Fundamental solutions for a class of Isaacs integral operators

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Dedicated to Louis Nirenberg with admiration

Abstract

In this article we study the existence of fundamental solutions for a class of Isaacs integral operators and we apply them to prove Liouville type theorems. In proving these theorems we use the comparison principle for non-local operators.

1 Introduction

In this article we study the existence of fundamental solutions for a class of Isaacs integral operators, which includes extremal operators of Caffarelli-Silvestre type [4], fractional Pucci operators and a class of non-convex (concave) operators. Then we apply these simple power type solutions, together with the Comparison Principle, to study (nonlinear) Liouville properties of the corresponding operator. In precise terms, given an operator $\mathcal{I}$ in the class we obtain results of existence/non-existence of solutions for the equation

$$\mathcal{I}u \leq 0, \qquad u \geq 0 \qquad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

and for the equation with a power nonlinearity

$$\mathcal{I}u + u^p \leq 0, \qquad u \geq 0 \qquad \text{in} \quad \mathbb{R}^N, \tag{1.2}$$

for $p > 1$. Our results generalize our earlier work [12] to nonconvex operators and simplify the proofs.

In what follows we describe in precise terms the class of operators to which our results apply. Given $N \geq 2$, $\alpha \in (0,1)$ and $\Lambda \geq 1$, we consider the set

$$\mathcal{A} = \{a \in L^\infty(S^{N-1}) \,/\, a(\omega) \in [1, \Lambda], \text{a.e. in } S^{N-1}\}$$

and for $a \in \mathcal{A}$ we define the linear operator $L_a(u)$ as

$$L_a(u)(x) = \int_{\mathbb{R}^N} \delta(u, x, y) \frac{a(\hat{y})\, dy}{|y|^{N+2\alpha}}, \quad x \in \mathbb{R}^N.$$

Here and in what follows we consider $\hat{y} = y/|y|$, for all $y \in \mathbb{R}^N \setminus \{0\}$ and

$$\delta(u, x, y) = u(x+y) + u(x-y) - 2u(x), \quad x, y \in \mathbb{R}^N.$$

We remark that this integral makes sense if we assume that the function u is such $y \to \delta(u, x, y)|y|^{-N-2\alpha}$ is integrable in $\mathbb{R}^N \setminus B(0, \varepsilon)$ for all $\varepsilon > 0$ and of class $C^{1,1}(x)$ in the sense defined by Caffarelli and Silvestre in [4], that is, there exists $v \in \mathbb{R}^N$ and $M > 0$ so that

$$|u(x+y) - u(x) - v \cdot y| \leq M|y|^2,$$

for y small. In particular, the linear operator L_a is well defined at x if u is bounded, continuous and of class $C^{1,1}(x)$.

Given sets of indices I and J, we consider the family $\mathcal{K} = \{a_{i,j} \in \mathcal{A} \,/\, (i,j) \in I \times J\}$ and we assume that $\mathcal{K}$ is $*$-weakly closed in $L^\infty(S^{N-1})$ and rotationally invariant, that is, if for a rotation matrix R in $\mathbb{R}^N$ and $a \in \mathcal{K}$ we define $a_R(x) = a(Rx)$ then $a_R \in \mathcal{K}$ for all $a \in \mathcal{K}$. We define the operator

$$\mathcal{I}(u)(x) = \inf_I \sup_J L_{a_{i,j}} u(x), \qquad x \in \mathbb{R}. \tag{1.3}$$

One important feature of the operators defined in this way is that, for every $x \in \mathbb{R}^N$ there is $(i^*, j^*) \in I \times J$ such that

$$\mathcal{I}(u)(x) = L_{a_{i^*,j^*}} u(x) = \int_{\mathbb{R}^N} \delta(u, x, y) \frac{a_{i^*,j^*}(\hat{y})\, dy}{|y|^{N+2\alpha}},$$

that is the infimum and supremum are achieved at every $x \in \mathbb{R}^N$.

Our first theorem is devoted to the existence of fundamental solutions for the class of operators just defined. We define the radially symmetric functions v_σ as follows

$$v_\sigma(r) = \begin{cases} r^\sigma & \text{if } -N < \sigma < 0 \\ -\log r & \text{if } \sigma = 0 \\ -r^\sigma & \text{if } 0 < \sigma < 2\alpha, \end{cases} \tag{1.4}$$

our goal is to find the value of the parameter σ so that this function solves the equation $\mathcal{I}(v_\sigma) = 0$. We prove the following

Theorem 1.1 *Under the assumptions given above, for every operator $\mathcal{I}$ as defined in (1.3), there is a unique $\sigma \in (-N, \min\{2\alpha, 1\})$ such that*

$$\mathcal{I}(v_\sigma) = 0.$$

Such a σ is denoted by $\sigma^+_{\mathcal{I}}$. If we define $w_\sigma := -v_\sigma$, there is a unique $\sigma \in (-N, min\{2\alpha, 1\})$ such that

$$\mathcal{I}(w_\sigma) = 0.$$

Such a σ is denoted by $\sigma^-_{\mathcal{I}}$.

The function $v_{\sigma^+_{\mathcal{I}}}$ and $w_{\sigma^-_{\mathcal{I}}}$ are fundamental solutions associated to the non-linear operators $\mathcal{I}$. The function $v_{\sigma^+_{\mathcal{I}}}$ is convex in r, corresponding to a upwards-pointing fundamental solutions, as defined by Armstrong, Sirakov and Smart in [1], for fully non-linear second order elliptic operators. Similarly, the function $w_{\sigma^-_{\mathcal{I}}}$, is a concave function of r and it is a downwards-pointing fundamental solutions.

Fundamental solutions for the extremal Pucci operator ($\alpha = 1$) were first defined by Labutin [15], [16] and were used for the study of removability of singularities for these operators. They were used later by Cutri and Leoni [9] for the study of Liouville type theorems and later for operators involving first order terms by Capuzzo-Dolcetta and Cutri in [6]. The results in [9] and [15] were generalized by the authors in [11] for a class of extremal operators with radial symmetry. Recently in [2], Armstrong, Sirakov and Smart obtained fundamental solutions for general, not necessarily radially invariant fully non-linear differential operators, and they were used recently by Armstrong and Sirakov [1] to prove Liouville type theorems for these differential operators.

In a recent paper, the authors consider in [12] similar results for the Caffarelli-Silvestre operators (see (2.3) and (2.4)). These extremal operators are convex and concave, depending if they are maximal or minimal. Moreover, they can be written in an explicit way, allowing the study of fundamental solutions in a very direct way. In this article we extend the results to a more general class of operators, including non-convex ones, using different and simpler arguments.

