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Abstract. We study systems of two elliptic equations, with right-hand sides with general power-like superlinear growth, and left-hand sides which are of Isaac's or Hamilton-Jacobi-Bellman type (however our results are new even for linear left-hand sides). We show that under appropriate growth conditions such systems have positive solutions in bounded domains, and that all such solutions are bounded in the uniform norm. We also get nonexistence results in unbounded domains.

1 Introduction.

We study positive solutions of nonvariational elliptic systems of the type

$$\left\{\begin{array}{rcll} -\mathcal{H}_1[u] &=& f_1(x,u_1,u_2) & \text{in} \quad \Omega \\ -\mathcal{H}_2[u] &=& f_2(x,u_1,u_2) & \text{in} \quad \Omega \\ u_1 &=& u_2 = 0 & \text{on} \quad \partial\Omega, \end{array}\right. \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $u = (u_1, u_2)$, f_1, f_2 are nonnegative locally Lipschitz functions defined in $\overline{\Omega} \times [0,\infty)^2$, and $\mathcal{H}_1, \mathcal{H}_2$ are uniformly elliptic linear or nonlinear operators.

In the last years there has been a lot of interest in superlinear elliptic systems with or without variational structure (we give various references below). Requiring a system to have such structure is clearly a strong assumption, as it means both that the elliptic operators are in divergence form, and that f_1 and f_2 are the derivatives of a given function. An important feature in most of the previous works on nonvariational systems is that only the latter of these two requirements was removed. On the other hand, operators in non-divergence form could be considered as in [3], [18], [37], provided they

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are linear, with the same principal part. It is our goal here to study the considerably more difficult case of *different* and *nonlinear* operators.

To accommodate the reader, we start by stating a consequence of our main result in a very particular model case. Let $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ be the Pucci operators: $\mathcal{M}^{+}_{\lambda,\Lambda}(M) = \sup_{A\in\mathcal{S}} \operatorname{tr}(AM)$, $\mathcal{M}^{-}_{\lambda,\Lambda}(M) = \inf_{A\in\mathcal{S}} \operatorname{tr}(AM)$ for any symmetric matrix M, where $\mathcal{S} = \mathcal{S}_N^{\lambda,\Lambda}$ is the set of symmetric matrices whose eigenvalues lie in the interval $[\lambda, \Lambda]$. These important fully nonlinear operators are an upper and a lower bound for all uniformly elliptic linear operators with ellipticity constant λ and bounded coefficients. Note $\mathcal{M}_{1,1}(D^2u) = \Delta u$.

Theorem 1.1 I. *Let $\lambda_1, \lambda_2, \Lambda_1, \Lambda_2 > 0$. Consider the system*

$$\begin{cases} \mathcal{M}_1[u_1] + u_2^p &= 0 \quad in \quad \Omega \\ \mathcal{M}_2[u_2] + u_1^q &= 0 \quad in \quad \Omega \\ \quad u_1 = u_2 &= 0 \quad on \quad \partial\Omega, \end{cases} \tag{2}$$

where $\mathcal{M}_k[u_k] = \mathcal{M}^{\pm}_{\lambda_k,\Lambda_k}(D^2u_k)$ (independently for $k = 1, 2$), and we set

$$\rho_k = (\lambda_k/\Lambda_k)^{\pm 1}, \qquad N_k = \rho_k(N-1) + 1.$$

Let $p, q \geq 1$ be such that $pq > 1$ and

$$\frac{2(p+1)}{pq-1} \geq N_2 - 2 \qquad or \qquad \frac{2(q+1)}{pq-1} \geq N_1 - 2.$$

Then there exists a positive classical solution of system (2). In addition, all such solutions are uniformly bounded in the L^∞-norm.

II. *Under the same hypotheses, system (2) (without the boundary condition) does not have positive solutions in the whole space, that is, if $\Omega = \mathbb{R}^N$.*

Remark 1. Systems like (2) have been extensively studied when $\mathcal{M}_1 = \mathcal{M}_2$ is the Laplacian (or other divergence form operators), see [1], [10], [15], [17], [29], [34], [35], [41], and the references in these papers.

Remark 2. When $\mathcal{M}_1 = \mathcal{M}_2$ and $p = q$ this theorem reduces to the known results for the scalar equation $-\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) = u^p$ (see [12], [30]).

More generally, $\mathcal{H}_k$ in (1) could be coupled second-order operators

$$\mathcal{H}_k[u] = L_k u = \sum_{i,j} a_{ij}^{(k)}(x)\partial_{ij}u_k + \sum_i b_i^{(k)}(x)\partial_i u_k + c_1^{(k)}(x)u_1 + c_2^{(k)}(x)u_2.$$

We will actually go much further, allowing $\mathcal{H}_1, \mathcal{H}_2$ to be nonlinear operators of Isaac's type, that is, sup-inf of coupled linear operators as above

$$\mathcal{H}_k[u] = \sup_{\alpha\in\mathcal{A}_k} \inf_{\beta\in\mathcal{B}_k} L_k^{(\alpha,\beta)}u, \tag{3}$$

where $\mathcal{A}_k$, $\mathcal{B}_k$ are arbitrary index sets. Note $\mathcal{H}_k$ is linear when $|\mathcal{A}_k| = |\mathcal{B}_k| = 1$ in (3). When $|\mathcal{B}_k| = 1$ (like for Pucci operators) the corresponding sup-operator is usually referred to as Hamilton-Jacobi-Bellman (HJB) operator. These operators are essential tools in control theory and in theory of large deviations, while general Isaac's operators are basic in game theory. We refer to [5], [24], and to the references in these papers, for a larger list of problems, where systems of type (1) appear.

Next, we give the hypotheses we make on $\mathcal{H}_k$, which we suppose to be in the form (3). We write $\mathcal{H}_i[u] = H_i(D^2u, Du, u, x)$, in order to distinguish between the way $\mathcal{H}_i$ depends on the derivatives of u. We assume all coefficients of the operators $L_k^{(\alpha,\beta)}$ in (3) are bounded measurable functions (say $|b_i^{(k)}| \le \gamma$, $|c_j^{(k)}| \le \delta$, we will no longer write the dependence in α, β), and

(H_1) for all $x \in \Omega$ the eigenvalues of $A_k(x) = (a_{ij}^{(k)}(x)) \in C(\overline{\Omega})$ are in $[\lambda_k, \Lambda_k]$, and there is an invertible matrix $R = R(x)$ such that the functions $H_k(R^T M R, 0, 0, x)$ depend only on the eigenvalues of M ;

(H_2) the functions $c_1^{(2)}(x)$, $c_2^{(1)}(x)$ are nonnegative in Ω.

Hypothesis (H_1) says $\mathcal{H}_k$ are uniformly elliptic and have an invariance property with respect to the second derivatives of u. Most operators which have geometrical meaning satisfy (H_1) with $R = I$ (Pucci operators are a trivial example). Of course for each $A \in \mathcal{S}$ there exists R such that $R^T A R = I$.

Hypothesis (H_2) means H_k is nondecreasing in the variable u_i, for $i \ne k$. Systems satisfying such a hypothesis are called quasimonotone.

If $\mathcal{H}_k$ are HJB operators, we suppose that

(H_3) there exist functions $\psi_1, \psi_2 \in W^{2,N}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$, such that $\psi_1 > 0, \psi_2 > 0$ in $\overline{\Omega}$, and $\mathcal{H}_k[\psi_1, \psi_2] \le 0$ in Ω, $k = 1, 2$.

If $\mathcal{H}_k$ are not HJB, we suppose they are bounded above by HJB operators, which satisfy (H_3), see Section 2 for a precise definition. Note we have (H_3) with $\psi_1 = \psi_2 \equiv 1$ provided $c_1^{(k)} + c_2^{(k)} \le 0$, for all α, β, k. We recently showed in [31], [32] (for scalar equations), [33] (for systems, the linear case was considered earlier in [5]) that hypothesis (H_3) is equivalent to supposing that the vector operator $(\mathcal{H}_1, \mathcal{H}_2)$ satisfies the comparison principle, and that under (H_3) the corresponding Dirichlet problem has a unique solution, see Section 2.

As we explain in Section 3, to any operator $\mathcal{H}_k$ as in (3) we can associate an explicitly given number $N_k = N(\mathcal{H}_k)$, which appears in the fundamental solution of a related nonlinear operator. To avoid introducing heavy notations at this stage, here we only note that $N(\mathcal{H}_k)$ is always in the interval $[(\lambda_k/\Lambda_k)(N-1)+1, (\Lambda_k/\lambda_k)(N-1)+1]$, where λ_k, Λ_k are as in (H_1).

We now come to the hypotheses on the functions f_i in (1). General nonlinearities with power growth were considered in [17]. We are going to use the same structural assumptions as in [17], strengthening them a little, to avoid technicalities. We suppose that f_i are locally bounded functions, for which there exist exponents $\alpha_{ij}, \gamma_{ij} \geq 0$ with $\gamma_{i1}\alpha_{i1}^{-1} + \gamma_{i2}\alpha_{i2}^{-1} = 1$ and nonnegative functions $d_{ij}, e_i \in C(\overline{\Omega})$, with either $\alpha_{11}, \alpha_{22} > 1$, $d_{11}, d_{22} > 0$ in $\overline{\Omega}$, or $\alpha_{12}, \alpha_{21} \geq 1, \alpha_{12}\alpha_{21} > 1$, $d_{12}, d_{21} > 0$ in $\overline{\Omega}$, such that

$$f_i = d_{i1}(x)u_1^{\alpha_{i1}} + d_{i2}(x)u_2^{\alpha_{i2}} + e_i(x)u_i^{\gamma_{i1}}u_2^{\gamma_{i2}} + g_i(x, u_1, u_2), \tag{4}$$

$$\lim_{|(u_1,u_2)|\to\infty} \left(d_{i1}(x)u_1^{\alpha_{i1}} + d_{i2}(x)u_2^{\alpha_{i2}}\right)^{-1} |g_i(x, u_1, u_2)| = 0, \tag{5}$$

$$\lim_{|(u_1,u_2)|\to 0} \left(|(u_1, u_2)|\right)^{-1} |f_i(x, u_1, u_2)| = 0, \quad i = 1, 2, \tag{6}$$

uniformly in $x \in \Omega$. Setting $a/0 = \infty$, $a_+ = \max\{a, 0\}$, for $a \in \mathbb{R}$, we define

$$\beta_i = \min\left\{ \frac{2}{(\alpha_{ii}-1)_+}, \frac{2\alpha_{jj}}{\alpha_{ji}(\alpha_{jj}-1)_+}, \frac{2(\alpha_{ij}+1)}{(\alpha_{ij}\alpha_{ji}-1)_+}, i, j = 1, 2,\ i \neq j \right\}.$$

We make the convention that when we speak of a solution (supersolution, subsolution) we mean L^N-viscosity solutions. We refer to [9] for a general review of this notion (we recall definitions and results that we need in the next section). Note that viscosity solutions are continuous and that any vector in $W^{2,N}_{\rm loc}(\Omega)$ satisfies (1) almost everywhere if and only if it is a L^N-viscosity solution. For HJB operators these solutions are always strong (that is, in $W^{2,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$, $p < \infty$), and are classical provided the dependence in x in (1) is C^α, see [39], [8]. Viscosity solutions are not an added complication – they provide a good framework in our setting, just like Sobolev spaces do for some variational problems, when one knows that any H^1-solution is classical.

Theorem 1.2 *Assume that system (1) satisfies* (H_1)*-*(H_3)*, (4)-(6), and*

$$\beta_1 \geq N_1 - 2 \qquad \text{or} \qquad \beta_2 \geq N_2 - 2. \tag{7}$$

Then there exists a positive solution of (1). In addition, all such solutions are uniformly bounded in the L^∞*-norm.*

The proof of the a priori bound in Theorem 1.2 is carried out with the help of a blow-up argument of Gidas-Spruck type, while the existence of a solution is a consequence of this bound combined with degree theory. Note that, taking again system (2) as an example, when $\mathcal{M}_1 = \mathcal{M}_2 = \Delta$ it is very well known how one can apply a fixed point theorem to get the existence of solution, once an a priori bound is proven. In our situation the known approach does not work, and our proof relies on a deep result of the theory of elliptic equations, the Krylov-Safonov improved strong maximum principle, see the proof of Proposition 6.2 and Theorem 7.1.

The blow-up method for obtaining a priori bounds was developed in [22] for the scalar case, and recently extended to some systems of equations in [17], [18], [37], [41] (see also the references in these works). This method is based on a contradiction argument, which in turn relies on Liouville (nonexistence) theorems for equations or systems in $\mathbb{R}^N$ or in a half-space of $\mathbb{R}^N$. Proving nonexistence results is usually the main difficulty in applying the Gidas-Spruck method.

In our situation we are led to proving Liouville results for systems involving a general class of extremal operators, related to ones recently defined in [20] and [21]. Specifically, we are going to consider the HJB operators

$$\mathcal{M}_J^+(M) = \sup_{\sigma(A)\in J} \mathrm{tr}(AM) \quad \text{and} \quad \mathcal{M}_J^-(M) = \inf_{\sigma(A)\in J} \mathrm{tr}(AM), \tag{8}$$

where J is any invariant with respect to permutations of coordinates subset of the cube $[\lambda,\Lambda]^N$ and $\sigma(A)$ is the set of eigenvalues of A. These operators reduce to classical Pucci type operators when $J=[\lambda,\Lambda]^N$, and to the Laplacian when $\lambda=\Lambda=1$. To each operator $\mathcal{M}_J^\pm$ in this class we associate a dimension-like number $N_J^\pm = N(\mathcal{M}_J^\pm)$ (see Section 3) :

$$N_J^+ = \min_{x\in J} \frac{\sum_{i=1}^N x_i}{\max x_i} \le \max_{x\in J} \frac{\sum_{i=1}^N x_i}{\min x_i} = N_J^-. \tag{9}$$

We will need to establish Liouville type theorems for the system

$$\left\{\begin{array}{rcll} \mathcal{M}_1(D^2u)+v^q &=& 0 & \text{in} \quad G,\\ \mathcal{M}_2(D^2v)+u^p &=& 0 & \text{in} \quad G,\end{array}\right. \tag{10}$$

where $G=\mathbb{R}^N$ or $G=\mathbb{R}^{N-1}\times\mathbb{R}_+$, and $\mathcal{M}_1$ and $\mathcal{M}_2$ are two extremal operators as in (8). Observe that the set J can be different for each operator and that one operator can be a sup and the other a inf. The next theorem is actually new even for Pucci operators. It also implies a new nonexistence result for different linear operators.

