Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 6, 1995, 371–397

POSITIVE SOLUTIONS OF NONLINEAR ELLIPTIC PROBLEMS APPROXIMATING DEGENERATE EQUATIONS

Monica Musso — Donato Passaseo

Dedicated to Louis Nirenberg on the occasion of his 70th birthday

1. Introduction

In recent years there has been an increasing interest in positive solutions of some nonlinear elliptic problems, where some concentration phenomena enable one to relate the number of positive solutions to the geometrical properties of the domain.

Phenomena of this type occur, for example, in some nonlinear problems involving critical or supercritical Sobolev exponents like the following:

$$\begin{cases} \Delta u + u^{p-1} = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth domain in $\mathbb{R}^N$, $N \geq 3$, and $p \geq 2N/(N-2)$ (the critical Sobolev exponent for the embedding $H_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$).

Many papers have been devoted to such problems (see [2], [5], [6], [10]–[12], [14], [15], [18]–[23], [25], [27], [28], and the references therein).

Here the lack of compactness, due to the presence of the critical exponent, is just associated with concentration phenomena and, when it is possible to overcome the difficulties due to the lack of compactness, one can often obtain multiplicity results for positive solutions.

1991 *Mathematics Subject Classification.* 35J65, 35J20, 35J70.

But these phenomena can also occur in subcritical problems: for example in the problem

$$\begin{cases} \Delta u - \lambda u + u^{p-1} = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

with $2 < p < 2N/(N-2)$, when $\lambda > 0$ is large enough (as pointed out in [3]), or in the problem

$$\begin{cases} \varepsilon\Delta u + g(x,u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

with g having subcritical growth, when $\varepsilon > 0$ is small enough (see [4]). These properties of concentration have been used in order to obtain the multiplicity results stated in [3], [4], [8], [9].

Analogous concentration phenomena are being investigated (see [14]) for the equation

$$\Delta u + a(x)u^{p-1} = 0$$

with $2 < p < 2N/(N-2)$ and $a(x)$ a positive function which behaves like $1/|x|^\alpha$ ($\alpha > 0$) near 0. Here the concentration properties are just due to the singular coefficient of the nonlinear term.

In [24] some concentration phenomena have been pointed out for degenerate elliptic problems like

$$\begin{cases} \operatorname{div}(\lambda(x)Du) + g(x,u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where λ is a positive function in Ω. Even when the nonlinear term is subcritical, these phenomena occur because of the degenerate character of the differential equation.

Many papers have been devoted to degenerate elliptic problems (see, for instance, [13], [17], [27] and the references therein). In [24] it is shown that the solutions of (1.4) tend to "concentrate" near the degeneration set of λ, that is, the subset of Ω where λ goes to zero.

So the following natural question arises: is it possible to relate the number of positive solutions of problems like (1.4) to the geometrical properties of the degeneration set? In particular, is it possible to show that problem (1.4) has several positive solutions if the degeneration set has several connected components?

In [24] an example was also given where the degeneration set consists of k spheres and the problem has at least $k+1$ distinct positive solutions.

In the present paper we answer the above question: we consider, for $\varepsilon > 0$, a family of problems

$$(P_\varepsilon(\Omega, g)) \qquad \begin{cases} \operatorname{div}(a_\varepsilon(x)Du) + g(x,u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $g(x,u)$ is a subcritical nonlinearity and, for all $\varepsilon > 0$ and almost all $x \in \Omega$, $a_\varepsilon(x)$ is a positive definite symmetric $N \times N$ matrix with coefficients in $L^\infty(\Omega)$.

We assume that there exist k pairwise disjoint subsets $\overline{\Omega'_1}, \dots, \overline{\Omega'_k}$ of Ω such that every connected component of $\overline{\Omega} \setminus \bigcup_{i=1}^k \overline{\Omega'_i}$ meets $\partial\Omega$ and that the matrix $a_\varepsilon(x)$ degenerates as $\varepsilon \to 0^+$ only in some subsets $\Omega_1, \dots, \Omega_k$ respectively of $\overline{\Omega'_1}, \dots, \overline{\Omega'_k}$. Then, under suitable assumptions on the nonlinear term $g(x,u)$, we obtain the existence of at least $k+1$ distinct positive solutions for problem $P_\varepsilon(\Omega, g)$ when $\varepsilon > 0$ is small enough.

The paper is organized as follows: we first consider the problem $P_\varepsilon(\Omega, g)$ in the particular case where $g(x,u) = u^{p-1}$ with $2 < p < 2N/(N-2)$. The multiplicity result given by Theorem (4.4) in this particular case obviously follows from Theorem (5.1) which concerns the case of a general nonlinearity $g(x,u)$, not necessarily homogeneous in u. Nevertheless we study the homogeneous case separately because it is, in some sense, the model problem and also because the proof, which in this case can be reduced to looking for the critical points of the energy functional constrained on the unit sphere of $L^p(\Omega)$, is used in the general case and seems to suggest better the behaviour of the solutions $u_{\varepsilon,1}, \dots, u_{\varepsilon,k+1}$ as $\varepsilon \to 0^+$.

The assumptions on the matrix $a_\varepsilon(x)$ are stated in Section 2; in Section 3 we introduce the main notations used in this paper; in Section 4 we prove the multiplicity result in the case of a homogeneous nonlinearity (Theorem (4.4)); in Section 5 we state the assumptions on the function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ and prove the multiplicity result in the general case (Theorem (5.1)). Finally, Proposition (5.13) provides some qualitative information on the behaviour as $\varepsilon \to 0^+$ of the solutions $u_{\varepsilon,1}, \dots, u_{\varepsilon,k+1}$ given by Theorem (5.1).

2. The homogeneous case

In this section we introduce our problem in the simplified case where the nonlinear term is homogeneous. Moreover, we state the assumptions which allow us to obtain a result on the multiplicity of positive solutions. The problem is the following:

$$(P_\varepsilon(\Omega,p))\qquad \begin{cases} \operatorname{div}(a_\varepsilon(x)Du)+u^{p-1}=0 & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N\ge 3$, $p\in\,]2,2N/(N-2)[$ and, for all $\varepsilon>0$, $a_\varepsilon(x)=(a_\varepsilon^{i,j}(x))$ is a positive definite symmetric $N\times N$ matrix with $a_\varepsilon^{i,j}(x)\in L^\infty(\Omega;\mathbb{R})$ for all $i,j=1,\dots,N$.

We make the following assumptions on $a_\varepsilon(x)$:

(a.1) for all $\varepsilon>0$ and for almost all $x\in\Omega$ there exist two positive constants $\Lambda_1(\varepsilon,x)$ and $\Lambda_2(\varepsilon,x)$ such that

$$\Lambda_1|\xi|^2\le a_\varepsilon^{i,j}(x)\xi_i\xi_j\le\Lambda_2|\xi|^2$$

for all $\xi\in\mathbb{R}^N$ (here and later on we write, as usual, $a_\varepsilon^{i,j}(x)\xi_i\xi_j$ instead of $\sum_{i,j=1}^N a_\varepsilon^{i,j}(x)\xi_i\xi_j$);

(a.2)

$$\liminf_{\varepsilon\to0^+}\big(\inf_{x\in\Omega}\Lambda_1(\varepsilon,x)/\varepsilon\big)>0;$$

(a.3) there exist $k>1$ subsets $\Omega_1,\dots,\Omega_k$ of Ω (the *degeneration subsets* for $a_\varepsilon(x)$) such that for all $i,j=1,\dots,k$,

$$\lim_{\varepsilon\to0^+}a_\varepsilon^{i,j}(x)/\varepsilon=a^{i,j}(x)\quad\text{uniformly in }\bigcup_{s=1}^k\Omega_s$$

with

$$\sup_{x\in\bigcup_{s=1}^k\Omega_s}|a^{i,j}(x)|<\infty$$

(see also Remark (5.14));

(a.4) for all $\eta>0$,

$$\liminf_{\varepsilon\to0^+}\left(\inf\left\{\Lambda_1(\varepsilon,x):x\in\Omega\setminus\bigcup_{s=1}^k\Omega_s(\eta)\right\}\right)>0,$$

where $\Omega_s(\eta)=\{x\in\Omega:d(x,\Omega_s)<\eta\}$.

Moreover, we require that the degeneration subsets $\Omega_1,\dots,\Omega_k$ satisfy the following condition:

(a.5) $\Omega_1,\dots,\Omega_k$ are smooth domains strictly contained in Ω, i.e. $\overline{\Omega}_s\subset\Omega$ for all $s=1,\dots,k$. For all $s=1,\dots,k$ let us denote by C_s the union of the connected components of $\overline{\Omega}\setminus\Omega_s$ which do not meet $\partial\Omega$, and set $\Omega'_s:=\Omega_s\cup C_s$. We require that the subsets $\overline{\Omega'_1},\dots,\overline{\Omega'_k}$ are pairwise disjoint, i.e.

$$\overline{\Omega'_s}\cap\overline{\Omega'_t}=\emptyset\quad\forall s,t\in\{1,\dots,k\}\text{ such that }s\ne t.$$

Notice that this condition implies, in particular, that every connected component of $\overline{\Omega} \setminus \bigcup_{s=1}^{k} \Omega'_s$ meets $\partial\Omega$.

3. Notations

Before stating the theorem that gives a multiplicity result for positive solutions of $P_\varepsilon(\Omega, p)$ if $\varepsilon > 0$ is small enough, we introduce some useful notation. Let $H_0^{1,2}(\Omega)$ denote the usual Sobolev space endowed with the norm

$$\|u\| = \left(\int_\Omega |Du|^2\, dx\right)^{1/2},$$

and let

$$\|u\|_p = \left(\int_\Omega |u(x)|^p\, dx\right)^{1/p}$$

denote the usual norm in the space $L^p(\Omega)$.

We denote by u^+ and u^- respectively the positive part and the negative part of a function $u \in H_0^{1,2}(\Omega)$.