In what follows we will only consider convex fundamental solutions so that we will drop + from the notation. In order to state our main theorems on entire solutions, it is convenient to define the dimension like numbers

$$N_{\mathcal{I}} = -\sigma_{\mathcal{I}} + 2\alpha. \tag{1.5}$$

Theorem 1.2 (The Liouville Property) *Assume that* $N_{\mathcal{I}} \leq 2\alpha$ *and* u *is a viscosity solution of*

$$\mathcal{I}(u) \leq 0, \quad and \quad u \geq 0, \quad in \quad \mathbb{R}^N,$$

then u *is a constant.*

In these theorems and in all the paper, by solution to an integral inequality or equation we mean solution in the viscosity sense as defined in [4], see also [12]. Our next result is a Liouville type theorem for the operator with a power non-linearity. We have

Theorem 1.3 (Liouville type Theorem) *Assume* $N_{\mathcal{I}} > 2\alpha$ *and that* u *is a viscosity solution of*

$$\mathcal{I}(u) + u^p \leq 0. \tag{1.6}$$

If $p \leq \frac{N_{\mathcal{I}}}{N_{\mathcal{I}}-2\alpha}$, *then* $u \equiv 0$. *Reciprocally, if* $p > \frac{N_{\mathcal{I}}}{N_{\mathcal{I}}-2\alpha}$ *then equation (1.6) has a nontrivial viscosity solution.*

At this point we observe that given the class of linear operators defined by the functions $\{a_{i,j} \,/\, (i,j) \in I \times J\}$, we can also define the nonlinear operator

$$\mathcal{J}(u)(x) = \sup_I \inf_J L_{a_{i,j}} u(x). \tag{1.7}$$

These sup-inf operators satisfy the same results as the inf-sup operators. In particular, Theorems 1.1, 1.2 and 1.3 hold.

It is important to say here that the non-existence Liouville type theorems are closely related with existence of positive solutions of related equations in bounded domains. In the case of second order differential operators, the well known blow-up technique introduced by Gidas and Spruck [14] allows to find a priori bounds for the positive solutions of the problem in a bounded domain, as a consequence of the non-existence theorem. Then classical degree theory is applicable to complete the existence arguments. Even though we do not investigate this line of research in this article, we believe that results of this sort are valid for non-local operators in the class considered here.

Our results are related to the central problem of existence and uniqueness of a positive solution for the semi-linear equation

$$\Delta^\alpha u + u^p = 0. \tag{1.8}$$

In the Sobolev critical case $p = (N + 2\alpha)/(N - 2\alpha)$ and in the sub-critical case, this was studied by Li [17] and Chen, Li and Ou [8]. In the case $\alpha = 1$ and $\Lambda = 1$, that is for the Laplacian, Theorem 1.3 is an extension of the classical result of Gidas [13]. Concerning results of classification of solution and Liouville type result for equation (1.8) and $\alpha = 1$ we mention the fundamental papers by Gidas and Spruck [14], Caffarelli, Gidas and Spruck [5] and Chen and Li [7].

The proofs of Theorems 1.2 and 1.3 are based on the fundamental solutions for the operator $\mathcal{I}$, found in Theorem 1.1, and the Comparison Principle. Here we follow closely the arguments in [12]. The difficulty in the use of the Comparison Principle is due to the fact that 'boundary values' have to be considered in all the complement of the domain, not only on the topological boundary. Consequently, the usual arguments based on the Comparison Principle through the Hadamard Three Spheres, need to be adapted weaking the intermediate results.

2 Comments about our class of operators

The class of operators defined in (1.3) includes various subclasses that we review in this section. Given any subset $\mathcal{B} \subset \mathcal{A}$, which is $*$-weakly closed in $L^\infty(S^{N-1})$ and rotationally invariant, we define maximal and minimal operators as follows

$$\mathcal{M}^+_{\mathcal{B}}(u)(x) = \sup_{a \in \mathcal{B}} L_a(u)(x) \tag{2.1}$$

and

$$\mathcal{M}^-_{\mathcal{B}}(u)(x) = \inf_{a \in \mathcal{B}} L_a(u)(x). \tag{2.2}$$

The Caffarelli-Silvestre operators defined in [4], are obtained considering $\mathcal{B} = \mathcal{A}$, and can be written as

$$\mathcal{M}^+_{\mathcal{A}}(u)(x) = \int_{\mathbb{R}^N} S_+(\delta(u, x, y)) \frac{dy}{|y|^{N+2\alpha}} \tag{2.3}$$

and

$$\mathcal{M}^-_{\mathcal{A}}(u)(x) = \int_{\mathbb{R}^N} S_-(\delta(u, x, y)) \frac{dy}{|y|^{N+2\alpha}}, \tag{2.4}$$

where

$$S_+(t) = \Lambda t_+ + \lambda t_- \quad \text{and} \quad S_-(t) = \lambda t_+ + \Lambda t_-, \qquad t \in \mathbb{R}.$$

These operators correspond to the overall maximal and minimal operators of the class $\mathcal{A}$ and they serve as an upper and lower bound for general operators, in the sense given below. If $\mathcal{I}$ is defined as in (1.3) then it satisfies the inequality

$$\mathcal{M}^-_{\mathcal{A}}(u-v) \leq \mathcal{I}(u) - \mathcal{I}(v) \leq \mathcal{M}^+_{\mathcal{A}}(u-v) \tag{2.5}$$

for all admissible functions u and v. It is important to notice that this inequality holds for any domain of integration, that is, for any $\Omega \subset \mathbb{R}^N$, we have

$$\begin{aligned} \int_\Omega S_-(\delta(u-v,x,y))\frac{dy}{|y|^{N+2\alpha}} &\leq \int_\Omega \{\delta(u,x,y)a^*_x(\hat{y}) - \delta(v,x,y)b^*_x(\hat{y})\}\frac{dy}{|y|^{N+2\alpha}} \\ &\leq \int_\Omega S_+\delta(u-v,x,y))\frac{dy}{|y|^{N+2\alpha}}, \end{aligned} \tag{2.6}$$

where $a^*_x, b^*_x \in \mathcal{A}$ are such that

$$\mathcal{I}(u) = \int_{\mathbb{R}^N} \delta(u,x,y)\frac{a^*_x(\hat{y})\,dy}{|y|^{N+2\alpha}} \quad \text{and} \quad \mathcal{I}(v) = \int_{\mathbb{R}^N} \delta(u,x,y)\frac{b^*_x(\hat{y})\,dy}{|y|^{N+2\alpha}}.$$

Another interesting class of operators is obtained by considering the set

$$\mathcal{B}_1 = \{a \in \mathcal{A} \,/\, a(\omega) = \omega^t A\omega, \omega \in S^{N-1}, A \in S_\Lambda\},$$

where S_Λ denotes the set of all symmetric matrices, such that $I \leq A \leq \Lambda I$, with I the identity matrix.