Theorem 1.3 *Suppose $pq>1$ and $N_1,N_2>2$, where N_1 and N_2 are the respective dimension-like numbers for $\mathcal{M}_1$ and $\mathcal{M}_2$, given by (9). Then*

I. there are no positive supersolutions to (10) with $G=\mathbb{R}^N$ if and only if

$$\frac{2(p+1)}{pq-1} \ge N_2-2 \qquad \text{or} \qquad \frac{2(q+1)}{pq-1} \ge N_1-2. \tag{11}$$

II. if $p,q\ge 1$ and (11) holds then there are no positive bounded solutions to (10) with $G=\mathbb{R}^{N-1}\times\mathbb{R}_+$, which vanish on $\{x_N=0\}$.

Liouville theorems for equations and systems in $\mathbb{R}^N$ or in $\mathbb{R}^N_+$ have a long history. Previous results for systems concern divergence form operators, see [3], [4], [17], [28], [36], and the references in these works.

It turns out that it is much more difficult to prove nonexistence results for fully nonlinear operators. The first result in this line is [12], for viscosity supersolutions in $\mathbb{R}^N$ of $\mathcal{M}(D^2u) = u^p$, where $\mathcal{M}$ is a Pucci operator. Results on nonexistence of radial solutions in $\mathbb{R}^N$ for the same equation were obtained in [19]. Nonexistence in a half-space for this equation is proved in [30]. Recently, Liouville results for scalar equations with general operators of type (8) were obtained in [20] and [21]. To our knowledge, Theorem 1.3 is the first result on nonexistence for systems involving fully nonlinear operators.

The paper is organized as follows. In Section 2 we restate our hypotheses in the general setting of fully nonlinear equations, and recall some recent results on solvability and properties of solutions of such equations. In Section 3 we define and study the dimension-like numbers $N(\mathcal{H})$, then in Section 4 we prove our Liouville result in the whole space. In Section 5 we obtain nonexistence results in a half-space, and in Section 6 we use the previous information to prove our existence theorems. Finally, in Section 7 we discuss an improved version of the strong maximum principle.

2 Preliminaries

In this section we give some precisions on the hypotheses in the introduction, and recall some known results which we shall need in the sequel.

We use the following notation. For some given positive constants λ, Λ, γ, and for all $M, N \in \mathcal{S}_N$ (as usual, $\mathcal{S}_N$ denotes the set of all symmetric matrices), $p, q \in \mathbb{R}^N$, we define the extremal operators

$$\mathcal{L}^-(M,p) = \mathcal{M}^-_{\lambda,\Lambda}(M) - \gamma\,|p|, \qquad \mathcal{L}^+(M,p) = \mathcal{M}^+_{\lambda,\Lambda}(M) + \gamma\,|p|.$$

The operators $\mathcal{L}^-, \mathcal{L}^+$ are extremal with respect to all linear uniformly elliptic operators with given ellipticity constant and L^∞-bounds for the coefficients. We set

$$m_i^-(u) := |u_i| + (u_j)_-, \qquad m_i^+(u) := |u_i| + (u_j)_+, \quad j \neq i,\ i,j = 1,2,$$

and, for some given $\delta > 0$,

$$F_i^*(M,p,u) = \mathcal{L}^+(M,p) + \delta m_i^+(u).$$

To facilitate what follows, we restate our hypotheses on system (1), with the above notation. First, we assume that the operators H_i are positively homogeneous of order 1, that is,

$(\widetilde{H}_0)$ $\qquad H_i(tM, tp, tu_1, tu_2, x) = t\, H_i(M, p, u_1, u_2, x), \quad$ for all $t \geq 0$.

Since we want to prove existence results, we need to suppose that

$(\widetilde{H}_1)$ $\qquad H_i(M, 0, 0, x)$ is continuous on $\mathcal{S}_N \times \overline{\Omega}$.

The following definition plays an important role. Given a second order operator $F = F(M, p, u, x)$ which is *convex* in (M, p, u) (note HJB operators are convex), we say that $H_i(M, p, u, x)$ satisfies condition (D_F) provided

$$(D_F) \qquad \begin{array}{rcl} -F(N-M, q-p, v-u, x) & \leq & H_i(M, p, u, x) - H_i(N, q, v, x) \\ & \leq & F(M-N, p-q, u-v, x). \end{array}$$

If F is positively homogeneous of order one in (M, p, u) then F is convex in (M, p, u) if and only if F satisfies (D_F), as can be easily proved.

Next, we suppose that the operators H_i are quasimonotone and uniformly elliptic with bounded measurable coefficients, that is, for some $\lambda, \Lambda, \gamma, \delta > 0$ and all $M, N \in \mathcal{S}_N$, $p, q \in \mathbb{R}^N$, $u, v \in \mathbb{R}^2$, $x \in \Omega$,

$(\widetilde{H}_2)$ $\qquad H_i$ satisfies $(D_{F_i^*})$, $i = 1, 2$.

Note the definition of m_i and $(\widetilde{H}_2)$ imply that the function $H_i(M, p, u, x)$ is nondecreasing in the variable u_j, for $j \neq i$.

Finally, we need a hypothesis which describes the admissible behaviour of the zero order terms in H_1, H_2. We shall give exact analogues of the known results on scalar equations, in the sense that we suppose only that the vector operator in the left-hand side of (1) satisfies a comparison principle. It turns out that this property can be described in terms of first eigenvalues of the system, as we explain next.

As an extension of earlier results on scalar equations in [31], [32], we showed in [33] (for the linear case see sections 13 and 14 in [5]) that if we have two operators $F_i(M, p, u_1, u_2, x)$, convex in (M, p, u), satisfying $(\widetilde{H}_0)$-$(\widetilde{H}_2)$, and if $F_1(0, 0, 0, 1, x) \not\equiv 0$ and $F_2(0, 0, 1, 0, x) \not\equiv 0$ in Ω then the vector operator $F = (F_1, F_2)$ has "first eigenvalues" $\lambda_1^+(F) \leq \lambda_1^-(F)$, such that the positivity of λ_1^+ is a necessary and sufficient condition for (F_1, F_2) to satisfy the comparison principle, and guarantees the unique solvability of the corresponding Dirichlet problem. If, on the contrary, we have $F_1(0, 0, 0, 1, x) \equiv 0$ or $F_2(0, 0, 1, 0, x) \equiv 0$ then the two scalar operators $F_1(M, p, t, 0, x)$ and $F_2(M, p, 0, t, x)$ have corresponding first eigenvalues $\lambda_{1,1}^{\pm}(F), \lambda_{1,2}^{\pm}(F)$, whose positivity has the same implications on (F_1, F_2). In this case we set $\lambda_1^+(F) = \min\{\lambda_{1,1}^+(F), \lambda_{1,2}^+(F)\}$, $\lambda_1^-(F) = \max\{\lambda_{1,1}^-(F), \lambda_{1,2}^-(F)\}$.

We shall suppose that there exist convex operators F_1, F_2, which satisfy $(\widetilde{H}_0) - (\widetilde{H}_2)$, such that H_i satisfies (D_{F_i}) (note in any case the extremal operators F_i^* can be used as such F_1, F_2), and

$(\widetilde{H}_3)$ $\qquad \lambda_1^+(F) > 0$.

Remark. We have shown in [33] that (H_3) and $(\widetilde{H}_3)$ are equivalent. Bounds on the eigenvalues in terms of the domain and the coefficients of the operators are given in [5] and [33]. These bounds can be used to verify $(\widetilde{H}_3)$.

Let us now recall the definition of a viscosity solution of (1), see [11], [9].

Definition 2.1 *We say that the vector $v \in C(\Omega)$ is a L^p-viscosity subsolution (supersolution) of $H_i(D^2 v_i, Dv_i, v, x) = f(x)$, $p \geq N$, provided for any $\varepsilon > 0$, any open ball $\mathcal{O} \subset \Omega$, and any $\varphi \in W^{2,p}(\mathcal{O})$ – we call φ a test function – such that $F_i(D^2\varphi(x), D\varphi(x), v(x), x) \leq f(x) - \varepsilon$ (resp. $F(D^2\varphi(x), D\varphi(x), v(x), x) \geq f(x) + \varepsilon$) a.e. in $\mathcal{O}$, the function $v_i - \varphi$ cannot achieve a local maximum (minimum) in $\mathcal{O}$.*

We say that v is a solution of (1) if v is at the same time a subsolution and a supersolution of (1).

Note that whenever a function v in $W^{2,N}_{\text{loc}}(\Omega)$ satisfies the inequality $F(D^2v, Dv, v, x) \geq (\leq) f$ a.e. in Ω then it is a viscosity solution.

Next, we recall some easy properties of Pucci operators.

Lemma 2.1 *Let $M, N \in \mathcal{S}_N$, $\phi(x) \in C(\overline{\Omega})$ be such that $0 < a \leq \phi(x) \leq A$. Then*

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = -\mathcal{M}^+_{\lambda,\Lambda}(-M),$$

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \lambda \sum_{\{\nu_i > 0\}} \nu_i + \Lambda \sum_{\{\nu_i < 0\}} \nu_i, \qquad \text{where } \{\nu_1, \ldots, \nu_N\} = \sigma(M),$$

$$\mathcal{M}^-_{\lambda,\Lambda}(M) + \mathcal{M}^-_{\lambda,\Lambda}(N) \leq \mathcal{M}^-_{\lambda,\Lambda}(M+N) \leq \mathcal{M}^-_{\lambda,\Lambda}(M) + \mathcal{M}^+_{\lambda,\Lambda}(N),$$

$$\mathcal{M}^-_{\lambda,\Lambda}(M) + \mathcal{M}^+_{\lambda,\Lambda}(N) \leq \mathcal{M}^+_{\lambda,\Lambda}(M+N) \leq \mathcal{M}^+_{\lambda,\Lambda}(M) + \mathcal{M}^+_{\lambda,\Lambda}(N),$$

$$\mathcal{M}^-_{\lambda a,\Lambda A}(M) \leq \mathcal{M}^-_{\lambda,\Lambda}(\phi M) \leq \mathcal{M}^-_{\lambda A,\Lambda a}(M),$$

We will also use the following simple fact.

Lemma 2.2 *Suppose $u \in C^2(B)$ is a radial function, say $u(x) = g(|x|)$, defined on a ball $B \subset \mathbb{R}^N$. Then $g''(|x|)$ is an eigenvalue of the matrix $D^2u(x)$, and $|x|^{-1}g'(|x|)$ is an eigenvalue of multiplicity $N-1$.*

We are going to use the Generalized Maximum Principle for elliptic equations, commonly known as the Alexandrov-Bakelman-Pucci (ABP) inequality. The following theorem follows from Proposition 3.3 in [9].

Theorem 2.1 *Suppose $u \in C(\overline{\Omega})$ is a L^N-viscosity solution of*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \gamma|Du| \geq f(x) \qquad \textit{in } \ \Omega \cap \{u > 0\},$$

where $f \in L^N(\Omega)$. *Then*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + \operatorname{diam}(\Omega).C_1 \|f_-\|_{L^N(\Omega^+)}, \tag{12}$$

where $\Omega^+ = \{x \in \Omega \ : \ u(x) > 0\}$ *and* C_1 *is a constant which depends on* N, λ, Λ, *diam*(Ω), *and* C_1 *remains bounded when these quantities are bounded.*

We have the following version of Hopf's boundary lemma (see for instance [2], where a more general result is given).

Theorem 2.2 *Let* Ω *be a regular domain and let* $u \in C(\overline{\Omega})$ *satisfy*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - \gamma|Du| - \delta u \leq 0 \quad in \quad \Omega, \tag{13}$$

for some $\gamma, \delta \geq 0$, *and* $u \geq 0$ *in* Ω. *Then either* u *vanishes identically in* Ω *or* $u(x) > 0$ *for all* $x \in \Omega$. *Moreover, in the latter case for any* $x_0 \in \partial\Omega$ *such that* $u(x_0) = 0$, *we have* $\liminf_{t \searrow 0} \dfrac{u(x_0 + t\nu) - u(x_0)}{t} > 0$, *where* ν *is the inner normal to* $\partial\Omega$.

We also recall the weak Harnack inequality for viscosity solutions of fully nonlinear equations, proved in [8] and [40].

Theorem 2.3 *Suppose* $u \in C(\Omega)$ *is a positive function satisfying*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - \gamma|Du| - \delta u \leq f \qquad in\ \Omega,$$

for some $f \in L^N(\Omega)$. *Then for any* $\Omega' \subset\subset \Omega$ *there exist positive constants* p_0 *and* C *depending on* $N, \lambda, \Lambda, \gamma, \delta$, *dist*$(\Omega', \partial\Omega)$ *such that*

$$\|u\|_{L^{p_0}(\Omega')} \leq C \left(\inf_{x \in \Omega'} u + \|f\|_{L^N(\Omega)} \right).$$

We shall use the following comparison principle, see [26] and [27].