For all $u \in H_0^{1,2}(\Omega)$, $\varepsilon > 0$ and $s \in \{1, \dots, k\}$ we set

$$A(\varepsilon, u, \Omega) := \int_\Omega \frac{a_\varepsilon^{i,j}(x)}{\varepsilon} \partial_{x_i} u \partial_{x_j} u\, dx, \quad A(s, u) := \int_{\Omega_s} a^{i,j}(x) \partial_{x_i} u \partial_{x_j} u\, dx.$$

Let $\lambda : \Omega \to \mathbb{R}$ be a strictly positive function with $\lambda \in L^\infty(\Omega)$ and $1/\lambda \in L^\infty(\Omega)$; then

$$\|u\|_{(\lambda,p)} = \left(\int_\Omega \lambda(x) |u(x)|^p\, dx\right)^{1/p}$$

is a norm in $L^p(\Omega)$ equivalent to the usual norm $\|u\|_p$.

For all $s = 1, \dots, k$ we set

$$\mu_{\varepsilon,s}^\lambda = \inf\left\{ \int_{\Omega'_s} \frac{a_\varepsilon^{i,j}(x)}{\varepsilon} \partial_{x_i} u \partial_{x_j} u\, dx : u \in H_0^{1,2}(\Omega'_s),\ \int_{\Omega'_s} \lambda(x)|u(x)|^p\, dx = 1 \right\};$$

indeed, $\mu_{\varepsilon,s}^\lambda$ is a minimum since $p \in\,]2, 2N/(N-2)[$ and it is strictly positive because of (a.1). Let the function $v_{\varepsilon,s}^\lambda(x)$ be a minimizing function for $\mu_{\varepsilon,s}^\lambda$.

For all $s = 1, \dots, k$ set

$$\mu_{0,s}^\lambda = \inf\left\{ \int_{\Omega_s} a^{i,j}(x) \partial_{x_i} u \partial_{x_j} u\, dx : u \in H_0^{1,2}(\Omega'_s), \right.$$
$$\left. \int_{\Omega'_s \setminus \Omega_s} |Du|^2\, dx = 0,\ \int_{\Omega'_s} \lambda(x)|u(x)|^p\, dx = 1 \right\};$$

as before, $\mu_{0,s}^\lambda$ is a minimum and it is a positive number. Let $v_{0,s}^\lambda(x)$ be a minimizing function for $\mu_{0,s}^\lambda$ for all $s = 1, \dots, k$. Set

$$\mu_{\mathrm{m}}^\lambda := \min_{s=1,\dots,k} \mu_{0,s}^\lambda \quad \text{and} \quad \mu_{\mathrm{M}}^\lambda := \max_{s=1,\dots,k} \mu_{0,s}^\lambda.$$

When $\lambda(x) = 1$ for all $x \in \Omega$, for simplicity of notation we will write $\mu_{\varepsilon,s}, v_{\varepsilon,s}, \mu_{0,s}$ and $v_{0,s}$ instead of $\mu^\lambda_{\varepsilon,s}, v^\lambda_{\varepsilon,s}, \mu^\lambda_{0,s}$ and $v^\lambda_{0,s}$ respectively.

4. Multiplicity of positive solutions of $P_\varepsilon(\Omega, p)$

We begin with two general results.

(4.1) Proposition. *Assume that the matrix $a_\varepsilon(x)$ satisfies the conditions* (a.1)–(a.5). *Let $\lambda \in L^\infty(\Omega)$ be a strictly positive function such that $1/\lambda \in L^\infty(\Omega)$. Then for all $s = 1, \dots, k$ we have* (see Notations)

$$\lim_{\varepsilon\to 0^+} \mu^\lambda_{\varepsilon,s} = \mu^\lambda_{0,s}.$$

Proof. From the definition of $\mu^\lambda_{\varepsilon,s}$ and $v^\lambda_{0,s}$ it follows that

$$\mu^\lambda_{\varepsilon,s} \le \int_{\Omega'_s} \frac{a^{i,j}_\varepsilon(x)}{\varepsilon} \partial_{x_i} v^\lambda_{0,s} \partial_{x_j} v^\lambda_{0,s}\, dx \quad \text{for all } s = 1, \dots, k. \tag{4.1}$$

By (a.3),

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \int_{\Omega'_s} \frac{a^{i,j}_\varepsilon(x)}{\varepsilon} \partial_{x_i} v^\lambda_{0,s} \partial_{x_j} v^\lambda_{0,s}\, dx &= \lim_{\varepsilon\to 0^+} \int_{\Omega_s} \frac{a^{i,j}_\varepsilon(x)}{\varepsilon} \partial_{x_i} v^\lambda_{0,s} \partial_{x_j} v^\lambda_{0,s}\, dx \\ &= \int_{\Omega_s} a^{i,j}(x) \partial_{x_i} v^\lambda_{0,s} \partial_{x_j} v^\lambda_{0,s}\, dx = \mu^\lambda_{0,s}\end{aligned}$$

and this implies that $\limsup_{\varepsilon\to 0^+} \mu^\lambda_{\varepsilon,s} \le \mu^\lambda_{0,s} < \infty$.

Since $(a_\varepsilon(x))$ is elliptic in Ω and $\liminf_{\varepsilon\to 0^+}(\inf\{\Lambda_1(\varepsilon,x)/\varepsilon : x \in \Omega\}) > 0$, for any sequence $(\varepsilon_n)_n$ of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$ the corresponding sequence $(v^\lambda_{\varepsilon_n,s})_n$ is bounded in $H^{1,2}_0(\Omega'_s)$; hence there exists a subsequence of $(v^\lambda_{\varepsilon_n,s})_n$ (which we shall call again $(v^\lambda_{\varepsilon_n,s})_n$) converging to a function $v_s \in H^{1,2}_0(\Omega'_s)$, weakly in $H^{1,2}_0(\Omega'_s)$, in $L^p(\Omega'_s)$ and almost everywhere in Ω'_s.

Moreover, since for all $\eta > 0$,

$$\liminf_{\varepsilon\to 0^+} \bigg(\inf \bigg\{ \frac{\Lambda_1(\varepsilon,x)}{\varepsilon} : x \in \Omega \setminus \bigcup_{t=1}^k \Omega_t(\eta) \bigg\} \bigg) = \infty,$$

we obtain $\int_{\Omega'_s \setminus \Omega_s} |Dv_s|^2\, dx = 0$. It follows that

$$\begin{aligned}\liminf_{n\to\infty} \mu^\lambda_{\varepsilon_n,s} &= \liminf_{n\to\infty} \int_{\Omega'_s} \frac{a^{i,j}_{\varepsilon_n}(x)}{\varepsilon_n} \partial_{x_i} v^\lambda_{\varepsilon_n,s} \partial_{x_j} v^\lambda_{\varepsilon_n,s}\, dx \\ &\ge \liminf_{n\to\infty} \int_{\Omega_s} \frac{a^{i,j}_{\varepsilon_n}(x)}{\varepsilon_n} \partial_{x_i} v^\lambda_{\varepsilon_n,s} \partial_{x_j} v^\lambda_{\varepsilon_n,s}\, dx \\ &\ge \int_{\Omega_s} a^{i,j}(x) \partial_{x_i} v_s \partial_{x_j} v_s\, dx \ge \mu^\lambda_{0,s}\end{aligned}$$

because $v_s \in H_0^{1,2}(\Omega'_s)$, $\int_{\Omega'_s\setminus\Omega_s} |Dv_s|^2\,dx = 0$ and $\int_{\Omega'_s} \lambda(x)|v_s(x)|^p\,dx = 1$. From the arbitrary choice of the sequence $(\varepsilon_n)_n$, the assertion follows. Notice that, like $v^\lambda_{0,s}$, also the function v_s realizes the minimum $\mu^\lambda_{0,s}$. □

(4.2) Lemma. *Assume that the subsets $\Omega_1,\dots,\Omega_k$ satisfy the assumptions introduced in Section 2. Let $\lambda : \Omega \to \mathbb{R}$ be a strictly positive function with $\lambda \in L^\infty(\Omega)$ and $1/\lambda \in L^\infty(\Omega)$. Suppose $u \in H_0^{1,2}(\Omega)$ is such that:*

(1) *$\int_\Omega \lambda(x)|u(x)|^p\,dx = 1$;*

(2) *$u = \sum_{t=1}^k u_t$ with $u_t \in H_0^{1,2}(\Omega'_t)$ and $\int_{\Omega'_t\setminus\Omega_t} |Du_t|^2\,dx = 0$ for all $t = 1,\dots,k$;*

(3) *there exists $s \in \{1,\dots,k\}$ such that $\int_{\Omega'_s} \lambda(x)|u(x)|^p\,dx = 1-\delta$ with*

$$0 < \delta < (\mu^\lambda_{\mathrm{m}}/\mu^\lambda_{\mathrm{M}})^{p/p-2}.$$

Then $\mu^\lambda_{0,s} < \sum_{t=1}^k A(t,u_t)$ (see Notations for μ^λ_{m}, μ^λ_{M}, $\mu^\lambda_{0,s}$ and $A(t,u_t)$).