The class of fractional Pucci operators is obtained by considering the maximal and minimal operators associated to the set $\mathcal{P} \subset \mathcal{A}$ given by

$$\mathcal{P} = \{a \in \mathcal{A} \,/\, a(\omega) = \frac{1}{|\det A^{\frac{1}{2}}||A^{-\frac{1}{2}}\omega|^{N+2\alpha}}, \, \omega \in S^{N-1}, \, A \in S_\Lambda\}. \tag{2.7}$$

More generally, if C is a closed subset of S_Λ, which is invariant under similarity transformations, that is, it satisfies

$$A \in C \text{ then } P^tAP \in C, \qquad \text{for all orthogonal metrix } P,$$

we consider the set

$$\mathcal{P}_C = \{a \in \mathcal{P} \,/\, A \in C\}.$$

The extremal operators defined with the set $\mathcal{P}_C$ is related to the second order operators studied by the authors in [10].

3 Fundamental solutions for Isaacs type integral operators

In this section we study the fundamental solutions for the operators of Isaacs type defined in (1.3), the main goal is to prove Theorem 1.1.

After some basic properties we concentrate in the analysis of sign of the coefficient we get when plugging in these operators a power function. Let us start observing the simple fact that if $\mathcal{I}$ is an operator of the form (1.3) then $\mathcal{I}$ is radially invariant, that is and if $v(x)$ is is radially symmetric then $\mathcal{I}v(x)$ is also radially symmetric.

We start describing the range of σ for which $\mathcal{I}(v_\sigma)$ makes sense.

Lemma 3.1 *For all $-N < \sigma < 2\alpha$, $\mathcal{I}(v_\sigma)(x)$ is well defined for $x \neq 0$. Moreover, for every $x \in \mathbb{R} \setminus \{0\}$,*

$$\lim_{\sigma\to -N} \mathcal{I}(v_\sigma)(x) = \infty \quad \textit{and} \quad \lim_{\sigma\to 2\alpha} \mathcal{I}(v_\sigma)(x) = -\infty. \tag{3.1}$$

Proof. Let $x \neq 0$ and $\sigma \in (-N, 2\alpha)$. By the properties of the class $\mathcal{K}$ we know that there is $(i^*, j^*) \in I \times J$ such that $\mathcal{I}(v_\sigma)(x) = L_{a_{i^*,j^*}}(v_\sigma)(x)$. Then, analyzing the integral definig the last term, we see that it has three singularities. Estimating the behavior of the integral at each of them we obtain the results. See [12]. □

Remark 3.1 *It is clear that this lemma holds also for maximal and minimal operators, even for linear operators.*

Next we obtain an explicit form for $\mathcal{I}(v_\sigma)$.

Lemma 3.2 *For any $-N < \sigma < 2\alpha$, we have*

$$\mathcal{I}v_\sigma(x) = c(\sigma)|x|^{\sigma-2\alpha}, \quad x \in \mathbb{R}^N \setminus \{0\},$$

where

$$c(\sigma) = \int_{\mathbb{R}^N} \delta_\sigma(y) \frac{a_\sigma(\hat{y})\, dy}{|y|^{N+2\alpha}}, \tag{3.2}$$

$a_\sigma := a_{i,j}$ for certain $(i,j) \in I \times J$ and

$$\delta_\sigma(y) = \begin{cases} |e_1 + y|^\sigma + |e_1 - y|^\sigma - 2 & \textit{if } \sigma \in (-N, 0) \\ -\log|e_1 + y| - \log|e_1 - y| & \textit{if } \sigma = 0, \\ -|e_1 + y|^\sigma - |e_1 - y|^\sigma + 2 & \textit{if } \sigma \in (0, 2\alpha). \end{cases} \tag{3.3}$$

Here, and in all what follows, $e_1 = (1, 0, ..., 0) \in \mathbb{R}^N$.

Proof. Let us consider first $\sigma \in (-N,0)\cup(0,2\alpha)$. We have that $\delta(v_\sigma,x,y) = |x|^\sigma \delta_\sigma(y/|x|)$ thus, for a certain $a_\sigma = a_{i,j}$, depending only on σ,

$$\mathcal{I}v_\sigma(x) = \int_{\mathbb{R}^N} |x|^\sigma \delta_\sigma(y/|x|) a_\sigma(\hat{y}) \frac{dy}{|y|^{N+2\alpha}},$$

and then, by a change of variables, (3.2) follows. When $\sigma = 0$ we proceed similarly, observing that using definition (3.3),

$$\delta(v_\sigma,x,y) = \delta_0(y/|x|). \quad \square$$

According to this lemma we need to analyze when the function c vanishes.

Lemma 3.3 *The function c has at most one zero in $(-N,2\alpha)\setminus\{0\}$.*

Proof. Given $(i,j)\in I\times J$ the function

$$c_{i,j}(\sigma) = \int_{\mathbb{R}^N} \delta_\sigma(y) \frac{a_{i,j}(\hat{y})\,dy}{|y|^{N+2\alpha}}$$

is convex in $(-N,0)$ and concave in $(0,2\alpha)$. Moreover, recalling (3.3),

$$\lim_{\sigma\to 0^-} c_{i,j}(\sigma) = 0 = \lim_{\sigma\to 0^+} c_{i,j}(\sigma)$$

and

$$c'_{i,j}(0^-) = \lim_{\sigma\to 0^-} \frac{c_{i,j}(\sigma)}{\sigma} = -\lim_{\sigma\to 0^+} \frac{c_{i,j}(\sigma)}{\sigma} = -c'_{i,j}(0^+).$$

We claim that the lemma holds for $c_{i,j}$. In fact, if $c_{i,j}$ has a zero in $(0,2\alpha)$, by concavity of $c_{i,j}$ and (3.1) (see Remark 3.1), we find that $c'_{i,j}(0^+)\geq 0$. From here $c'_{i,j}(0^-)\leq 0$ and then, by convexity, $c_{i,j}$ does not vanishes in $(-N,0)$. If $c_{i,j}$ has a zero in $(-N,0)$, by a similar argument it has no other zero, completing the proof of the claim.

Our second step is to prove that for every $i\in I$, the function

$$c_i(\sigma) = \sup_{j\in J} \int_{\mathbb{R}^N} \delta_\sigma(y) \frac{a_{i,j}(\hat{y})\,dy}{|y|^{N+2\alpha}},$$

satisfies the lemma. For this purpose we first assume that c_i has a zero $\sigma_i \in (0,2\alpha)$ and that σ_i is the rightest one, that is, for every other zero $\sigma_0 \in (0,2\alpha)$ we have $\sigma_0 \leq \sigma_i$ and let $j^* = j^*(\sigma_i)\in J$ be such that

$$c_i(\sigma_i) = c_{i,j^*}(\sigma_i) = \int_{\mathbb{R}^N} \delta_{\sigma_i}(y) \frac{a_{i,j^*}(\hat{y})\,dy}{|y|^{N+2\alpha}}.$$

Since the function c_{i,j^*} satisfies the lemma, we have that $c_{i,j^*}(\sigma) > 0$ for all $\sigma \in (0, \sigma_i)$ and then, by definition, $c_i(\sigma) \geq c_{i,j^*}(\sigma) > 0$ for all $\sigma \in (0, \sigma_i)$, concluding that σ_i is the only zero of c_i in $(0, \sigma_i)$. But as c_{i,j^*} satisfies the lemma, $c_{i,j^*}(\sigma) > 0$, for all $\sigma \in (-N, 0)$ and then, by definition, $c_i(\sigma) > 0$, for all $\sigma \in (-N, 0)$, completing the proof in this case. Assume next that c_i has a zero σ_i in the interval $(-N, 0)$ and that it is the rightest one, then we choose j^* as above and we conclude that c_i has only one zero in $(-N, 0)$. To complete this second step, we just observe that c_i does not have a zero in $(0, 2\alpha)$, since then, by the argument given above, it should be positive in $(-N, 0)$.

