (3.1)-(3.2) possesses only one solution.*

Proof. We define the field f_0 as

$$f(x,y) = -a_0 y + b_0 x + f_0(x,y),$$

where

$$a_0 = \lim_{t\to-\infty} \tilde{a}(t) \quad \text{and} \quad b_0 = \lim_{t\to-\infty} \tilde{b}(t).$$

The linear system

$$\dot{x} = y, \quad \dot{y} = -a_0 y + b_0 x,$$

has the origin as an hyperbolic critical point by our hypothesis on p. In the first quadrant it has an unstable direction with eigenvalue $\lambda_1 = \alpha$ and on the fourth quadrant a stable direction with eigenvalue $\lambda_2 = -N_0 + 2 + \alpha$.

The idea is to consider the field f as the linear system perturbed by f_0 and apply the stable-unstable manifold theorem as in Theorem 4.1 in [9]. However f_0 is not differentiable, it is not even well defined in any neighborhood of the origin. So we need to modify it, but we do away from all orbits coming out from the origin, corresponding to all solutions of the initial value problem (3.1)-(3.2).

This modification, that we call $\tilde{f}_0$, has to satisfy the following condition, in order to apply the above cited result: for all $\varepsilon > 0$ there exists $\delta > 0$ so that

$$|\tilde{f}_0(x,y) - \tilde{f}_0(x',y')| \le \varepsilon|(x,y) - (x',y')|, \tag{3.16}$$

for all $|(x,y)|, |(x',y')| \le \delta$. In other words, $\tilde{f}_0$ should have vanishing Lipschitz constants. We first modify the function f_0 for all (x,y) such that

$$y \ge \alpha x - \frac{x^p}{(N-1)Q_0},$$

simply defining

$$\tilde{f}_0(x,y) = f_0(x, \alpha x - \frac{x^p}{(N-1)Q_0}).$$

Next we see that an orbit $(x(t), \dot{x}(t))$, associated to a solution of the initial value problem (3.1)-(3.2), can be described as a function $y = y(x)$, near the origin. We then define

$$y_0(x) = \inf\{y(x) \,/\, y(x) \text{ is a solution of (3.1)-(3.2)}\}.$$

It is clear that y_0 also describe a solution of (3.1)-(3.2). Then, according to Lemma 3.3 and Proposition 3.1, we have that

$$q(x) = \frac{x^p}{y_0(x) - \alpha x}$$

satisfies $\lim_{x\to 0} q(x) = Q_0$. Let $\tilde{q}(x)$ be a function such that $\tilde{q}(x) < q(x)$ and $\lim_{x\to 0} \tilde{q}(x) = Q_0$. Then for every (x, y) such that $y \le \tilde{q}(x)$ we define

$$\tilde{f}_0(x, y) = f_0(x, \alpha x - \frac{x^p}{(N-1)\tilde{q}(x)}).$$

The function $\tilde{f}_0(x, y)$ is completely defined in $[0, x_0) \times I\!R$ for some $x_0 > 0$. This function is Lipschitz continuous and, by construction and the properties of f, it satisfies (3.16). Uniqueness of solutions for the initial value problem (3.1)-(3.2) follows now from the stable-unstable manifold theorem. □

Remark 3.5 For our purpose, the proof of Theorem 1.1 and Theorem 6.1 in Section 6, we only need to have uniqueness in the range of p as in Proposition 3.2. Outside of this range we do not know if uniqueness holds.

4 Emden-Fowler Analysis: Asymptotic behavior at infinity

In this section we study the asymptotic behavior of the solution of the initial value problem (3.1)-(3.2), as $r \to \infty$. We do this by analyzing the Emden-Fowler transformation in the phase plane.

Our next two lemmas give important information on the asymptotic behavior of the orbits as t approaches infinity. In what follows it is convenient to define the set

$$\mathcal{R} = \{(x, y) \,/\, x > 0, \alpha x - y > 0\},$$

that contains all orbits associated to solutions of our initial value problem.

Lemma 4.1 *Assume that $N_\infty > 2$ then we have:*

1) The field f possesses exactly one critical point $P = (\bar{x}, 0) \in \mathcal{R}$ if and only if $p > N_\infty/(N_\infty - 2)$.

2) If $p > N_\infty/(N_\infty - 2)$ then (3.10) has an ingoing direction in the fourth quadrant.

Proof. 1) The coefficient $\tilde{b}$ is always positive on the positive real axis, because our hypothesis on p. Then the equation

$$f(x, 0) = \tilde{b}x - \frac{x^p}{\theta} = 0 \tag{4.1}$$

has at least one solution. Notice that both $\tilde{b}$ and θ are bounded away from zero and infinity on the positive real axis.

Even though the function $f(x,0)$ is not differentiable, it has one-sided derivatives for all $x>0$, which are given by

$$\frac{\partial f_\pm}{\partial x}(x,0) = \tilde b_\pm(1-p).$$

Both one-sided derivatives are positive so that (4.1) possesses exactly one solution.

2) We can do a perturbation analysis at the origin using Theorem 4.1 in [9]. In fact, if we write

$$f(x,y) = -a_\infty y + b_\infty x + f_\infty(x,y),$$

then the function f_∞ satisfies (3.16) in the fourth quadrant. The hypothesis on p implies that the linear system has a negative eigenvalue associated with direction $(1,-(N_\infty-2))$. From here we obtain that the origin has an ingoing direction in the fourth quadrant. $\square$

Remark 4.1 Using similar arguments as in the lemma we can prove that the origin is unstable on the fourth quadrant if $p < N_\infty/(N_\infty-2)$.

Lemma 4.2 *Assume that $N_\infty > 2$ and $p > N_\infty/(N_\infty-2)$. If $x(t)$ is a solution of (3.10) with (3.11), such that, as t goes to ∞, $(x(t),\dot x(t))$ approaches $(0,0)$ through the fourth quadrant, then the associated solution u satisfies*

$$\lim_{r\to\infty} r^{N_\infty-2}u(r) = C \quad \textit{and} \quad \lim_{r\to\infty} r^{N_\infty-1}u'(r) = -(N_\infty-2)C,$$

for certain constant $C>0$. Moreover,

$$\lim_{r\to\infty} r^{-(N_\infty-1)}e^{\int_{r_0}^r \frac{\tilde N-1}{s}ds} = l, \tag{4.2}$$

where $r_0>0$ is an appropriate constant and $l>0$.