Proof. Suppose that $s = 1$. Define $c_t := \int_{\Omega'_t} \lambda(x)|u_t(x)|^p\,dx$ and $\overline{u}_t(x) = u_t(x)/c_t^{1/p}$ for all $t = 1,\dots,k$. As $0<\delta<1$, $c_1 = 1-\delta$ and $\sum_{t=1}^k c_t = 1$, there exists $\overline{t} \in \{2,\dots,k\}$ such that $c_{\overline{t}} \neq 0$. Let us compute:

$$\begin{aligned}
\sum_{t=1}^k & A(t,u) - \mu^\lambda_{0,1} \\
&= \sum_{t=1}^k A(t,\overline{u}_t)c_t^{2/p} - \mu^\lambda_{0,1} \ge \sum_{t=1}^k \mu^\lambda_{0,t}c_t^{2/p} - \mu^\lambda_{0,1} \\
&= \mu^\lambda_{0,1}(c_1^{2/p}-1) + \sum_{t=2}^k \mu^\lambda_{0,t}c_t^{2/p} \ge \mu^\lambda_{\mathrm{M}}(c_1^{2/p}-1) + \mu^\lambda_{\mathrm{m}}\bigg(\sum_{t=2}^k c_t^{2/p}\bigg) \\
&\ge \mu^\lambda_{\mathrm{M}}(c_1-1) + \mu^\lambda_{\mathrm{m}}\bigg(\sum_{t=2}^k c_t^{2/p}\bigg) = \mu^\lambda_{\mathrm{M}}\bigg(-\sum_{t=2}^k c_t\bigg) + \mu^\lambda_{\mathrm{m}}\bigg(\sum_{t=2}^k c_t^{2/p}\bigg) \\
&= \sum_{t=2}^k (\mu^\lambda_{\mathrm{m}}c_t^{2/p} - \mu^\lambda_{\mathrm{M}}c_t).
\end{aligned}$$

Now it suffices to remark that, since we have

$$0 < \sum_{t=2}^k c_t < (\mu^\lambda_{\mathrm{m}}/\mu^\lambda_{\mathrm{M}})^{p/(p-2)},$$

it follows that $0 < c_{\overline{t}} < (\mu^\lambda_{\mathrm{m}}/\mu^\lambda_{\mathrm{M}})^{p/(p-2)}$ for some $\overline{t} = 2,\dots,k$ and so

$$\mu^\lambda_{\mathrm{m}}c_{\overline{t}}^{2/p} - \mu^\lambda_{\mathrm{M}}c_{\overline{t}} > 0,$$

while for $t \neq \overline{t}$ we have $\mu^\lambda_{\mathrm{m}}c_t^{2/p} - \mu^\lambda_{\mathrm{M}}c_t \ge 0$, because $0 \le c_t < (\mu^\lambda_{\mathrm{m}}/\mu^\lambda_{\mathrm{M}})^{p/(p-2)}$ for all $t = 2,\dots,k$.

(4.3) Remark. Under the same assumptions of Lemma (4.2), if $u \in H_0^{1,2}(\Omega)$ is such that $\|u\|_{(\lambda,p)} = 1$, $u = \sum_{t=1}^k u_t$ with $u_t \in H_0^{1,2}(\Omega_t')$, $\int_{\Omega_t'\setminus\Omega_t} |Du_t|^2\,dx = 0$ and

$$1 - (\mu_{\mathrm{m}}^\lambda/\mu_{\mathrm{M}}^\lambda)^{p/(p-2)} < \int_{\Omega_s'} \lambda(x)|u_s(x)|^p\,dx < 1,$$

then $\mu_{0,s}^\lambda < \sum_{t=1}^k A(t,u_t)$. □

We can now formulate our main result on multiplicity of positive solutions for problem $P_\varepsilon(\Omega,p)$.

(4.4) Theorem. *Assume that the domains $\Omega,\Omega_1,\ldots,\Omega_k$ and the matrix $a_\varepsilon(x)$ satisfy the conditions introduced in Section 2. Then there exists $\overline{\varepsilon} > 0$ such that for all $\varepsilon \in \,]0,\overline{\varepsilon}[$ problem $P_\varepsilon(\Omega,p)$ has at least $k+1$ solutions $u_{\varepsilon,1},\ldots,u_{\varepsilon,k+1}$. Moreover, these solutions have the following properties:*

(I) $\lim_{\varepsilon\to 0^+} \|u_{\varepsilon,s}\|^2 = 0$ *for all* $s = 1,\ldots,k+1$;

(II) *there exists $\delta \in \,]0,1[$ such that for all $s = 1,\ldots,k$ and for all $\varepsilon \in \,]0,\overline{\varepsilon}[$ the solution $u_{\varepsilon,s}$ minimizes the functional*

$$u \mapsto \int_\Omega a_\varepsilon^{i,j}(x)\partial_{x_i}u\partial_{x_j}u\,dx$$

in the set

$$\left\{u \in H_0^{1,2}(\Omega) : \|u\|_p = \|u_{\varepsilon,s}\|_p,\ \int_{\Omega_s'} |u(x)|^p\,dx > (1-\delta)\int_\Omega |u(x)|^p\,dx\right\}$$

(the subsets Ω_s' have been introduced in Section 2);

(III) *for all* $s = 1,\ldots,k$,

$$\lim_{\varepsilon\to 0^+} \frac{\int_{\Omega_s'} |u_{\varepsilon,s}(x)|^p\,dx}{\int_\Omega |u_{\varepsilon,s}(x)|^p\,dx} = 1;$$

(IV) *for all* $s = 1,\ldots,k$,

$$\lim_{\varepsilon\to 0^+} \frac{\int_{\Omega_s'} a_\varepsilon^{i,j}(x)\partial_{x_i}u_{\varepsilon,s}\partial_{x_j}u_{\varepsilon,s}\,dx}{\varepsilon\|u_{\varepsilon,s}\|_p^2} = \mu_{0,s}$$

(see Notations for $\mu_{0,s}$);

(V)

$$\begin{aligned}\mu_{\mathrm{M}} &\le \liminf_{\varepsilon\to 0^+} \frac{\int_\Omega a_\varepsilon^{i,j}(x)\partial_{x_i}u_{\varepsilon,k+1}\partial_{x_j}u_{\varepsilon,k+1}\,dx}{\varepsilon\|u_{\varepsilon,k+1}\|_p^2} \\ &\le \limsup_{\varepsilon\to 0^+} \frac{\int_\Omega a_\varepsilon^{i,j}(x)\partial_{x_i}u_{\varepsilon,k+1}\partial_{x_j}u_{\varepsilon,k+1}\,dx}{\varepsilon\|u_{\varepsilon,k+1}\|_p^2} \\ &\le 2^{(p-2)/p}\mu_{\mathrm{M}},\end{aligned}$$

where $\mu_{\mathrm{M}} = \max_{s=1,\ldots,k} \mu_{0,s}$.

Proof. The solutions of problem $P_\varepsilon(\Omega,p)$ correspond to the positive functions u which are critical points of the functional

$$f_\varepsilon(u)=\int_\Omega \frac{a_\varepsilon^{i,j}(x)}{\varepsilon}\partial_{x_i}u\partial_{x_j}u\,dx$$

constrained to lie upon the manifold

$$V_p=\left\{u\in H_0^{1,2}(\Omega):\int_\Omega (u^+)^p\,dx=1\right\}.$$

In fact, a function u which is a constrained critical point for f_ε on V_p is a weak solution of the equation

$$\operatorname{div}(a_\varepsilon(x)Du)+\mu_\varepsilon(u^+)^{p-1}=0;$$

multiplying the last equation by u^- we see that $u\geq 0$ and it solves the equation

$$\operatorname{div}(a_\varepsilon(x)Du)+\mu_\varepsilon u^{p-1}=0$$

with $\mu_\varepsilon=\varepsilon f_\varepsilon(u)$, so that $[\varepsilon f_\varepsilon(u)]^{1/(p-2)}u$ is a solution of $P_\varepsilon(\Omega,p)$ and it is strictly positive by the maximum principle.

We now prove that for all $s=1,\dots,k$,

$$\liminf_{\varepsilon\to 0^+}\left(\inf\left\{f_\varepsilon(u):u\in V_p,\ \int_{\Omega'_s}|u(x)|^p\,dx=1-\delta\right\}\right)>\mu_{0,s}, \tag{4.2}$$

where $0<\delta<(\mu_{\mathrm{m}}/\mu_{\mathrm{M}})^{p/(p-2)}$ with

$$\mu_{\mathrm{m}}=\min_{t=1,\dots,k}\mu_{0,t}\quad\text{and}\quad \mu_{\mathrm{M}}=\max_{t=1,\dots,k}\mu_{0,t}.$$

By contradiction, assume that there exist a sequence $(\varepsilon_n)_n\to 0$ of positive numbers and a sequence $(u_n)_n$ of functions in V_p such that

$$\int_{\Omega'_s}|u_n(x)|^p\,dx=1-\delta\quad\text{for all } n\in\mathbb{N}, \tag{4.3}$$

$$\lim_{n\to\infty}f_{\varepsilon_n}(u_n)\leq\mu_{0,s}. \tag{4.4}$$

Since $a_\varepsilon(x)$ is elliptic in Ω and $\liminf_{\varepsilon\to 0^+}(\inf\{\Lambda_1(\varepsilon,x)/\varepsilon : x\in\Omega\})>0$, the sequence $(u_n)_n$ has to be bounded in $H_0^{1,2}(\Omega)$; hence there exists a subsequence of $(u_n)_n$ (which we shall denote again by $(u_n)_n$) converging to a function $u\in H_0^{1,2}(\Omega)$, weakly in $H_0^{1,2}(\Omega)$, in $L^p(\Omega)$ and almost everywhere in Ω.

Moreover, since for all $\eta>0$

$$\liminf_{\varepsilon\to 0^+}\left(\inf\left\{\frac{\Lambda_1(\varepsilon,x)}{\varepsilon}:x\in\Omega\setminus\bigcup_{j=1}^k\Omega_j(\eta)\right\}\right)=\infty,$$

we must have $\int_{\Omega\setminus\bigcup_{j=1}^k\Omega_j}|Du(x)|^2\,dx=0$, which implies $u(x)=0$ for all $x\in\Omega\setminus\bigcup_{j=1}^k\Omega'_j$ because $u\in H_0^{1,2}(\Omega)$ and every connected component of $\overline{\Omega}\setminus\bigcup_{j=1}^k\Omega'_j$ meets $\partial\Omega$ (see Section 2).

Since the subsets $\Omega'_1, \dots, \Omega'_k$ are pairwise disjoint, we have $u(x) = u_1(x) + \dots + u_k(x)$, where $u_j \in H_0^{1,2}(\Omega'_j)$, $1 \le j \le k$, and $\int_{\Omega'_s} |u(x)|^p\,dx = 1 - \delta$.