Theorem 2.4 *Assume* Ω *is a bounded domain in* $\mathbb{R}^N$. *Let* $F : \mathcal{S}_N \to \mathbb{R}$ *be a continuous function such that there are positive constants* λ, Λ, *for which*

$$\lambda\, tr(B) \leq F(M + B) - F(M) \leq \Lambda\, tr(B), \tag{14}$$

for all $M, B \in \mathcal{S}_N$, $B \geq 0$ *(this is equivalent to saying* F *satisfies* $(\widetilde{H}_2)$ *with* $\gamma = \delta = 0$*). Then if* $u, v \in C(\bar{\Omega})$ *are subsolution and supersolution of*

$$F(D^2w) = f(x) \quad in\ \Omega, \quad f \in C(\Omega),$$

such that $u(x) \leq v(x)$ *for all* $x \in \partial\Omega$, *then* $u(x) \leq v(x)$ *for all* $x \in \Omega$.

We are going to use the following standard convergence result from general theory of viscosity solutions (see Theorem 3.8 in [9]).

Theorem 2.5 *Suppose $u_n \in C(\overline{\Omega})$, $g_n \in L^N(\Omega)$, and $H_n(M,p,u,x)$ are operators satisfying $(\widetilde{H}_2)$, such that u_n is a solution (or subsolution, or supersolution) in Ω of the equation*

$$H_n(D^2u_n, Du_n, u_n, x) = g_n(x)$$

in a domain Ω. Suppose $u_n \to u$ in $C(\overline{\Omega})$, $g_n \to g$ in $L^N(\Omega)$, and H_n converges to an operator H in the sense that for any ball $B \subset\subset \Omega$ and any $\phi \in W^{2,N}(B)$ we have

$$H_n(D^2\phi, D\phi, u_n, x) \to H(D^2\phi, D\phi, u, x) \quad \text{in } L^N(B).$$

Then u is a solution (or subsolution, or supersolution) in Ω of

$$H(D^2u, Du, u, x) = g(x).$$

We shall also need the C^α-estimate proved in [40] (we refer to [38] for more general results and simple self-contained proofs of the C^α-estimates).

Theorem 2.6 *Let Ω be a regular domain and $f \in L^N(\Omega)$. If $u \in C(\overline{\Omega})$ satisfies the inequalities*

$$\begin{aligned} \mathcal{M}^-_{\lambda,\Lambda}(D^2u) - \gamma|Du| - \delta|u| &\leq -f \\ \mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \gamma|Du| + \delta|u| &\geq f \end{aligned}$$

in Ω, then for any $\Omega' \subset\subset \Omega$ there exist constants $\alpha, A > 0$ depending only on $N, \lambda, \Lambda, \gamma, \delta$, $\|f\|_{L^N(\Omega)}$, $\|u\|_{L^\infty(\Omega)}$, Ω', Ω, such that $u \in C^\alpha(\Omega')$ and $\|u\|_{C^\alpha(\Omega')} \leq A$. If in addition $u|_{\partial\Omega} \in C^\beta(\partial\Omega)$ for some $\beta > 0$ then $u \in C^\alpha(\Omega)$ and $\|u\|_{C^\alpha(\Omega)} \leq A$ (with constants depending also on β).

3 Extremal operators and definition of $N(\mathcal{H}_k)$

In this section we are going to discuss in more details the class of extremal operators $\mathcal{M}^\pm_J$ we consider for the Liouville theorems.

Let $\mathcal{J}$ denote the set of subsets of $[\lambda, \Lambda]^N$ which are invariant with respect to permutations of coordinates. First, obviously there is a one to one correspondence between elements of $\mathcal{J}$ and subsets $\mathcal{A}$ of $\mathcal{S}_N^{\lambda,\Lambda}$ (the set of symmetric matrices whose eigenvalues lie in $[\lambda, \Lambda]$), and such that $P\mathcal{A}P^T = \mathcal{A}$, for each orthogonal matrix P. Namely, for each $J \in \mathcal{J}$ we can take $\mathcal{A}$ to be the set of matrices whose eigenvalues are in J, and for each such $\mathcal{A}$ we can take J to contain the eigenvalues of the matrices in $\mathcal{A}$. So we can indifferently write

$$\mathcal{M}_J^+(M) = \sup_{\sigma(A)\in J} \mathrm{tr}(AM) = \mathcal{M}_{\mathcal{A}}^+(M) = \sup_{A\in\mathcal{A}} \mathrm{tr}(AM),$$

respectively $\mathcal{M}_J^- = \mathcal{M}_{\mathcal{A}}^-$.

The following lemma shows that the set of these operators is nothing but the set of convex or concave positively homogeneous uniformly elliptic operators $F(M)$, which depend only on the eigenvalues of M.

Lemma 3.1 *Suppose $F : \mathcal{S}_N \to \mathbb{R}$ is an operator satisfying $(\widetilde{H}_2)$, that is,*

$$\lambda \mathrm{tr}(B) \leq F(M+B) - F(M) \leq \Lambda \mathrm{tr}(B),$$

for all $M, B \in \mathcal{S}_N$, $B \geq 0$. Suppose also that $F(tM) = tF(M)$ for all $t \geq 0$ and that $F(PMP^T) = F(M)$ for any orthogonal matrix P. If F is convex (resp. concave) then there exists a set $J \in \mathcal{J}$, such that $F = \mathcal{M}_J^+$ (resp. $F = \mathcal{M}_J^-$).

Proof. Since F is convex, it is a supremum of affine functions. Since F is homogeneous, these functions can be taken to be linear, that is,

$$F(M) = \sup_{A\in\mathcal{A}_1} \mathrm{tr}(AM),$$

where $\mathcal{A}_1$ is some set of matrices such that $\mathcal{A}_1 \subset \mathcal{S}_N^{\lambda,\Lambda}$, by the hypothesis on F. Then we can take $\mathcal{A} = \cup(P\mathcal{A}_1P^T)$, where the union is taken over all orthogonal matrices P. □

Next, for a fixed operator $\mathcal{M}_{\mathcal{A}}^\pm$ in this class, we compute the quantity $\mathcal{M}_{\mathcal{A}}^\pm(D^2|x|^\alpha)$, for any $\alpha \in \mathbb{R}$. By Lemma 2.2 we have

$$\sigma(D^2|x|^\alpha) = |x|^{\alpha-2}\{\alpha, \alpha, \ldots, \alpha, \alpha(\alpha-1)\}$$

so clearly for each x there exist an orthogonal matrix P such that

$$D^2|x|^\alpha = \alpha|x|^{\alpha-2} P E_\alpha P^t,$$

where $E_\alpha = I + (\alpha-2)e_{nn}$, and e_{nn} is the unitary matrix, whose last coefficient is one, and all other coefficients are zero. Hence for $\alpha \neq 0$

$$\begin{aligned} c(\alpha) &:= |\alpha|^{-1}|x|^{2-\alpha}\mathcal{M}_{\mathcal{A}}^+(D^2|x|^\alpha) \\ &= \begin{cases} -\inf_{A\in\mathcal{A}} \left\{\sum_{i=1}^{N-1} a_{ii} + a_{nn}(\alpha-1)\right\} & \text{if} \quad \alpha < 0 \\ \sup_{A\in\mathcal{A}} \left\{\sum_{i=1}^{N-1} a_{ii} + a_{nn}(\alpha-1)\right\} & \text{if} \quad \alpha > 0, \end{cases} \end{aligned}$$

and the same for $\mathcal{M}_{\mathcal{A}}^-$, with inf and sup reversed (here $A = (a_{ij})$).

By noting that $c(\alpha)$ is continuous and monotonous in $\mathbb{R}_+$ and $\mathbb{R}_-$, and by looking at its sign at minus infinity, zero, and one, we immediately get the following lemma. It permits to define the dimension-like numbers for the operators $\mathcal{M}_{\mathcal{A}}^{\pm}$.

Lemma 3.2 *Suppose $\mathcal{A}$ is a set of matrices such that $P\mathcal{A}P^T = \mathcal{A}$ for all orthogonal matrices P. Then*

1. *There exists $\alpha_0 \neq 0$ such that $\mathcal{M}_{\mathcal{A}}^+(D^2|x|^{\alpha_0}) = 0$ if and only if*

$$\sum_{i=1}^{N-1} a_{ii} - a_{nn} > 0 \qquad \text{for all } A \in \mathcal{A}.$$

In this case $\alpha_0 < 0$ and we denote $N_{\mathcal{A}}^+ = 2 - \alpha_0$.

2. *There always exists $\alpha_0 < 0$ such that $\mathcal{M}_{\mathcal{A}}^-(D^2|x|^{\alpha_0}) = 0$, except if $N = 2$, $\lambda = \Lambda$, and $\mathcal{A} = \{\lambda I\}$ (that is, we are working with the Laplacian in dimension two). We denote $N_{\mathcal{A}}^- = 2 - \alpha_0$.*

In addition, there exists $\alpha_1 \in (0, 1)$ such that $\mathcal{M}_{\mathcal{A}}^-(D^2|x|^{\alpha_1}) = 0$ if and only if

$$\sum_{i=1}^{N-1} a_{ii} - a_{nn} < 0 \qquad \text{for some } A \in \mathcal{A}.$$

Further, we have $\mathcal{M}_{\mathcal{A}}^-(D^2 \log r) = 0$ if and only if

$$\max_{A\in\mathcal{A}} \left\{ \sum_{i=1}^{N-1} a_{ii} - a_{nn} \right\} = 0.$$

So, since $a_{nn} \neq 0$,

$$\sum_{i=1}^{N-1} a_{ii} - a_{nn} = a_{nn} \left(\frac{\operatorname{tr}(A)}{a_{nn}} - 2 \right),$$

and by observing that for all $A \in \mathcal{S}_N$ we have $\min \sigma(A) \leq a_{ii} \leq \max \sigma(A)$ for all i, we get the following result.

Lemma 3.3 *Let J be a subset of $[\lambda, \Lambda]^N$ which is invariant with respect to permutations of coordinates. Let $\mathcal{A}$ be the corresponding set of matrices whose eigenvalues are in J. Then we have*

$$N_J^+ = N_{\mathcal{A}}^+ = \min_{A\in\mathcal{A}} \frac{\operatorname{tr}(A)}{\max \sigma(A)} \leq \max_{A\in\mathcal{A}} \frac{\operatorname{tr}(A)}{\min \sigma(A)} = N_{\mathcal{A}}^- = N_J^-$$

(of course the first two of these are valid if $N^+_{\mathcal{A}}$ is defined), or, equivalently,

$$N^+_{\mathcal{A}} = N^+_J = \min_{x\in J} \frac{\sum_{i=1}^N x_i}{\max x_i} \leq \max_{x\in J} \frac{\sum_{i=1}^N x_i}{\min x_i} = N^-_J = N^-_{\mathcal{A}}.$$

We now define the numbers $N(\mathcal{H})$ which appear in Theorem 1.2.

Definition 3.1 *For any $\mathcal{H}[u] = H(M,p,u,x)$ satisfying $(\widetilde{H}_0)-(\widetilde{H}_2)$ we set*

$$\mathcal{J}^\pm_H = \{J \in \mathcal{J} \;:\; \mathcal{M}^\pm_J(M) \leq H(M,0,0,x) \text{ for all } M \in \mathcal{S}_N, x \in \overline{\Omega}\}$$

(note $\mathcal{J}^+_H$ can be empty but $[\lambda,\Lambda]^N$ is always in $\mathcal{J}^-_H$), and

$$N(\mathcal{H}) = \min\left\{ \min_{J\in\mathcal{J}^+_H} N^+_J, \min_{J\in\mathcal{J}^-_H} N^-_J \right\}.$$

4 Liouville results. Proof of Theorem 1.3 I

We start by proving a version, for the extremal operators of the previous section, of the Hadamard Three Spheres Theorem which can be found in [21]. For completeness we give the proof here.

For a given $u \in C(B_R)$, we define $m_u(r) = \min_{|x|\leq r} u(x)$, for $0 \leq r < R$. Theorem 2.2 clearly implies that if $\mathcal{M}^\pm_J(D^2u) \leq 0$ then $m_u(r) = \min_{|x|=r} u(x)$.

Theorem 4.1 *Let $J \in \mathcal{J}$ and set $N^+_J = \min_{x\in J} \frac{\sum_{i=1}^N x_i}{\max x_i}$. If $u \in C(B_R)$ is a viscosity solution of*

$$\mathcal{M}^+_J(D^2u) \leq 0, \quad in \quad B_R, \tag{15}$$

then for any $0 < r_1 < r < r_2 < R$ we have

$$\begin{aligned} m(r) &\geq \frac{m(r_1)(r^{2-N^+_J} - r_2^{2-N^+_J}) + m(r_2)(r_1^{2-N^+_J} - r^{2-N^+_J})}{r_1^{2-N^+_J} - r_2^{2-N^+_J}} && \text{if} \quad N^+_J \neq 2 \\ m(r) &\geq \frac{m(r_1)\log(r_2/r) + m(r_2)\log(r/r_1)}{\log(r_2/r_1)} && \text{if} \quad N^+_J = 2. \end{aligned} \tag{16}$$

The same result holds for $\mathcal{M}^-_J$, replacing N^+_J by N^-_J.