Proof. From the hypothesis we see that $\lim_{t\to\infty}\theta(t) = \theta_\infty$ and $\lim_{t\to\infty}\tilde N(t) = N_\infty$. Moreover, by the previous lemma, the orbit approaches the origin in the direction $(1,-(N_\infty-2))$, with an eigenvalue $\lambda_1 = -(N_\infty-2-\alpha)$. This implies that, for a certain constant $C>0$,

$$\lim_{t\to\infty} x(t)e^{(N_\infty-2-\alpha)t} = C \quad \text{and} \quad \lim_{t\to\infty} \dot x(t)e^{(N_\infty-2-\alpha)t} = -C(N_\infty-2).$$

From here we obtain the desired behavior for u and u'. Next we use the Lipschitz continuity of d to obtain a constant $c>0$ such that

$$|\tilde N(r) - N_\infty| \le c\left|\frac{u^p r}{u'}\right|.$$

Then, using the asymptotic behavior of u and u', together with the fact that $p > N_\infty/(N_\infty-2)$, we find an exponent $a>0$ such that

$$|\tilde N(r) - N_\infty| \le cr^{-a},$$

from where (4.2) follows. □

At this point we introduce some terminology to describe possible behavior that the solutions of the initial value problem (3.1)-(3.2) may have, as r approaches infinity. We say that:

i) u is a *crossing solution* if there exists $\bar{r}$ such that $u(\bar{r}) = 0$ and $u(r) > 0$ for all $r \in (0, \bar{r})$.
ii) u is a *slow decaying solution* if there is a constant c^* such that

$$\lim_{r\to\infty} r^{\alpha} u(r) = c^*.$$

iii) u is a *pseudo slow decaying solution* if there are constants $C_1, C_2 > 0$ such that

$$C_1 = \liminf_{r\to\infty} r^{\alpha} u(r) < \limsup_{r\to\infty} r^{\alpha} u(r) = C_2.$$

iv) u is a *fast decaying solution* if there is a constant $C > 0$ such that

$$\lim_{r\to\infty} r^{N_\infty - 2} u(r) = C.$$

With these definitions we consider the following subsets of $(1, \infty)$, according to the behavior of the solutions of (3.1)-(3.2). We define:

$$\begin{array}{lllll}
\mathcal{C} & = & \{p & \mid & p > 1, \quad u(r, p, \gamma) \quad \text{has a finite zero}\}. \\
\mathcal{P} & = & \{p & \mid & p > 1, \quad u(r, p, \gamma) \quad \text{is positive and is pseudo-slow decaying}\} \\
\mathcal{S} & = & \{p & \mid & p > 1, \quad u(r, p, \gamma) \quad \text{is positive and is slow decaying}\} \\
\mathcal{F} & = & \{p & \mid & p > 1, \quad u(r, p, \gamma) \quad \text{is positive is and fast decaying}\}.
\end{array}$$

In view of Remark 3.2, we notice that these sets do not depend on the particular value of $\gamma > 0$. Our goal is to prove that $(1, \infty) = \mathcal{C} \cup \mathcal{P} \cup \mathcal{S} \cup \mathcal{F}$ and that $\mathcal{F}$ is a singleton. Now we have our first main step.

Proposition 4.1 *Assuming that $N_\infty > 2$ and $p_0 < p_\infty$ (that is $N_0 > N_\infty$, see (1.7)) then we have:*
1) If $p \geq p_\infty$ then $p \in \mathcal{S}$.
2) If $p \leq \max\{N_\infty/(N_\infty - 2), p_0\}$ then $p \in \mathcal{C}$.

Proof. Let u be a solution of (3.1)-(3.2) and x its Emden-Fowler transformation. We first claim that in both cases the solution $x(t)$ is bounded. In fact, by the analysis in Lemma 3.3 we know that $x(t)$ and $\dot{x}(t)$ are bounded above. To see that $\dot{x}(t)$ is bounded below, we notice that if this is not the case then there is a constant $M > 0$ so that $\dot{x}(t) \leq -M$ for all $t \geq t_0$ and then x eventually become zero.

1) First assume that $(x, \dot{x})$ is a homoclinic orbit. Then, denoting by C the corresponding closed curve and by n the outward normal, we have

$$I = \int_C (y, f(x, y)) \cdot n ds = 0.$$

But, if Ω is the region surrounded by C, by the divergence Theorem we find that

$$I = \int_\Omega \frac{\partial f}{\partial y} dxdy = \int_\Omega -\tilde{a} dxdy.$$

By hypothesis on p we have $\tilde{a} > 0$, reaching a contradiction. We notice that f is differentiable a.e. by Proposition 2.1. This argument is known as Dulac's criterion in dynamical systems.

Assume now that u is a crossing solution. Since $p > N_\infty/(N_\infty - 2)$, by Lemma 4.1 the origin has an ingoing direction and there is a unique critical point $P = (\bar{x}, 0) \in \mathcal{R}$. Then, by the Poincaré-Bendixon Theorem, following the ingoing orbit backwards, we find a point $\hat{O} = (\hat{x}, 0) \in \mathcal{R}$, $\hat{x} \leq \bar{x}$, so that this orbit connects $\hat{O}$ with the origin O. Let call C the closed curve obtained by joining this orbit and the line $O\hat{O}$. Since $f(x, 0)$ is positive on the line $O\hat{O}$ we find

$$I = \int_C (y, f(x, y)) \cdot n ds > 0,$$

from where a contradiction follows, using the divergence Theorem again.

Since u is not crossing and not fast decaying, the orbit associated to u has to approach either a periodic orbit or the critical point P as a consequence of the Poincaré-Bendixon theorem. However, the former cannot occur by the argument given above. We conclude then that $p \in \mathcal{S}$.