So, the assumptions of Proposition (4.1) (with $\lambda(x) = 1$ for all $x \in \Omega$) are satisfied and, by the choice of δ, we have

$$\mu_{0,s} < \sum_{t=1}^k A(t,u) \le \liminf_{n\to\infty} \sum_{t=1}^k A(\varepsilon_n, u_n, \Omega_t) \le \lim_{n\to\infty} f_{\varepsilon_n}(u_n),$$

contrary to (4.4). So (4.2) is proved.

Now, let us verify that, for all $s = 1, \dots, k$,

$$\limsup_{\varepsilon\to 0^+} \left(\inf \left\{ f_\varepsilon(u) : u \in V_p,\ \int_{\Omega'_s} |u(x)|^p\,dx > 1-\delta \right\} \right) \le \mu_{0,s}; \tag{4.5}$$

in fact, let $v_{0,s}$ be the positive function which realizes the minimum $\mu_{0,s}$, i.e. $v_{0,s} \in H_0^{1,2}(\Omega'_s)$, $v_{0,s} > 0$ in Ω'_s, $\int_{\Omega'_s} |v_{0,s}|^p\,dx = 1$, $\int_{\Omega'_s \setminus \Omega_s} |Dv_{0,s}|^2\,dx = 0$ and

$$\int_{\Omega_s} a^{i,j}(x)\partial_{x_i} v_{0,s} \partial_{x_j} v_{0,s}\,dx = \mu_{0,s}.$$

It is easy to verify that $\lim_{\varepsilon\to 0^+} f_\varepsilon(v_{0,s}) = \mu_{0,s}$, which implies (4.5), for all $s = 1, \dots, k$.

From (4.2) and (4.5) we infer that for all $\varepsilon > 0$ small enough and for all $s = 1, \dots, k$ there exists a function $\overline{u}_{\varepsilon,s}$ which is a minimum for the functional f_ε in the set

$$\left\{ u \in V_p : \int_{\Omega'_s} |u(x)|^p\,dx > 1-\delta \right\}.$$

The solutions $u_{\varepsilon,s} = [\varepsilon f_\varepsilon(\overline{u}_{\varepsilon,s})]^{1/(p-2)} \overline{u}_{\varepsilon,s}$ have property (II) with $0 < \delta < (\mu_{\mathrm{m}}/\mu_{\mathrm{M}})^{p/(p-2)}$ by construction. Moreover, it is evident that

$$\limsup_{\varepsilon\to 0^+} f_\varepsilon(u_{\varepsilon,s}/\|u_{\varepsilon,s}\|_p) \le \mu_{0,s} \quad \text{for all } s = 1, \dots, k. \tag{4.6}$$

We now prove that there exists another critical point $\overline{u}_{\varepsilon,k+1}$ for f_ε on V_p. Suppose that $\mu_{\mathrm{M}} = \mu_{0,1}$; let $\gamma : [0,1] \to V_p$ be a continuous path joining the functions $v_{0,1}$ and $v_{0,2}$:

$$\gamma(\tau) = \frac{\tau v_{0,1} + (1-\tau) v_{0,2}}{\|\tau v_{0,1} + (1-\tau) v_{0,2}\|_p}.$$

One can verify that

$$\limsup_{\varepsilon\to 0^+} \{ f_\varepsilon \circ \gamma(\tau) : \tau \in [0,1] \} \le 2^{(p-2)/2} \mu_{\mathrm{M}}. \tag{4.7}$$

Let $\overline{\mu} > \mu_{\mathrm{M}}$ be such that (see (4.2)) $\overline{\mu} < \liminf_{\varepsilon\to 0^+} (\inf\{ f_\varepsilon(u) : u \in V_p, \int_{\Omega'_1} |u(x)|^p\,dx = 1-\delta \})$. For ε small enough, $v_{0,1}$ and $v_{0,2}$ belong to $\{u \in V_p : f_\varepsilon(u) \le \overline{\mu}\}$ and they are not connected in that sublevel, which does not meet the set $\{u \in V_p : \int_{\Omega'_1} |u(x)|^p\,dx = 1-\delta\}$; while the two functions $v_{0,1}$ and $v_{0,2}$

are connected in the sublevel $\{u \in V_p : f_\varepsilon(u) \le \mu_\varepsilon\}$, with $\mu_\varepsilon = \max\{f_\varepsilon \circ \gamma(\tau) : \tau \in [0,1]\}$, in which the curve γ lies (see (4.7)).

Moreover, as $\liminf_{\varepsilon\to 0^+}(\inf_{x\in\Omega} \Lambda_1(\varepsilon,x)/\varepsilon) > 0$ and $p \in]2, 2N/(N-2)[$, the functional f_ε constrained on V_p satisfies the well-known Palais–Smale condition. Hence, by the Mountain Pass Theorem of Ambrosetti–Rabinowitz, there is a critical value for f_ε on V_p in the interval $]\overline{\mu}, \mu_\varepsilon]$ if ε is small enough.

Let $\overline{u}_{\varepsilon,k+1}$ be the corresponding critical point; then the function

$$u_{\varepsilon,k+1} = [\varepsilon f_\varepsilon(\overline{u}_{\varepsilon,k+1})]^{1/(p-2)} \overline{u}_{\varepsilon,k+1}$$

is a solution of $P_\varepsilon(\Omega,p)$ and it is distinct from the previous ones because it corresponds to a greater critical level.

Let us prove (I). Since

$$\limsup_{\varepsilon\to 0^+} f_\varepsilon(\overline{u}_{\varepsilon,s}) < \infty \quad \text{for all } s = 1,\dots,k+1$$

and $\liminf_{\varepsilon\to 0^+}(\inf_{x\in\Omega} \Lambda_1(\varepsilon,x)/\varepsilon) > 0$, it follows that, if we choose $\lambda_1 > 0$ such that $\inf_{x\in\Omega} \Lambda_1(\varepsilon,x)/\varepsilon \ge \lambda_1 > 0$ for all $\varepsilon > 0$ small enough, then, for all $s = 1,\dots,k+1$,

$$\begin{aligned}
\|u_{\varepsilon,s}\|^2 &= \int_\Omega |Du_{\varepsilon,s}|^2\,dx = \int_\Omega \frac{\Lambda_1(\varepsilon,x)}{\varepsilon}\cdot\frac{\varepsilon}{\Lambda_1(\varepsilon,x)} |Du_{\varepsilon,s}|^2\,dx \\
&\le \frac{1}{\lambda_1}\int_\Omega \frac{\Lambda_1(\varepsilon,x)}{\varepsilon}|Du_{\varepsilon,s}|^2\,dx \le \frac{1}{\lambda_1} f_\varepsilon(u_{\varepsilon,s}) \\
&= \frac{1}{\lambda_1}[\varepsilon f_\varepsilon(\overline{u}_{\varepsilon,s})]^{2/(p-2)} f_\varepsilon(\overline{u}_{\varepsilon,s}) \to 0 \quad \text{as } \varepsilon \to 0^+.
\end{aligned}$$

Let us prove (III). Since $\overline{u}_{\varepsilon,s} = u_{\varepsilon,s}/\|u_{\varepsilon,s}\|_p$ for all $s = 1,\dots,k$, we have

$$\int_{\Omega'_s} |\overline{u}_{\varepsilon,s}|^p\,dx \le 1.$$

By contradiction, assume that there exists an infinitesimal sequence $(\varepsilon_n)_n$ of positive numbers such that

$$\lim_{n\to\infty}\int_{\Omega'_s} |\overline{u}_{\varepsilon_n,s}|^p\,dx < 1. \tag{4.8}$$

Since $\limsup_{n\to\infty} f_{\varepsilon_n}(\overline{u}_{\varepsilon_n,s}) \le \mu_{0,s}$ and $\liminf_{\varepsilon\to 0^+}(\inf_{x\in\Omega}\Lambda_1(\varepsilon,x)/\varepsilon) > 0$, the sequence $(\overline{u}_{\varepsilon_n,s})_n$ is bounded in $H_0^{1,2}(\Omega)$ and then $(\overline{u}_{\varepsilon_n,s})_n$, or a subsequence, converges in $H_0^{1,2}(\Omega)$ weakly, in $L^p(\Omega)$ and almost everywhere in Ω to a function $\overline{u}$. Since $\overline{u} \in H_0^{1,2}(\Omega)$ and since for all $\eta > 0$ we have

$$\liminf_{n\to\infty}\left(\inf\left\{\frac{\Lambda_1(\varepsilon_n,x)}{\varepsilon_n} : x \in \Omega \setminus \bigcup_{t=1}^k \Omega_t(\eta)\right\}\right) = \infty,$$

it follows that $\overline{u}(x) = 0$ for all $x \in \Omega\setminus\bigcup_{t=1}^k \Omega'_t$ and $\int_{\Omega'_t\setminus\Omega_t} |D\overline{u}(x)|^2\,dx = 0$ for all $t = 1,\dots,k$. As $\int_{\Omega'_s} |\overline{u}(x)|^p\,dx \ge 1-\delta$ and, if (4.8) holds, then $\int_{\Omega'_s} |\overline{u}(x)|^p\,dx < 1$, there is $t \in \{1, \dots, k\}$, $t \neq s$, such that $\int_{\Omega'_t} |\overline{u}_t(x)|^p \, dx \neq 0$. By the choice of δ (see Remark (4.3)),

$$\mu_{0,s} < \sum_{t=1}^{k} A(t, \overline{u}) \leq \liminf_{n\to\infty} \sum_{t=1}^{k} A(\varepsilon_n, \overline{u}_{\varepsilon_n,s}, \Omega'_t) \leq \lim_{n\to\infty} f_{\varepsilon_n}(\overline{u}_{\varepsilon_n,s}) \leq \mu_{0,s},$$

which is a contradiction.

