Boundary blow up solutions for fractional elliptic equations

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Abstract

In this article we study existence of boundary blow up solutions for some fractional elliptic equations including

$$\begin{aligned}(-\Delta)^{\alpha} u + u^p &= f \text{ in } \Omega,\\ u &= g \text{ on } \Omega^c,\\ \lim_{x\in\Omega, x\to\partial\Omega} u(x) &= \infty,\end{aligned}$$

where Ω is a bounded domain of class C^2, $\alpha \in (0,1)$ and the functions $f : \Omega \to \mathbb{R}$ and $g : \mathbb{R}^N \setminus \bar{\Omega} \to \mathbb{R}$ are continuous. We obtain existence of a solution u when the *boundary value* g blows up at the boundary and we get explosion rate for u under an additional assumption on the rate of explosion of g. Our results are extended for an ample class of elliptic fractional nonlinear operators of Isaacs type.

1 Introduction

In the study of reaction diffusion equations, a well known problem is the existence and asymptotic behavior of boundary blow up solutions. In its simplest form, the problem is to find solutions of the equation

$$\begin{aligned} -\Delta u + u^p &= f \quad \text{in} \quad \Omega, & (1.1)\\ \lim_{x\in\Omega, x\to\partial\Omega} u(x) &= \infty, & (1.2) \end{aligned}$$

where $p > 1$ and Ω is a smooth bounded domain. There is a vast literature on this problem and its extensions in various directions, like more general divergence form or even non-divergence form second order operators, general non-linearities and the Keller-Osserman condition, and analysis of the behavior of the solution near the boundary and uniqueness. Without being exhaustive, we would like to mention the pioneering papers by Keller [16] and Osserman [20] and the work by Bandle and Marcus [1], [2], Loewner and Nirenberg [18], Kondrat'ev and V. Nikishkin [17], Diaz and Letelier [15], Diaz and Diaz [12], Del Pino and Letelier [11], Marcus and Veron [19] and Esteban Felmer and Quaas [10]. We refer the reader to the review by Rădulescu [21] for a more complete account on the literature.

The simplest case presented in (1.1)-(1.2) has various interesting features, like for example the existence of solutions regardless of the behavior of the function f near the boundary. In fact, the strong absorption term represented by the super-linear non-linearity allows to prove local estimates on the solution u that depend on the values of f locally, see for example [4], [15] and [10]. Another interesting characteristic is given by the fact that, even for $f \equiv 0$ boundary blow up solutions exists and they are unique. Finally we mention that many questions on the existence, uniqueness and boundary behavior of solutions may be addressed using appropriate super and sub-solutions and the comparison principle.

During the last years there has been a renewed and increasing interest in the study of linear and nonlinear integral operators, including the fractional laplacian, motivated by many applications and by important advances on the theory of nonlinear partial differential equations. In this line, one is interested in understanding the structure of the solution set of equations involving these fractional operators in some simple situations. This is the case of the existence of boundary blow up solutions to (1.1) -(1.2) with the fractional laplacian instead of the laplacian.

It is the purpose of this article to formulate and study existence and asymptotic behavior of boundary blow up solutions, also called large solutions, for equations involving the fractional laplacian and other more general integral operators. We consider the boundary value problem

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= f \quad \text{in} \quad \Omega, &(1.3)\\ u &= g \quad \text{on} \quad \bar{\Omega}^c, &(1.4)\\ \lim_{x\in\Omega, x\to\partial\Omega} u(x) &= \infty, &(1.5)\end{aligned}$$

where we assume that Ω is a bounded domain of class C^2, $\alpha \in (0,1)$ and the functions $f : \Omega \to \mathbb{R}$ and $g : \bar{\Omega}^c \to \mathbb{R}$ are continuous. We readily observe that in this non-local setting, the prescribed values of the solution outside the domain Ω will definitely play a role. Another simple observation at this early stage is that the explosive solution, and the external values g, cannot have an arbitrary behavior near the boundary if the fractional laplacian is going to be well defined.

In order to state our main theorems in precise terms, we describe first our assumptions on the functions f and g. On the external values or *boundary values* g, we assume:

(G0) For each open set O containing $\bar{\Omega}$, the function $g : \bar{\Omega}^c \to \mathbb{R}$ has a global modulus of continuity in O^c.

(G1) The function g explodes at the boundary:

$$\lim_{x\notin\Omega, x\to\partial\Omega} g(x) = \infty. \tag{1.6}$$

(G2) The explosion of g at the boundary is controlled by:

$$\limsup_{x\notin\Omega, x\to\partial\Omega} g(x)\,\mathrm{dist}(x,\partial\Omega)^{\frac{2\alpha}{p-1}} < \infty. \tag{1.7}$$

(G3) For certain $R > 0$ such that $\bar{\Omega} \subset B(0,R)$, we have

$$\int_{\mathbb{R}^N\setminus B(0,R)} \frac{|g(y)|}{|y|^{N+2\alpha}} dy < \infty. \tag{1.8}$$

On the function f, the forcing term, we assume:

(F0) $f : \Omega \to \mathbb{R}$ is a continuous function.

(F1) The behavior of f near the boundary is controlled by

$$\limsup_{x\in\Omega, x\to\partial\Omega} f(x)\mathrm{dist}(x,\partial\Omega)^{\frac{2\alpha p}{p-1}} < \infty. \tag{1.9}$$

(F2) There is $m \in \mathbb{R}$ such that

$$f(x) \geq m \qquad \forall x \in \Omega.$$

Now we state our existence theorem:

Theorem 1.1 *Assume that Ω is a bounded domain of class C^2 and*

$$0 < \alpha < 1, \quad p > 2\alpha + 1. \tag{1.10}$$

Further we assume that the functions $f : \Omega \to \mathbb{R}$ and $g : \bar{\Omega}^c \to \mathbb{R}$ satisfy [G0]-[G3] *and* [F0]-[F2]*, respectively. Then there is at least one solution u of equation (1.3)-(1.5) and there is a constant $c > 0$ such that u satisfies*

$$u(x)\,\mathrm{dist}(x, \partial\Omega)^{\frac{2\alpha}{p-1}} \leq c, \qquad \forall x \in \Omega. \tag{1.11}$$

We observe that the exponent p has to satisfy condition (1.10), which does not appear in the second order case, where $p > 1$ is only required. This condition in p is needed so that the solution is integrable near the boundary and the fractional laplacian is well defined. Therefore, we believe that the condition $p > 2\alpha + 1$ is not optimal and to avoid it an other notion of solution needs to be define since our notion of viscosity solution (see below) needs integrability.

We notice that the explosion of the solution is driven by the boundary values g, since the forcing term f is not necessarily explosive. One may expect that blow up solution driven by f exists, when g is bounded for example, however it is not obvious how to prove it.

We have preferred to state our main theorems for the fractional laplacian since the results are new even in this case. In Section §4 and §5 we state and prove more general versions of our theorems including more general non-linear fractional operators, see Theorem 4.1 and Theorem 5.1.

In Theorem 1.1 we consider the equation is satisfied in the viscosity sense. However, since f, g and the solution u itself are unbounded we need to make some precisions in Section §2. Naturally, this linear problem could be approached using other notions of solutions, but we prefer the viscosity framework in order to include in our results more general elliptic fractional operators. We do not discuss regularity properties of the solutions, but we expect the known regularity results for bounded solutions with continuous bounded data could be applied with small changes, see [5], [6] and [22]. In particular when f is further assumed to be Hölder continuous, we expect the solutions are of class $C^{2\alpha,\gamma}$ for some $\gamma > 0$, see [7] and [22].