2) We first observe that when $p \leq N_\infty/(N_\infty - 2)$ then $f(x, 0) < 0$ for all $x > 0$. In case $p < N_\infty/(N_\infty - 2)$, this observation together with Remark 4.1 implies u is a crossing solution. If $p = N_\infty/(N_\infty - 2)$ then we can prove that $f(x, y) < 0$ in a small neighborhood of $(0, 0)$ in the fourth quadrant. Thus, the solution must also leave the fourth quadrant, crossing the negative y-axis.

If $N_\infty/(N_\infty - 2) < p \leq p_0$ the we can use again the Dulac's criterion we get a contradiction if the solution x returns to the x-axis from below . $\square$

Corollary 4.1 *There is $p^* \in \mathcal{F}$ such that if $p_0 = p_\infty$ then $p^* = p_\infty$ and if $p_0 < p_\infty$, then*

$$p^* \in (\max\{N_\infty/(N_\infty - 2), p_0\}, p_\infty).$$

Proof. We first see that, using the Poincaré-Bendixon theorem, we have $(1, \infty) = \mathcal{C} \cup \mathcal{P} \cup \mathcal{S} \cup \mathcal{F}$. On the other hand, the sets $\mathcal{C}$ and $\mathcal{P} \cup \mathcal{S}$ are open, so that $\mathcal{F}$ cannot be empty. The rest follows from Proposition 4.1. $\square$

5 Coffman-Kolodner Analysis

In this section we study the solutions obtained near a fast decaying solution, the idea is to vary p in order to classify them. We differentiate the solution of (3.1)-(3.2) with respect to p, keeping the initial condition fixed. The resulting function φ has valuable information on the solutions near the fast decaying one.

This idea was introduced by Coffman and Kolodner in studying uniqueness questions for semi-linear equations. They rather differentiate with respect to the initial condition though.

The idea of differentiate the solution of (3.1)-(3.2) with respect to p was used in [15], to establish the uniqueness of the critical exponent for $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$.

By analyzing the function φ we will prove in this section the following two propositions, that are crucial in the proof of our main results.

Proposition 5.1 *If $p^* \in \mathcal{F}$, then for $p < p^*$ close to p^* we have $p \in \mathcal{C}$.*

Proposition 5.2 *If $p^* \in \mathcal{F}$, then for $p > p^*$ close to p^* we have $p \in \mathcal{S} \cup \mathcal{P}$.*

For the proof of these propositions we need some preliminary lemmas. Since in our analysis γ is kept fixed, so we do not make explicit mention of it. Its value will be chosen later.

Let $p^* \in \mathcal{F}$ and let $u(r, p^*)$ be the solution of (3.1)-(3.2). In view of Proposition 4.1, in order to prove our results we may assume that $p > N_\infty/(N_\infty - 2)$. Our first goal is to study the differentiability of u with respect to p.

Lemma 5.1 *The function $p \to u(r,p)$ is differentiable with respect to p at $p = p^*$. Moreover if we define $\varphi(r,p) = \partial u(r,p)/\partial p$, then φ is of class C^1 and it satisfies*

$$\varphi'' + \frac{\tilde{N}-1}{r}\varphi' + p^* \frac{u^{p^*-1}}{\theta}\varphi + \frac{u^{p^*}}{\theta}\log u = 0 \quad in \quad (0,+\infty) \quad a.e., \tag{5.1}$$

with initial conditions $\varphi(0) = 0$ and $\varphi'(0) = 0$. Here $u(r) = u(r,p^*)$, $\theta(r)$ and $\tilde{N}(r)$ are associated to $u(r,p^*)$.

Proof. It is easy to see that equation (5.1), with initial conditions $\varphi(0) = 0$ and $\varphi'(0) = 0$, has a unique C^1 solution for all $r > 0$. In fact, we need to consider the following integral equation

$$\varphi'(r) = -\int_0^r e^{-\int_s^r \frac{\tilde{N}(\tau)-1}{\tau} d\tau} \left[p\frac{u^{p-1}}{\theta}\varphi + \frac{u^p}{\theta}\log u \right] ds.$$

We just notice that the functions $\tilde{N}$ and θ are fixed bounded measurable functions, thus the contraction mapping principle can be used to obtain existence and uniqueness of solution for small r.

Then we consider, for notational convenience, $u(r) = u(r,p^*)$, $v(r) = u(r,p)$ and

$$\phi(r,p) = \frac{v(r) - u(r)}{p - p^*} - \varphi(r).$$

Thus, considering the equations satisfied for u, v we obtain

$$(v-u)'' + \frac{(N-1)}{r}(v-u)'d(Q_*) + \frac{(N-1)}{r}v'(o_1(p) + \frac{1}{\theta})(Q - Q_*) = 0,$$

where
$$Q(r) = \frac{v^p}{(N-1)(v'/r)}, \qquad Q_*(r) = \frac{u^{p^*}}{(N-1)(u'/r)}$$
and
$$o_1(p) = \frac{d(Q) - d(Q_*)}{Q - Q_*} - \frac{1}{\theta}.$$
Since $Q_*(r)$ is a strictly increasing function, we have that $Q_*(r) \notin \{Q_i \,/\, i \in \mathcal{I}\}$ a.e., thus $\lim_{p \to p^*} o_1(p) = 0$ a.e. We also see that
$$Q(r) - Q_*(r) = \frac{1}{(N-1)(v'/r)}(v^p - u^{p^*}) - \frac{Q_*}{v'}(v-u)'.$$
Thus, using the equation for φ and the optimality condition $d(Q_*)\theta - S(\theta) = Q_*$, which holds a.e., we obtain
$$\phi'' + \left[\frac{\tilde{N} - 1 + (N-1)o_1(p)}{r}\right]\phi' + \left[\frac{p^* u^{p^*-1}}{\theta} + o_2(p)\right]\phi + o_3(p) = 0,$$
with boundary condition $\phi(0) = \phi'(0) = 0$. Here $o_i(p)$, $i = 2, 3$ denotes a bounded function of (r, p) such that $o_i(p) \to 0$ as $p \to p^*$, a.e. in r.