Let us prove (IV). By construction, the functions $\overline{u}_{\varepsilon,s} = u_{\varepsilon,s}/\|u_{\varepsilon,s}\|_p$ are such that

$$\limsup_{\varepsilon\to 0^+} f_\varepsilon(\overline{u}_{\varepsilon,s}) \leq \mu_{0,s} \quad \text{for all } s = 1, \dots, k.$$

If, by contradiction, there exists an infinitesimal sequence $(\varepsilon_n)_n$ of strictly positive numbers such that

$$\lim_{n\to\infty} A(\varepsilon_n, \overline{u}_{\varepsilon_n,s}, \Omega'_s) < \mu_{0,s}, \tag{4.9}$$

then it is easy to prove that the sequence $(\overline{u}_{\varepsilon_n,s})_n$ is bounded in $H_0^{1,2}(\Omega)$ and that one of its subsequences converges in $L^p(\Omega)$ and almost everywhere in Ω to a function $\overline{u}_s$ which is zero in $\Omega \setminus \Omega'_s$. It follows that

$$\mu_{0,s} \leq A(s, \overline{u}_s) \leq \lim_{n\to\infty} A(\varepsilon_n, \overline{u}_{\varepsilon_n}, \Omega'_s),$$

contrary to (4.9).

Property (V) is a simple consequence of the fact that $f_\varepsilon(\overline{u}_{\varepsilon,k+1}) \in]\overline{\mu}, \mu_\varepsilon]$ with $\overline{\mu} > \mu_{\mathrm{M}}$ and $\limsup_{\varepsilon\to 0^+} \mu_\varepsilon \leq 2^{(p-2)/2}\mu_{\mathrm{M}}$ (see (4.7)).

5. The nonhomogeneous case

This section is devoted to the more general semilinear problem having the nonlinear term not necessarily homogeneous.

We are concerned with the following problem:

$$(P_\varepsilon(\Omega, g)) \qquad \begin{cases} \operatorname{div}(a_\varepsilon(x) Du) + g(x, u(x)) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, and, for all $\varepsilon > 0$, $a_\varepsilon(x) = (a_\varepsilon^{i,j}(x))$ is a positive definite symmetric $N \times N$ matrix ($a_\varepsilon^{i,j}(x) \in L^\infty(\Omega; \mathbb{R})$ for all $i, j = 1, \dots, N$).

Both Ω and $a_\varepsilon(x)$ satisfy the assumptions required in Section 2 for the homogeneous problem $P_\varepsilon(\Omega, p)$.

The requirements on the function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ are the following:

(g.1) for all $t \in \mathbb{R}$, $g(x, t)$ is measurable with respect to x; for almost all $x \in \Omega$, $g(x, t)$ is a C^1-function with respect to t;

(g.2) there exist a positive constant $a > 0$ and $q \in]2, 2N/(N-2)[$ such that for all $t > 0$ and for almost all $x \in \Omega$,

$$|g(x,t)| \le a + at^{q-1} \quad \text{and} \quad |g_t'(x,t)| \le a + at^{q-2},$$

where $g_t'(x,t)$ denotes the derivative of g with respect to t;

(g.3) there exist $p \in]2, 2N/(N-2)[$ and a strictly positive function $\lambda : \Omega \to \mathbb{R}$ with $\lambda \in L^\infty(\Omega)$ and $1/\lambda \in L^\infty(\Omega)$ such that

$$\lim_{t \to 0^+} g(x,t)/t^{p-1} = \lambda(x) \quad \text{uniformly on } \Omega;$$

(g.4) there exists $\theta \in]0, 1/2[$ such that

$$G(x,t) \le \theta t g(x,t)$$

for all $t \ge 0$ and for almost all $x \in \Omega$, where

$$G(x,t) = \begin{cases} \int_0^t g(x,\tau)\,d\tau & \text{if } t \ge 0, \\ 0 & \text{otherwise;} \end{cases}$$

(g.5) for all $t > 0$ and for almost all $x \in \Omega$,

$$\frac{d}{dt}\left[\frac{g(x,t)}{t}\right] > 0.$$

We have the following result on the number of solutions of problem $P_\varepsilon(\Omega, g)$.

(5.1) Theorem. *If the domains $\Omega, \Omega_1, \dots, \Omega_k$ and the matrix $a_\varepsilon(x)$ satisfy the conditions introduced in Section 2 and if the above conditions on g are satisfied, then there exists $\overline{\varepsilon} > 0$ such that for all $\varepsilon \in]0, \overline{\varepsilon}[$ problem $P_\varepsilon(\Omega, g)$ has at least $k+1$ solutions $u_{\varepsilon,1}, \dots, u_{\varepsilon,k+1}$.*

Let us observe that a positive function $u_\varepsilon \in H_0^{1,2}(\Omega)$ is a solution of P_ε if and only if u_ε is a critical point for the functional $F_\varepsilon : H_0^{1,2}(\Omega) \to \mathbb{R}$,

$$F_\varepsilon(u) = \frac{1}{2} \int_\Omega \frac{a_\varepsilon^{i,j}(x)}{\varepsilon} \partial_{x_i} u \partial_{x_j} u\,dx - \frac{1}{\varepsilon} \int_\Omega G(x, u(x))\,dx.$$

Define the set

$$M_\varepsilon = \{u \in H_0^{1,2}(\Omega) : u \ne 0 \text{ in } \Omega \text{ and } J_\varepsilon(u) = 0\},$$

where

$$J_\varepsilon(u) = F_\varepsilon'(u)[u] = \int_\Omega \frac{a_\varepsilon^{i,j}(x)}{\varepsilon} \partial_{x_i} u \partial_{x_j} u\,dx - \frac{1}{\varepsilon} \int_\Omega g(x, u(x)) u(x)\,dx.$$

We will now prove some properties of M_ε which we need for proving Theorem (5.1).

(5.2) Proposition. *For all ε small enough, M_ε is a C^1-manifold of codimension 1 in $H_0^{1,2}(\Omega)$.*

Proof. First observe that for all $u \in M_\varepsilon$,

$$\operatorname{meas}\{x \in \Omega : u(x) > 0\} > 0.$$

In fact, if $u \le 0$ in Ω, since it is not restrictive to assume that $g(x,t) = 0$ for all $t \le 0$ and for almost all $x \in \Omega$ and $J_\varepsilon(u) = 0$, we have $\int_\Omega a_\varepsilon^{i,j}(x)\partial_{x_i}u\partial_{x_j}u\,dx = 0$, which implies $u = 0$.

Therefore, by (g.5), for all $u \in M_\varepsilon$,

$$\begin{aligned} J_\varepsilon'(u)[u] &= 2\int_\Omega \frac{a_\varepsilon^{i,j}(x)}{\varepsilon}\partial_{x_i}u\partial_{x_j}u\,dx - \frac{1}{\varepsilon}\int_\Omega g_t'(x,u(x))u(x)^2\,dx \\ &\quad - \frac{1}{\varepsilon}\int_\Omega g(x,u(x))u(x)\,dx \\ &= \frac{1}{\varepsilon}\int_\Omega g(x,u(x))u(x)\,dx - \frac{1}{\varepsilon}\int_\Omega g_t'(x,u(x))u(x)^2\,dx < 0. \end{aligned}$$

So the assertion follows as $J_\varepsilon \in C^1$. □

(5.3) Proposition. *For all $\varepsilon > 0$ small enough there exists $r > 0$ such that $\|u\| > r$ for all $u \in M_\varepsilon$.*

Proof. It is sufficient to prove that there exists a constant $r > 0$ such that $J_\varepsilon(u) > 0$ for all $u \ne 0$ with $\|u\| \le r$. Since $F_\varepsilon \in C^2(H_0^{1,2}(\Omega);\mathbb{R})$ and $F_\varepsilon'(0) = 0$, it suffices to prove that $F_\varepsilon''(0)[v][v] > 0$ for all $v \in H_0^{1,2}(\Omega)\setminus\{0\}$. By calculation, we have

$$F_\varepsilon''(u)[v][v] = 2\int_\Omega \frac{a_\varepsilon^{i,j}(x)}{\varepsilon}\partial_{x_i}v\partial_{x_j}v\,dx - \frac{1}{\varepsilon}\int_\Omega g_t'(x,u(x))v(x)^2\,dx.$$

Since $g_t'(x,0) = 0$ for almost all $x \in \Omega$ from (g.1) and (g.3) and since we have $\inf_{x\in\Omega}\Lambda_1(\varepsilon,x) > 0$ for $\varepsilon > 0$ small enough, we can conclude that $F_\varepsilon''(0)[v][v] > 0$ for all $v \in H_0^{1,2}(\Omega)\setminus\{0\}$. □

(5.4) Proposition. *For all $\varepsilon > 0$ small enough,*

$$\inf\{F_\varepsilon(u) : u \in M_\varepsilon\} > 0.$$

Proof. Let $u \in M_\varepsilon$ with ε fixed. By (g.4) we obtain

$$\begin{aligned} F_\varepsilon(u) &= \frac{1}{2}\int_\Omega \frac{a_\varepsilon^{i,j}(x)}{\varepsilon}\partial_{x_i}u\partial_{x_j}u\,dx - \frac{1}{\varepsilon}\int_\Omega G(x,u(x))\,dx \\ &\ge \frac{1}{2}\int_\Omega \frac{a_\varepsilon^{i,j}(x)}{\varepsilon}\partial_{x_i}u\partial_{x_j}u\,dx - \frac{1}{\varepsilon}\theta\int_\Omega g(x,u(x))u(x)\,dx \\ &= \left(\frac{1}{2}-\theta\right)\int_\Omega \frac{a_\varepsilon^{i,j}(x)}{\varepsilon}\partial_{x_i}u\partial_{x_j}u\,dx \ge \left(\frac{1}{2}-\theta\right)\|u\|^2\frac{\inf_{x\in\Omega}\Lambda_1(\varepsilon,x)}{\varepsilon}. \end{aligned}$$