In our second theorem we consider the asymptotic behavior of the blow up solutions of (1.3)-(1.5). For this purpose we need a new hypothesis on the behavior of g near the boundary

(G4) There are constants $c > 0$ and $\beta \in [-2\alpha/(p-1), 0)$ such that

$$\liminf_{x \notin \Omega, x \to \partial\Omega} g(x)\,\mathrm{dist}(x, \partial\Omega)^{-\beta} > c > 0. \qquad (1.12)$$

Then we have our second main result:

Theorem 1.2 *Assuming u is a solution of the equation (1.3)-(1.5), that the function g satisfies* (G0), (G1) *and* (G4) *and that f satisfies* (F0) *and* (F1). *Then the solution u satisfies*

$$u(x) \geq c\mathrm{dist}(x, \partial\Omega)^{\beta}, \quad \text{for all } x \in A_\delta, \qquad (1.13)$$

where $c > 0$ and $\delta > 0$ are constants and $A_\delta = \{y \in \Omega \,/\, d(y, \partial\Omega) < \delta\}$.

Theorems 1.2 can also be proved for much more general fractional elliptic operators as we see in Section §5. Regarding estimate (1.13), we observe that if $\beta = -2\alpha/(p-1)$ then the solution u satisfies

$$c \leq u(x)\mathrm{dist}(x, \partial\Omega)^{2\alpha/(p-1)} \leq C, \quad \text{for all } x \in A_\delta,$$

for some constant $0 < c \leq C$. Thus, we have found the exact rate of explosion for the solution u, which is the first step in a more careful analysis of the asymptotic behavior of u, where one would like to prove that $c = C$. In case one can prove this, that is, that

$$\lim_{x \to \partial\Omega} u(x)\mathrm{dist}(x, \partial\Omega)^{2\alpha/(p-1)}$$

exists, then the uniqueness question is in order, as in the second order case.

Even though we use the classical super and sub-solution approach to prove our main theorems, the novelty resides on the estimates needed to find these super and sub-solutions and the formulation of the problem itself. We use different function involving powers of the distance function $d(x) = d(x, \partial\Omega)$ in the construction of super and sub-solutions and barrier functions. The point is that even though the computation may be difficult, near the boundary we can obtain effective estimates. In this respect, regarding the nonlinearity we recall that in the second order case, functions satisfying the Keller-Osserman may replace the power function u^p in the limiting case $p = 1$. In our problem a similar limiting case occurs at $p = 2\alpha + 1$ and certainly we would expect a Keller-Osserman condition here, however it is not obvious how does it look like and how super and sub-solutions could be obtained.

This article is organized as follows. In Section §2 we present some preliminaries defining the class of operators to which our results apply, extending the notion of viscosity solutions, comparison and stability theorems to unbounded data on the boundary and blow up solutions. Section §3 is devoted to the existence theorems for the non-linear problem with linear and power first order term and bounded data. In Section §4 we state and prove a more general version of Theorem 1.1 for more general fractional operators. For this purpose we construct an appropriate super-solution and use the comparison theorem. In Section §5 we state and prove a more general version of Theorem 1.2. We construct an appropriate sub-solution and use the comparison theorem.

2 Preliminaries

We start this section defining the class of operators we consider in our article. Let $K : \mathbb{R}^N \to \mathbb{R}$ be a positive even function satisfying

$$\frac{\lambda}{|y|^{N+2\alpha}} \leq K(y) \leq \frac{\Lambda}{|y|^{N+2\alpha}}, \tag{2.1}$$

where $N \geq 2$, $\Lambda \geq \lambda > 0$ and $\alpha \in (0,1)$. For such a K and for a suitable function u we define the linear operator $L_K(u)$ as

$$L_K(u)(x) = \int_{\mathbb{R}^N} \delta_L(u,x,y)K(y)dy, \qquad x \in \mathbb{R}^N,$$

where $\delta_L(u,x,y) = u(x+y) + u(x-y) - 2u(x)$. If we denote by $\mathcal{L}_0$ the class of all these linear operators then we define the operators

$$\mathcal{M}^+u(x) = \sup_{L\in\mathcal{L}_0} L(u)(x) \quad \text{and} \quad \mathcal{M}^-u(x) = \inf_{L\in\mathcal{L}_0} L(u)(x),$$

the maximal and the minimal operator for the class $\mathcal{L}_0$, respectively. We remark that $\mathcal{L}_0$, $\mathcal{M}^+$ and $\mathcal{M}^+$ depend on the parameters Λ, λ and α, but we make that explicit to not overcharge the notation.

In this paper we consider more general operators defined as

$$\mathcal{F}(u) = \inf_{a\in A} \sup_{b\in B} L_{K^{a,b}}(u),$$

where A, B are index sets and for each $a \in A$ and $b \in B$ the function $K^{a,b}$ satisfies (2.1). We denote by $\mathcal{E}$ the class of all these operators. In what follow we briefly review some basic definitions and comparison theorems for integral operators $\mathcal{F} \in \mathcal{E}$. Before doing that we have a preliminary lemma needed to consider unbounded solutions in the viscosity sense.

We consider the function $W(x) = cd^\beta(x, \partial\Omega)$, for $c > 0$ and $\beta > -1$.

Lemma 2.1 *When Ω is of class C^2 and $\beta > -1$, the function W is integrable in any open bounded set containing $\partial\Omega$.*

Proof. We introduce a parametrization of the region near the boundary $\partial\Omega$. Since the boundary of Ω is of class C^2, for every $\bar{x} \in \partial\Omega$ there is a diffeomorphism

$$\varphi : [-1, 1] \times B_1 \to B(\bar{x}),$$

where $B_1 = \{z \in \mathbb{R}^{N-1} \,/\, |z| \leq 1\}$ and $B(\bar{x})$ is such that $\varphi((0, 1] \times B_1) := B_+(\bar{x}) \subset \Omega$, $\varphi([0, 1] \times B_1) = \bar{\Omega} \cap B(\bar{x})$ and $\varphi(\{0\} \times B_1) = \partial\Omega \cap B(\bar{x})$. We write $\varphi([-1, 0) \times B_1) = B_-(\bar{x}) \subset \mathbb{R}^N \setminus \Omega$ and $B(\bar{x}) = \bar{B}_+(\bar{x}) \cup \bar{B}_-(\bar{x})$. The function φ further satisfies $\varphi(0, 0) = \bar{x}$ and

$$s = \text{sign}(s)\,\text{dist}(\varphi(s, z), \partial\Omega), \quad \forall (s, z) \in [-1, 1] \times B_1,$$

where $\text{sign}(s) = 1$ if $s \geq 0$ and $\text{sign}(s) = -1$ if $s < 0$. Moreover, there are constants $0 < c \leq C$ such that, for all $(s, z) \in [-1, 0) \cup (0, 1] \times B_1$

$$c \leq |D\varphi(s, z)| \leq C, \quad c \leq \frac{|\varphi(s, z)|}{|(s, z)|} \leq C \tag{2.2}$$

and

$$|D\varphi(s, z)w| \geq c|w|, \quad \forall w \in \mathbb{R}^N. \tag{2.3}$$

We observe that the diffeomorphism φ can be extended to an open set containing $[0, 1] \times B_1$ and that, even though φ depends on $\bar{x} \in \partial\Omega$ the constants c, C are independent of $\bar{x}$, by compactness of the boundary of $\partial\Omega$. Moreover, there is $\delta > 0$ so that the annulus $\{x \in \mathbb{R}^N \,/\, d(x, \partial\Omega) < \delta\} \subset \bigcup_{\bar{x} \in \partial\Omega} B(\bar{x})$.

Now, for every $\bar{x} \in \partial\Omega$, we use (2.2) to find that

$$\int_{B(\bar{x})} W(x)dx \leq c \int_{[-1,1] \times B_1} s^\beta ds dz < \infty,$$

since $\beta + 1 > 0$. We may cover $\partial\Omega$ by a finite number of sets of the form $B(\bar{x})$ and then it is clear then, that for every bounded, open set O such that $\partial\Omega \subset O$, we have

$$\int_O W(x)dx < \infty,$$

completing the proof. □

In what follows we recall and extend the definition of viscosity solution, comparison theorem and stability properties for unbounded functions. We first define the notion of solution for the equation

$$\mathcal{F}(u) + h(u) = f(x) \quad \text{in} \quad \Omega, \tag{2.4}$$

$$u = g \quad \text{in} \quad \bar{\Omega}^c, \tag{2.5}$$

when $\mathcal{F} \in \mathcal{E}$, $h : \mathbb{R} \to \mathbb{R}$, $f : \Omega \to \mathbb{R}$ and $g : \bar{\Omega}^c \to \mathbb{R}$ are continuous, bounded from below and g satisfies (G2)-(G3).