We can rewrite this equation in the form of an integral equation
$$\phi'(r) = -\int_0^r e^{-\int_s^r \frac{\tilde{N}(\tau) - 1 + (N-1)o_1(p)}{\tau} d\tau} \left[p^* \frac{u^{p^*-1}}{\theta} + o_2(p)\right]\phi + o_3(p) ds.$$
Here we can use the Gronwall inequality, recalling the convergence properties of the functions o_i, as p approaches p^*, to conclude that
$$\lim_{p \to p^*} \phi(r) = \lim_{p \to p^*} \phi'(r) = 0,$$
for all $r > 0$. This proves the differentiability of $u(r, p)$ and $u'(r, p)$ and that
$$\frac{\partial u}{\partial p} = \varphi \quad \text{and} \quad \frac{\partial u'}{\partial p} = \varphi' \qquad \forall r > 0.\square$$

In the discussion to follow we will keep $p = p^*$ fixed. Then, for notational convenience, we will write p instead of p^*. In the proof of Proposition 5.1 we will come back to the regular notation.

Now we fix the constant $\gamma > 0$, the initial condition in (3.1)-(3.2), in such a way that $\dot{x}(T) = 0$ implies $u(e^T) = 1$. This is possible since a change in γ implies time translation in the dynamical system. The next lemma provides two identities that are very important in the sequel. These type of identities where introduced in [26], for a related problem.

Lemma 5.2 *Let $u(r,p)$ and $\varphi(r)$ as above and $r_0>0$. Then the following identities hold:*

$$\{e^{\int_{r_0}^r \frac{\tilde N-1}{s}ds}[(ru)''\varphi-(ru)'\varphi']\}' = \frac{e^{\int_{r_0}^r \frac{\tilde N-1}{s}ds}}{\theta}[(p-3)u^p\varphi+ru'u^p\log u+u^{p+1}\log u], \tag{5.2}$$

$$\{e^{\int_{r_0}^r \frac{\tilde N-1}{s}ds}(u'\varphi-u\varphi')\}' = \frac{e^{\int_{r_0}^r \frac{\tilde N-1}{s}ds}}{\theta}[(p-1)u^p\varphi+u^{p+1}\log u], \tag{5.3}$$

for all $r\in(0,\infty)$ a.e.

Proof. The proof is obtained by a routine calculation, starting from the equations satisfied by φ and u. We omit the details.□

The next lemma is a key step in our arguments.

Lemma 5.3 *It is not possible to have simultaneously that*

$$\lim_{r\to\infty}\varphi(r)=c_1\le 0 \quad \textit{and} \quad \lim_{r\to\infty} r\varphi'(r)=0.$$

Proof. Since u is a fast decaying solution, using Lemma 4.2 we find that

$$\lim_{r\to\infty} e^{\int_{r_0}^r \frac{\tilde N-1}{s}ds}(u'\varphi-u\varphi')=(2-N_\infty)Cc_1l, \tag{5.4}$$

for C and l as in the lemma. On the other hand, using the equation for u we find

$$e^{\int_{r_0}^r \frac{\tilde N-1}{s}ds}[(ru)''\varphi-(ru)'\varphi'] = e^{\int_{r_0}^r \frac{\tilde N-1}{s}ds}[((3-\tilde N)u'-\frac{ru^p}{\theta})\varphi-(u+ru')\varphi'].$$

Since we are assuming $p>N_\infty/(N_\infty-2)$, from here and Lemma 4.2, we obtain

$$\lim_{r\to\infty} e^{\int_{r_0}^r \frac{\tilde N-1}{s}ds}[(ru)''\varphi-(ru)'\varphi']=(2-N_\infty)(3-N_\infty)Cc_1l. \tag{5.5}$$

Now we integrate identities (5.3) and (5.2) and use (5.4) and (5.5) to find

$$\int_0^\infty \frac{e^{\int_{r_0}^r \frac{\tilde N-1}{s}ds}}{\theta}[(p-1)u^p\varphi+u^{p+1}\log u]=(2-N_\infty)Cc_1l$$

and

$$\int_0^\infty \frac{e^{\int_{r_0}^r \frac{\tilde N-1}{s}ds}}{\theta}[(p-3)u^p\varphi+ru'u^p\log u+u^{p+1}\log u]=(2-N_\infty)(3-N_\infty)Cc_1l$$

We multiply the first integral by $(p-3)/(p-1)$ and subtract the second getting

$$\int_0^\infty \frac{e^{\int_{r_0}^r \frac{\tilde N-1}{s}ds}}{\theta}[(\alpha u+ru')u^p\log u] \quad = \quad (3-N_\infty-\frac{p-3}{p-1})(2-N_\infty)Cc_1l. \tag{5.6}$$

We notice that $\dot{x}(t) = r^\alpha(\alpha u + ru')$, then $\alpha u + ru'$ change the sign when $\dot{x}$ does. But we have chosen γ so that $\log u$ change sign when $\dot{x}$ does. Thus $(\alpha u+ru')u^p \log u > 0$, for all $r \geq 0$. On the other hand, since $p > N_\infty/(N_\infty-2)$ we have that the right-hand side in (5.6) is negative or zero, providing a contradiction. $\square$

Continuing with our analysis we define the function

$$w = w_\eta(r) = r^\eta u(r,p),$$

for $\eta > 0$ chosen so that $\eta = (N_\infty - 1)/2$ if $N_\infty > 3$ and $\eta = (N_\infty - 2)/2$ if $2 < N_\infty \leq 3$. This function was introduced by Erbe and Tang in [12], for a related problem. The function w satisfies the equation

$$w'' + \frac{(\tilde{N} - 1 - 2\eta)}{r} w' + \frac{\eta(\eta + 2 - \tilde{N})}{r^2} w + r^\eta \frac{u^p}{\theta} = 0. \tag{5.7}$$

Next we define

$$y(r) = r^\eta \varphi.$$

When $N_\infty > 3$, the function y satisfies the equation

$$y'' + \frac{(\tilde{N} - N_\infty)}{2r} y' + (\frac{(N_\infty - 1)(3 + N_\infty - 2\tilde{N})}{4r^2} + \frac{pu^{p-1}}{\theta})y + r^\eta \frac{u^p}{\theta} \log u = 0. \tag{5.8}$$

Since u is a fast decaying solution we can use Lemma 4.2 to find that

$$\lim_{r\to\infty} r^{(N_\infty-2)(p-1)} u^{p-1}(r) = C^{p-1}. \tag{5.9}$$

But, since $p > N_\infty/(N_\infty - 2)$, we have that $(N_\infty - 2)(p - 1) > 2$. Thus the coefficient in the third term of (5.8) is negative for r large.