The conclusion follows from Proposition (5.3) and from (a.2). □

(5.5) Proposition. *Set* $S := \{u \in H_0^{1,2}(\Omega) : u^+ \neq 0\}$. *Then for every* $u \in S$ *there exists a unique* $\alpha_\varepsilon \in \mathbb{R}^+$ *such that*

$$\alpha_\varepsilon \int_\Omega a_\varepsilon^{i,j}(x)\partial_{x_i} u \partial_{x_j} u \, dx = \int_\Omega g(x, \alpha_\varepsilon u(x)) u(x) \, dx.$$

Proof. Let $u \in S$ and consider the map $\mathbb{R} \ni z \mapsto F_\varepsilon(zu)$. It is continuous; it achieves its maximum as, by (g.3) and (g.4), $z = 0$ is a strictly local minimum and $\lim_{z\to\infty} F_\varepsilon(zu) = -\infty$. Let $z = \alpha_\varepsilon$ be such a maximum point. Then necessarily

$$\frac{\partial}{\partial z} F_\varepsilon(zu)\bigg|_{z=\alpha_\varepsilon} = \alpha_\varepsilon \int_\Omega \frac{a_\varepsilon^{i,j}(x)}{\varepsilon} \partial_{x_i} u \partial_{x_j} u \, dx - \frac{1}{\varepsilon} \int_\Omega g(x, \alpha_\varepsilon u(x)) u(x) \, dx = 0,$$

from which the existence of α_ε follows. The uniqueness of α_ε follows by observing that (g.5) means that $g(x, zu(x))/(zu(x))$ is a strictly increasing function of z, for all $x \in \Omega$ such that $u(x) > 0$. □

Roughly speaking, Proposition (5.5) says that, if we fix $\varepsilon > 0$ and $u \in S$, the half-line connecting the origin in $H_0^{1,2}(\Omega)$ with u meets the manifold M_ε in a unique point $\alpha_\varepsilon u$.

(5.6) Remark. By the implicit function theorem, the function $\psi_\varepsilon : S \to \mathbb{R}$ defined by $\psi_\varepsilon(u) = \alpha_\varepsilon$ if and only if $F_\varepsilon(\alpha_\varepsilon u)$ is the maximum value of the function $\mathbb{R} \ni z \mapsto F_\varepsilon(zu)$ (i.e. $\alpha_\varepsilon u \in M_\varepsilon$) is continuous ($C^1$) for all $\varepsilon > 0$ small enough.

(5.7) Proposition. *For all* $\varepsilon > 0$ *and for every pair of functions* $(u_s, u_t) \in M_\varepsilon \times M_\varepsilon$ *such that* $u_s \in H_0^{1,2}(\Omega'_s)$, $u_t \in H_0^{1,2}(\Omega'_t)$ (see Section 2) *there exists a continuous path* $\gamma_\varepsilon : [0,1] \to M_\varepsilon$ *connecting* u_s *to* u_t,

$$\gamma_\varepsilon(\tau) = \alpha_\varepsilon(\tau)[\tau u_t + (1-\tau)u_s]$$

with $\alpha_\varepsilon(\tau)$ *a real number, depending on* ε *and on* τ, *such that* $\alpha_\varepsilon(\tau) \geq 1$ *for all* $\tau \in [0,1]$.

Proof. Existence and continuity of γ_ε follow from Proposition (5.5) and Remark (5.6).

Let us show that $\alpha_\varepsilon(\tau) \geq 1$ for all $\tau \in [0,1]$. Observe that $\alpha_\varepsilon(0) = 1$ and $\alpha_\varepsilon(1) = 1$. Consider the functional

$$\mathbb{R} \ni z \mapsto F_\varepsilon(z(\tau u_t + (1-\tau)u_s));$$

if $\frac{\partial}{\partial z} F_\varepsilon(z(\tau u_t + (1-\tau)u_s))|_{z=1}$ is greater than 0, then $\alpha_\varepsilon(\tau) > 1$ for all $\tau \in]0,1[$. Let us compute, by using the definition of M_ε:

$$\begin{aligned}
\frac{\partial}{\partial z} & F_\varepsilon(z(\tau u_t + (1-\tau)u_s))\Big|_{z=1} \\
&= \tau^2 \int_{\Omega'_t} \frac{a_\varepsilon^{i,j}(x)}{\varepsilon} \partial_{x_i} u_t \partial_{x_j} u_t \, dx + (1-\tau)^2 \int_{\Omega'_s} \frac{a_\varepsilon^{i,j}(x)}{\varepsilon} \partial_{x_i} u_s \partial_{x_j} u_s \, dx \\
&\quad - \frac{1}{\varepsilon} \int_{\Omega'_t} g(x, \tau u_t) \tau u_t \, dx - \frac{1}{\varepsilon} \int_{\Omega'_s} g(x, (1-\tau)u_s)(1-\tau) u_s \, dx \\
&= \frac{\tau^2}{\varepsilon} \int_{\Omega'_t} \left[g(x, u_t) - \frac{g(x, \tau u_t)}{\tau} \right] u_t \, dx \\
&\quad + \frac{(1-\tau)^2}{\varepsilon} \int_{\Omega'_s} \left[g(x, u_s) - \frac{g(x, (1-\tau)u_s)}{1-\tau} \right] u_s \, dx > 0
\end{aligned}$$

where the last inequality is a consequence of (g.5). □

(5.8) Lemma. *The following statements are equivalent:*

(i) $u_\varepsilon \geq 0$, $u_\varepsilon \neq 0$ *is a critical point of* F_ε;

(ii) $u_\varepsilon \geq 0$, $u_\varepsilon \in M_\varepsilon$ *is a critical point of* F_ε *constrained on* M_ε.

Proof. (i)⇒(ii). Since $J_\varepsilon(u_\varepsilon) = F'_\varepsilon(u_\varepsilon)[u_\varepsilon] = 0$ it follows that u_ε belongs to M_ε and it is a constrained critical point of F_ε constrained on M_ε.

(ii)⇒(i). By assumption there exists $\mu \in \mathbb{R}$ such that $F'_\varepsilon(u_\varepsilon) = \mu J'_\varepsilon(u_\varepsilon)$ and $u_\varepsilon \in M_\varepsilon$. So we have

$$0 = J_\varepsilon(u_\varepsilon) = F'_\varepsilon(u_\varepsilon)[u_\varepsilon] = \mu J'_\varepsilon(u_\varepsilon)[u_\varepsilon]$$

with $J'_\varepsilon(u_\varepsilon)[u_\varepsilon] < 0$ (see the proof of Proposition (5.2)). Consequently, $\mu = 0$. □

(5.9) Definition. For all $\varepsilon > 0$ and for all $s = 1, \ldots, k$, we set

$$\overline{\mu}_{\varepsilon,s} := \inf\{F_\varepsilon(u) : u \in H_0^{1,2}(\Omega'_s) \cap M_\varepsilon\}$$

and we denote by $\omega_{\varepsilon,s} \in H_0^{1,2}(\Omega'_s) \cap M_\varepsilon$ a function which realizes $\overline{\mu}_{\varepsilon,s}$. For simplicity of notation, we write $\overline{\mu}_{0,s}$ and $\overline{v}_{0,s}$ instead of $\mu^\lambda_{0,s}$ and $v^\lambda_{0,s}$ respectively (see Notations), where $\lambda(x)$ is the positive function appearing in the assumption (g.3). Let $\overline{\mu}_{\varepsilon,\mathrm{m}} = \min_{1\leq s\leq k} \overline{\mu}_{\varepsilon,s}$ and $\overline{\mu}_{\varepsilon,\mathrm{M}} = \max_{1\leq s\leq k} \overline{\mu}_{\varepsilon,s}$.

(5.10) Lemma. *For all* $s = 1, \ldots, k$,

$$\limsup_{\varepsilon\to 0^+} \frac{\overline{\mu}_{\varepsilon,s}}{\varepsilon^{2/(p-2)}} \leq \left(\frac{1}{2} - \frac{1}{p}\right) \overline{\mu}_{0,s}^{\,p/(p-2)},$$

where p *is the number in* $]2, 2N/(N-2)[$ *which appears in* (g.3).

Proof. By Proposition (5.5), for all $\varepsilon > 0$ there exists a unique positive constant α_ε such that $\alpha_\varepsilon \overline{v}_{0,s} \in M_\varepsilon$, where $\overline{v}_{0,s}$ is the function in $H_0^{1,2}(\Omega'_s)$ that realizes $\overline{\mu}_{0,s}$.

Let us first observe that $\lim_{\varepsilon\to 0^+}\alpha_\varepsilon = 0$; in fact, by contradiction suppose that there is an infinitesimal sequence $(\varepsilon_n)_n$ such that $\lim_{n\to\infty}\alpha_{\varepsilon_n} = \alpha > 0$. Since $\lim_{\varepsilon\to 0^+} a^{i,j}_\varepsilon(x)/\varepsilon = a^{i,j}(x)$ uniformly in Ω_s and $\int_{\Omega'_s\setminus\Omega_s}|D\overline{v}_{0,s}|^2\,dx = 0$, we have

$$\begin{aligned}\lim_{n\to\infty} & F_{\varepsilon_n}(\alpha_{\varepsilon_n}\overline{v}_{0,s})\\ &= \lim_{n\to\infty}\left[\frac{\alpha^2_{\varepsilon_n}}{2}\int_{\Omega'_s}\frac{a^{i,j}_{\varepsilon_n}(x)}{\varepsilon_n}\partial_{x_i}\overline{v}_{0,s}\partial_{x_j}\overline{v}_{0,s}\,dx - \frac{1}{\varepsilon_n}\int_{\Omega'_s}G(x,\alpha_{\varepsilon_n}\overline{v}_{0,s})\,dx\right]\\ &= \frac{\alpha^2}{2}\int_{\Omega_s}a^{i,j}(x)\partial_{x_i}\overline{v}_{0,s}\partial_{x_j}\overline{v}_{0,s}\,dx - \lim_{n\to\infty}\frac{1}{\varepsilon_n}\int_{\Omega'_s}G(x,\alpha\overline{v}_{0,s})\,dx = -\infty,\end{aligned}$$

which contradicts the fact that $F_{\varepsilon_n}(\alpha_{\varepsilon_n}\overline{v}_{0,s}) \geq 0$ for all $n \geq 1$ since $\alpha_{\varepsilon_n}\overline{v}_{0,s}$ belongs to M_{ε_n} and it is the maximum point for the function $\mathbb{R}^+ \ni z \mapsto F_{\varepsilon_n}(z\overline{v}_{0,s})$ (see Proposition (5.5)).