Our final step is the general case. Assume that c has a zero $\sigma_0 \in (-N, 0)$ and that it is the leftmost one. Let i^* be such that $0 = c(\sigma_0) = c_{i^*}(\sigma_0)$. Since c_{i^*} satisfies the lemma, by definition of c, we conclude that c does not have other zero in $(-N, 0) \cup (0, 2\alpha)$. Assume now that c has a zero in $\sigma_0 \in (0, 2\alpha)$, we assume it is the leftmost one and we reapeat the argument. $\Box$

Proof of Theorem 1.1. In view of Lemma 3.3, we just need to assume that c does not have a zero in $(-N, 2\alpha) \setminus \{0\}$ and prove that $\mathcal{I}(v_0) = 0$.

We see that for every $\sigma \in (-N, 0)$ we have $c(\sigma) > 0$ and then

$$\mathcal{I}(\frac{v_\sigma - 1}{-\sigma}) = r^{\sigma - 2\alpha} \frac{c(\sigma)}{-\sigma} \geq 0$$

and similarly,

$$\mathcal{I}(\frac{v_\sigma + 1}{\sigma}) = r^{\sigma - 2\alpha} \frac{c(\sigma)}{\sigma} \leq 0.$$

Thus $u_\sigma^- = (v_\sigma - 1)/-\sigma$ is a subsolution for $\sigma \in (-N, 0)$ and $u_\sigma^+ = (v_\sigma + 1)/\sigma$ is a supersolution for $\sigma \in (0, 2\alpha)$. It is clear that

$$\lim_{\sigma \to 0^-} u_\sigma^-(r) = \lim_{\sigma \to 0^+} u_\sigma^+(r) = -\log r,$$

uniformly on every compact subset of $\mathbb{R}^N \setminus \{0\}$. Then, by stability properties (see remark below), we find that $v_0(r) = -\log r$, is simultaneously a supersolution and a sub-solution of

$$\mathcal{I}(u)(r) = 0.$$

We conclude that v_0 is a solution of the equation.

To complete the proof we just need to prove that $\sigma_{\mathcal{I}} < 1$, whenever $2\alpha \geq 1$. In this case, we notice that $c(1) \leq \int_{\mathbb{R}^N} S_+(\delta_1(y)) \frac{dy}{|y|^{N+2\alpha}}$, since $\mathcal{M}_{\mathcal{A}}^+$ is the maximal operator in the whole class. But then we just need to observe that $\delta_1(y) \leq 0$, by definition, so that $c(1) < 0$. $\Box$

Remark 3.2 *Caffarelli and Silvestre proved in [4] a stability result for integral operators, see Lemma 4.5. Eventhough this lemma does not apply directly here, since the functions u_σ^+ and u_σ^- are not bounded, a carefull look at its proof allows to obtain what we need, slightly changing the arguments.*

Remark 3.3 *It is interesting to notice that rotationally invariant second order elliptic operators have fundamental solutions that are power functions, that is, they are like $\Phi(x) = r^\sigma$, $r = |x|$. This was proved by Armstrong, Sirakov and Smart [2]. In contrast, for integral operators this is no loger true, as the following example shows. We consider the kernel*

$$K(y) = \frac{1+\chi(y)}{|y|^{N+2\alpha}},$$

where χ is the characteristic function of $B_1(0)$, the ball of radius 1 and centered at the origin. Then, the operator

$$J(u)(x) = \int_{\mathbb{R}^N} \delta(u,x,y)K(y)dy$$

is rotationally invariant. However J cannot have a power as a fundamental solution, since

$$\begin{aligned} J(\Phi)(x) &= c(\sigma)r^{\sigma-2\alpha} + \int_{B_1(0)} \frac{|x+y|^\sigma + |x-y|^\sigma - 2|x|^\sigma}{|y|^{N+2\alpha}}dy \qquad (3.4)\\ &= r^{\sigma-2\alpha}\left\{c(\sigma) - \int_{B_{1/r}(0)} \frac{|e_1+y|^\sigma + |e_1-y|^\sigma - 2}{|y|^{N+2\alpha}}dy\right\}, \qquad (3.5) \end{aligned}$$

is equal to zero for certain $\sigma \neq 0$, only if the integral above is constant, which is not. Similar conclusion is reached when $\phi(x) = -\log r$.

In this article we are further assuming that the kernels are homogenous.

We conclude this section with a partial result on uniqueness of fundamental solutions. We start with a form of strong maximum principle

Lemma 3.4 *If u is a solution of*

$$\mathcal{I}(u) \leq 0, \quad u \geq 0, \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},$$

regular in $\mathbb{R}^N \setminus \{0\}$. Then, either $u \equiv 0$ or $u(x) > 0$ for all $x \in \mathbb{R}^N \setminus \{0\}$.

Proof. If u has a global minimum at $x_0 \neq 0$ and u is not constant, then $\mathcal{I}(u)(x_0) > 0$, getting a contradiction. □

Theorem 3.1 *Assume that $\mathcal{I}$ is an operator as in (1.3). Then the fundamental solution $v_{\sigma_\mathcal{I}}$ is unique in the class of $\sigma_\mathcal{I}$-homogeneous regular functions.*

Proof. First assume that $\sigma_\mathcal{I} < 0$ and suppose that that $v(x) = |x|^{\sigma_\mathcal{I}}\phi(\hat{x})$ is a fundamental solution and define $w_s = |x|^{\sigma_\mathcal{I}}(s - \phi(\hat{x}))$. Then for s large w_s is positive and by (2.5)

$$\mathcal{M}^-_\mathcal{A}(w_s) \leq 0.$$

Let

$$s_0 := inf\{s > 0 : w_s > 0 \text{ in } I\!R^N \setminus \{0\}\},$$

then there exists $x_0 \in \mathbb{R}^N \setminus \{0\}$ such that $w_{s_0}(x_0) = 0$ and $w_{s_0} \geq 0$, getting a contradiction, unless $w_{s_0} \equiv 0$, which implies $\hat{v} \equiv 1$ that means uniqueness.