When $2 < N_\infty \leq 3$, then y satisfies the equation

$$y''+(\tilde{N}-N_\infty+1)\frac{y'}{r}+(\frac{(N_\infty - 2)(N_\infty + 2 - 2\tilde{N})}{4r^2}+\frac{pu^{p-1}}{\theta})y+r^\eta \frac{u^p}{\theta} \log u = 0 \tag{5.10}$$

Since, again we have (5.9), the coefficient of the third term in (5.10) is also negative for r large.

Now we can prove the following lemma on the asymptotic behavior of y.

Lemma 5.4 *The function y defined above satisfies $y(r) > 0$ for r large.*

Proof. Suppose, for contradiction, that there exists a $\bar{r}$ large such that $y(\bar{r}) < 0$, then we have the following two possibilities:

a) $y(r) \leq 0$ for all $r > \bar{r}$ or **b)** there exists $r^* > \bar{r}$ such that $y(r^*) > 0$.

In case **a)** we have that $\varphi(r) < 0$ for all $r > \bar{r}$. From 5.1 and (5.3) we have then that for r large, a.e.,

$$\{e^{\int_{r_0}^r \frac{\tilde{N}-1}{s} ds}(u'\varphi - u\varphi')\}' < 0 \quad \text{and} \quad \{e^{\int_{r_0}^r \frac{\tilde{N}-1}{s} ds}\varphi'\}' > 0. \tag{5.11}$$

Again there are two possibilities:

i) There exists $\tilde{r} > \bar{r}$ such that $u'(\tilde{r})\varphi(\tilde{r}) - u(\tilde{r})\varphi'(\tilde{r}) \leq 0$ or

ii) $u'(r)\varphi(r) - u(r)\varphi'(r) > 0$ for all $r \geq \bar{r}$.

If i) is true, from (5.11) we have $u'(r)\varphi(r) - u(r)\varphi'(r) < 0$ for all $r > \tilde{r}$, from where it follows that the function u/φ is strictly decreasing for all $r > \tilde{r}$. Thus there is a number c_∞, possibly $-\infty$, such that

$$\lim_{r\to\infty} \frac{u(r)r^{N_\infty-2}}{\varphi(r)r^{N_\infty-2}} = c_\infty,$$

and then $\lim_{r\to\infty} \varphi r^{N_\infty-2} = C/c_\infty \leq 0$, where C is given in Lemma 4.2. From the fact that $\{e^{\int_{r_0}^r \frac{\tilde{N}-1}{s}ds}\varphi'\}' > 0$ for r large, then there is a positive constant c_1, possibly $+\infty$, so that

$$\lim_{r\to\infty} \varphi' r^{N_\infty-1} = c_1.$$

Hence by the L'Hopital's rule we get

$$\lim_{r\to\infty} \varphi' r^{N_\infty-1} = (2-N_\infty)\lim_{r\to\infty} \varphi r^{N_\infty-2} = (2-N_\infty)\frac{C}{c_\infty}.$$

From here we obtain that $\varphi(r) \to 0$ and $r\varphi'(r) \to 0$ as $r \to \infty$, contradicting Lemma 5.3.

If ii) is true, we have

$$u'(r)\varphi(r) - u(r)\varphi'(r) > 0 \qquad \text{for all} \quad r \geq \tilde{r}. \tag{5.12}$$

From (5.11) there exists $c_2 \in (-\infty, +\infty]$ such that $\lim_{r\to\infty} \varphi'(r)r^{N_\infty-1} = c_2$.

In case $c_2 \leq 0$ we have $\varphi'(r) < 0$ for all r large, consequently there exists $c_1 \in [-\infty, 0)$ such that $\lim_{r\to\infty} \varphi(r) = c_1$. We claim that c_1 is finite. In fact, we first observe that, since $\varphi'(r)r^{N_\infty-1} = r^{N_\infty-2}(r\varphi'(r))$ converges to a finite limit, we have necessarily that $\lim_{r\to\infty} r\varphi'(r) = 0$. Then from (5.11) and (5.12) we find a finite constant $c \geq 0$ such that

$$\lim_{r\to\infty} r^{N_\infty-1}(u'(r)\varphi(r) - u(r)\varphi'(r)) = c, \tag{5.13}$$

from where it follows that c_1 is finite. Thus we get a contradiction with Lemma 5.3.

In case $c_2 > 0$ then $\varphi'(r) > 0$ for all r large, so that there exists a constant $c_1 \in (-\infty, 0]$ such that $\lim_{r\to\infty} \varphi(r) = c_1$.

Again we have (5.13) from where we find a non-negative constant c_3 such that $\lim_{r\to\infty} r\varphi'(r) = c_3$. If $c_3 > 0$ then integrating this last limit we conclude that φ is unbounded, which is impossible. Thus we again contradict Lemma 5.3.

In case **b)**, we claim that $y(r) > 0$ for all $r > r^*$. In fact, the contrary would imply that y has a local maximum point in $r_2 \in (r^*, \infty)$. But from (5.8) or from (5.10) we see that

$$(e^{\int_{r_0}^r \frac{\tilde{N}-N_\infty}{2s}ds}y')' > 0 \quad \text{or} \quad (e^{\int_{r_0}^r \frac{\tilde{N}-N_\infty+1}{2s}ds}y')' > 0,$$

for all r near r_2 a.e., respectively. But integrating these conditions, we see they are incompatible with the fact that r_2 is a local maximum point. $\square$

Corollary 5.1 *The function y defined above, satisfies $y'(r) > 0$ for r large.*

Proof. From Lemma 4.5 we have $y(r) > 0$ for r large, so in the case $N_\infty > 3$, by (5.8) we see that $\{e^{\int_{r_0}^{r} \frac{\tilde{N}-N_\infty}{2s} ds} y'\}'(r) > 0$.