Proof. Let us define

$$\phi(x) = \begin{cases} C_1|x|^{2-N_J^+} + C_2 & \text{if} \quad N_J^+ \neq 2 \\ C_1 \log(|x|) + C_2 & \text{if} \quad N_J^+ = 2, \end{cases}$$

where C_1 and C_2 are such that $\phi(x) = m(r_1)$ on ∂B_{r_1} and $\phi(x) = m(r_2)$ on ∂B_{r_2}. It is trivial to check that $C_1 > 0$ if $N_J^+ > 2$ and $C_1 < 0$ if $N_J^+ \leq 2$. Since $\mathcal{M}_{\mathcal{J}}^+(D^2\phi) = 0$ in $\overline{B}_{r_2} \setminus B_{r_1}$ (here we have to recall that $\mathcal{M}_J^+(-M) = -\mathcal{M}_J^-(M)$, and to note that when $N_J^+ < 2$ we have $\alpha_1 = 2 - N_J^+$ in the second part of Lemma 3.2), by using Theorem 2.4 we get $u(x) \geq \phi(x)$ in $B_{r_2} \setminus \overline{B}_{r_1}$, which implies the result. □

Proposition 4.1 *Let $J \in \mathcal{J}$ and set $N_J^+ = \min_{x \in J} \dfrac{\sum_{i=1}^N x_i}{\max x_i}$. Let $u \in C(\mathbb{R}^N)$ be a viscosity solution of*

$$\mathcal{M}_J^+(D^2u) \leq 0 \quad in \quad \mathbb{R}^N.$$

Then

1) if $N_J^+ > 2$ and u is positive then the function $r^{N_J^+ - 2} m_u(r)$ is increasing.

2) if $N_J^+ \leq 2$ and u is bounded below then u is constant.

The same result holds for $\mathcal{M}_J^-$, replacing N_J^+ by N_J^- (note once more that $N_J^- > 2$ except for the Laplacian in dimension two).

Proof. 1) First, (16) implies $m(r)(r_1^{2-N_J^+} - r^{2-N_J^+}) \geq m(r_1)(r^{2-N_J^+} - r_2^{2-N_J^+})$, since u is positive. This inequality holds for all r_2, so we can take $r_2 \to \infty$, and conclude, by using $m(r) \geq 0$.

2) We take $r_2 \to \infty$ in (16), and obtain that $m(r) \geq m(r_1)$, therefore the decreasing function $m(r)$ is constant. Then by the strong maximum principle (Theorem 2.2) u is constant. □

Proof of Theorem 1.3 I. We argue by contradiction, and suppose that $u, v \geq 0$ are nontrivial supersolutions to system (10). First, if $N_1 \leq 2$ or $N_2 \leq 2$ we immediately conclude, by the second statement in Proposition 4.1. So we shall suppose $N_1 > 2, N_2 > 2$. For given $r_1 > 0$ fixed, we consider the function

$$g(r) = m_u(r_1/2)\left(1 - \frac{(r - r_1/2)_+^3}{(r_1/2)^3}\right),$$

similarly to [12].

If we define $\phi(x) = g(|x|)$, we have constructed ϕ so that $u \geq \phi$ in $B_{r_1/2}$, $\phi \leq 0 < u$ in $\mathbb{R}^N \setminus B_{r_1}$, and $u(x_0) = \phi(x_0)$ for some x_0 with $|x_0| = r_1/2$. Hence the minimum of $u - \phi$ in $\mathbb{R}^N$ is non-positive and achieved at a point

$\bar{x} = x(r_1)$, such that $r_1/2 \le |\bar{x}| < r_1$. Thus we can use ϕ as test function in the first equation of (10) at $\bar{x}$ (recall Definition 2.1) and get

$$\mathcal{M}_1(D^2\phi(\bar{x})) + v(\bar{x})^q \le 0.$$

Then, by a simple computation and the definition of $\mathcal{M}_1 = \mathcal{M}_{\mathcal{A}_1}$,

$$v(\bar{x})^q \le \frac{3m_u(r_1/2)}{(r_1/2)^3}\left[a_{nn} + \hat{a}\frac{(|\bar{x}| - r_1/2)^+}{|\bar{x}|}\right](|\bar{x}| - r_1/2)^+,$$

for some $A = (a_{ij}) \in \mathcal{A}_1 \subset \mathcal{S}_N^{\lambda,\Lambda}$, where we have set $\hat{a} := \sum_{i=1}^{N-1} a_{ii}$. If $|\bar{x}| = r_1/2$, then we get $v(|\bar{x}|) = 0$, which is impossible. Thus, we necessarily have $r_1/2 < \bar{x} < r_1$. Noting that the expression in the large brackets is bounded, by $m_v(r_1) \le v(\bar{x})$ we get

$$(m_v(r_1))^q \le C\frac{m_u(r_1/2)}{(r_1/2)^2},$$

for some positive constant C. Now we use the first statement in Proposition 4.1 to obtain

$$(m_v(r_1))^q \le C\frac{m_u(r_1)}{(r_1)^2}, \tag{17}$$

with C possibly different, but independent of r_1. Arguing in the same way we get from the second equation in (10) that

$$(m_u(r_1))^p \le C\frac{m_v(r_1)}{(r_1)^2}. \tag{18}$$

Thus, by combining the last two inequalities we conclude that

$$m_v(r_1) \le C_1\frac{1}{(r_1)^{2(p+1)/(pq-1)}}, \quad \text{and} \quad m_u(r_1) \le C_2\frac{1}{(r_1)^{2(q+1)/(pq-1)}}.$$

Hence

$$r_1^{N_2-2}m_v(r_1) \le C_1\frac{1}{(r_1)^{\alpha_2-(N_2-2)}}\,, \tag{19}$$

$$r_1^{N_1-2}m_u(r_1) \le C_2\frac{1}{(r_1)^{\alpha_1-(N_1-2)}}\,, \tag{20}$$

where $\alpha_1 = 2(q+1)/(pq-1)$ and $\alpha_2 = 2(p+1)/(pq-1)$.

Therefore if $\alpha_2 > N_2 - 2$ or $\alpha_1 > N_1 - 2$ then one of increasing functions $r_1^{N_2-2}m_v(r_1)$ or $r_1^{N_1-2}m_u(r_1)$ goes to 0 as $r_1 \to \infty$, providing a contradiction.

In case, say, $\alpha_2 = N_2 - 2$ we need a supplementary logarithmic lower bound. In this case we define, for fixed $0 < r_1 < r_2$,

$$h(r) = c_1 \frac{\log(1+r)}{r^{N_2-2}} + c_2,$$

where $c_1 > 0$ (to be chosen later) and $c_2 \in \mathbb{R}$ are such that $h(r_1) \leq m(r_1)$ and $h(r_2) = m(r_2)$. More specifically, we take

$$0 < c_1 \leq (m(r_1) - m(r_2)) \,/\, \left[\frac{\log(1+r_1)}{r_1^{N_2-2}} - \frac{\log(1+r_2)}{r_2^{N_2-2}} \right],$$

$$c_2 = m(r_2) - c_1 \frac{\log(1+r_2)}{r_2^{N_2-2}}.$$

We may choose and fix r_1 large enough so that $h''(r) > 0$ and $h'(r) < 0$, for $r > r_1$. Let $w(x) = h(|x|)$, then

$$\mathcal{M}_2(D^2 w(x)) = a_{nn} h''(r) + \hat{a} \frac{h'(r)}{r}, \quad r_1 \leq r \leq r_2,$$

for some $A = (a_{ij}) \in \mathcal{A}_2$ (which may depend on r).

Then, if $\mathcal{M}_2 = \mathcal{M}^+_{\mathcal{A}_2}$ is a supremum operator by its definition we have

$$\mathcal{M}_2(D^2 w(x)) \geq a^*_{nn} h''(r) + \hat{a}^* \frac{h'(r)}{r},$$

where $A^* = (a^*_{ij}) \in \mathcal{A}_2$ is the matrix at which $N_2 = N_{\mathcal{A}_2}$ is achieved (recall Lemma 3.3). We recall that $N_2 - 1 = \hat{a}^*/a^*_{nn}$. A simple computation gives

$$h''(r) + (N_2 - 1) \frac{h'(r)}{r} \geq -C \frac{c_1}{|x|^{N_2}},$$

for some positive constant C, hence

$$\mathcal{M}_2(D^2 w(x)) \geq -C \frac{c_1}{|x|^{N_2}}. \tag{21}$$

If $\mathcal{M}_2 = \mathcal{M}^-_{\mathcal{A}_2}$ is an infimum operator the optimality condition is reversed. So we use a different argument to get (21). In case $h''(r) + \frac{\hat{a}}{a_{nn}} h'(r)/r \geq 0$ we have trivially (21). In case $h''(r) + \frac{\hat{a}}{a_{nn}} h'(r)/r < 0$, since $\frac{\hat{a}}{a_{nn}} \leq N_2 - 1$, $N_2 = N^-_{\mathcal{A}_2}$, we have

$$a_{nn} h''(r) + \hat{a} \frac{h'(r)}{r} \geq a_{nn} (h''(r) + (N_2 - 1) \frac{h'(r)}{r}),$$

from which (21) follows.

On the other hand, by Proposition 4.1 we have

$$m_v(x) \geq \frac{m_v(r_1) r_1^{N_2-2}}{|x|^{N_2-2}} = \frac{C_1}{|x|^{N_2-2}},$$

for $|x| \geq r_1$, and then from the equation for v, (17) and $\alpha_2 = N_2 - 2$ we get

$$\mathcal{M}_2(D^2 v(x)) \leq -m_u^p(x) \leq -|x|^{2p} m_v^{pq} \leq -\frac{C_2}{|x|^{N_2}},$$

for some positive C_2. Therefore, we can use Theorem 2.4, choosing a smaller c_1 if necessary, to conclude $w(x) \leq v(x)$ in $B_{r_2} \setminus B_{r_1}$. Letting $r_2 \to \infty$ we finally conclude that $c_2 \geq 0$ at the limit and

$$v(x) \geq \frac{C \log(1 + |x|)}{|x|^{N_2-2}},$$

for large $|x|$ and some positive C. This is a contradiction with (19).

In the case $\alpha_1 = N_1 - 2$, we can argue in the same way. This proves that system (10) has no positive supersolutions provided (11) holds.

It remains to construct a super-solution of (10), when

$$\alpha_2 < N_2 - 2 \quad \text{and} \quad \alpha_1 < N_1 - 2.$$

Note the solution of the linear system $s + 1 = pt$, $t + 1 = qs$ is $t = \frac{\alpha_1}{2}$ and $s = \frac{\alpha_2}{2}$. So for these s, t we have

$$s < \frac{N_2 - 2}{2} \quad \text{and} \quad t < \frac{N_1 - 2}{2}$$

We define the functions

$$v(r) = A(1 + r^2)^{-s} \quad \text{and} \quad u(r) = B(1 + r^2)^{-t},$$

and claim that (u, v) is a radial super-solution of (10), for an appropriate choice of the positive constants A and B. We observe that $v''(r) \geq v'(r)/r$ for all $r > 0$, and for the function $w(x) = v(|x|)$ we have

$$\mathcal{M}_2(D^2 w(x)) = a_{nn}(v''(r) + \frac{\hat{a}}{a_{nn}} \frac{v'(r)}{r}),$$

for some $A = (a_{ij}) \in \mathcal{A}_2$. In the case $\mathcal{M}_2$ is a supremum operator we have as above $\frac{\hat{a}}{a_{nn}} \geq N_2 - 1$, by the definition of N_2. We also recall that $\lambda \leq a_{nn} \leq \Lambda$. Then using $v' \leq 0$, we get

$$\mathcal{M}_2(D^2 w(x)) \leq \bar{C}(v''(r) + (N_2 - 1)\frac{v'(r)}{r}).$$

In the case of an infimum operator we get the same just from the definition of $\mathcal{M}_2$. Thus, by a simple computation,

$$\mathcal{M}_2(D^2w) + u^p \le -C_1 \frac{A(N_2 - 2(s+1))}{(1+r^2)^{s+1}} + \frac{B^p}{(1+r^2)^{pt}}.$$

By arguing in the same way we see that $z(x) = u(|x|)$ satisfies

$$\mathcal{M}_1(D^2z) + v^q \le -C_2 \frac{B(N_1 - 2(t+1))}{(1+r^2)^{t+1}} + \frac{A^p}{(1+r^2)^{qs}}.$$

Recall that by the definition of s and t we have $N_2 - 2(s+1) > 0$, $N_1 - 2(t+1) > 0$, and $s+1 = pt$, $t+1 = qs$. Finally, we remark that $pq > 1$ permits to choose A and B such that the right hand sides of the last two inequalities are equal to zero. □

5 A Liouville theorem in a half-space

In this section we prove the second part of Theorem 1.3. We shall actually need a stronger variant of this theorem.

We use an idea by Dancer [13], which consists in the following : if there is a solution of the problem in $\{x_N > 0\}$, and if one is able to show that any such solution is increasing in the x_N-direction, then, after eventually some supplementary work, one should be able to pass at the limit as $x_N \to \infty$ and thus get a solution of the same problem in $\mathbb{R}^{N-1}$, which in turn permits to use the nonexistence result for the whole space, that we already proved. Note that the numbers N_1, N_2 from Theorem 1.3 (which we defined in the previous sections) are strictly increasing in N, so the nonexistence results in $\mathbb{R}^N_+$ hold for a larger range of p, q than the nonexistence result in $\mathbb{R}^N$.

Monotonicity results for systems of two equations were proved in [17]. The approach there relies on a moving planes argument and on some extensions of the Harnack-Krylov-Safonov estimates for nonlinear elliptic systems obtained in [5]. Note that the moving planes method was recently extended to viscosity solutions and fully nonlinear equations in [14].

When trying to get extensions of the Harnack estimates from [5], which are needed in the moving planes argument, a crucial role is played by an extension to viscosity solutions of a basic "quantitative strong maximum principle" of Krylov and Safonov, which was one of the cornerstones of their theory of linear equations in nondivergence form, see [25]. We give such an extension in the appendix (Theorem 7.1), see also the comments there. We use this result also in Section 6.