If $\sigma_\mathcal{I} < 0$, the proof is similar. Finally, in the case $\sigma_\mathcal{I} = 0$, we consider a fundamental solution of the form $v(x) = -\phi(\hat{x}) - log|x|$ and then we use as above $w(x) = \hat{v}(\hat{x})$ which has his maximum achieved on rays of $\mathbb{R}^N$ so we have a contradiction, except if it is constant. □

4 Some remarks about limits

In this section we briefly consider the dependence of $\sigma_\mathcal{I}$ on α and, in particular, we discuss the limit as α approaches one, and the integral operator becomes a second order differential operator.

Given a fixed family of kernels and $\alpha \in (0, 1)$, we define by $\mathcal{I}_\alpha$ the integral operator given in (1.3). We have

$$\begin{aligned} \lim_{\alpha\to1}(1-\alpha)\mathcal{I}_\alpha(u)(x) &= \lim_{\alpha\to1}\inf_I\sup_J(1-\alpha)\int_{\mathbb{R}^N}\delta(u,x,y)\frac{a_{i,j}(\hat{y})\,dy}{|y|^{N+2\alpha}} \\ &= C\inf_I\sup_J\int_{S^{N-1}}\omega^tD^2u(x)\omega\ a_{i,j}(\omega)d\omega, \qquad (4.1)\end{aligned}$$

where C depends on N and the operator $\mathcal{I}$. It is well known that any fully nonlinear second order elliptic operator depending only on the second derivative can be recovered with operators of the form (4.1). In fact, it is sufficient to consider kernels of the form given by (2.7), as proved in [4]. Notice that given $a \in \mathcal{P}$ then

$$\int_{S^{N-1}}\omega^tD^2u(x)\omega\ a(\omega)d\omega = C_N\text{tr}(AD^2u(x)) = C_N\sum_{i,j}a_{i,j}u_{i,j},$$

where A is the matrix defining a.

Given the operator $\mathcal{I}_\alpha$ we may use Theorem 1.1 to get fundamental solutions, that is, $\sigma_\mathcal{I} := \sigma_\mathcal{I}(\alpha)$. We have

$$\mathcal{I}_\alpha(v_{\sigma_{\mathcal{I}(\alpha)}})(x) = 0, \qquad \text{for all} \quad x \in \mathbb{R}^N \setminus \{0\}, \alpha \in (0,1).$$

Next we prove the following continuity result.

Proposition 4.1 *$\sigma_\mathcal{I}(\alpha)$ is a continuous function of $\alpha \in (0,1)$.*

Proof. Let α_n be a sequence such that $\alpha_n \to \alpha_0 \in (0,1)$, then up to a subsequence, $\sigma(\alpha_n) \to \sigma_0$. We assume that $\sigma_0 \neq 0$, then we obviously have that $v_n := v_{\sigma_{\mathcal{I}(\alpha_n)}} \to v_{\sigma_0}$ uniformly on every compact set. This information, together with the stability properties of the operators, implies

$$0 = \lim_{n\to\infty} \mathcal{I}_{\alpha_n}(v_n)(x) = \mathcal{I}_{\alpha_0}(v_{\sigma_0})(x), \qquad x \in \mathbb{R}^N \setminus \{0\}.$$

Thus, v_{σ_0} is a fundamental solution of $\mathcal{I}_{\alpha_0}$. By the uniqueness property of $\sigma_{\mathcal{I}(\alpha_0)}$, we conclude that $\sigma_{\mathcal{I}(\alpha_0)} = \sigma_0$, completing the proof of the continuity if $\sigma_0 \neq 0$. In case $\sigma_0 = 0$ we sligthly modify the arguments. □

Remark 4.1 *Similarly, if $\sigma_{\mathcal{I}(\alpha_n)} \geq \sigma_0 > -N$, for all $n \in \mathbb{N}$ and $\alpha_n \to 1$ then we can argue that $\sigma_{\mathcal{I}(\alpha_n)}$ converges to the unique exponent of the differential operator $\mathcal{I}$, that was proved to exists in Theorem 3 in [2].*

We observe that, in case the dimension like number $N_\mathcal{I}$ is larger than N, then $N_\mathcal{I}$ cannot be achieved as limit of the numbers $N_{\mathcal{I}_n}$, nor the corresponding fundamental solutions. However, it would be interesting to understand the limit case $N_\mathcal{I} = N$.

5 Proof of Liouville type theorems

In this section we prove Theorems 1.2 and 1.3 basing our arguments on the fundamental solutions found in Section §3 and the Comparison Principle for integral operators. Since most of the arguments were already given in the simpler, but quite similar case of Caffarelli-Silvestre operators in [12], we will be sketchy at certain points.

For integral operators we have a Comparison Principle that in some sense works as in the case of a second order elliptic differential operator, however it has an important difference. The boundary condition for a bounded set Ω does not make really sense for integral operators when considered only on $\partial\Omega$ since sets with vanishing measure are negligible, instead we have to give boundary conditions in the whole complement of the domain Ω. We recall the comparison principle proved in [4] (Theorem 5.2), that we use later in the section.

Theorem 5.1 *Assume u and v are super-solution and sub-solutions of the equation*

$$\mathcal{I}(u) = g,$$

in $\bar{\Omega}$, where Ω is a bounded open subset of $\mathbb{R}^N$ and g is a continuous function in $\bar{\Omega}$. Moreover, assume that $u \leq v$ in $\mathbb{R}^N \setminus \Omega$. Then $u \leq v$ in Ω.

Next we apply the Comparison Principle together with the fundamental solutions found in Section §3 to prove the Liouville Property, that is Theorem 1.2. The idea is to use the fundamental solution $v_{\sigma_{\mathcal{I}}}$ that satisfies $\mathcal{I}(v_{\sigma_{\mathcal{I}}}) = 0$ in $\mathbb{R} \setminus \{0\}$, properly compared with the supersolution u, in order to prove that u possesses a global minimun, which is impossible according to Lemma 3.4.

A difficulty arises in the use of the Comparison Principle when $\sigma_{\mathcal{I}} \leq 0$, since the function $v_{\sigma_{\mathcal{I}}}$ is unbounded, actually $v_{\sigma_{\mathcal{I}}}(r) \to \infty$ as $r \to 0$. Consequently, it is impossible to have it below u in any set including a neighborhood of the origin. Even if $\sigma_{\mathcal{I}} > 0$, case where it is bounded at the origin, the fundamental solution $v_{\sigma_{\mathcal{I}}}$ is not a solution at the origin, not even a sub-solution. In order to overcome these difficulties we consider this fundamental solution slightly perturbed taking σ near $\sigma_{\mathcal{I}}$ and we truncate it near the origin.