We next prove that $\lim_{\varepsilon\to 0^+}\alpha_\varepsilon/\varepsilon^{1/(p-2)} = \overline{\mu}^{1/(p-2)}_{0,s}$. In fact, since $\alpha_\varepsilon\overline{v}_{0,s}$ belongs to M_ε for all $\varepsilon > 0$, α_ε satisfies the equation

$$\int_{\Omega'_s}\frac{a^{i,j}_\varepsilon(x)}{\varepsilon}\partial_{x_i}\overline{v}_{0,s}\partial_{x_j}\overline{v}_{0,s}\,dx = \frac{1}{\varepsilon}\int_{\Omega'_s}\frac{g(x,\alpha_\varepsilon\overline{v}_{0,s}(x))\alpha_\varepsilon\overline{v}_{0,s}(x)}{{\alpha_\varepsilon}^2}\,dx. \tag{5.1}$$

We are now going to evaluate the limits as $\varepsilon \to 0^+$ of both sides of the above equation. By (a.3),

$$\lim_{\varepsilon\to 0^+}\int_{\Omega'_s}\frac{a^{i,j}_\varepsilon(x)}{\varepsilon}\partial_{x_i}\overline{v}_{0,s}\partial_{x_j}\overline{v}_{0,s}\,dx = \lim_{\varepsilon\to 0^+}\int_{\Omega_s}\frac{a^{i,j}_\varepsilon(x)}{\varepsilon}\partial_{x_i}\overline{v}_{0,s}\partial_{x_j}\overline{v}_{0,s}\,dx = \overline{\mu}_{0,s}.$$

In order to evaluate the other limit, we write

$$\frac{1}{\varepsilon}\int_{\Omega'_s}\frac{g(x,\alpha_\varepsilon\overline{v}_{0,s}(x))\alpha_\varepsilon\overline{v}_{0,s}(x)}{\alpha^2_\varepsilon}\,dx = \frac{\alpha^{p-2}_\varepsilon}{\varepsilon}\int_{\Omega'_s}\frac{g(x,\alpha_\varepsilon\overline{v}_{0,s}(x))\overline{v}_{0,s}(x)^p}{(\alpha_\varepsilon\overline{v}_{0,s}(x))^{p-1}}\,dx.$$

Since $\lim_{\varepsilon\to 0^+}\alpha_\varepsilon = 0$ and (g.3) holds,

$$\lim_{\varepsilon\to 0^+}\frac{g(x,\alpha_\varepsilon\overline{v}_{0,s}(x))\overline{v}_{0,s}(x)^p}{(\alpha_\varepsilon\overline{v}_{0,s}(x))^{p-1}} = \lambda(x)\overline{v}_{0,s}(x)^p \quad \text{for almost all } x \in \Omega'_s.$$

Moreover, from (g.2) and (g.3) (where we can assume $q \geq p$), there exist $C > 0$ and $\eta > 0$ such that

$$\begin{aligned}&\left|\frac{g(x,\alpha_\varepsilon\overline{v}_{0,s}(x))\overline{v}_{0,s}(x)^p}{(\alpha_\varepsilon\overline{v}_{0,s}(x))^{p-1}}\right|\\ &\qquad\leq \begin{cases}(\lambda(x)+C)\overline{v}_{0,s}(x)^p & \text{if } x\in\Omega'_s,\ |\alpha_\varepsilon\overline{v}_{0,s}(x)|\leq\eta,\\ (a/\eta^{p-1} + a(\alpha_\varepsilon\overline{v}_{0,s}(x))^{q-p})\overline{v}_{0,s}(x)^p & \text{if } x\in\Omega'_s,\ |\alpha_\varepsilon\overline{v}_{0,s}(x)|>\eta,\end{cases}\end{aligned}$$

where a is the positive constant which appears in (g.2). From the Lebesgue Theorem we deduce that

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_{\Omega_s'}\frac{g(x,\alpha_\varepsilon\overline{v}_{0,s}(x))\alpha_\varepsilon\overline{v}_{0,s}(x)}{\alpha_\varepsilon^2}\,dx=\bigg(\lim_{\varepsilon\to 0^+}\frac{\alpha_\varepsilon^{p-2}}{\varepsilon}\bigg)\int_{\Omega_s'}\lambda(x)|\overline{v}_{0,s}(x)|^p\,dx$$
$$=\lim_{\varepsilon\to 0^+}\frac{\alpha_\varepsilon^{p-2}}{\varepsilon}.$$

Passing to the limit as $\varepsilon\to 0^+$ in (5.1), we see that

$$\lim_{\varepsilon\to 0^+}\alpha_\varepsilon/\varepsilon^{1/(p-2)}=\overline{\mu}_{0,s}^{1/(p-2)}.$$

The conclusion is quite near; in fact, by definition of $\overline{\mu}_{\varepsilon,s}$,

$$\begin{aligned}\frac{\overline{\mu}_{\varepsilon,s}}{\varepsilon^{2/(p-2)}}&\le\frac{F_\varepsilon(\alpha_\varepsilon\overline{v}_{0,s})}{\varepsilon^{2/(p-2)}}\\&=\frac{\alpha_\varepsilon^2}{2\varepsilon^{2/(p-2)}}\int_{\Omega_s'}\frac{a_\varepsilon^{i,j}(x)}{\varepsilon}\partial_{x_i}\overline{v}_{0,s}\partial_{x_j}\overline{v}_{0,s}\,dx\\&\quad-\frac{1}{\varepsilon^{p/(p-2)}}\int_{\Omega_s'}G(x,\alpha_\varepsilon\overline{v}_{0,s}(x))\,dx\\&=\frac{\alpha_\varepsilon^2}{2\varepsilon^{2/(p-2)}}\int_{\Omega_s'}\frac{a_\varepsilon^{i,j}(x)}{\varepsilon}\partial_{x_i}\overline{v}_{0,s}\partial_{x_j}\overline{v}_{0,s}\,dx\\&\quad-\frac{\alpha_\varepsilon^p}{\varepsilon^{p/(p-2)}}\int_{\Omega_s'}\frac{G(x,\alpha_\varepsilon\overline{v}_{0,s}(x))\overline{v}_{0,s}(x)^p}{(\alpha_\varepsilon\overline{v}_{0,s}(x))^p}\,dx.\end{aligned}\tag{5.2}$$

Since $\lim_{\varepsilon\to 0^+}\alpha_\varepsilon=0$ and (g.3) holds, we see that

$$\lim_{\varepsilon\to 0^+}\frac{G(x,\alpha_\varepsilon\overline{v}_{0,s}(x))\overline{v}_{0,s}(x)^p}{(\alpha_\varepsilon\overline{v}_{0,s}(x))^p}=\frac{\lambda(x)}{p}\overline{v}_{0,s}(x)^p\quad\text{for almost all } x\in\Omega_s';$$

(g.2) and (g.3) ensure that there exist constants $C,\eta,b>0$ such that

$$\bigg|\frac{G(x,\alpha_\varepsilon\overline{v}_{0,s}(x))\overline{v}_{0,s}(x)^p}{(\alpha_\varepsilon\overline{v}_{0,s}(x))^p}\bigg|\le\begin{cases}(\lambda(x)/p+C)\overline{v}_{0,s}(x)^p & \text{if } x\in\Omega_s',\ |\alpha_\varepsilon\overline{v}_{0,s}(x)|\le\eta,\\(b/\eta^p+b(\alpha_\varepsilon\overline{v}_{0,s}(x))^{q-p})\overline{v}_{0,s}(x)^p & \text{if } x\in\Omega_s',\ |\alpha_\varepsilon\overline{v}_{0,s}(x)|>\eta.\end{cases}$$

From (a.3) and the Lebesgue Theorem, if we pass to the limit $\varepsilon\to 0^+$ in (5.2) we finally obtain

$$\limsup_{\varepsilon\to 0^+}\frac{\overline{\mu}_{\varepsilon,s}}{\varepsilon^{2/(p-2)}}\le\bigg(\frac{1}{2}-\frac{1}{p}\bigg)\overline{\mu}_{0,s}^{p/(p-2)}.$$ □

(5.11) Corollary. *For $\delta \in (0,1)$ and for all $s = 1, \dots, k$ we have*

$$\limsup_{\varepsilon\to 0^+} \frac{\inf\{F_\varepsilon(u) : u \in H_0^{1,2}(\Omega) \cap M_\varepsilon, \int_{\Omega'_s} \lambda(x) \frac{|u(x)|^p}{\|u\|^p_{(\lambda,p)}}\, dx > 1-\delta\}}{\varepsilon^{2/(p-2)}} \le \Big(\frac{1}{2} - \frac{1}{p}\Big)\overline{\mu}_{0,s}^{p/(p-2)}.$$

(see Notations for $\|u\|_{(\lambda,p)}$).