Here are the precise monotonicity and (non-)existence statements for systems in a half-space. Suppose we have an autonomous system of the type

$$\left\{ \begin{array}{rcl} F_1(D^2u_1) + f_1(u_1,u_2) & = & 0 \\ F_2(D^2u_2) + f_2(u_1,u_2) & = & 0, \end{array} \right. \tag{22}$$

where F_1, F_2 satisfy (14), $f_i \in C^1(\mathbb{R}^2), i = 1,2$, and f_i is increasing in u_j, $i \neq j$. We shall suppose in addition that we can write

$$f_1(u_1,u_2) = u_2^p + g_1(u_1,u_2)u_1, \qquad f_2(u_1,u_2) = u_1^q + g_2(u_1,u_2)u_2, \tag{23}$$

for some $p, q \geq 1$ and some nonnegative continuous functions g_1, g_2, which have polynomial growth in u_1, u_2.

Let J_1, J_2 be the subsets at which the dimension-like numbers for F_1 and F_2 are attained, see Definition 3.1. Set $\widetilde{N}_1 = N(F_1)$, $\widetilde{N}_2 = N(F_2)$. Let $\mathcal{M}_{J_1}$, $\mathcal{M}_{J_2}$ be the corresponding extremal operators. We consider the system

$$\left\{ \begin{array}{rcl} \mathcal{M}_{J_1}(D^2u_1) + u_2^p & \leq & 0 \\ \mathcal{M}_{J_2}(D^2u_2) + u_1^q & \leq & 0. \end{array} \right. \tag{24}$$

Theorem 5.1 *Suppose we have a system of type (22) which satisfies (23). If problem (24) has no positive bounded solutions in $\mathbb{R}^{N-1}$, then (22) has no nontrivial nonnegative bounded solutions in $\mathbb{R}^N_+$ which vanish on $\{x_N = 0\}$.*

In the case when F_1 and F_2 are the Laplacian, this theorem appeared in [17]. Its proof is divided into two parts. In the first part we prove that all bounded solutions of (22) in a half-space are strictly increasing in the x_N-direction, and in the second part we pass to the limit as $x_N \to \infty$.

The monotonicity result is proved like in [17], once we have the corresponding Harnack estimates for fully nonlinear systems, together with a variant of the moving planes method for viscosity solutions. We are going to sketch the arguments in Section 5.2, for completeness.

On the other hand, the passage to the limit cannot be done like in [17], since the approach there, based on multiplication by cut-off functions and integration, is not applicable to operators in non-divergence form. We give a different and simpler proof in Section 5.3.

5.1 Harnack type estimates for systems

The argument in the next section requires some Harnack estimates for systems which we state in this section. Such results were obtained in [5]. The first theorem below is a particular case of Theorem 3.2 and Proposition 3.1 in [5]. We include it here for the reader's convenience.

In this section G denotes an arbitrary domain in $\mathbb{R}^N$ and Q_l $(l = 1,2)$ are concentric cubes with side l, properly included in G.

Theorem 5.2 ([5]) *Assume $f_1(u_1,u_2), f_2(u_1,u_2)$ are globally Lipschitz continuous functions, with Lipschitz constant A, such that f_i is nondecreasing in u_j for $i \neq j$. Let (u_1,u_2) be a nonnegative solution of (22) in a domain G. We suppose that the system is fully coupled, in the sense that $f_1(0,v) > 0$ for all $v > 0$, and $f_2(u,0) > 0$ for $u > 0$. Then for any cube Q in G there exists a function $\Phi(t)$ (depending on $N, \lambda, \Lambda, A, Q$ and G), continuous on $[0,\infty)$, such that $\Phi(0) = 0$ and*

$$\sup_{x\in Q} \max\{u_1,u_2\} \leq \Phi(\inf_{x\in Q} \min\{u_1,u_2\}).$$

In particular, if any of u_1, u_2 vanishes at one point in G then both u_1 and u_2 vanish identically in G.

Further, if $Q_1 \subset Q_2 \subset G$, and u is a subsolution of (22) then for each $p > 0$ there exists a constant C depending only on p, N, λ, Λ, and A such that

$$\sup_{x\in Q_1} \max\{u_1,u_2\} \leq C\| \max\{u_1,u_2\}\|_{L^p(Q_2)}$$

The same results hold if f_i depend on x and the constant A is uniform in x.

Next, we state two Harnack inequality for systems, which play an important role in the moving planes argument. The proofs of these theorems are the same as the proofs of Theorems 3.6 and 1.4 from [17], making essential use of Theorem 7.1 in the Appendix.

Theorem 5.3 *Let F_1, F_2 satisfy (14). Suppose the functions $a, b, c, d \in L^\infty(Q_2)$ are such that $|a|, |d| \leq A$, and $0 \leq b \leq A, 0 \leq c \leq A$ in Q_2. Suppose (u_1,u_2) is a positive solution of*

$$\begin{cases} F_1(D^2u_1) + a(x)u_1 + b(x)u_2 &= 0 \\ F_2(D^2u_2) + c(x)u_1 + d(x)u_2 &= 0 \end{cases}$$

in Q_2. Assume in addition that $b(x)$ is bounded below by a positive constant on Q_1. Then

$$\sup_{x\in Q_1} u_1 \leq C \inf_{x\in Q_1} u_1. \tag{25}$$

where the constant C depends on N, A, λ, Λ, and on upper bound for $\dfrac{\sup_{Q_2} b}{\inf_{Q_1} b}$.

Theorem 5.4 *Let (u_1,u_2) be a positive solution of (22) in some domain G and suppose (23) holds. Suppose K is a compact set properly included in G and $\max\left\{\inf_{x\in K} u_1, \inf_{x\in K} u_2\right\} \leq 1$, $\max\left\{\sup_{x\in G} u_1, \sup_{x\in G} u_2\right\} \leq M$. Then*

$$\sup_{x\in K} \max\{u_1,u_2\} \leq C \min\left\{\left(\inf_{x\in K} u_1\right)^{\frac{1}{p}}, \left(\inf_{x\in K} u_2\right)^{\frac{1}{q}}\right\},$$

where C depends only on $N, \lambda, \Lambda, M, K, G$.

5.2 A monotonicity result

This section is devoted to the following theorem.

Theorem 5.5 *Suppose we have a nontrivial nonnegative* bounded *solution (u_1, u_2) of system (22) in $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N \mid x_N > 0\}$, such that $u_1 = u_2 = 0$ on $\partial\mathbb{R}^N_+$. Suppose (23) is satisfied. Then*

$$\frac{\partial u_i}{\partial x_N} > 0 \qquad in \;\; \mathbb{R}^N_+, \quad i = 1, 2. \tag{26}$$

In the the proof of this theorem we shall use an extension to viscosity solutions of fully nonlinear systems a maximum principle in narrow domains, proved by Cabre [6] in the case of strong solutions of a scalar equation.

We recall the following definition of [6]. For a given domain $\Omega \subset \mathbb{R}^N$, the quantity $R(\Omega)$ is defined to be the smallest positive constant R such that

$$\text{meas}\,(B_R(x) \setminus \Omega) \geq \frac{1}{2}\,\text{meas}\,(B_R(x))\,, \qquad \text{for all} \;\; x \in \Omega.$$

If no such radius R exists, we define $R(\Omega) = +\infty$. It is easy to see that whenever the domain Ω is contained between two parallel hyperplanes at a distance d, we have $R(\Omega) \leq 2^N d\omega_N^{-1}$, where ω_N is the volume of the unit ball in $\mathbb{R}^N$.

We have the following easy extension of the results in [6] (see also [7]).

Proposition 5.1 *Let Ω be a domain with $R(\Omega) < \infty$. Suppose $u \in C(\Omega)$ and $f \in L^\infty(\Omega)$ satisfy $\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \gamma|Du| - \delta u \geq f$ in Ω, for some $\gamma, \delta \geq 0$, and $\sup_\Omega u < \infty$. Then*

$$\sup_\Omega u \leq \limsup_{x \to \partial\Omega} u(x) + CR(\Omega)^2 ||f||_{L^\infty(\Omega)},$$

where C is a constant depending only on $N, \lambda, \Lambda, \gamma, \delta, R(\Omega)$, and C is bounded when these quantities are bounded.

The proof of this result is identical to the proof of Theorem 5.3 in [7], by using the weak Harnack inequality for viscosity solutions.

It is not difficult to deduce from Proposition 5.1 a maximum principle for systems in domains with small $R(\Omega)$.

Theorem 5.6 *Let $H_i(M, p, u_1, u_2, x)$ be operators satisfying $(\widetilde{H}_2)$. Then there exists a number $\bar{R}$ depending only on $N, \lambda, \Lambda, \gamma, \delta,$, such that $R(\Omega) \le \bar{R}$ implies that each solution $u \in C(\overline{\Omega})$ of*

$$\begin{cases} H_1(D^2u_1, Du_1, u_1, u_2, x) & \ge & 0 & in & \Omega \\ H_2(D^2u_2, Du_2, u_1, u_2, x) & \ge & 0 & in & \Omega \\ u_1 \le 0, \quad u_2 & \le & 0 & on & \partial\Omega, \end{cases} \tag{27}$$

satisfies $u_i \le 0$ in Ω, $i = 1, 2$.

Proof. From $\widetilde{H}_2$ and (27) we get

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u_i) + \gamma|Du_i| - \delta u_i \ge -2\delta u_i^+ - \delta u_j^+, \quad i = 1, 2, \quad j \ne i.$$

By applying Proposition 5.1 to these equations we obtain

$$\sup_\Omega u_i^+ \le C_1 R(\Omega)^2 (\sup_\Omega u_1^+ + \sup_\Omega u_2^+), \quad i = 1, 2.$$

We sum up these two equations and take $\bar{R} = (\sqrt{4C_1})^{-1}$. □

Proof of Theorem 5.5. Set $M = \max\left\{\sup_{\mathbb{R}^N_+} u_1, \sup_{\mathbb{R}^N_+} u_2\right\}$. We can suppose that the functions f_1 and f_2 are globally Lipschitz continuous. Indeed, if they are not, we can replace them by $f_1\varphi$ and $f_2\varphi$, where φ is a cut-off function such that $\varphi = 1$ on the positive cube with side M, and $\varphi = 0$ outside a cube with side $M + 1$, containing properly the previous one.

Hence system (22) satisfies the hypotheses of Theorem 5.2, from which we deduce that either both functions u_1 and u_2 vanish identically on $\mathbb{R}^N_+$ or both u_1 and u_2 are strictly positive on $\mathbb{R}^N_+$. The first case is excluded by hypothesis. So we can assume that u_1, u_2 are strictly positive in $\mathbb{R}^N_+$.

For each $\lambda > 0$ we denote $T_\lambda = \{x \in \mathbb{R}^N \mid x_N = \lambda\}$, $\Sigma_\lambda = \{x \in \mathbb{R}^N \mid 0 < x_N < \lambda\}$, and introduce the functions

$$v_i^{(\lambda)}(x) = u_i(x', 2\lambda - x_N), \qquad w_i^{(\lambda)}(x) = v_i^{(\lambda)}(x) - u_i(x), \qquad i = 1, 2,$$

defined in Σ_λ. Since both (u_1, u_2) and $(v_1^{(\lambda)}, v_2^{(\lambda)})$ satisfy system (22) we obtain by subtracting the corresponding equations and by using Proposition 2.1 from [14] (recall u_1, u_2 are only continuous)

$$\begin{cases} \mathcal{M}^-_{\lambda,\Lambda}(D^2w_1^{(\lambda)}) + c_{11}^{(\lambda)}(x)w_1^{(\lambda)} + c_{12}^{(\lambda)}(x)w_2^{(\lambda)} & \le & 0 \\ \mathcal{M}^-_{\lambda,\Lambda}(D^2w_2^{(\lambda)}) + c_{21}^{(\lambda)}(x)w_1^{(\lambda)} + c_{22}^{(\lambda)}(x)w_2^{(\lambda)} & \le & 0 \end{cases} \tag{28}$$

in Σ_λ, where $c_{ij}^\lambda(x)$ is the partial derivative of f_i with respect to u_j, evaluated at some point between $u_j(x)$ and $v_j^{(\lambda)}(x)$. Note that $c_{ij}^{(\lambda)}$ are bounded by a Lipschitz constant of $\vec{f} = (f_1, f_2)$ on $[0, M]^2$, and $c_{12}^{(\lambda)}, c_{21}^{(\lambda)} \geq 0$.

Obviously $\vec{w}^{(\lambda)} = (w_1^{(\lambda)}, w_2^{(\lambda)}) \equiv 0$ on T_λ and $\vec{w}^{(\lambda)} > 0$ on T_0 (recall that $u_i = 0$ on T_0 and $u_i > 0$ on T_λ, $\lambda > 0$). By Theorem 5.6, if λ is small enough, then $\vec{w}^{(\lambda)} \geq 0$ in Σ_λ. Hence

$$\lambda^* = \sup\{\lambda \mid \vec{w}^{(\mu)} \geq 0 \ \text{ in } \ \Sigma_\mu, \ \forall\, \mu < \lambda\} > 0.$$

We see that for each $0 < \lambda \leq \lambda^*$ the function $w_i^{(\lambda)} \geq 0$ satisfies the inequality $\mathcal{M}^-_{\lambda,\Lambda}(D^2 w_i^{(\lambda)}) + c_{ii}^{(\lambda)} w_i^{(\lambda)} \leq 0$ in Σ_λ. Hence Hopf's lemma (Theorem 2.2) implies $w_i^{(\lambda)} > 0$ and $\frac{\partial u_i}{\partial x_N} = -\frac{1}{2}\frac{\partial w_i^{(\lambda)}}{\partial x_N} > 0$ on T_λ. Therefore, the theorem is proved if we show that $\lambda^* = +\infty$.