It will be convenient to consider the following definitions. Given $r_0 > 0$ and a radially symmetric function v defined in $\mathbb{R}\setminus\{0\}$ we define the truncation of v near the origin as

$$T(v, r_0, r) = \begin{cases} v(r_0) & \text{if } 0 \leq r \leq r_0, \\ v(r) & \text{if } r_0 \leq r. \end{cases} \tag{5.1}$$

Upon this function and for $R > r_0$ we define its normalized version

$$NT(v, r_0, R, r) = \begin{cases} \frac{T(v,r_0,r)-v(R)}{v(r_0)-v(R)} & \text{if } 0 \leq r \leq R, \\ 0 & \text{if } R \leq r. \end{cases} \tag{5.2}$$

We observe that $NT(v)$ takes the value 1 in the ball B_{r_0} and the value 0 on the exterior of the ball B_R. We also define

$$m(r) = \min_{|x| \leq r} u(x),$$

where u is a non-negative function.

Now we prove the Liouville property.

Proof of Theorem 1.2. By Lemma 3.4, we may assume that $u(x) > 0$ for all x. Let us consider first the case $N_I < 2\alpha$, that is, $\sigma_{\mathcal{I}} \in (0, 2\alpha)$, and consider the function $w(r) = T(-r^\sigma, \varepsilon, r)$, where $\varepsilon > 0$ and $\sigma \in (0, \sigma_I)$. Then we have that

$$\mathcal{I}w(x) \geq 0 \qquad \text{for all} \quad r_1 < |x|, \tag{5.3}$$

if $r_1 \geq 1$ and ε are chosen large and small enough, respectively. In fact, by the choice of σ and Lemma 3.2 we have

$$\mathcal{I}w(r) \geq r^{\sigma-2\alpha} \left(c(\sigma) - I(r, \varepsilon) \right),$$

where

$$I(r, \varepsilon) = r^{-\sigma+2\alpha} \int_{\hat{B}_\varepsilon(x)} S_+(\delta(r^\sigma - w, x, y)) \frac{dy}{|y|^{N+2\alpha}}.$$

Here $c(\sigma) > 0$, $r = |x|$ and $\hat{B}_\varepsilon(x) = B_\varepsilon(x) \cup B_\varepsilon(-x)$. Finally, using the definition of w and estimating the corresponding integrals we find, as in Theorem 1.2 of [12], that for $\varepsilon > 0$ small enough $I(r, \varepsilon) < c(\sigma)$, for all $r \geq r_1$, proving (5.3). We define now the function $\phi(x) = NT(w, \varepsilon, R, r)$, with $R > r_1$ and we see that

$$\mathcal{I}\phi \geq 0, \quad \text{for all} \quad r_1 < |x| < R,$$

and $u(x) \geq \phi(x)$ for all $r_1 \leq |x|$ or $|x| \geq r_2$. Then we use comparison Theorem 5.1 to obtain that $u(x) \geq \phi(x)$ for $R \geq |x| \geq r_1$. If we take limit when $R \to \infty$, noticing that $w(R) \to -\infty$, we obtain that

$$u(x) \geq m(r_1) \quad \text{for all} \quad r_1 < |x|. \tag{5.4}$$

But then u has a global minimum point in $B(0, r_1)$. Unless u is a constant function, we get a contradicting since $\mathcal{I}u \leq 0$ in all $\mathbb{R}^N$.

To conclude we analyze the case $\sigma_{\mathcal{I}} = -N_{\mathcal{I}} + 2\alpha = 0$. Here we consider the minimal operator $\mathcal{M}^-_{\mathcal{A}}$ and the corresponding fundamental solution r^{σ^-}, with $-N < \sigma^- < 0$. Then $\mathcal{M}^-_{\mathcal{A}}(r^\sigma) = c^-(\sigma) r^{\sigma-2\alpha}$, with $c^-(\sigma) > 0$.

Notice that by (2.5) we have

$$I(r^\sigma - \log r) \geq \mathcal{I}(-\log r) + \mathcal{M}^-_{\mathcal{A}}(r^\sigma) = c^-(\sigma) r^{\sigma-2\alpha}.$$

We define now the truncated function $w(r) = T(r^\sigma - \log r, \varepsilon, r)$ and we observe that $w(r) \to -\infty$ when $r \to \infty$. From here, we can proceed as above and find ε and r_1 appropriates so that

$$\mathcal{I}w \geq 0, \quad \text{for all} \quad |x| \geq r_1$$

and from here we conclude as before. □

In order to prove the Liouville type theorems for nonlinear equations, Hadamard Three Spheres Theorem is usually applied. However, such a theorem does not seem to be true, since the boundary values need to be taken in the complement of the given annulus. Fortunately we can prove a weaker version of Hadamard Three Sphere Theorem which is enough for our purposes.

We prove three preliminary lemmas, whose proof are based on the fundamental solution $v_{\sigma_{\mathcal{I}}}$ that satisfies $\mathcal{I}(v_{\sigma_{\mathcal{I}}}) = 0$ in $\mathbb{R} \setminus \{0\}$, properly compared with the supersolution u.

Lemma 5.1 *Assume that $N_{\mathcal{I}} > 2\alpha$. Then, for all $\sigma \in (-N, \sigma_{\mathcal{I}})$ and $r_1 \geq 1$, there exists $c > 0$ such that for every non-negative viscosity solution of $u \neq 0$ of*

$$\mathcal{I}u(x) \leq 0 \quad \text{in} \quad \mathbb{R}^N \tag{5.5}$$

we have

$$m(r) \geq cm(r_1)r^{\sigma}, \quad \text{for all} \quad r \geq r_1. \tag{5.6}$$

Proof. We recall that $N_{\mathcal{I}} = 2\alpha - \sigma_{\mathcal{I}}$. Let $R > r_1$, $\varepsilon > 0$ and $\sigma \in (-N, \sigma_{\mathcal{I}})$. Then we define the comparison function $\phi(x) = m(r_1)\, NT(w, \varepsilon, R, r)$, where $w(r) = T(r^{\sigma}, \varepsilon, r)$ is the truncated perturbed fundamental solution. If we proceed as in the proof of Theorem 1.2 we can prove that

$$\mathcal{I}\phi \geq 0, \quad \text{for all} \quad r_1 < |x| < R$$

and $u(x) \geq \phi(|x|)$ in $B_{r_1} \cup B_R^c$. Here we use the fact that $\sigma < \sigma_{\mathcal{I}}$, allowing to control the truncated part. From here we may use Theorem 5.1 to obtain $u(x) \geq \phi(|x|)$, for all $r_1 \leq |x| \leq R$, that is

$$u(x) \geq m(r_1)\frac{r^{\sigma} - R^{\sigma}}{\varepsilon^{\sigma} - R^{\sigma}}.$$

Then, taking the limit as $R \to \infty$ we obtain (5.6) with $c = \varepsilon^{-\sigma}$. □