Consequently $e^{\int_{r_0}^{r} \frac{\tilde{N}-N_\infty}{2s} ds} y'$ is increasing, and then there exists $L \in (-\infty, +\infty]$ such that

$$\lim_{r\to\infty} y'(r) = L.$$

If $L < \infty$ we have

$$\lim_{r\to\infty} (\frac{N_\infty - 1}{2}) r^{\frac{N_\infty - 3}{2}} \varphi(r) + r^{\frac{N_\infty - 1}{2}} \varphi'(r) = L.$$

Then $\varphi(r) \to 0$ and $r\varphi'(r) \to 0$ as $r \to \infty$, but this contradicts Lemma 5.3. Hence $L = \infty$ and then $y'(r) > 0$ for r large.

In the case $2 < N_\infty \leq 3$, by (5.10) we have $\{e^{\int_{r_0}^{r} \frac{\tilde{N}-N_\infty+1}{s} ds} y'\}'(r) > 0$. So a similar argument as the before case can be use to get the conclusion. $\square$

Now we are prepared for proving Proposition 5.1. From now on we come back to our notation $p^* \in \mathcal{F}$.

Proof of Proposition 5.1. Let $p^* \in \mathcal{F}$ and $p < p^*$ sufficiently close to p^*. Here, and in what follows, we assume that $u(0, p) = u(0, p^*) = \gamma$, where γ was chosen before. Suppose first that $N_\infty > 3$. Let us define

$$w(r) = r^{(N_\infty - 1)/2} u(r, p), \qquad w_*(r) = r^{(N_\infty - 1)/2} u(r, p^*)$$

and $v = w_* - w$. By the extremal condition $u = u(r, p)$ satisfies

$$\theta_* u'' + S(\theta_*) \frac{N-1}{r} u' + u^p \leq 0,$$

where θ_* is the function associated to $u(., p^*)$. Therefore v satisfies

$$\begin{aligned} v''(r) + \frac{(\tilde{N}_* - N_\infty)}{2r} v' &+ (\frac{(N_\infty - 1)(3 + N_\infty - 2\tilde{N}_*)}{4r^2}) v \\ &+ r^{(N_\infty - 1)/2} \frac{(u(r, p^*)^{p^*} - u(r, p)^p)}{\theta_*} \geq 0. \end{aligned} \tag{5.14}$$

By the mean value theorem we have

$$u(r, p^*)^{p^*} - u(r, p)^p = p^* (\xi(r))^{p^* - 1} (u(r, p^*) - u(r, p)) + u(r, p)^{p^*} - u(r, p)^p, \tag{5.15}$$

where

$$\xi(r) \in (\min\{(u(r, p^*), u(r, p)\}, \max\{(u(r, p^*), u(r, p)\}).$$

Next we use continuity of the solution of (3.1)-(3.2) with respect to the parameter p and the fact that $u'(r,p) < 0$ for all $r > 0$, to find $\bar{r}$ and $\varepsilon > 0$ such that $u(r,p) < 1$, for all $r \geq \bar{r}$ and for all $p \in (p^* - \varepsilon, p^*)$. Then v satisfies

$$v'' + \frac{(\tilde{N}_* - N_\infty)}{2r} v' + \left(\frac{(N_\infty - 1)(3 + N_\infty - 2\tilde{N}_*)}{4r^2} + \frac{p^*(\xi(r))^{p^*-1}}{\theta_*} \right) v \geq 0 \quad (5.16)$$

Using (5.9) and that $p > N_\infty/(N_\infty - 2)$ we conclude the existence of r^* such that

$$\frac{(N_\infty - 1)(3 + N_\infty - 2\tilde{N}_*)}{4r^2} + \frac{p^*(u(r,p^*))^{p^*-1}}{\theta_*} < 0 \quad \text{for all} \quad r \geq r^*. \quad (5.17)$$

On the other hand, by Lemma 5.4 and Corollary 5.1, there exists $\tilde{r}$ such that $y(\tilde{r}) > 0$ and $y'(\tilde{r}) > 0$, for $\tilde{r} > \max\{r^*, \bar{r}\}$. Thus $v(\tilde{r}) > 0$ and $v'(\tilde{r}) > 0$ for a fix $p \in (p^* - \varepsilon, p^*)$ close to p^*. Suppose now by contradiction that $p \in \mathcal{F} \cup \mathcal{P} \cup \mathcal{S}$, then $v(r) \to 0$ as $r \to \infty$ or $v < 0$ negative for r large. Thus v has a positive maximum, let us say in $\hat{r}$. Since $v(\hat{r}) > 0$, we get $u(\hat{r},p) < u(\hat{r},p^*)$, hence $u(\hat{r},p^*) > \xi(\hat{r})$. Thus, from (5.17) and (5.16) and the fact that $\hat{r}$ is a maximum of v we get a contradiction.

In case $2 < N_\infty \leq 3$ we proceed slightly different. Define

$$w(r) = r^{(N_\infty - 2)/2} u(r,p), \quad w_*(r) = r^{(N_\infty - 2)/2} u(r,p^*)$$

and $v = w_* - w$. Then we use a similar argument and we get a contradiction again.□

Proof of Proposition 5.2. Let us assume that $p \in \ C \cup \mathcal{F}$ and $p > p^*$. We will proceed similar to the previous proposition. This time v satisfies the reverse inequality in (5.16) with $\tilde{N}$ and θ related to $u(.,p)$ in stead of $\tilde{N}_*$ θ_*, for $N_\infty > 3$. Moreover, there exists $\tilde{r}$ large such that $v(\tilde{r}) < 0$ and $v'(\tilde{r}) < 0$. Then, the contradiction comes from the fact that v must have a minimum. A similar argument work for $2 < N_\infty \leq 3$.□

6 Proof of Main Theorems and examples

In this concluding section we prove Theorem 1.1 on the existence of a critical exponent for maximal operators in the radial case. We will further discuss the existence of pseudo-slow decaying solutions and a class of singular solutions for p near the critical exponents. We give examples where these situations occurr. We end the section with results for the equation associated to minimal operators, that are in analogy with maximal operators.