Suppose for contradiction that λ^* is finite. By Theorem 5.6 we can fix $\varepsilon_0 > 0$ such that the matrix operator $\mathcal{M}^-_{\lambda,\Lambda} + C_\lambda(x)$ satisfies the maximum principle in the domain $\Sigma_{\lambda^*+\varepsilon_0} \setminus \Sigma_{\lambda^*-\varepsilon_0}$ (here $C_\lambda(x)$ denotes the matrix of the coefficients in (28)). For instance, we can take $\varepsilon_0 = \frac{\omega_N}{2^{N+1}}\overline{R}$, where $\overline{R}$ is the number from Theorem 5.6.

Lemma 5.1 *There exists $\delta_0 \in (0, \varepsilon_0]$, such that for each $\delta \in (0, \delta_0)$ we have*

$$w_i^{(\lambda^*+\delta)} \geq 0 \qquad \text{in } \ \Sigma_{\lambda^*-\varepsilon_0} \setminus \Sigma_{\varepsilon_0}, \qquad i = 1, 2.$$

The proof of this lemma is the same as the proof of Lemma 3.1 in [17]. Then we can apply Theorem 5.6 to (28) in $\Sigma_{\lambda^*+\delta} \setminus \Sigma_{\lambda^*-\varepsilon_0}$ and in Σ_{ε_0} (these domains are narrow enough) to conclude that $w_i^{(\lambda^*+\delta)} \geq 0$ in $\Sigma_{\lambda^*+\delta}$ for each $\delta \in (0, \delta_0)$. This contradicts the maximal choice of λ^*. □

5.3 Proof of Theorem 5.1

Suppose $u = (u_1, u_2)$ is a solution of (22), $u \not\equiv 0$, $0 \leq u_1, u_2 \leq M$. For each $x = (y, x_N)$ in the strip $\Sigma_1 = \{x \in \mathbb{R}^N \ : \ 0 < x_N < 1\}$ we set

$$u_i^{(n)}(y, x_N) = u_i(y, x_N + n), \quad i = 1, 2.$$

Now $u^{(n)}$ satisfies the same system as u, since (22) is autonomous. Then, using once more the C^α-regularity, Theorem 2.6, we see that $\{u_n\}$ is bounded in C^α and hence a subsequence of it converges uniformly on compact subsets of Σ_1 to a vector $\widetilde{u}$. By Theorem 2.5 this vector satisfies

$$\begin{cases} F_1(D^2\widetilde{u}_1) + f_1(\widetilde{u}_1, \widetilde{u}_2) &= 0 \quad \text{in } \Sigma_1 \\ F_2(D^2\widetilde{u}_2) + f_2(\widetilde{u}_1, \widetilde{u}_2) &= 0 \quad \text{in } \Sigma_1, \end{cases} \tag{29}$$

so, by the definition of $\mathcal{M}_{J_1}, \mathcal{M}_{J_2}$ and the hypotheses on f_i,

$$\left\{ \begin{array}{lcll} \mathcal{M}_{J_1}(D^2\widetilde{u}_1) + \widetilde{u}_2^p & \leq & 0 & \text{in } \Sigma_1 \\ \mathcal{M}_{J_2}(D^2\widetilde{u}_2) + \widetilde{u}_1^q & \leq & 0 & \text{in } \Sigma_1. \end{array} \right. \tag{30}$$

However, the monotonicity result of Theorem 5.5 trivially implies that $\widetilde{u}$ is strictly positive and independent of the x_N-variable. This means that the last line and column of $D^2\widetilde{u}_1, D^2\widetilde{u}_2$ contain only zeros, so the N-dimensional extremal operators $\mathcal{M}_{J_1}, \mathcal{M}_{J_2}$ applied to these matrices are actually $(N-1)$-dimensional, and we have (30) in $\mathbb{R}^{N-1}$.

6 Proof of Theorem 1.2

6.1 The setting

The proof of our existence theorem is an application of degree theory for compact operators in cones. This theory, essentially developed by Krasnoselskii, has often been used to show that such operators possess fixed points. We are going to use an extension of Krasnoselskii results, due to Benjamin and Nussbaum, in the form that appeared in [16].

We start by recalling the abstract setting in [16]. Let K be a closed cone with non-empty interior in the Banach space $(E, \|\cdot\|)$. Let $\Phi : K \to K$ and $\Psi : K \times [0, \infty) \to K$ be compact operators such that $\Phi(0) = 0$ and $\Psi(x, 0) = \Phi(x)$ for all $x \in K$. Then the following theorem holds (see Proposition 2.1 and Remark 2.1 in [16]).

Theorem 6.1 *Assume there exist numbers $R_1 > 0, R_2 > 0$ and $T > 0$ such that $R_1 \neq R_2$, and*

(i) $x \neq \beta\Phi(x)$ for all $0 \leq \beta \leq 1$ and $\|x\| \leq R_1$,

(ii) $\Psi(x, t) \neq x$ for all $\|x\| = R_2$ and all $t \in [0, +\infty)$,

(iii) $\Psi(x, t) \neq x$ for all $x \in \overline{B_{R_2}}$ and all $t \geq T$.

Then Φ has a fixed point $x \in K$ such that $\|x\|$ is between R_1 and R_2.

Note that (i) implies that $i_C(\Phi, B_{R_1}) = 1$, while (ii) and (iii) imply $i_C(\Phi, B_{R_2}) = 0$, where i_C is the Krasnoselskii index and $B_R = \{x \in K : \|x\| < R\}$, so Theorem 6.1 follows from the excision property of the index.

We set $E = \{u = (u_1, u_2) \in C(\overline{\Omega}) \times C(\overline{\Omega}) \mid u_i = 0 \text{ on } \partial\Omega,\ i = 1, 2\}$ and $K = \{u \in E \,|\, u_i \geq 0 \text{ in } \Omega,\ i = 1, 2\}$. It is clear that solving (1) is equivalent to finding a fixed point in K of $\Phi : K \to K$, defined by

$$\Phi(u_1, u_2)(x) := S(f_1(u_1, u_2, x), f_2(u_1, u_2, x)), \quad x \in \Omega,$$

where for any $(h_1, h_2) \in K$ we define $S(h_1, h_2)$ as the solution of the Dirichlet problem

$$\begin{cases} -H_1(D^2u_1, Du_1, u_1, u_2, x) &= h_1(x) \quad \text{in} \quad \Omega \\ -H_2(D^2u_2, Du_2, u_1, u_2, x) &= h_2(x) \quad \text{in} \quad \Omega \\ u_1 = u_2 &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{31}$$

Lemma 6.1 *The operator $S : K \to K$ is well defined, continuous and compact. In addition, $S(0,0) = (0,0)$.*

Proof. It follows from the results in [33] (which extend the earlier results in [5], [31], [32]) that under $(\widetilde{H}_0)$-$(\widetilde{H}_3)$ system (31) is uniquely solvable for any $h \in L^N(\Omega)^2$ and satisfies the maximum principle, that is, $h \geq (\leq)0$ implies $u \geq (\leq)0$ in Ω. Hence S is well defined and $S(0,0) = (0,0)$. In addition, the Alexandrov-Bakelman-Pucci estimate (which extends Theorem 2.1 to systems)

$$\|u_i\|_{L^\infty(\Omega)} \leq C \max_{i=1,2} \|H_i\|_{L^N(\Omega)} \tag{32}$$

is valid (here C depends on $N, \lambda, \Lambda, \gamma, \delta$, diam$(\Omega)$ and $\lambda_1^+(F) > 0$). Hence S is continuous.

Further, it follows from (32) and from the C^α-estimates, Theorem 2.6, that if u_n is a sequence of solutions of (31) then u_n is uniformly bounded in $C^\alpha(\Omega)$, for some $\alpha \in (0,1)$. Therefore the compactness of S follows from the compactness of the embedding $C^\alpha(\Omega) \hookrightarrow C(\overline{\Omega})$. $\square$

In our case we define the operator Ψ as follows : for any $u \in K$, $t \in [0, \infty]$,

$$\Psi(u_1, u_2, t)(x) = S(f_1(u_1 + t, u_2 + t, x), f_2(u_1 + t, u_2 + t, x)).$$

First we show that condition (i) in Theorem 6.1 is satisfied. This is the content of the following proposition.

Proposition 6.1 *There is $R_1 > 0$ so that for all $t \in [0,1]$ the system*

$$\begin{cases} -H_1(D^2u_1, Du_1, u_1, u_2, x) &= tf_1(u_1, u_2, x) \quad \text{in} \quad \Omega \\ -H_2(D^2u_2, Du_2, u_1, u_2, x) &= tf_2(u_1, u_2, x) \quad \text{in} \quad \Omega \\ u_1 = u_2 &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{33}$$

has no solution (u_1, u_2) with $0 < \|u\| := \max(\|u_1\|_{L^\infty(\Omega)}, \|u_2\|_{L^\infty(\Omega)}) \leq R_1$.

Proof. We argue by contradiction. Let $\{(u_1^{(n)}, u_2^{(n)}, t_n)\}_{n\in\mathbb{N}}$ be a sequence of positive solutions to (33) such that $\|u_i^{(n)}\|_{L^\infty(\Omega)} \to 0$ as $n \to +\infty$, $i = 1, 2$, and $t_n \in [0,1]$. Define

$$v_i^{(n)}(x) = \frac{u_i^{(n)}(x)}{\|u^{(n)}\|}.$$

Then we have, by $(\widetilde{H}_0)$,

$$
\begin{aligned}
-H_1(D^2 v_1^{(n)}, Dv_1^{(n)}, v_1^{(n)}, v_2^{(n)}, x) &= \frac{t_n}{\|u^{(n)}\|} f_1(u_1^{(n)}, u_2^{(n)}, x) \\
-H_2(D^2 v_2^{(n)}, Dv_2^{(n)}, v_1^{(n)}, v_2^{(n)}, x) &= \frac{t_n}{\|u^{(n)}\|} f_2(u_1^{(n)}, u_2^{(n)}, x).
\end{aligned}
$$

Note $v_1^{(n)}, v_2^{(n)} \le 1$ in Ω, and $v_i^{(n)}(x_n) = 1$ for some i and some $x_n \in \Omega$. However, the right hand sides of the last two equalities go to zero uniformly in Ω, by hypothesis (6). Then, by Lemma 6.1 $v_i^{(n)}$ converges uniformly to some function v_i. Applying Theorem 2.5 and then Lemma 6.1 yields $v \equiv 0$, a contradiction. □

Remark. Note that if the left hand side of the system is decoupled, that is, H_1 does not depend on u_2 and H_2 does not depend on u_1 (like in Theorem 1.1) then we can allow one of the functions f_i to have a linear growth in u_j, $i \neq j$, since then the equalities

$$
\begin{aligned}
-H_1(D^2 v_1, Dv_1, v_1, x) &= v_2 \\
-H_2(D^2 v_2, Dv_2, v_2, x) &= 0.
\end{aligned}
$$

would still imply $v \equiv 0$. This remark shows that we can allow $p = 1$ or $q = 1$ in Theorem 1.1, which in particular gives an existence result for higher order equations involving iterated fully nonlinear operators. □

In order to prove condition (iii) in Theorem 6.1 we state the following proposition.

Proposition 6.2 *There exists a constant $T > 0$ so that if the system*

$$
\left\{
\begin{aligned}
-H_1(D^2u_1, Du_1, u_1, u_2, x) &= f_1(u_1 + t, u_2 + t, x) \quad in \quad \Omega \\
-H_2(D^2u_2, Du_2, u_1, u_2, x) &= f_2(u_1 + t, u_2 + t, x) \quad in \quad \Omega \\
u_1 = u_2 &= 0 \qquad\qquad\qquad\qquad on \quad \partial\Omega,
\end{aligned}
\right. \tag{34}
$$

possesses a solution $u = (u_1, u_2) \in K$, then $0 \le t \le T$.

Proof. First, by $(\widetilde{H}_2)$, the positivity of f_i and the strong maximum principle (Theorem 2.2) we have $u_i \equiv 0$ in Ω or $u_i > 0$ in Ω, $i = 1, 2$.

By the hypotheses we made on the functions f_1, f_2, for any $A > 0$ we can find $T = T(A) \ge 1$ such that for all $t \ge T$ we have either (case I)

$$
f_1(u_1 + t, u_2 + t, x) \ge A(u_1 + t), \qquad f_2(u_1 + t, u_2 + t, x) \ge A(u_2 + t),
$$

or (case II)

$$
f_1(u_1 + t, u_2 + t, x) \ge A(u_2 + t), \qquad f_2(u_1 + t, u_2 + t, x) \ge a(u_1 + t),
$$

or the last two inequalities with A and a interchanged (here a is a positive lower bound for d_{21} or d_{12} from (4))[3].

In case I we see that the nonpositive functions $v_1 = -u_1$, $v_2 = -u_2$ satisfy

$$\begin{array}{rcl} F_1^*(D^2v_1, Dv_1, v_1, 0, x) & \geq & F_1^*(D^2v_1, Dv_1, v_1, v_2, x) \geq -Av_1 + 1 \\ F_2^*(D^2v_2, Dv_2, 0, v_2, x) & \geq & F_2^*(D^2v_2, Dv_2, v_1, v_2, x) \geq -Av_2 + 1 \end{array} \quad (35)$$

in Ω, by $(\widetilde{H}_2)$ (recall $(\widetilde{H}_2)$ implies the system is quasimonotone). Clearly this implies $v_1, v_2 < 0$ in Ω, by Theorem 2.2. We recall the following corollary from Proposition 4.2 in [32].