Lemma 5.2 *Assume that $N_{\mathcal{I}} > 2\alpha$. Then, there is $r_1 > 0$ and a constant c such that for every non-negative viscosity solution of (5.5) we have*

$$m(R/2) \leq cm(R), \quad \text{for all} \quad R \geq r_1. \tag{5.7}$$

Proof. Here we use the fundamental solution truncated near the origin and we compensate it with a truncation near infinity. The balance is obtained since we apply the Comparison Principle away from the annulus $R < |x| < R/2$, which has comparable radia. Given $\varepsilon > 0$ and $R > 0$, we define

$$R_0 = R\left[\frac{\varepsilon}{1+\varepsilon 2^{\sigma_{\mathcal{I}}}}\right]^{-1/\sigma_{\mathcal{I}}}$$

and assume that ε is such that $R_0 < R/2$. We consider the functions $w(r) = T(r^{\sigma_{\mathcal{I}}}, R_0, r)$ and

$$\phi(r) = m(R/2)TN(r^{\sigma_{\mathcal{I}}}, R_0, 2R, r).$$

We observe that $u(x) \geq \phi(|x|)$ for all $|x| \leq R/2$ or $|x| \geq 2R$. Next we claim that

$$\mathcal{I}\phi(|x|) \geq 0 \quad \text{for all} \quad R/2 < |x| < 2R. \tag{5.8}$$

Assuming the claim for the moment, we may apply the comparison principle, Theorem 5.1, to obtain that $u(x) \geq \phi(|x|)$ for all $R/2 < |x| < 2R$, from where we obtain, by taking the minimum of u in $0 < |x| \leq R$, that

$$m(R) \geq \varepsilon m(R/2)(1 - 2^{\sigma_{\mathcal{I}}}).$$

The result follows taking $c = \varepsilon(1 - 2^{\sigma_{\mathcal{I}}})$. Next we show that the claim (5.8) holds if we choose $\varepsilon > 0$ small enough. For this purpose we define the doubly truncated fundamental solution $w_R(r) = r^{\sigma_{\mathcal{I}}}$ if $R_0 < |x| < 2R$ with $w_R(r) = R_0^{\sigma_{\mathcal{I}}}$ if $0 \leq |x| \leq R_0$ and $w_R(r) = (2R)^{\sigma_{\mathcal{I}}}$ if $|x| \geq 2R$. We see that in order to prove (5.8) we just need to check that $\mathcal{I}w_R(|x|) \geq 0$ if $R/2 < |x| < 2R$. By definition of σ_I we have that

$$0 = \mathcal{I}(r^{\sigma_{\mathcal{I}}}) \geq \mathcal{I}w(r) + I(\varepsilon, r), \tag{5.9}$$

where $r = |x|$ and

$$I(\varepsilon, r) = \int_{\hat{B}_{R_0}(x)} S_+(\delta(r^{\sigma_{\mathcal{I}}} - w(r), x, y))\frac{dy}{|y|^{N+2\alpha}}.$$

Estimating this integral, as in Lemma 4.2 of [12], we find

$$I(\varepsilon, r) \leq CR^{-N_{\mathcal{I}}}\varepsilon^{-(\sigma_{\mathcal{I}}+N)/\sigma_{\mathcal{I}}},$$

where we notice that $\sigma_{\mathcal{I}} + N > 0$ and the constant C does not depend on ε nor R. On the other hand we have $\mathcal{I}w \geq \mathcal{I}w_R - E(\varepsilon, r)$, where

$$E(\varepsilon, r) = \int_{\mathbb{R}^N} S_-(w_R(r) - w(r), x, y))\frac{dy}{|y|^{N+2\alpha}}.$$

We estimate the value of $E(\varepsilon, r)$ from below, using the estimates in Lemma 4.2 of [12], and we get

$$E(\varepsilon, r) \geq CR^{-N_{\mathcal{I}}}.$$

Since

$$\mathcal{I}w_R \geq E(\varepsilon, r) - I(\varepsilon, r),$$

for all $R/2 \leq r \leq 2R$, from the above estimates the result follows if we choose ε small enough. $\square$

We still need to prove another lemma that will be used in the critical case, that is when $p = \frac{N_{\mathcal{I}}}{N_{\mathcal{I}}-2\alpha}$. We define the function $\Gamma(x) = \eta(x)h(x)$ for $x \neq 0$, where

$$\eta(x) = \log(1 + |x|) \quad \text{and} \quad h(x) = |x|^{-N_{\mathcal{I}}+2\alpha},$$

then the following lemma allows to use Γ as a comparison function:

Lemma 5.3 *There exists a constant $C >$ such that*

$$\mathcal{I}(\Gamma)(x) \geq -C|x|^{-N_{\mathcal{I}}}, \quad x \neq 0. \tag{5.10}$$

Proof. Since $\mathcal{I}h = 0$ we have by (2.5)

$$\begin{aligned} \mathcal{I}(\Gamma)(x) &= \mathcal{I}(\eta h)(x) - \eta(x)\mathcal{I}(h^-)(x) \\ &\geq \mathcal{M}^-_{\mathcal{A}}(\eta h^- - \eta(x)h^-)(x), \end{aligned} \tag{5.11}$$

where $\eta(x)$ is considered constant regarding the integral defining $\mathcal{M}^-_{\mathcal{A}}$.

From here, we use exactly the same argument as in Lemma 6.1 of [12] to find

$$\mathcal{M}^-_{\mathcal{A}}(\eta h - \eta(x)h)(x) \geq -C|x|^{-N_{\mathcal{I}}},$$

for x large. $\square$

Now we start with the proof of the nonlinear Liouville type theorem.

Proof of Theorem 1.3 (The sub-critical and critical case). The proof of this theorem follows the lines of the proof of Theorem 1.3 in [12]. The idea is to use the equation and the lemmas just proved in order to analyze the asymptotic behavior of $m(r)$. In the sub-critical case, the first step is to use a proper test function and the scaling property of the equation to obtain

$$m(R) \leq \frac{C}{R^{\frac{2\alpha}{p-1}}}, \tag{5.12}$$

for large R. Then, using Lemma 5.1 we get another estimate for $m(R)$, which is incompatible with (5.12), if the function u is not constant. See [12].