Definition 2.1 *We say that a function $u : (\partial\Omega)^c \to I\!R$ is W-admissible if u is continuous, bounded from below, it satisfies (G3) and there exists a bounded open set O containing $\partial\Omega$ such that*

$$u(x) \le W(x) \qquad \forall x \in O.$$

We have the following corollary of Lemma 2.1

Corollary 2.1 *If K satisfies* (2.1) *then, for every W-admissible function u, $x \in \Omega$ and $\varepsilon < d(x, \partial\Omega)$, the integral*

$$\int_{B(0,\varepsilon)^c} (u(x+y) + u(x-y) - 2u(x))K(y)dy$$

is well defined.

We define the notion of viscosity solution in this setting.

Definition 2.2 *A W-admissible function u is a viscosity super-solution (sub-solution) of (2.4)-(2.5) if*

$$u \ge g \text{ (resp. } u \le g\text{) in } \bar{\Omega}^c$$

and for every point $x_0 \in \Omega$ and any neighborhood V of x_0 with $\bar{V} \subset \Omega$ and for any $\varphi \in C^2(\bar{V})$ such that $u(x_0) = \varphi(x_0)$ and

$$u(x) > \varphi(x) \quad (\text{resp. } u(x) < \varphi(x)) \quad \text{for all} \quad x \in V \setminus \{x_0\}$$

the function v defined by

$$v(x) = u(x) \quad \text{if } x \in \mathbb{R}^N \setminus V \quad \text{and} \quad v(x) = \varphi(x) \quad \text{if } x \in V$$

satisfies

$$\mathcal{F}(v)(x_0) + h(v(x_0)) \le f(x_0) \quad (\text{resp. } \mathcal{F}(v)(x_0) + h(v(x_0)) \ge f(x_0)).$$

Remark 2.1 *As in the usual definition, we may consider inequality instead of strict inequality*

$$u(x) \ge \varphi(x) \quad \text{for all} \quad x \in V \setminus \{x_0\},$$

and 'in some neighborhood V of x_0' instead of 'in all neighborhoods'.

Next we prove a stability theorem that applies to unbounded solutions.

Theorem 2.1 *Assume that $\{u_n\}$ is a sequence of W-admissible functions, bounded bellow, satisfying (G3) uniformly and such that $u_n \to u$ uniformly in any open set O such that $O \cap \partial\Omega = \emptyset$. Assume further $\{f_n\}$ is a sequence of continuous functions in Ω converging to the continuous function f uniformly on every compact subset of Ω. Then, if all u_n are super-solutions (resp. sub-solutions) of (2.4)-(2.5) with f_n then u is a super-solutions (resp. sub-solutions) of (2.4)-(2.5) with f.*

Proof. The proof goes as in the usual case, just noticing that the integrals used to evaluate the operator $\mathcal{F}$ in the test functions v are convergent (see Definition 2.2). □

Next we prove a comparison theorem for unbounded solutions.

Theorem 2.2 *Assume u and v are super-solution and sub-solutions of (2.4)-(2.5) with an increasing h. Assume further that u is of class C^2 in $(\partial\Omega)^c$, v can be extended continuously to $\mathbb{R}^N$. Then $u \geq v$ in Ω. In the case u is bounded and not C^2 in $(\partial\Omega)^c$ the result also holds.*

Proof. Since u is of class C^2 in $(\partial\Omega)^c$, we only need to care on the fact that u may be unbounded. We first see that from our hypotheses $u - v$ satisfies

$$-\mathcal{M}^-(u-v) \geq -(h(u) - h(v)) \quad \text{in} \quad \Omega. \tag{2.6}$$

Under our assumptions $u - v$ has a global minimun in Ω achieved at $x_0 \in \Omega$. Let $\Omega_- = \{x \in \mathbb{R}^N \,/\, u(x) - v(x) < 0\}$ and observe that $x_0 \in \Omega_-$, $\bar{\Omega}_- \subset \Omega$ and $u(x) - v(x) \geq 0$ for all $x \in \Omega_-^c$. We see that then

$$-\mathcal{M}^-(u-v) \geq 0 \quad \text{in} \quad \Omega_-$$

and we can argue as in Lemma 5.10 in [5] to conclude that in fact $u(x) - v(x) \geq 0$ in Ω_-, providing a contradiction.

In the case u is only bounded by Lemma 5.8 of [5] we find (2.6). Then we argue in the same way. □

Lemma 2.1 *Let u_1 and u_2 be super-solutions of*

$$-\mathcal{F}(u) + h(u) = f, \tag{2.7}$$

u_1 in A_δ and u_2 in Ω, respectively. Suppose further that $u_1(x) > u_2(x)$ for $x \in \partial A_\delta \setminus \partial\Omega$, then the function

$$\bar{W}(x) = \begin{cases} u_2(x) & \text{if } x \in \Omega \setminus A_\delta \\ \min\{u_2(x), u_1(x)\} & \text{if } x \in (\Omega \setminus A_\delta)^c, \end{cases} \tag{2.8}$$

is a super-solution of (2.7) in Ω. The analogous results holds for sub-solution.

Proof. Let φ be a test function for $\bar{W}$ at $x_0 \in \Omega$, then $\varphi(x_0) = \bar{W}(x_0)$ and $\varphi < \bar{W}$ in $V \setminus \{x_0\}$. Suppose first that $\bar{V} \subset A_\delta$ and $\bar{W}(x_0) = u_1(x_0)$, then $\varphi < u_1$ in $V \setminus \{x_0\}$. Thus, if we define

$$v_1(x) = u_1(x) \quad \text{if } x \in \mathbb{R}^N \setminus V \quad \text{and} \quad v_1(x) = \varphi(x) \quad \text{if } x \in V$$

then, since u_1 is a super-solution, v_1 satisfies

$$-\mathcal{F}(v_1)(x_0) + h(v_1(x_0)) \geq f(x_0).$$

Then we conclude that v defined as

$$v(x) = \bar{W}(x) \quad \text{if } x \in \mathbb{R}^N \setminus V \quad \text{and} \quad v(x) = \varphi(x) \quad \text{if } x \in V$$

satisfies

$$-\mathcal{F}(v)(x_0) + h(v(x_0)) \geq f(x_0),$$

that is $\bar{W}$ is a super-solution. In case $\bar{W}(x_0) = u_2(x_0)$, the same argument proves that $\bar{W}$ is a super-solution. We are only left with the case $x_0 \in \Omega \setminus A_\delta$. Using $u_1 > u_2$ in $\partial A_\delta \setminus \partial\Omega$ we find $\bar{W}(x_0) = u_2(x_0)$, then we can use the same argument again to get that $\bar{W}$ is a super-solution.□

3 The Dirichlet problem.

In this section we study the existence of solutions for the Dirichlet problem for equations involving $\mathcal{F} \in \mathcal{E}$, both in the homogeneous and nonlinear case. In the homogeneous case, including a linear zero order term, an existence result is proved by Barles, Chasseigne and Imbert in [3] using and indirect method based on an existence result in $\mathbb{R}^N$. Here we provide a direct approach that relies on Perron's method together with appropriate barrier functions similar to those used in the case of local operators (see, for example, Section 3 in [9]).

In this section we consider bounded external values g. We assume

($\tilde{\mathbf{G}}$0) $g : \Omega^c \to \mathbb{R}^N$ is bounded and has a global modulus of continuity in Ω^c.

Our first theorem of this section is

Theorem 3.1 *Suppose g satisfies ($\tilde{G}$0), f is continuous in $\bar{\Omega}$ and $C \geq 0$. Then there exists a viscosity solution of*

$$\begin{aligned} -\mathcal{F}(u) + Cu &= f \quad \text{in} \;\; \Omega, \\ u &= g \quad \text{on} \;\; \Omega^c. \end{aligned}$$

Before the proof we give some preliminary lemmas.