Lemma 6.2 *If $F(M, p, t, 0, x)$ satisfies $(\widetilde{H}_0)$-$(\widetilde{H}_2)$ and is convex in (M, p, t) then the quantity*

$$\lambda_1^-(F, \Omega) = \sup\{\lambda \mid \Psi^-(F, \Omega, \lambda) \neq \emptyset\}, \qquad \textit{where}$$

$$\Psi^-(F, \Omega, \lambda) = \{\psi \in C(\overline{\Omega}) \mid \psi < 0 \ \ \textit{in } \Omega, \ F(D^2\psi, D\psi, \psi, x) + \lambda\psi \geq 0 \ \ \textit{in } \Omega\},$$

is bounded by constants which depend only on $N, \lambda, \Lambda, \gamma, \delta, \Omega$.

So (35) is a contradiction, since A can be taken arbitrarily large.

In case II we get, by $(\widetilde{H}_2)$ and $\mathcal{M}^+(M) = -\mathcal{M}^-(-M)$,

$$\begin{array}{rcl} -\mathcal{M}^-_{\lambda,\Lambda}(D^2u_1) + \gamma|Du_1| + \delta u_1 & \geq & Au_2 + A > Au_2, \\ -\mathcal{M}^-_{\lambda,\Lambda}(D^2u_2) + \gamma|Du_2| + \delta u_2 & \geq & au_1 + a > au_1, \end{array}$$

in Ω. This implies $u_1, u_2 > 0$ in Ω, by Theorem 2.2. Fix $\varepsilon > 0$, depending only on the geometry of Ω, such that

$$\Omega' = \{x \in \Omega \ : \ \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$$

is a domain which is not empty, and fix two concentric cubes $Q_1 \subset\subset Q_2 \subset \Omega'$. Then Theorem 7.1 implies

$$\inf_{Q_1} u_1 \geq \kappa A \inf_{Q_1} u_2 \qquad \text{and} \qquad \inf_{Q_1} u_2 \geq \kappa a \inf_{Q_1} u_1,$$

where κ depends only on $N, \lambda, \Lambda, \gamma, \delta$, and Ω. This is clearly a contradiction, if we fix A larger than $(\kappa\sqrt{a})^{-2}$. □

Finally, condition (ii) in Theorem 6.1 is a consequence of the following proposition.

[3]Note that if H_1, H_2 are linear, then it is trivial to conclude at this stage, simply by adding up the two equations and by using known results on scalar inequalities, for the sum $u_1 + u_2$.

Proposition 6.3 *For each t_0 there exists a constant C, such that if $u = (u_1, u_2)$ is a solution of system (34) with $0 \le t \le t_0$, then*

$$\|u\|_{L^\infty(\Omega)} \le C.$$

Proof. We argue by contradiction, using the widely employed blow-up method of Gidas and Spruck [22], [17]. Suppose there exists a sequence $(u_{1,n}, u_{2,n})$ of positive solutions of (34) with $t = t_n \in [0, t_0]$, such that at least one of the sequences $u_{1,n}$ and $u_{2,n}$ tends to infinity in the L^∞-norm. Without restricting the generality we suppose $t_0 = 0$ (the argument below remains the same for $t_0 > 0$). Let β_1, β_2 be the numbers from Theorem 1.2. We set

$$\lambda_n = ||u_{1,n}||_{L^\infty(\Omega)}^{-\beta_1},$$

if $||u_{1,n}||_{L^\infty(\Omega)}^{\beta_2} \ge ||u_{2,n}||_{L^\infty(\Omega)}^{\beta_1}$ (up to a subsequence), and $\lambda_n = ||u_{2,n}||_{L^\infty(\Omega)}^{-\beta_2}$ otherwise. Say we are in the first of these two situations.

Note that we have $\lambda_n \to 0$ as $n \to \infty$. Let $x_n \in \Omega$ be a point where $u_{1,n}$ assumes its maximum. The functions

$$v_{i,n}(x) = \lambda_n^{\beta_i} u_{i,n}(\lambda_n x + x_n),$$

are such that $v_{1,n}(0) = 1$ and $0 \le v_{i,n} \le 1$ in Ω. One easily verifies that the functions $v_{1,n}$ and $v_{2,n}$ satisfy

$$\begin{array}{rcl} -H_{1,n}(D^2 v_{1,n}, Dv_{1,n}, v_{1,n}, v_{2,n}, \cdot) & = & a_1(\cdot)\lambda_n^{\beta_1+2-\beta_1\alpha_{11}}\, v_{1,n}^{\alpha_{11}} + \\ b_1(\cdot)\lambda_n^{\beta_1+2-\beta_2\alpha_{12}}\, v_{2,n}^{\alpha_{12}} & + & c_1(\cdot)\lambda_n^{\beta_1+2-\beta_1\gamma_{11}-\beta_2\gamma_{12}}\, v_{1,n}^{\gamma_{11}} v_{2,n}^{\gamma_{12}} + \widetilde{g}_{1,n} \\ & & \\ -H_{2,n}(D^2 v_{2,n}, Dv_{2,n}, v_{1,n}, v_{2,n}, \cdot) & = & a_2(\cdot)\lambda_n^{\beta_2+2-\beta_1\alpha_{21}}\, v_{1,n}^{\alpha_{21}} + \\ b_2(\cdot)\lambda_n^{\beta_2+2-\beta_2\alpha_{22}}\, v_{2,n}^{\alpha_{22}} & + & c_2(\cdot)\lambda_n^{\beta_2+2-\beta_1\gamma_{21}-\beta_2\gamma_{22}}\, v_{1,n}^{\gamma_{21}} v_{2,n}^{\gamma_{22}} + \widetilde{g}_{2,n} \end{array} \tag{36}$$

in the domain $\Omega_n = \dfrac{1}{\lambda_n}(\Omega - x_n)$, where the dot stands for $\lambda_n x + x_n$, and we have set $\widetilde{g}_{i,n} = \lambda_n^{\beta_i+2} g_i\left(\cdot, \lambda_n^{-\beta_1} v_{1,n}, \lambda_n^{-\beta_2} v_{2,n}\right)$,

$$H_{i,n}(M, p, u_1, u_2, x) := H_i(M, \lambda_n p, \lambda_n^2 u_1, \lambda_n^2 u_2, x_n + \lambda_n x).$$

By compactness we can assume that $\{x_n\}$ tends to some point $x_0 \in \overline{\Omega}$. It is a very standard fact that the domain Ω_n converges either to $\mathbb{R}^N$ or to a half-space in $\mathbb{R}^N$.

With the choice of β_1, β_2 that we made in the introduction, we have that all powers of λ_n in (36) are non-negative (see for example lemma 2.2 in [17]). Thus the right hand side of (36) is bounded in $L^\infty(\Omega)$, so by compactness –

see Lemma 6.1 – we find that, up to a subsequence, $v_{i,n}$ converges to some function v_i uniformly in compact sets of $\mathbb{R}^N$ (or $\mathbb{R}^N_+$). In order to pass to the limit in (36), we use the fact that the sequence of operators $H_{i,n}$ satisfies the hypothesis of Theorem 2.5. Indeed, as can be easily verified with the help of $(\widetilde{H}_2)$ and $\lambda_n \to 0$, for any fixed ball B and any $\phi \in W^{2,N}(B)$ we have

$$H_{i,n}(D^2\phi, D\phi, v_{1,n}, v_{2,n}, x) \to H_1(D^2\phi, 0, 0, 0, x_0) \qquad \text{in } \ L^N(B).$$

Thus we can pass to the limit in (36). Note that in the passage to the limit the terms in the right hand side of (36) which contain strictly positive powers of λ_n disappear, as well as $\widetilde{h}_{i,n}$, while the terms where the power of λ_n is zero remain. Actually, this observation has dictated the choice of β_1, β_2 (note this choice depends only on the exponents α_{ij}) – more details on this can be found in [17].

In this way we obtain a nontrivial (since $v_1(0) = 1$) bounded solution of the system

$$\left\{ \begin{array}{rcl} -H_1(D^2v_1, 0, 0, 0, x_0) & = & c_{11}v_1^{\alpha_{11}} + c_{12}v_2^{\alpha_{12}} + c_{13}v_1^{\gamma_{11}}v_2^{\gamma_{12}} \\ -H_2(D^2v_2, 0, 0, 0, x_0) & = & c_{21}v_1^{\alpha_{21}} + c_{22}v_2^{\alpha_{22}} + c_{23}v_1^{\gamma_{21}}v_2^{\gamma_{22}} \end{array} \right. \tag{37}$$

in $\mathbb{R}^N$ or $\mathbb{R}^N_+$ (with a Dirichlet boundary condition), where $c_{ij} \geq 0$ are constants. There are several cases now, depending on the possible values of c_{ij}. We shall only list them and give the contradiction in each case, referring to [17] for an explanation on how these cases appear. Note in any case $c_{12} = 0$ implies $c_{13} = 0$ and $c_{21} = 0$ implies $c_{23} = 0$.

Let J_1, J_2 be the sets at which the dimension-like numbers N_1, N_2 corresponding to the operators in the left-hand side of (37) are attained, and let $\mathcal{M}_1, \mathcal{M}_2$ be the corresponding extremal operators, see Definition 3.1. So (v_1, v_2) is a supersolution of (37) with H_1, H_2 replaced by $\mathcal{M}_1, \mathcal{M}_2$, according to Definition 3.1. When the domain for (37) is $\mathbb{R}^N$ we are going to use Theorem 1.3 to get a contradiction, while in the case when the domain is $\mathbb{R}^N_+$ we are going to apply Theorem 5.1 directly to (37). If $N_1 \leq 2$ or $N_2 \leq 2$ we just use Proposition 4.1 part 2.

Case 1. $c_{11} > 0$ and $c_{22} > 0$. If the domain is $\mathbb{R}^N$ then we have a contradiction with the Liouville theorem for scalar inequalities from [20], which is a particular case of Theorem 1.3 with $p = q$ and $\mathcal{M}_1 = \mathcal{M}_2$. If the domain is $\mathbb{R}^N_+$ we have two subcases. If all $c_{ij} > 0$ we have a contradiction with our Liouville theorem in half-space, see Theorem 5.1. If $c_{21} = c_{23} = 0$ then the second equality in (37) implies $v_2 \equiv 0$ by Theorem 5.1 (with $p = q = \alpha_{22}$). Then the first equation in (37) becomes scalar in v_1 and we apply the same theorem to it.

Case 2. $c_{12} > 0$ and $c_{21} > 0$. Then we have a contradiction with Theorems 1.3 and 5.1.

Case 3. $c_{21} = c_{22} = c_{23} = 0$, $c_{11}, c_{12} > 0$. Then the second equation in (37) implies that $v_2 \equiv c_0$, a constant (this is a consequence from the Harnack inequality, see for instance [8]). If $c_0 = 0$ (this is the only case for a half-space, because of the Dirichlet boundary condition) again the first equation is scalar. If $c_0 > 0$ by the first equation we get a positive bounded solution to $-\mathcal{M}^-_{\lambda,\Lambda}(D^2 v_1) \geq c_0 c_{21} = \widetilde{c} > 0$ in $\mathbb{R}^N$ which is easily seen to be impossible. Indeed, if there were such a function, by the comparison Theorem 2.4 we would have $v_1(0) \geq w_R(0)$, where w_R is the solution of the Dirichlet problem $-\mathcal{M}^-_{\lambda,\Lambda}(D^2 w) = \widetilde{c}$ in the ball B_R (this problem is solvable, see for instance Proposition 7.1). Then if $v_R(y) = w_R(Ry)$, we have $-\mathcal{M}^-_{\lambda,\Lambda}(D^2 v_R) = \widetilde{c}R^2$ in B_1. So, by Theorem 7.1 we have $v_1(0) \geq cR^2$ for all R, a contradiction. □

Theorem 1.2 is proved, since it is a consequence of Theorem 6.1.

7 Appendix

Here we prove the following result. To fix notations, again Q_l will denote a cube with size l. Any two cubes such that one is obtained by doubling the size of the other are supposed to be concentric. For any measurable set E we denote with $\text{meas}(E) = |E|$ its Lebesgue measure.

Theorem 7.1 *Let $\omega \subset Q_2$ be a closed set with positive measure and suppose $u \in C(Q_2)$ is a positive function satisfying*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2 u) - \gamma|Du| - \delta u \leq -\alpha\chi_\omega \quad \text{in } Q_2, \tag{38}$$

for some $\alpha > 0$ (here χ_ω denotes the characteristic function of ω). Then there exists a constant $m > 0$, depending only on $N, \lambda, \Lambda, \gamma, \delta$, and on a positive lower bound of the measure of ω, such that

$$\inf_{Q_1} u \geq m\alpha. \tag{39}$$

This theorem extends a result by Krylov and Safonov, concerning linear elliptic operators and strong solutions, that is, $u \in W^{2,N}_{\text{loc}}(Q_2)$. For such operators and solutions Theorem 7.1 follows from Theorem 2, page 118, in the book [25].

As we will show below, Theorem 7.1 can be reduced to the result in [25]. However, since this reference is not easy to work with, and since Theorem 7.1

plays an essential role in our arguments, we are going to give a full self-contained proof of this theorem, which is based only on the Alexandrov-Bakelman-Pucci inequality, convergence properties of viscosity solutions and on a result from measure theory – Egoroff's theorem.

First proof of Theorem 7.1. We use the following well-known results from the theory of fully nonlinear operators.