This analysis cannot longer be done in the critical case, since no contradiction arises in the behavior of m. In what follows we provide the details in this case, which is more interesting. The starting point for the study of the critical case is the following inequality that recover an estimate obtained in the second order differential case directly from the Hadamard Three Spheres Theorem. We claim that for certain $r_1 > 0$ and $c > 0$ we have

$$u(x) \geq cm(r_1)r^{\sigma_{\mathcal{I}}}, \quad \text{for } r \geq r_1. \tag{5.13}$$

In fact, from equation (1.6) and Lemma 5.1 we have that, for any $\sigma < \sigma_{\mathcal{I}}$,

$$\mathcal{I}u(x) = -u^p \leq c(m(r_1))^p r^{p\sigma}, \quad \text{for } r \geq r_1. \tag{5.14}$$

On the other hand we consider the function $w(r) = T(r^{\sigma^+}, \varepsilon, r)$ where $0 < \varepsilon < r_1/2$. Since $r^{\sigma_{\mathcal{I}}}$ is a fundamental solution for $\mathcal{I}$ and using (2.6) we have

$$\mathcal{I}w(r) \geq \int_{\hat{B}_\varepsilon(x)} S_+(\delta(w - r^{\sigma_{\mathcal{I}}}, x, y))\frac{dy}{|y|^{N+2\alpha}}$$

and then, estimating this integral, we obtain

$$\mathcal{I}w(r) \geq -c\frac{\varepsilon^{\sigma_I + N}}{|x|^{N+2\alpha}}. \tag{5.15}$$

If we define $\phi(r) = m(r_1)\, NT(r^{\sigma_{\mathcal{I}}}, \varepsilon, r_2, r)$ and we use (5.15) we get

$$\mathcal{I}\phi(r) \geq \frac{m(r_1)}{w(\varepsilon) - w(r_2)}\mathcal{I}w(r) \geq -\frac{c}{|x|^{N+2\alpha}}, \tag{5.16}$$

for all $r \geq r_1$. On the other hand, we recall that $\sigma_I + N > 0$ and we choose $\sigma < \sigma_{\mathcal{I}}$ such that $-\sigma p < N + 2\alpha$. Then, using (5.14), (5.16) and taking r_1 large enough, by the choice of σ, we find that

$$\mathcal{I}u \leq -\frac{c}{|x|^{-p\sigma}} \leq -\frac{c}{|x|^{-N+2\alpha}} \leq \mathcal{I}\phi$$

and $u(x) \geq \phi(x)$ for all $r = |x|$ such that $0 \leq r \leq r_1$ or $r \geq r_2$. Thus, by Comparison Principle Theorem 5.1 we have that $u(x) \geq \phi(r)$ for all $r_1 \leq r = |x| \leq r_2$. Taking the limit as $r_2 \to \infty$, we find (5.13).

At this point we have to distinguish two cases, depending on the value of σ_I. The first case corresponds to $\sigma_{\mathcal{I}} \in (-N, -1]$. Here we observe that the function Γ is decreasing for all $r > 0$, with a singularity at the origin if $\sigma_{\mathcal{I}} \in (-N, -1)$ and bounded if $\sigma_{\mathcal{I}} = -1$. We consider $\varepsilon > 0$ and define the function $w(r) = T(\Gamma, \varepsilon, r)$. Using Lemma 5.3 and proper estimates we find that

$$\mathcal{I}w(x) \geq -\frac{c}{|x|^{N_\mathcal{I}}} - \frac{o(1)}{|x|^{N+2\alpha}} \geq -\frac{c}{|x|^{N_\mathcal{I}}}, \quad \text{for all} \quad |x| \geq r_1, \tag{5.17}$$

where we used the fact that $N_\mathcal{I} < N + 2\alpha$. Then we define the comparison function $\phi(x) = m(r_1)\, NT(\Gamma, \varepsilon, r_2, r)$. We observe that $\phi(x) \leq u(x)$ for all x such that $|x| \leq r_1$ or $|x| \geq r_2$. Moreover

$$\mathcal{I}\phi(x) \geq -\frac{c}{|x|^{N_\mathcal{I}}} \quad \text{for all} \quad r_1 \leq |x| \leq r_2.$$

Then, from here, the equation for u and (5.13) we can use the Comparison Principle Theorem 5.1 to obtain $u(x) \geq \phi(x)$ for all $r_1 < |x| < r_2$. Taking limit as $r_2 \to \infty$ we find that

$$u(x) \geq c\frac{\log(1+|x|)}{|x|^{N_\mathcal{I}-2\alpha}} \quad \text{for all} \quad r_1 < |x|.$$

From here and estimate (5.12) we find that

$$\frac{C}{|x|^{N_\mathcal{I}-2\alpha}} \geq m(r) \geq c\frac{\log(1+|x|)}{|x|^{N_\mathcal{I}-2\alpha}}$$

for all $|x|$ large, a contradiction.

The case $\sigma_\mathcal{I} \in (-1, 0)$ still needs some work, but it follows similar lines. See [12]. □

In order to complete the proof of Theorem 1.3 we will use the following inequality proved in [12].

Lemma 5.4 *Let $\alpha \in (0,1)$ and $q > 0$, then for all $s \in [0,1)$, $t \geq 0$ and $u \geq 0$ the following inequality holds*

$$\begin{aligned} \left(1 - s + ((s+t)^2 + u^2)^{1/2}\right)^{-2\alpha q} &+ \left(1 - s + ((s-t)^2 + u^2)^{1/2}\right)^{-2\alpha q} \\ \leq \left((1+t)^2 + u^2\right)^{-\alpha q} &+ \left((1-t)^2 + u^2\right)^{-\alpha q}. \end{aligned}$$

Proof of Theorem 1.3 (The super-critical case). We define the function

$$v(x) = \frac{1}{(1+|x|)^{2\alpha q}}, \quad \text{with} \quad \frac{1}{p-1} < q < \frac{N_\mathcal{I} - 2\alpha}{2\alpha},$$

and we prove next that v satisfies (1.6). As a direct consequence of Lemma 5.4 we have

$$\delta(v, x, y) \leq \frac{1}{(1+|x|)^{2\alpha q}} \left\{ \frac{1}{|e_1 + \tilde{y}|^{2\alpha q}} + \frac{1}{|e_1 - \tilde{y}|^{2\alpha q}} - 2 \right\},$$

where $\tilde{y} = Ry/(1+r)$, with R an appropriate rotation matrix. Recalling the definition of δ_σ in (3.3), from here we get

$$\begin{aligned} \mathcal{I}(v) &\leq \frac{1}{(1+|x|)^{2\alpha q}} \inf_I \sup_J \int_{\mathbb{R}^N} \frac{a_{i,i}(\delta_{2\alpha q}(\tilde{y}))dy}{|y|^{N+2\alpha}} \\ &= \frac{1}{(1+|x|)^{2\alpha(q+1)}} c(-2\alpha q) = \frac{-C}{(1+|x|)^{2\alpha(q+1)}}. \end{aligned}$$

From the inequalities satisfied by q we see that $-C = c(-2\alpha q) < 0$. Then we have that

$$\mathcal{I}(cv) + (cv)^p \leq \frac{-cC}{(1+|x|)^{2\alpha(q+1)}} + \frac{c^p}{(1+|x|)^{2\alpha qp}},$$

and we choose c small enough to finally obtain

$$\mathcal{I}(cv) + (cv)^p \leq 0,$$

completing the proof.□

Acknowledgements: P.F. was partially supported by Fondecyt Grant # 1070314, FONDAP, BASAL-CMM projects and CAPDE, Anillo ACT-125.

A. Q. was partially supported by Fondecyt Grant # 1070264 and USM Grant # 12.09.17. and Programa Basal, CMM. U. de Chile and CAPDE, Anillo ACT-125.

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