Lemma 3.1 *Under the hypothesis of Theorem 3.1 there exist a super and sub-solution of*

$$-\mathcal{F}(w) + Cw = f \quad \textit{in} \quad \Omega, \tag{3.1}$$

with $w = 0$ in Ω^c.

From now on we denote by $d(x)$ the distance of x to $\partial\Omega$, that is,

$$d(x) := \mathrm{dist}(x, \partial\Omega), \qquad x \in \Omega.$$

Here we will always assume that $\delta > 0$ is small enough so that $d(x)$ is well defined and C^2 in A_δ. Then we define our barrier function as follows

$$\xi(x) = \begin{cases} d(x)^\beta & \text{if } x \in A_\delta \\ \ell(x) & \text{if } x \in \Omega \setminus A_\delta \\ 0 & \text{if } x \in \Omega^c, \end{cases} \tag{3.2}$$

for $\beta > 0$ and a function ℓ such that ξ is positive and C^2 in Ω. We prove:

Lemma 3.2 *There exist $\delta > 0$, $\beta \in (0, 2\alpha)$ and $C > 0$ such that*

$$\mathcal{M}^+(\xi) \leq -Cd(x)^{\beta - 2\alpha} \quad \textit{in} \quad A_\delta.$$

Proof. We first prove the result for linear operators. In all the proof we assume $x \in A_\delta$. We first see that there is a constant c_δ such that

$$L_{B^c}\xi(x) = \int_{B^c(0,\delta)} |\xi(x+y) + \xi(x-y) - 2\xi(x))| \, K(y)dy \leq c_\delta. \tag{3.3}$$

Then we need to estimate the integral over $B(0, \delta)$. We write

$$\begin{aligned} L_B\xi(x) &= \int_{B(0,\delta)} (\xi(x+y) + \xi(x-y) - 2\xi(x))K(y)dy \\ &= A(x) + B_+(x) + B_-(x) + I(x), \end{aligned} \tag{3.4}$$

where

$$A(x) = \int_{D_A} - 2d^\beta(x)K(y)dy < 0, \tag{3.5}$$

$$B_\pm(x) = \int_{D_B} (d^\beta(x \pm y) - 2d^\beta(x))K(y)dy \tag{3.6}$$

and

$$I(x) = \int_{D_I} (d^\beta(x+y) + d^\beta(x-y) - 2d^\beta(x))K(y)dy \tag{3.7}$$

with the domains of integration given by

$$
\begin{aligned}
D_A &= \{y \in B(0,\delta)\,/\,x+y \notin \Omega \quad \text{and} \quad x-y \notin \Omega\}, &(3.8)\\
D_{B_\pm} &= \{y \in B(0,\delta)\,/\,x \pm y \in \Omega \quad \text{and} \quad x \mp y \notin \Omega\} \quad \text{and} &(3.9)\\
D_I &= \{y \in B(0,\delta)\,/\,x+y \in \Omega \quad \text{and} \quad x-y \in \Omega\}. &(3.10)
\end{aligned}
$$

For notational convenience we write $d = d(x)$, whenever no confusion arises. We observe that for $\mu > 0$

$$
\mu \text{dist}(x, \partial\Omega) = \text{dist}(\mu x, \mu\partial\Omega). \tag{3.11}
$$

We estimate $B_+(x)$ ($B_-(x)$ is analogous). With a change of variable we find

$$
B_+(x) = d^{\beta-2\alpha} \int_{d^{-1}D_{B_+}} (\text{dist}(d^{-1}x+y, d^{-1}\partial\Omega)^\beta - 2) K_d(y)dy = d^{\beta-2\alpha}\, C(x),
$$

where $K_d = d^{N+2\alpha}K(yd)$. Now we write

$$
d^{-1}D_{B_+} = (d^{-1}D_{B_+} \cap B(0,R)) \cup (d^{-1}D_{B_+} \cap B(0,R)^c) := B_{1,R} \cup B_{2,R}
$$

and find that there exists a positive constant C_0 such that, for all R large

$$
\int_{B_{1,R}} K_d(y)dy \geq C_0.
$$

On the other hand, we find that for $< \beta \leq \beta_0 < 2\alpha$ we have

$$
\text{dist}(d^{-1}x+y, d^{-1}\partial\Omega)^\beta - 2 \leq |y|^{\beta_0}.
$$

Therefore if R is large

$$
\int_{B_{2,R}} (\text{dist}(d^{-1}x+y, d^{-1}\partial\Omega)^\beta - 2) K_d(y)dy \leq \frac{C_0}{2}
$$

independent of β. To conclude, we see that for fixed R we have

$$
\lim_{\beta\to 0} \int_{B_{1,R}} (\text{dist}(d^{-1}x+y, d^{-1}\partial\Omega)^\beta - 2) K_d(y)dy = -\int_{B_{1,R}} K_d(y)dy \leq -C_0,
$$

from where we see that $C(x) \leq -\frac{C_0}{4}$ for all β small.

Now we estimate $I(x)$. For that purpose we observe that

$$
\delta_L(\xi, x, y) = \int_0^1 \int_{-1}^1 \sum_{i,j=1}^N \frac{\partial^2 \xi}{\partial x_i \partial x_j}(x+sty)ty_iy_j dsdt, \tag{3.12}
$$

Using the definition of ξ we find that

$$\frac{\partial^2 \xi}{\partial x_i \partial x_j}(w) = \beta(\beta-1)d(w)^{\beta-2}A_{ij}(w), \tag{3.13}$$

where

$$A_{ij}(w) = \frac{\partial d}{\partial x_i}(w)\frac{\partial d}{\partial x_j}(w) + \frac{d(w)}{\beta-1}\frac{\partial^2 d}{\partial x_i \partial x_j}(w). \tag{3.14}$$

We observe that for δ small we have $\delta_L(\xi, x, y) \leq 0$ in D_I. Moreover, for $y \in B(0, d/2)$, $s \in [0,1]$, $t \in [-1,1]$ and $w = x + sty$ we have that $A_{ij}(w)$ is bounded and $d/2 \leq d(w) \leq 3d/2$. Thus, from (3.12)-(3.14), we find that

$$\delta_L(\xi, x, y) \leq -cd(x)^{\beta-2}|y|^2. \tag{3.15}$$

From here we obtain

$$\begin{aligned} I(x) \leq \int_{B(0,d/2)} \delta_L(\xi, x, y)K(y)dy &\leq -cd^{\beta-2}\int_{B(0,d/2)} \frac{|y|^2}{|y|^{N+2\alpha}}dy \\ &\leq -cd^{\beta-2\alpha}\int_{B(0,1/2)} |y|^{-N-2\alpha+2}dy \\ &\leq -Cd^{\beta-2\alpha}. \end{aligned} \tag{3.16}$$

We finally conclude that there exists $C_1 > 0$ such that

$$L\xi(x) \leq d(x)^{\beta-2\alpha}(-C + d(x)^{2\alpha+\beta}c_\delta) \leq -C_1 d(x)^{\beta-2\alpha} \quad \text{in} \quad A_\delta.$$

To finish the proof we observe that C_1 depends on K only through λ and Λ, so we can take the supremum over all linear operators in $\mathcal{L}_0$ to conclude.$\Box$

Proof of Lemma 3.1. Choose a point $x_0 \in \Omega^c$ such that $1 < d(x_0)$ and take r_1 and R such that $1 < r_1 < d(x_0) < R$ and $\bar{\Omega} \subset \{r_1 < |x - x_0| < R\}$ Let $\varepsilon > 0$ and $\sigma \in (-N, -N + \eta)$, with η small, and define the function

$$w(r) = \begin{cases} \varepsilon^\sigma & \text{if } 0 < r \leq \varepsilon, \\ r^\sigma & \text{if } \varepsilon \leq r. \end{cases}$$

We see from the proof of Lemma 4.1 in [13] that for ε small there exists $c > 0$ such that $\mathcal{M}^-(w(|x - x_0|)) \geq c|x - x_0|^{\sigma-2\alpha}$ for all $r_1 < |x - x_0| < R$. Next we define $G(x) = r_0^\sigma - w(|x - x_0|)$ and observe that if r_0 is small $G > 0$ in $\bar{\Omega}$ and using that $\mathcal{M}^+(-w) = -\mathcal{M}^-(w)$ we find $\mathcal{M}^-(G(x)) \leq -c$ in Ω, for some other positive constant c. From here, defining $u_2 = aG$ and taking a large, we find

$$-\mathcal{F}(u_2) + Cu_2 \geq f \quad \text{in} \quad \Omega.$$

On the other hand, defining $u_1 = b\xi$ and taking b large, we find from Lemma 3.2 that

$$-\mathcal{F}(u_1) + Cu_1 \geq f \quad \text{in} \quad A_\delta,$$

and $u_1 > u_2$ in $\partial A_\delta \setminus \partial\Omega$. Thus, from Lemma 2.1 we get that $\bar{W}$ defined in (2.8) is a super-solution of (3.1). The sub-solution can be constructed similarly by changing the sing of u_1 and u_2.