Lemma 7.1 *Let $w \in W^{2,N}_{\text{loc}}(\Omega)$. There exists a scalar* linear *uniformly elliptic second order operator L_0 (depending on w) with bounded measurable coefficients, such that*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2w) - \gamma|Dw| = L_0 w$$

The ellipticity constant of L_0 and the L^∞-bounds for the coefficients of L_0 depend only on $N, \lambda, \Lambda, \gamma$.

Proof. This is very standard. Recall $\mathcal{M}^-_{\lambda,\Lambda}(M) = \inf_{A\in\mathcal{S}_N^{\lambda,\Lambda}} \text{tr}(AM)$. This infimum is attained (since $\mathcal{S}_N^{\lambda,\Lambda}$ is compact), for any fixed M. Then we take

$$L_0 w(x) = \text{tr}\left(A_0(x)D^2w(x)\right) - \vec{b}(x).Dw(x),$$

where $x \longrightarrow A_0(x)$ is a measurable selection of elements of $\mathcal{S}_N^{\lambda,\Lambda}$ at which the infimum above is attained, and

$$\vec{b}(x) = \begin{cases} \gamma\dfrac{Dw(x)}{|Dw(x)|}, & \text{if} \quad Dw(x) \neq 0 \\ 0, & \text{if} \quad Dw(x) = 0. \end{cases}$$

Proposition 7.1 *Let $c, f \in L^\infty(\Omega)$, and $c \geq 0$ in Ω. Then there exists a unique solution $v \in W^{2,N}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ of the following problem*

$$\begin{cases} \mathcal{M}^-_{\lambda,\Lambda}(D^2v) - \gamma|Dv| - cv &= f \quad \textit{a.e. in } \Omega \\ v &= 0 \quad \textit{on } \partial\Omega. \end{cases} \tag{40}$$

Proof. When $c \equiv 0$ this result was proved in [9] (Corollary 3.10 in that paper). Exactly the same proof works for $c \geq 0$, since the authors use Theorem 17.17 in [23] and the ABP estimate, which both hold when $c \geq 0$.

In order to prove Theorem 7.1 we apply Proposition 7.1 to get a solution in $W^{2,N}_{\text{loc}} \cap C(\overline{\Omega})$ of

$$\begin{cases} \mathcal{M}^-_{\lambda,\Lambda}(D^2v) - \gamma|Dv| - \delta v &= -\alpha\chi_\omega \quad \text{a.e. in } Q_2 \\ v &= 0 \quad \text{on } \partial Q_2. \end{cases} \tag{41}$$

Set $w = v - u$. By Lemma 2.1 we have

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2 w) + \gamma|Dw| - \delta w & \geq & 0 & \text{in } Q_2 \\ v & = & 0 & \text{on } \partial Q_2. \end{cases}$$

so Theorem 2.1 implies $w \leq 0$ in Q_2, that is,

$$u \geq v \qquad \text{in } \ Q_2. \tag{42}$$

By Lemma 7.1 equation (41) can be recast as a linear one, in which the coefficients depend on v but their L^∞ bounds do not. So we can apply the result in [25] to this equation and v, and conclude, by (42). □

Second proof of Theorem 7.1. Here we give a self-contained proof of the theorem. We start with the following basic proposition.

Proposition 7.2 *There exists a number $\rho_0 \in (0,1)$ depending only on N, λ, Λ, γ, δ, such that if for some $\rho \in (0, \rho_0]$ and some cube $Q_{2\rho} \subset\subset Q_2$, the function $u \in C(Q_2)$ satisfies*

$$\begin{aligned} G[u] := \mathcal{M}^-_{\lambda,\Lambda}(D^2 u) - \gamma|Du| - \delta u & \leq 0 \qquad \text{in } Q_{2\rho} \\ u & \geq 0 \qquad \text{in } Q_{2\rho}, \end{aligned}$$

then for any $\nu, a > 0$ there exists $\overline{\kappa} > 0$ depending on ν, N, λ, Λ, γ, δ, such that

$$\text{meas}\,\{x \in Q_\rho \ : \ u(x) \geq a\} \geq \nu\,|Q_\rho| \qquad \text{implies} \qquad u \geq \overline{\kappa} a \quad \text{in } \ Q_\rho.$$

Before proving this proposition, let us show how Theorem 7.1 follows from it. We also need a well-known result from measure theory, usually referred to as Egoroff's theorem.

Theorem 7.2 *Suppose $\{u_n\}$ is a sequence of functions which converges locally in measure in a domain G, $|G| < \infty$, to a function u, that is, for any compact set $K \subset G$ and any $\varepsilon > 0$*

$$\text{meas}\{x \in K \ : \ |u_n(x) - u(x)| \geq \varepsilon\} \to 0 \quad \text{as} \ \ n \to \infty.$$

Then there exists a subsequence of $\{u_n\}$ which converges to u almost everywhere in G, and for any $\delta > 0$ and any open bounded set $E \subset \overline{E} \subset G$ there exists an open subset $E_1 \subset E$ such that $|E_1| < \delta$ and u_n converges uniformly to u in $\overline{E} \setminus E_1$.

Proof of Theorem 7.1. We are going to suppose that $\omega = Q_1$ (we actually apply the theorem only in this case). The full strength of Theorem 7.1 can then be obtained by a covering argument.

Replacing u by u/α we can suppose $\alpha = 1$. Let ρ_0 be the number from Proposition 7.2.

Claim. There exist $\nu, a > 0$ such that for any cube $Q \subset Q_1$ with size ρ_0, and for any solution u of (38)

$$\text{meas}\,\{x \in Q \;:\; u(x) \geq a\} \geq \nu.$$

If this claim is true then Theorem 7.1 follows from Proposition 7.2. So suppose the claim is false, that is, for all $n \in \mathbb{N}$ there exists a cube $Q^{(n)}$ with fixed size ρ_0, $Q^{(n)} \subset Q_1$, and a solution u_n of (38), such that

$$\text{meas}\left\{x \in Q^{(n)} \;:\; u_n(x) \geq \frac{1}{n}\right\} \leq \frac{1}{n}.$$

Then, clearly, there exists a subsequence of $\{Q^{(n)}\}$ which contains a fixed cube $\overline{Q}$ with size larger than $\rho_0/2$. The above inequality implies that $\{u_n\}$ converges in measure to zero in this cube. Then by Egoroff's theorem $\{u_n\}$ converges uniformly to zero in some subset of $\overline{Q}$ with positive measure. This implies, by Theorem 2.5, that zero is a solution of (38) in some subset of Q_1, a contradiction. □

We now turn to Proposition 7.2. First we prove a weaker result.

Proposition 7.3 *There exist numbers $\beta, \kappa, \rho_0 \in (0,1)$ depending only on N, λ, Λ, γ, δ, such that if for some $\rho \in (0, \rho_0]$ and some cube $Q_{2\rho} \subset Q_2$, the function $u \in C(Q_2)$ satisfies*

$$\begin{aligned} G[u] := \mathcal{M}^-_{\lambda,\Lambda}(D^2u) - \gamma|Du| - \delta u &\leq 0 \qquad \text{in } Q_{2\rho} \\ u &\geq 0 \qquad \text{in } Q_{2\rho}, \end{aligned}$$

then for any $a > 0$

$$\text{meas}\,\{x \in Q_\rho \;:\; u(x) \geq a\} \geq (1-\beta)\,|Q_\rho| \qquad \text{implies} \qquad u \geq \kappa a \quad \text{in } Q_\rho.$$

Proof. Without restricting the generality we can suppose $a = 1$ (replace u by u/a). To simplify some of the following computations, we suppose that Q stands for a ball instead of a cube in this lemma (this is obviously equivalent).

Set $v(x) = 1 - \frac{|x|^2}{\rho^2}$. Then, by Lemmas 2.1 and 2.2, for any $x \in Q_\rho$

$$\begin{aligned} \mathcal{M}^+_{\lambda,\Lambda}(D^2(v-u)) + \gamma|D(v-u)| &\geq \mathcal{M}^-_{\lambda,\Lambda}(D^2v) - \gamma|Dv| - G[u] - \delta u \\ &\geq -\frac{2}{\rho^2}\left(N\Lambda + \gamma|x| + \delta\rho^2 u\right) \\ &\geq -\frac{C}{\rho^2}\left(1 + \gamma\rho + \delta\rho^2 u\right), \end{aligned}$$

provided $u \in W^{2,N}(Q_{2\rho})$. Extending this inequality to u only continuous is easy (and standard, since $v \in C^2$), by using Definition 2.1 and test functions.

Since $v - u \leq 0$ on ∂Q_ρ, by applying Theorem 2.1 to the last inequality we obtain

$$\begin{aligned}\sup_{Q_\rho}(v-u) &\leq C\rho^{-1}\|1+\gamma\rho+\delta\rho^2 u\|_{L^N(Q_\rho\cap\{v-u>0\})}\\ &\leq C\rho^{-1}(1+\gamma\rho+\delta\rho^2\sup_{v-u>0}u)\,|Q_\rho\cap\{v-u>0\}|^{1/N}.\end{aligned}$$

Note that $\{v-u>0\}\subset\{u<1\}$, so $\mathrm{meas}(Q_\rho\cap\{v-u>0\})\leq C(N)\beta\rho^N$, by hypothesis. Then

$$\sup_{Q_\rho}(v-u)\leq C\beta^{1/N}(1+\gamma\rho+\delta\rho^2).$$

By choosing β sufficiently small and $\rho_0 \leq 1$ we get

$$\frac{3}{4}-\inf_{Q_{\frac{\rho}{2}}}u=\inf_{Q_{\frac{\rho}{2}}}v-\inf_{Q_{\frac{\rho}{2}}}u\leq\sup_{Q_{\frac{\rho}{2}}}(v-u)\leq\sup_{Q_\rho}(v-u)\leq\frac{1}{4}$$

for $\rho \leq 1$, so $u \geq \dfrac{1}{2}$ in $Q_{\frac{\rho}{2}}$.

Now set, for $s > 0$ and $x \in Q_{2\rho}\setminus Q_{\frac{\rho}{2}}$,

$$w(x)=\frac{1}{4}\,\frac{|x|^{-s}-(2\rho)^{-s}}{(\rho/2)^{-s}-(2\rho)^{-s}}\,.$$

It is easy to compute, with the help of Lemma 2.2, that

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2(|x|^{-s}))-\gamma|D(|x|^{-s})|=s(\lambda(s+1)-\Lambda(N-1)-\gamma|x|)|x|^{-s-2},$$

and hence, fixing s such that $\lambda(s+1)=\Lambda(N-1)$,

$$\begin{aligned}\mathcal{M}^+_{\lambda,\Lambda}(D^2(w-u))+\gamma|D(w-u)| &\geq \mathcal{M}^-_{\lambda,\Lambda}(D^2w)-\gamma|Dw|-G[u]-\delta u\\ &\geq -C\rho^s|x|^{-s-1}-\delta u\\ &\geq -C\rho^{-1}(1+\rho u)\end{aligned}$$

in the set $Q_{2\rho}\setminus Q_{\frac{\rho}{2}}$. Since $w-u\leq 0$ on $\partial(Q_{2\rho}\setminus Q_{\frac{\rho}{2}})$ and $u<1$ on the set $\{w-u>0\}$, Theorem 2.1 yields

$$\sup_{Q_\rho\setminus Q_{\frac{\rho}{2}}}(w-u)\leq\sup_{Q_{2\rho}\setminus Q_{\frac{\rho}{2}}}(w-u)\leq C(1+\rho)\|1\|_{L^N(Q_{2\rho})}\leq C|Q_{2\rho}|^{1/N}=C\rho$$

so, by taking ρ_0 sufficiently small, we have, for $\rho \le \rho_0$,

$$u(x) \ge \inf_{Q_\rho \setminus Q_{\frac{\rho}{2}}} w - C\rho \ge (2^{-s-4} - C\rho) \ge 2^{-s-5}\,, \qquad \text{for } x \in Q_\rho \setminus Q_{\frac{\rho}{2}},$$

which finishes the proof of Proposition 7.3. □

Now we can carry out the proof of Proposition 7.2 with the help of an argument which goes back to Krylov. It uses the following well-known measure theoretic result.

Lemma 7.1 *Let G be a cube and K be some measurable subset of G, such that $|K| \le \eta|G|$, for some $\eta \in (0,1)$. Let $\mathcal{F}$ be the set of all cubes B contained in G, and such that $|B \cap K| \ge \eta|B|$. Then, setting $\zeta = \frac{1-\eta}{\eta} > 0$,*

$$\text{meas}(\cup_{B \in \mathcal{F}} B) \ge (1+\zeta)\text{meas}(K).$$

Proof. This is inequality (9.20) from [23], setting f to be the indicator function of K in the reasoning there. □

Proof of Proposition 7.2. Set $K_a = \{x \in Q_\rho : u(x) \ge a\}$. We know that $|K_a| \ge \nu\,|Q_\rho|$. If $|K_a| \ge (1-\beta)|Q_\rho|$, where β is the number from Proposition 7.3 then we conclude, by that Proposition.

If, on the other hand, $|K_a| < (1-\beta)|Q_\rho|$, we apply Lemma 7.1, with $\eta = 1-\beta$. By Proposition 7.3 we have $u \ge \kappa a$ in each cube in $\mathcal{F}$ (defined in Lemma 7.1), for some $\kappa > 0$, depending on the appropriate quantities. Hence, by Lemma 7.1,

$$|K_{\kappa a}| \ge (1+\zeta)|K_a| \ge \nu(1+\zeta)\,|Q_\rho|.$$

We repeat the same reasoning and get either Proposition 7.2 or

$$|K_{\kappa^2 a}| \ge \nu(1+\zeta)^2\,|Q_\rho|.$$

This process stops after at most n iterations, where n is a number such that $\nu(1+\zeta)^n \ge 1$. □

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