Proof of Theorem 1.1 By Corollary 4.1 we only need to prove that $\mathcal{F}$ is a singleton, which a consequence of Proposition 5.1 and 5.2. □

Remark 6.1 If we further assume in Theorem 1.1 that $p_0 < p_\infty$ then the critical exponent p^* satisfies the strict inequalities

$$\max\{\frac{N_\infty}{N_\infty - 2}, p_0\} < p^* < p_\infty. \tag{6.1}$$

Proof of Theorem 1.2 . The proof is a direct consequence of our analysis for the crossing solution. $\square$

Remark 6.2 If the operator $\mathcal{M}$ in equation (1.12) is given by (1.8) then all solutions of (1.12) are radially symmetric, as proved by Da Lio and Sirakov in [11], using the moving planes technique, as developed in Serrin [32], Gidas, Ni and Nirenberg [19] and, Berestycki and Nirenberg [2].

Next we discuss possible solutions of equation (1.5), for p near the critical exponent p^*. Besides the classification of solutions given in Section 4. that considers only regular solutions, here we also allow singular solutions:

We say that a positive solution u of (1.5) is *singular* if

$$\lim_{r\to\infty} u(r) = 0 \quad \text{and} \quad \lim_{r\to 0} u(r) = \infty.$$

We may further classify the singular solutions according to a more precise asymptotic behavior as seen in the statement of the following theorem. In order to state this theorem let us consider the Emden-Fowler transformation of (1.5) when $p = p^*$, the critical exponent. In our analysis, a crucial role is played by the coefficient $\tilde{a}$ in equation (3.10), which is obtained as $\tilde{a} = \tilde{N} - 2 - 2\alpha$, where $\tilde{N}$ is defined in (2.8). We are interested in the values that $\tilde{a}$ may take in a neighborhood of the unique critical point $(x^*, 0) = (x(p^*), 0)$ of (3.10), which is characterized by

$$f(x(p^*), 0) = 0.$$

If we consider the curve

$$\frac{x^{p^*}}{(N-1)(y - \alpha^* x)} = -\frac{(x^*)^{p^*-1}}{\alpha^*(N-1)}, \tag{6.2}$$

where $\alpha^* = 2/(p^* - 1)$. We define

$$a^*_- = \lim_{(x,y)\to(x^*,0)^-} \tilde{a}(x, y) \quad \text{and} \quad a^*_+ = \lim_{(x,y)\to(x^*,0)^+} \tilde{a}(x, y),$$

where the first limit is taken from left of the curve (6.2) and the second limit is taken from the right of curve (6.2). In general we have $a^*_- \leq a^*_+$ and, depending on the geometry of D we may have a strict inequality. Now we state our second main theorem

Theorem 6.1 *Assume $N \geq 2$ and D satisfies condition D). Further assume that $N_\infty > 2$ and $p_0 < p^* < p_\infty$, where p^* is given by Theorem 1.1.*

Then we have the following two possibilities:

a) If $a_-^* > 0$, then for $p < p^*$, p close to p^*, equation (1.5) possesses at least four singular solutions u_i, $i = 1, ..., 4$, such that $u_1(r) = cr^{-\alpha}$,

$$c_{12} = \liminf_{r \to 0,\infty} r^\alpha u_2(r) < \limsup_{r \to 0,\infty} r^\alpha u_2(r) = c_{22},$$

$$\lim_{r \to \infty} r^{N_\infty - 2} u_3(r) = c_f \quad \text{and} \quad \lim_{r \to \infty} r^\alpha u_4(r) = c_s,$$

and, for i=3,4

$$c_{1i} = \liminf_{r \to 0} r^\alpha u_i(r) < \limsup_{r \to 0} r^\alpha u_i(r) = c_{2i},$$

for certain positive constants c, c_f, c_s, c_{1i} and c_{2i}, where $i = 2, 3, 4$.

b) If $a_+^* < 0$ then for $p > p^*$, p close to p^*, equation (1.5) possesses at least three singular solutions u_i, $i = 1, 2, 3$, such that $u_1(r) = cr^{-\alpha}$,

$$c_{12} = \liminf_{r \to 0,\infty} r^\alpha u_2(r) < \limsup_{r \to 0,\infty} r^\alpha u_2(r) = c_{22},$$

$$\lim_{r \to 0} r^\alpha u_3(r) = c_s \quad \text{and} \quad c_{13} = \liminf_{r \to \infty} r^\alpha u_3(r) < \limsup_{r \to \infty} r^\alpha u_3(r) = c_{23},$$

for certain positive constants c, c_s, c_{1i} and c_{2i}, $i = 2, 3$. Moreover, the regular solution u, whose behavior at infinity is like $r^{-\alpha}$ has a pseudo-slow decay, that is

$$c_1 = \liminf_{r \to \infty} r^\alpha u(r) < \limsup_{r \to \infty} r^\alpha u(r) = c_2,$$

in other words $p \in \mathcal{P}$.

Proof of Theorem 6.1 . a) Under the given hypothesis we claim that the critical point $P = (x(p), 0)$, characterized as $f(x(p), 0) = 0$, is stable. Let us assume the claim for the moment. Since $p < p^*$ and p is close to p^*, we have that the solution x_0 such that

$$\lim_{t \to -\infty} (x_0(t), \dot{x}_0(t)) = (0, 0)$$

is a crossing solution. This implies that the orbit of the solution x_∞ such that

$$\lim_{t \to +\infty} (x_\infty(t), \dot{x}_\infty(t)) = (0, 0), \tag{6.3}$$

stays in the first and fourth quadrant bounded by the orbit of x_0. Then, by the stability of P and the Poincaré-Bendixon theorem, this orbit approaches a periodic orbit. On the other hand, the orbit coming into P must wrap around a periodic orbit (possibly different), as $t \to -\infty$. Using the Emden-Fowler back we obtain the asymptotic behavior.

Next we prove the claim. Under the assumption, there is a neighborhood V of the critical point P such that $\tilde{a} > 0$ in V, as follows by continuity (semi-continuity of $\tilde{a}$.)