Proof of Theorem 3.1. Let $\{g_n\}$ be a sequence of smooth functions, with a common modulus of continuity with g in Ω^c and so that $g_n \to g$ uniformly in Ω^c. Since Ω is of class C^2 we may assume that g_n is of class $C^2(\mathbb{R}^N)$. Using the sub and super-solution given in Lemma 3.1 and Theorem 2.2 we can apply Perron's Method (see [14]) to find v_1 and v_2 satisfying

$$\begin{aligned} -\mathcal{M}^+(v_1) + Cv_1 &= f + \mathcal{M}^+(g_n) - Cg_n \ \text{ in } \ \Omega, & (3.17)\\ v_1 &= 0 \ \text{ on } \ \Omega^c & (3.18) \end{aligned}$$

and

$$\begin{aligned} -\mathcal{M}^-(v_2) + Cv_2 &= f + \mathcal{M}^-(g_n) - Cg_n \ \text{ in } \ \Omega, & (3.19)\\ v_2 &= 0 \ \text{ on } \ \Omega^c. & (3.20) \end{aligned}$$

Now we define $\bar{u}_n = v_1 + g_n$, $\underline{u}_n = v_2 + g_n$ and we use that

$$\mathcal{F}(\bar{u}_n) \leq \mathcal{M}^+(\bar{u}_n) \leq \mathcal{M}^+(v_1) + \mathcal{M}^+(g_n)$$

and

$$\mathcal{F}(\underline{u}_n) \geq \mathcal{M}^-(\underline{u}_n) \geq \mathcal{M}^-(v_2) + \mathcal{M}^-(g_n)$$

to find that $\underline{u}_n$ and $\bar{u}_n$ are sub and super-solution of (3.1) with $u = g_n$ on Ω^c, respectively. Therefore using again Perron's Method we find a solution u_n of (3.1) with $u = g_n$ on Ω^c.

Using the regularity results of [6] (see Theorem 3.3), which also holds for a C^2 domain, we obtain uniform Hölder regularity for u_n, hence equicontinuity. Thus, the Arzela-Ascoli Theorem provides a subsequence such that $u_n \to u$ in $C^\gamma(\bar{\Omega})$, $\gamma > 0$. Finally, by stability property (Corollary 4.7 in [5]), we get that u is a solution of (3.1) with $u = 0$ on Ω^c.□

We conclude this section studying the existence of a solution to

$$\left\{ \begin{aligned} -\mathcal{F}(u) + u^p &= f \quad \text{in } \Omega,\\ u &= g \quad \text{on } \Omega^c, \end{aligned} \right. \tag{3.21}$$

for $p > 1$. We have

Theorem 3.2 *Suppose g satisfies* $(\tilde{\mathbf{G0}})$ *and f is a bounded continuous function defined on Ω. Then there exists a viscosity solution of (3.21).*

Proof. We observe first that there exist a large constant $M > 0$ such that $-M^p \leq f(x) \leq M^p$ for all $x \in \Omega$ and $-M \leq g(x) \leq M$ for all $x \in \Omega^c$. Hence, $-M$ and M are sub-solution and a super-solution of (3.21), respectively.

Then we define $v_0 = -M$ and we use Theorem 3.1 to define iteratively the sequence of functions v_n so that

$$\begin{cases} -\mathcal{F}(v_{n+1}) + Cv_{n+1} = f + Cv_n - v_n^p & \text{in } \Omega, \\ v_{n+1} = g & \text{on } \Omega^c. \end{cases} \tag{3.22}$$

Here C is a positive constant so that the function $r(t) = Ct - t^p$ is increasing in the interval $[-M, M]$. FromTheorem 2.2 we see that

$$-M \leq v_n \leq v_{n+1} \leq M \quad \text{in} \quad \Omega \quad \forall n \in \mathbb{N}.$$

Now we use the regularity results of [6] to get equicontinuity of v_n in $\bar{\Omega}$. Then by the Arzela-Ascoli Theorem there exists a subsequence such that $v_n \to v$ in $C^\gamma(\bar{\Omega})$. Finally, by stability properties (see Corollary 4.7 in [5]) we get that v is a solution of (3.21).□

4 Construction of a super-solution and proof of Theorem 1.1 and generalization.

In this section we prove our main Theorem 1.1 on the existence of solutions for our equation (1.3)-(1.5). Actually, our method of proof allows the treatment of a large class of problems including some Isaacs type integral operators. Let $\mathcal{F} \in \mathcal{E}$ and consider the equation

$$-\mathcal{F}(u) + u^p = f \quad \text{in } \Omega, \tag{4.1}$$

$$u = g \quad \text{on } \mathbb{R}^N \setminus \Omega, \tag{4.2}$$

$$\lim_{x \in \Omega, x \to \partial\Omega} u(x) = \infty. \tag{4.3}$$

We will prove the following theorem

Theorem 4.1 *Assume that Ω is a bounded domain of class C^2 and α satisfies (1.10). Further assume that the functions $f : \Omega \to \mathbb{R}$ and $g : \bar{\Omega}^c \to \mathbb{R}$ satisfy* [G0]-[G3] *and* [F0]-[F2], *respectively. Then there is at least one solution u of equation (4.1)-(4.3) that satisfies*

$$u(x)\,\mathrm{dist}(x, \partial\Omega)^{\frac{2\alpha}{p-1}} \leq c, \qquad \forall x \in \Omega, \tag{4.4}$$

for certain constant $c > 0$.

The idea of the proof is to solve the corresponding equation with bounded data. Then, by using an appropriate super-solution, the solution is obtained as a limit. The super-solution we consider is just a power of the distance to the boundary of Ω. Given $\beta < 0$ and $\delta > 0$ small, we define the function

$$\xi(x) = \begin{cases} d(x)^\beta & \text{if } x \in A_\delta \\ \ell(x) & \text{if } x \in \Omega \setminus A_\delta \\ g(x) & \text{if } x \in \Omega^c, \end{cases} \tag{4.5}$$

where ℓ is such that ξ is of class C^2 and positive in Ω. In this section we are only interested in the case $\beta = -\frac{2\alpha}{(p-1)}$, but we prefer to consider a general $\beta < 0$ in order to use pieces of the proof in the following section. We start analyzing ξ in the case of linear operators.

Proposition 4.1 *Assume that $\beta < 0$, g satisfies (G0)-(G3) and $L \in \mathcal{L}_0$, then the function $G : \Omega \to \mathbb{R}$ defined as*

$$G(x) = \xi^{-\beta+2\alpha}(x)L\xi(x), \quad x \in \Omega, \tag{4.6}$$

is bounded above, that is, the exists $C \in \mathbb{R}$ such that $G(x) \leq C$ for all $x \in \Omega$. Notice that when $\beta = -\frac{2\alpha}{(p-1)}$ we have $p = \beta - 2\alpha$.

Proof. We notice that $L\xi(x)$ is a bounded function in $\Omega \setminus A_\delta$, so we only need to consider $x \in A_\delta$. Next we observe that for certain constant $c_\delta > 0$

$$L_{B^c}\xi(x) = \int_{B(0,\delta)^c} (\xi(x+y) + \xi(x-y) - 2\xi(x))K(y)dy \leq c_\delta d(x)^\beta, \tag{4.7}$$

so we only need to analyze $L_B\xi(x)$, the integral over $B(0,\delta)$. We see that

$$L_B\xi(x) = A_+(x) + A_-(x) + B_+(x) + B_-(x) + I(x), \quad x \in A_\delta, \tag{4.8}$$

where

$$A_\pm(x) = \int_{D_{A_\pm}} (g(x \pm y) - \xi(x))K(y)dy, \tag{4.9}$$

$$B_\pm(x) = \int_{D_{B_\pm}} (\xi(x \pm y) - \xi(x))K(y)dy \tag{4.10}$$

and

$$I(x) = \int_{D_I} (\xi(x+y) + \xi(x-y) - 2\xi(x))K(y)dy, \tag{4.11}$$

where $D_{B_\pm}$, D_I are defined in (3.9) and (3.10) and

$$D_{A_\pm} = \{y \in B(0,\delta) \,/\, x \pm y \notin \Omega\}.$$

In what follows we study each of these integrals in detail, starting the analysis with $A_+(x)$ ($A_-(x)$ is similar). Since $D_{A_-} \cap B(0) = \emptyset$ we obtain that

$$\begin{aligned}\int_{D_{A_-}} K(y)dy &\leq \Lambda \int_{B(0,d)^c} \frac{dy}{|y|^{N+2\alpha}} \\ &\leq \Lambda d^{-2\alpha} \int_{B(0,1)^c} \frac{dy}{|y|^{N+2\alpha}} \leq cd^{-2\alpha}, \qquad (4.12)\end{aligned}$$

where we used that $d^{-1}D_{A_-} \subset B(0,1)^c$. To continue we use hypothesis (G2) and decrease $\delta > 0$, if necessary, to have

$$g(w) \leq c\text{dist}(w, \partial\Omega)^\beta, \quad w \in B(x,\delta), \quad x \in A_\delta. \qquad (4.13)$$

Since $d = \text{dist}(x, \partial\Omega) \leq \text{dist}(x+y, \partial\Omega)$ for all $y \in D_{A_-}$, using (4.12) and (4.13), we find that

$$\int_{D_{A_-}} g(x+y)K(y)dy \leq c\Lambda d^{\beta-2\alpha}. \qquad (4.14)$$

From (4.12) and (4.14) we get the desired inequality for $A_+(x)$.

We continue with the analysis of $B_+(x)$ ($B_-(x)$ is similar). We first observe that if $x - y \notin \Omega$ then $|y| > d > d/2$ and we have

$$|B_+(x)| \leq \Lambda \int_{\delta > |w-x| \geq d/2} \frac{|\xi(w) - \xi(x)|}{|w-x|^{N+2\alpha}} dw. \qquad (4.15)$$

Since $x \in A_\delta$ there is a unique $\bar{x} \in \partial\Omega$ such that $d = |x - \bar{x}|$ and we can use the change of variable φ defined in Section §2. We may assume that $B(x,\delta) \subset B(\bar{x})$ and then, for $(s,z) \in [0,1] \times B_1$ such that $w = \varphi(s,z)$ we have

$$d/2 \leq |w - x| \leq c|(s-d, z)|.$$

Consequently, using (2.2), (2.3) and changing variables with φ we obtain

$$\begin{aligned}|B_+(x)| &\leq c\Lambda \int_{\substack{[0,1]\times B_1 \cap \\ \{|(s-d,z)| \geq d/2c\}}} \frac{|s^\beta - d^\beta|}{|(s-d,z)|^{N+2\alpha}} dsdz \\ &\leq c\Lambda d^{\beta-2\alpha} \int_{\substack{[0,d^{-1}]\times B_{d^{-1}} \cap \\ \{|(s-1,z)| \geq 1/2c\}}} \frac{|s^\beta - 1|}{|(s-1,z)|^{N+2\alpha}} dsdz \\ &\leq Cd^{\beta-2\alpha}, \qquad (4.16)\end{aligned}$$

where the last integral is bounded, since the singularity has been removed and $p > 2\alpha + 1$. From here we obtain the desired inequality for $B_+(x)$.

We finally study the integral $I(x)$. We will consider separately the integral over $B(0,d/2)$ and $D_I \setminus B(0,d/2)$. We first see that

$$\begin{aligned} & |\int_{D_I\cap B(0,d/2)^c} (\xi(x+y)-\xi(x))K(y)dy| \\ = & |\int_{D_I\cap B(0,d/2)^c} (\xi(x-y)-\xi(x))K(y)dy| \qquad (4.17) \\ \leq & \Lambda \int_{\delta>|w-x|\geq d/2} \frac{|\xi(w)-\xi(x)|}{|w-x|^{N+2\alpha}}dw, \end{aligned}$$

where this last integral was estimated above in (4.15) and (4.16). Next we look at the integral over $B(0,d/2)$. Recalling (3.12)-(3.14) we see that, for $y\in B(0,d/2)$, $s\in[0,1]$, $t\in[-1,1]$ and $w=x+sty$

$$d/2 \leq d(w) \leq 3d/2 \quad \text{and} \quad A_{ij}(w)\leq C,$$

for some constant C. Thus, from (3.12)-(3.14), we find that

$$|\delta_L(\xi,x,y)| \leq cd(x)^{\beta-2}|y|^2 \qquad (4.18)$$

and we conclude as in (3.16) that

$$|\int_{B(0,d/2)} \delta(\xi,x,y)K(y)dy| \leq Cd^{\beta-2\alpha}, \qquad (4.19)$$

completing the estimate of $I(x)$ as required. $\Box$

As a direct consequence we have

Corollary 4.1 *Under the hypothesis of Proposition 4.1, assuming that* $\beta = -2\alpha/(p-1)$ *and* $\mathcal{F}\in\mathcal{E}$ *we have*

$$\mathcal{F}(\xi(x)) = \xi^p(x)G(x), \quad x\in\Omega,$$

with the function G *bounded from above over* Ω.

Corollary 4.2 *Under the hypothesis of Proposition 4.1,* $\beta=-2\alpha/(p-1)$*,* $\mathcal{F}\in\mathcal{E}$ *and* f *satisfying (F0)-(F2) there is* $\rho>0$ *so that the function* $W(x)=\rho\xi(x)$ *is a super-solution of (4.1)-(4.2), that is, it satisfies*

$$\begin{aligned} -\mathcal{F}(W)+W^p &\geq f, \quad in\ \Omega \\ W &\geq g \quad in\ \bar{\Omega}^c. \end{aligned}$$

Proof. By Corollary 4.1 we have

$$-\mathcal{F}(\rho\xi(x)) + (\rho\xi(x))^p = -\rho\xi^p(x)G(x) + \rho^p\xi^p(x) \geq \xi^p(x)\rho(-C + \rho^{p-1}).$$

We complete the proof using assumption (F1) to find ρ large enough, so that

$$\xi^p(x)\rho(-C + \rho^{p-1}) \geq f(x) \quad \text{in} \quad \Omega.\square$$

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Using Theorem 3.2 we find a solution u_n of (3.21) with $g_n = \min\{n, g\}$ in Ω^c and $f_n = \min\{n, f\}$ in Ω. By Theorem 2.2 we have $u_n \leq u_{n+1}$. Now we define $w_n = \min\{n+1, w\}$, were w is the super-solution found Corollary 4.2 and we notice that w_n is still a super-solution by Lemma 2.1. Then, we use again Theorem 2.2 to obtain $u_n \leq w_n \leq w$ in Ω. Using a diagonal argument and the C^γ interior estimate of [5], we find a subsequence that converges to a function u uniformly over compact sets. By the stability Theorem 2.1 u is a solution of our problem. Moreover, since $u \geq u_n$ in Ω, for all n, u satisfies (4.4).$\square$

5 Construction of a sub-solution and proof of Theorem 1.2 and generalization.

In this section we prove Theorem 1.2 on the behavior of the blow up solutions to our problem. Since our method is based on sub-solutions and comparison theorems, our result can be extended to a more general class of fractional operators. Here is the statement:

Theorem 5.1 *Assume $\mathcal{F} \in \mathcal{E}$ and u is a solution of (4.1)-(4.3). Assume further that that the function g satisfies* (G0), (G1) *and* (G4) *and that f satisfies* (F0) *and* (F1). *Then the solution u satisfies*

$$u(x) \geq c d^\beta(x, \partial\Omega), \qquad \text{for all } x \in A_\delta, \tag{5.1}$$

where $c > 0$ and $\delta > 0$ are constants. Here and in all what follows β is the constant appearing in (G4).