Suppose by contradiction that P is not stable. Then there is an orbit leaving the neighborhood V and we have several cases:

i) If the orbit spirals out, then there is $x_1 < x_2 < x(p)$ such that the orbit goes from $(x_2, 0)$ to $(x_1, 0)$ as t increases, staying all time in V. The orbit, together with the segment from $(x_1, 0)$ to $(x_2, 0)$, describes a closed curve on which we may use the integral argument as in Proposition 4.1, having in mind that $\tilde{a}$ is positive. This case is so impossible.

ii) If the orbit goes out of P through the first quadrant, then at some $\bar{t}$ the orbit crosses the x-axis at a point $(\bar{x}, 0)$. We see then that for every x_1 such that $x(p) < x_1 < \bar{x}$ the orbit passing through $(x_1, 0)$ approaches P as $t \to -\infty$. Choosing x_1 close enough we see that the corresponding orbit stays inside V. The orbit, together with the segment from $(x_1, 0)$ to P defines a closed curve where the integral argument again can be applied.

iii) If the orbit leaves P through the fourth quadrant. Then it may happen that it crosses the x-axis at a point $(\bar{x}, 0)$, $\bar{x} \geq 0$. Then we may repeat the arguments given above to get again an impossible. It may still happen that the orbit leave the fourth quadrant through the negative y-axis. Then we see that the orbit of x_∞ must approach P as $t \to -\infty$. We repeat the argument given above again.

These three cases exhaust the possibilities, so that the claim is proved.

b) In this situation we argue similarly, but here the critical point is unstable. $\square$

In what follows we provide two examples where the situation described in the theorem takes place.

Example 1. Suppose the set D is such that, for a certain $e > 0$, $\partial S(\theta_0) = (-\infty, e]$ and $N_0/2 \leq e(N-1) < N_\infty - 1$. Then condition a) of Theorem 6.1 holds. Moreover, in this case the coefficient $\tilde{a}$ is positive in a neighborhood of the critical point $P = (x(p), 0)$ for all p such that $\max\{N_\infty/(N_\infty - 2), p_0\} < p < p^*$ and consequently the conclusions of Theorem 6.1 part a) holds for all such a p. This follows by the fact that for all in such a range we have $(p+1)/((p-1)(N-1)) \in \partial S(\theta_0)$, so the critical point P is always in the region where $\tilde{a} = a_0$. Notice that when $p = p_0$ then P is locally a center.

Example 2. Suppose that D is such that, for a certain $e > 0$, $\partial S(\theta_\infty) = [e, +\infty)$ and $N_\infty/2 \geq e(N-1)$. Then condition b) of Theorem 6.1 holds. Moreover, in this case the coefficient $\tilde{a}$ is negative in a neighborhood of the critical point P for all p such that $p^* < p < p_\infty$ and consequently the conclusions of Theorem 6.1 part b) holds for all these p. This follows by the fact that for all $p^* < p < p_\infty$ we have $(p+1)/((p-1)(N-1)) \in \partial S(\theta_\infty)$, so the critical point is always in the region where $\tilde{a} = a_\infty$. Notice that when $p = p_\infty$. then P is locally a center.

In this case we also have that P is a center.

Remark 6.3 In Theorem 6.1, it may occur that $0 \in [a_-^*, a_+^*]$. In this situation we do not know what behavior we have near the critical exponent.

We end this article by briefly discussing the case of a minimal operator. Given a set D satisfying D) and a C^2 radially symmetric function we define a minimal operator on $D^2 u$ as

$$\mathcal{M}^-(D^2 u) = \inf_{(a_1,a_2)\in D} \frac{(N-1)}{r} u' a_1 + u'' a_2. \tag{6.4}$$

We are interested in the study of the nonlinear equation

$$\mathcal{M}^-(D^2 u) + u^p = 0, \quad u > 0 \quad \text{in} \quad I\!R^N. \tag{6.5}$$

We proceed in analogy with the case of maximal operators. Here we are interested in the right half of the set D, which is described by the function $\tilde{S}$, as mentioned in Section 2. We define the extreme dimension numbers, which in this case appear in reverse order. We consider the equations

$$\partial \tilde{S}(\theta_0^-) \ni -\frac{1}{N-1} \quad \text{and} \quad \partial \tilde{S}(\theta_\infty^-) \ni \frac{\tilde{S}(\theta_\infty^-)}{\theta_\infty^-},$$

If the first equation has more than one solution, we consider θ_0^- as the rightist one, and if the second equation has more than one solution, we consider leftist one. Then we define as usual

$$N_0^- = (N-1)\frac{\tilde{S}(\theta_0^-)}{\theta_0^-} + 1 \quad \text{and} \quad N_\infty^- = (N-1)\frac{\tilde{S}(\theta_\infty^-)}{\theta_\infty^-} + 1.$$

We notice that it this case $N_\infty^- \geq N_0^-$. For equation (6.5) we have the following

Theorem 6.2 *Assume $N \geq 2$ and D satisfies condition D). Then there are two dimension like numbers $0 < N_0^- \leq N_\infty^-$, depending only on D and N, such that if $N_\infty^- > 2$ then there is a critical exponent p_-^* such that*

$$p_\infty^- \leq p_-^* \leq p_0^-, \tag{6.6}$$

where

$$p_0^- = \frac{N_0^- + 2}{N_0^- - 2} \;\; \textit{if} \;\; N_0^- > 2, \quad p_0^- = \infty \;\; \textit{if} \;\; 0 < N_0^- \leq 2, \quad p_\infty^- = \frac{N_\infty^- + 2}{N_\infty^- - 2},$$

and such that:

i) If $1 < p < p_-^*$ then there is no radial solution to (6.5).

ii) If $p = p_-^*$ then there is a unique radial solution of (6.5) whose behavior at infinity is like $r^{-(N_\infty^- - 2)}$.

iii) If $p_-^* < p$ then there is a unique radial solution to (6.5) whose behavior at infinity is like $r^{-\alpha}$. In ii) and iii) uniqueness is meant up to scaling.

Remark 6.4 The proof of this theorem follows the lines of that of Theorem 1.1, with some minor changes. In addition to this theorem, we may also state and prove a result analogous to Theorem 6.1, for which we do not give details.

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