In order to prove this theorem we construct a sub-solution based on the distance function to a set slightly larger than Ω. We consider $\varepsilon > 0$, $\sigma > 0$ and the set $\Omega_\varepsilon = \{y \in \mathbb{R}^N \,/\, d(y, \Omega) < \varepsilon\}$. We define the function η_ε as follows

$$\eta_\varepsilon(x) = \begin{cases} \min\{\sigma d^\beta(x, \partial\Omega_\varepsilon), g(x)\} & \text{if } x \in \Omega_\varepsilon \setminus \Omega \\ \sigma d^\beta(x, \partial\Omega_\varepsilon) & \text{if } x \in A_\delta \\ \sigma \ell_\varepsilon(x) & \text{if } x \in \Omega \setminus A_\delta \\ g(x) & \text{if } x \in \Omega_\varepsilon^c, \end{cases} \tag{5.2}$$

where the function ℓ_ε is chosen so η_ε is of class C^2 in Ω and the constant $\sigma > 0$ will be chosen later. We state the following proposition

Proposition 5.1 *Assume that* $\beta \in [-\frac{2\alpha}{(p-1)}, 0)$, *g satisfies* (G4) *and* $L \in \mathcal{L}_0$. *Let* $G_\varepsilon : \Omega \to \mathbb{R}$ *be defined as*

$$G_\varepsilon(x) = \text{dist}(x, \partial\Omega_\varepsilon)^{-\beta+2\alpha} L\eta_\varepsilon(x), \quad x \in \Omega. \tag{5.3}$$

Then the function G_ε *is bounded in* $\Omega \setminus A_\delta$ *and there exist* $c > 0$ *and* $\delta > 0$ *such that*

$$G_\varepsilon(x) \geq c \quad \textit{for all} \quad x \in A_\delta \quad \textit{and} \quad \textit{for all} \quad \varepsilon > 0. \tag{5.4}$$

The constants c *and* δ *can be chosen independent of* $L \in \mathcal{L}_0$ *and* ε.

Proof. We let $\delta > 0$ so that the distance function $d(\cdot, \partial\Omega)$ is well defined in $A_{2\delta}$ and in all the proof assume that $x \in A_\delta$. We see that there is a constant c_δ independent of $\varepsilon \in (0, \varepsilon_0)$ and $\sigma \in (0, \sigma_0)$, such that

$$|L_{B^c}\eta_\varepsilon(x)| = |\int_{B^c(0,\delta)} \eta_\varepsilon(x+y)+\eta_\varepsilon(x-y)-2\eta_\varepsilon(x))K(y)dy| \leq c_\delta d^\beta(x). \tag{5.5}$$

Our goal is to get a lower estimate for $L_B\eta_\varepsilon(x)$, the integral over $B(0,\delta)$. For that purpose we consider the hypothesis (G4) and decrease the value of δ if necessary so we have

$$g(z) \geq c\text{dist}(z, \partial\Omega)^\beta, \quad z \notin \Omega, \quad d(z, \partial\Omega) < 2\delta. \tag{5.6}$$

Next we define the function $\bar\eta_\varepsilon$ by

$$\bar\eta_\varepsilon(z) = \begin{cases} \min\{\sigma\text{dist}(z, \partial\Omega_\varepsilon)^\beta, c\text{dist}(z, \partial\Omega)^\beta\} & \text{if } z \in \Omega_\varepsilon \setminus \Omega \\ \sigma\text{dist}(z, \partial\Omega_\varepsilon)^\beta & \text{if } z \in A_\delta \\ \sigma\ell_\varepsilon(z) & \text{if } z \in \Omega \setminus A_\delta \\ c\text{dist}(z, \partial\Omega)^\beta & \text{if } z \in \Omega_\varepsilon^c, \end{cases} \tag{5.7}$$

and we easily see that $\eta_\varepsilon(z) \geq \bar\eta_\varepsilon(z)$ if $z \in B(x,\delta) \cap \Omega^c$ and $\eta_\varepsilon(z) = \bar\eta_\varepsilon(z)$ if $z \in B(x,\delta) \cap \Omega$. Next we define

$$\tilde\Omega_\varepsilon = \Omega \cup \{z \in \Omega_\varepsilon \,/\, c\text{dist}(z, \partial\Omega)^\beta \geq \sigma\text{dist}(z, \partial\Omega_\varepsilon)^\beta\}.$$

We see that $\tilde\Omega_\varepsilon = \Omega_{\tilde\varepsilon}$, where $0 < \tilde\varepsilon = \varepsilon/(1 + (c/\sigma)^{1/\beta}) < \varepsilon$. Now we have

$$\begin{aligned} L_B\eta_\varepsilon(x) &= \int_{B(0,\delta)} (\eta_\varepsilon(x+y) + \eta_\varepsilon(x-y) - 2\eta_\varepsilon(x))K(y)dy \\ &\geq B(x) + \sigma(A(x) + I(x)), \end{aligned} \tag{5.8}$$

where

$$A(x) = -2\int_{B(0,\delta)\setminus B(0,d_\varepsilon)} d_\varepsilon^\beta K(y)dy, \tag{5.9}$$

$$B(x) = \int_{D_B^\varepsilon} c\,\mathrm{dist}(x-y,\Omega)^\beta K(y)dy \tag{5.10}$$

and

$$I(x) = \int_{B(0,d_\varepsilon)} (\mathrm{dist}(x+y,\Omega_\varepsilon)^\beta + \mathrm{dist}(x-y,\Omega_\varepsilon)^\beta - 2d_\varepsilon)K(y)dy. \tag{5.11}$$

Here and in what follows we have considered

$$D_B^\varepsilon = \{y \in B(0,\delta) \,/\, x+y \in \tilde{\Omega}_\varepsilon \text{ and } x-y \notin \tilde{\Omega}_\varepsilon\}.$$

and $d_\varepsilon = \mathrm{dist}(x,\Omega_\varepsilon)$.

Using the same estimates as in (4.12) and (4.19) we directly find that

$$|A(x)| \le C d_\varepsilon^{\beta-2\alpha} \quad \text{and} \quad |I(x)| \le C d_\varepsilon^{\beta-2\alpha}, \tag{5.12}$$

where the constant C above is independent of ε. Next we analyze $B(x)$.

Lemma 5.1 *Under the hypothesis of Proposition 5.1, there exists a constant* $c > 0$, *so that*

$$B(x) \ge c d_\varepsilon^{\beta-2\alpha}, \quad \text{for all } x \in A_\delta. \tag{5.13}$$

Proof. We start using (3.11) and changing variables to get

$$B(x) \ge c\lambda d_\varepsilon^{\beta-2\alpha} b(\varepsilon, x), \tag{5.14}$$

where

$$b(\varepsilon, x) = \int_{d_\varepsilon^{-1} D_B^\varepsilon} \mathrm{dist}(y, d_\varepsilon^{-1}(x-\Omega))^\beta \frac{dy}{|y|^{N+2\alpha}}. \tag{5.15}$$

It is clear that $b(\varepsilon, x) > 0$ for all $(\varepsilon, x) \in (0, \epsilon_0) \times A_\delta$. If $x \in A_\delta$ is fixed, we have that $d_\varepsilon \to d = d(x, \partial\Omega)$ and the domains $\Omega \subset \tilde{\Omega}_\varepsilon \subset \Omega_\varepsilon$ all converge to Ω as $\varepsilon \to 0$ so that

$$\lim_{\epsilon\to 0} b(\epsilon, x) = \int_{d^{-1} D_B} \mathrm{dist}(y, d^{-1}(x-\Omega))^\beta \frac{dy}{|y|^{N+2\alpha}},$$

where $D_B = \{y \in B(0,\delta) \,/\, x+y \in \Omega \text{ and } x-y \notin \Omega\}$. If $d < \delta/2$, this last integral is positive for all x, since D_B has a positive measure. On the other hand, if we assume $\varepsilon > 0$ is fixed, then $d_\varepsilon \ge \varepsilon = \lim_{x\to\partial\Omega} d_\varepsilon$ for all $x \in A_\delta$ so

$$\lim_{x\to\partial\Omega} b(\epsilon, x) = \int_{\varepsilon^{-1} D_B^\varepsilon} \mathrm{dist}(y, \varepsilon^{-1}(x-\Omega))^\beta \frac{dy}{|y|^{N+2\alpha}}.$$

Again this integral is positive since D_B^ε has positive measure. Finally, we notice that $(\epsilon, x) \to \{0\} \times \partial\Omega$ is equivalent to $d_\varepsilon \to 0$. In this situation we see that $d_\varepsilon^{-1}(-x+\Omega) \subset d_\varepsilon^{-1}(-x+\tilde{\Omega}_\varepsilon) \subset d_\varepsilon^{-1}(-x+\Omega_\varepsilon)$ and all these domains converge to a semi-space S and the set $d_\varepsilon^{-1} D_B^\varepsilon$ converges to S^c. The semi-space S contains the origin, which is at a distance 1 from the boundary of S. Clearly we get

$$\lim_{d_\varepsilon \to 0} b(\epsilon, x) = \int_{S^c} \mathrm{dist}(y, S)^\beta \frac{dy}{|y|^{N+2\alpha}},$$

which is clearly positive. This completes the proof. □

Proof of Proposition 5.1 continued. From (5.5), (5.8), (5.12) and the result of Lemma 5.1 we can find σ small enough so that (5.4) is achieved.□

Proposition 5.1 readily extends for all nonlinear operator in the class $\mathcal{E}$.

Corollary 5.1 *Under the hypothesis of Proposition 5.1, assuming that $\beta \in [-2\alpha/(p-1), 0)$ and $\mathcal{F} \in \mathcal{E}$, the function $G_\varepsilon : \Omega \to \mathbb{R}$ defined as*

$$G_\epsilon(x) = \eta_\epsilon^{-\beta+2\alpha}(x)\mathcal{F}(\eta_\epsilon(x)), \quad x \in \Omega,$$

is bounded in $\Omega \setminus A_\delta$ and there exist $c > 0$ and $\delta > 0$ such that

$$G_\varepsilon(x) \geq c \quad \textit{for all} \quad x \in A_\delta \quad \textit{and} \quad \textit{for all} \quad \varepsilon > 0. \tag{5.16}$$

Now we can construct a sub-solution for our equation. We have

Proposition 5.2 *Assume the hypothesis of Proposition 5.1, $\beta \in [-2\alpha/(p-1), 0)$, $\mathcal{F} \in \mathcal{E}$ and f satisfies* (F1) *and* (F2). *Then, for all $\varepsilon > 0$ there is a function $w_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ that is a sub-solution of (4.1)-(4.2), that is, it satisfies*

$$\begin{aligned} -\mathcal{F}(w_\varepsilon(x)) + w_\varepsilon^p(x) &\leq f(x), \quad x \in \Omega \\ w_\varepsilon(x) &\leq g(x) \quad x \in \Omega^c. \end{aligned}$$

Moreover w_ε satisfies

$$w_\varepsilon(x) \geq c\,\mathrm{dist}(x, \partial\Omega_\varepsilon)^\beta \quad \textit{for all } x \in A_\delta, \tag{5.17}$$

for certain $c > 0$, independent of $\varepsilon > 0$.

Proof. We proceed as in Lemma 3.1 choosing a point $x_0 \in \Omega^c$ such that $d(x_0, \partial\Omega) > 1$ and let $\varepsilon > 0$ and $R > 0$ such that $d(x_0, \partial\Omega) - \varepsilon > 1$ and $\bar{\Omega} \subset \{1 < |x - x_0| < R\}$. For $\sigma \in (-N, -N+\gamma)$, with $\gamma > 0$ small, we define the function

$$w(r) = \begin{cases} \varepsilon^\sigma & \text{if } 0 < r \leq \varepsilon, \\ r^\sigma & \text{if } \varepsilon \leq r. \end{cases}$$

We see from the proof of Lemma 4.1 in [13] that for ε small there exists $c > 0$ such that $\mathcal{M}^-(w(|x-x_0|)) \geq c|x-x_0|^{\sigma-2\alpha}$ for all $1 < |x-x_0| < R$. Define now $G = w(|x-x_0|) - \varepsilon^\sigma$ and notice that $G < 0$ in $\bar{\Omega}$. Then we define $w_\varepsilon = \lambda_1 \eta_\varepsilon + \lambda_2 G$, for $\lambda_1 > 0$, $\lambda_2 > 0$ and we obtain

$$\begin{aligned} -\mathcal{M}^-(w_\varepsilon) + (w_\varepsilon)^p &\leq \lambda_1 G_\varepsilon d_\varepsilon^{\beta-2\alpha} - \lambda_2 c|x-x_0|^{\sigma-2\alpha} + (\lambda_1 \eta_\varepsilon)^p \\ &\leq -\lambda_1 d_\varepsilon^{\beta-2\alpha}(G_\varepsilon - \lambda_1^{p-1} d_\varepsilon^{p\beta-\beta+2\alpha}) - \lambda_2 c|x-x_0|^{\sigma-2\alpha}. \end{aligned}$$

By Corollary 5.1, G_ε is bounded away from 0 in A_δ and by hypothesis $p\beta - \beta + 2\alpha \geq 0$. Then, choosing λ_1 small enough we make the first term above negative and then making λ_2 large enough we obtain that

$$-\lambda_1 d_\varepsilon^{\beta-2\alpha}(G_\varepsilon - \lambda_1^{p-1} d_\varepsilon^{p\beta-\beta+2\alpha}) - \lambda_2 c|x-x_0|^{\sigma-2\alpha} \leq f \qquad \text{in} \quad A_\delta. \quad (5.18)$$

We recall that f is bounded from below by hypothesis (F2). To complete the proof we just need to make λ_2 larger, if necessary, to get

$$-\lambda_1 d_\varepsilon^{\beta-2\alpha}(G_\varepsilon - \lambda_1^{p-1} d_\varepsilon^{p\beta-\beta+2\alpha}) - \lambda_2 c|x-x_0|^{\sigma-2\alpha} \leq f \qquad \text{in} \quad \Omega \setminus A_\delta. \quad (5.19)$$

From (5.18) and (5.19) we obtain finally that

$$-\mathcal{F}(w_\varepsilon(x)) + w_\varepsilon^p(x) \leq -\mathcal{M}^-(w_\varepsilon) + (w_\varepsilon)^p \leq f(x), \quad x \in \Omega.$$

On the other hand, since $G \leq 0$ in $\mathbb{R}^N$ we have that

$$w_\varepsilon \leq g \qquad \text{in} \quad \Omega^c.$$

Finally we observe that (5.17) is also satisfied for ε and δ small enough. □

Now we are in a position to prove our general Theorem 5.1.

Proof of Theorem 5.1. By Theorem 2.2 we simply have that, for every $\varepsilon > 0$,

$$u(x) \geq w_\varepsilon(x) \geq c\,\mathrm{dist}(x, \partial\Omega_\varepsilon)^\beta \quad \text{for all } x \in A_\delta.$$

Since the inequality holds for every $\varepsilon > 0$, the result follows. □

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