(5.12) Lemma. *There exists $\delta \in (0,1)$ such that for all $s = 1, \dots, k$,*

$$\liminf_{\varepsilon\to 0^+} \frac{\inf\big\{F_\varepsilon(u) : u \in H_0^{1,2}(\Omega) \cap M_\varepsilon, \int_{\Omega'_s} \lambda(x) \frac{|u(x)|^p}{\|u\|^p_{(\lambda,p)}}\, dx = 1-\delta\big\}}{\varepsilon^{2/(p-2)}} > \Big(\frac{1}{2} - \frac{1}{p}\Big)\overline{\mu}_{0,s}^{p/(p-2)}.$$

Proof. By contradiction assume that there exist a sequence $(\varepsilon_n)_n \to 0$ of positive numbers and a sequence $(u_n)_n$ of functions such that $u_n \in M_{\varepsilon_n}$,

$$\int_{\Omega'_s} \lambda(x) \frac{|u_n(x)|^p}{\|u_n\|^p_{(\lambda,p)}}\, dx = 1-\delta \quad \text{for all } n = 1, 2, \dots \tag{5.3}$$

and

$$\lim_{n\to\infty} \frac{F_{\varepsilon_n}(u_n)}{\varepsilon_n^{2/(p-2)}} \le \Big(\frac{1}{2} - \frac{1}{p}\Big)\overline{\mu}_{0,s}^{p/(p-2)}. \tag{5.4}$$

Let $\overline{u}_n(x) := u_n(x)/\|u_n\|_{(\lambda,p)}$ and $\widetilde{u}_n(x) := u_n(x)/\|u_n\|_q$ (we can assume $q \ge p$ without any loss of generality). The proof consists of the following three steps.

Step 1. We prove that the sequence $(\widetilde{u}_n)_n$ is bounded in $H_0^{1,2}(\Omega)$. In fact, if we set $\beta_n = \|u_n\|_q$, from (g.4) we see that

$$\frac{F_{\varepsilon_n}(u_n)}{\varepsilon_n^{2/(p-2)}} \ge \frac{1}{\varepsilon_n^{2/(p-2)}}\Big(\frac{1}{2} - \theta\Big)\beta_n^2 \int_\Omega \frac{a^{i,j}_{\varepsilon_n}(x)}{\varepsilon_n} \partial_{x_i}\widetilde{u}_n \partial_{x_j}\widetilde{u}_n\, dx, \tag{5.5}$$

which, in particular, implies $\beta_n \to 0$. Since $u_n \in M_{\varepsilon_n}$ and $\int_\Omega |\widetilde{u}_n(x)|^q\, dx = 1$ for all $n = 1, 2, \dots$, there exist two positive constants C_1 and C_2 such that

$$\begin{aligned} 0 < C_1 &\le \int_\Omega \frac{a^{i,j}_{\varepsilon_n}(x)}{\varepsilon_n} \partial_{x_i}\widetilde{u}_n \partial_{x_j}\widetilde{u}_n\, dx = \frac{1}{\varepsilon_n \beta_n^2} \int_\Omega g(x, \beta_n \widetilde{u}_n)\beta_n \widetilde{u}_n\, dx \\ &= \frac{\beta_n^{p-2}}{\varepsilon_n} \int_\Omega \frac{g(x, \beta_n \widetilde{u}_n(x))\widetilde{u}_n(x)^p}{\beta_n^{p-1}\widetilde{u}_n(x)^{p-1}}\, dx \le C_2 \frac{\beta_n^{p-2}}{\varepsilon_n}; \end{aligned} \tag{5.6}$$

the last inequality is due to (g.2) and (g.3). Since $\liminf_{\varepsilon\to 0^+}(\inf_{x\in\Omega} \Lambda_1(\varepsilon, x)/\varepsilon) > 0$, if we choose $\lambda_1 > 0$ such that $\inf_{x\in\Omega} \Lambda_1(\varepsilon, x)/\varepsilon \ge \lambda_1 > 0$, from (5.5) and

(5.6) it follows that

$$A \geq \frac{F_{\varepsilon_n}(u_n)}{\varepsilon_n^{2/(p-2)}} \geq \left(\frac{1}{2} - \theta\right) B \int_\Omega \frac{a^{i,j}_{\varepsilon_n}(x)}{\varepsilon_n} \partial_{x_i} \widetilde{u}_n \partial_{x_j} \widetilde{u}_n \, dx$$
$$\geq \left(\frac{1}{2} - \theta\right) B \lambda_1 \int_\Omega |D\widetilde{u}_n|^2 \, dx$$

with suitable positive constants A and B. Then $(\widetilde{u}_n)_n$ is bounded in $H^{1,2}_0(\Omega)$; hence there is a subsequence of $(\widetilde{u}_n)_n$ (which we still call $(\widetilde{u}_n)_n$) converging to a function $\widetilde{u} \in H^{1,2}_0(\Omega)$, weakly in $H^{1,2}_0(\Omega)$, in $L^p(\Omega)$, in $L^q(\Omega)$ and almost everywhere in Ω.

Since $\widetilde{u} \neq 0$ (because $\|\widetilde{u}\|_q = 1$), we have $\|\widetilde{u}\|_{(\lambda,p)} \neq 0$ and so $\overline{u}_n = \widetilde{u}_n / \|\widetilde{u}_n\|_{(\lambda,p)}$ converges in $L^p(\Omega)$ to $\overline{u} := \widetilde{u}/\|\widetilde{u}\|_{(\lambda,p)} \in H^{1,2}_0(\Omega)$. We can also assume that for this subsequence there exists

$$\lim_{n\to\infty} \int_\Omega \frac{a^{i,j}_{\varepsilon_n}(x)}{\varepsilon_n} \partial_{x_i} \overline{u}_n \partial_{x_j} \overline{u}_n \, dx = \overline{\mu};$$

then $\overline{\mu} < \infty$ and, if we set $\alpha_n := \|u_n\|_{(\lambda,p)}$, since $\alpha_n \overline{u}_n \in M_{\varepsilon_n}$ for all $n = 1, 2, \ldots$, we get, arguing as in the proof of Lemma (5.10),

$$\lim_{n\to\infty} \alpha_n^{p-2}/\varepsilon_n = \overline{\mu}. \tag{5.7}$$

Step 2. We prove that $\overline{\mu} \leq \overline{\mu}_{0,s}$. By using (5.4), for n large enough we have

$$\left(\frac{1}{2} - \frac{1}{p}\right) \overline{\mu}_{0,s}^{p/(p-2)} \geq \frac{F_{\varepsilon_n}(\alpha_n \overline{u}_n)}{\varepsilon_n^{2/(p-2)}} \tag{5.8}$$
$$= \frac{1}{2} \frac{\alpha_n^2}{\varepsilon_n^{2/(p-2)}} \int_\Omega \frac{a^{i,j}_{\varepsilon_n}(x)}{\varepsilon_n} \partial_{x_i} \overline{u}_n \partial_{x_j} \overline{u}_n \, dx$$
$$- \frac{1}{\varepsilon_n^{p/(p-2)}} \int_\Omega G(x, \alpha_n \overline{u}_n(x)) \, dx;$$

now, since $\overline{u}_n \to \overline{u}$ in $L^p(\Omega)$ and in $L^q(\Omega)$, and (g.2), (g.3) hold, we can use the Lebesgue Theorem as in the proof of Lemma (5.10) to obtain, as $n \to \infty$,

$$\left(\frac{1}{2} - \frac{1}{p}\right) \overline{\mu}_{0,s}^{p/(p-2)} \geq \limsup_{n\to\infty} \frac{F_{\varepsilon_n}(\alpha_n \overline{u}_n)}{\varepsilon_n^{2/(p-2)}} \geq \left(\frac{1}{2} - \frac{1}{p}\right) \overline{\mu}^{p/(p-2)}$$

and consequently

$$\overline{\mu} \leq \overline{\mu}_{0,s}. \tag{5.9}$$

Step 3. We arrive at a contradiction. Since for all $\eta > 0$

$$\liminf_{\varepsilon\to 0^+} \left(\inf \left\{ \frac{\Lambda_1(\varepsilon, x)}{\varepsilon} : x \in \Omega \setminus \bigcup_{t=1}^k \Omega_t(\eta) \right\} \right) = \infty,$$

we must have $\int_{\Omega\setminus\bigcup_{t=1}^k\Omega_t}|D\overline{u}|^2\,dx=0$. Since $\overline{u}\in H_0^{1,2}(\Omega)$ and every connected component of $\overline{\Omega}\setminus\bigcup_{t=1}^k\Omega_t'$ meets $\partial\Omega$, it follows that $\overline{u}(x)=0$ for almost all $x\in\Omega\setminus\bigcup_{t=1}^k\Omega_t'$.

As $\overline{\Omega_{t_1}'}\cap\overline{\Omega_{t_2}'}=\emptyset$ for all $t_1,t_2\in\{1,\dots,k\}$, $t_1\neq t_2$, $\overline{u}$ can be written in the form $\overline{u}(x)=\overline{u}_1(x)+\dots+\overline{u}_k(x)$ with $\overline{u}_t\in H_0^{1,2}(\Omega_t')$ for all $t=1,\dots,k$. Since $\int_\Omega\lambda(x)|\overline{u}(x)|^p\,dx=1$ and $\int_{\Omega_s'}\lambda(x)|\overline{u}(x)|^p\,dx=1-\delta$, there exists at least one $t\in\{1,\dots,k\}$, $t\neq s$, such that $\int_{\Omega_t'}\lambda(x)|\overline{u}(x)|^p\,dx>0$. Moreover, $\int_{\Omega_t'\setminus\Omega_t}|D\overline{u}(x)|^2\,dx=0$ for $t=1,\dots,k$. Hence $\overline{u}$ satisfies the assumptions of Lemma (4.1) if $\delta\in\,]0,(\overline{\mu}_{\mathrm{m}}/\overline{\mu}_{\mathrm{M}})^{p/(p-2)}[$ and so

$$\begin{aligned}\overline{\mu}_{0,s}&<\sum_{t=1}^k\int_{\Omega_t}a^{i,j}(x)\partial_{x_i}\overline{u}_t\partial_{x_j}\overline{u}_t\,dx\le\liminf_{n\to\infty}\int_\Omega\frac{a^{i,j}_{\varepsilon_n}(x)}{\varepsilon_n}\partial_{x_i}\overline{u}_n\partial_{x_j}\overline{u}_n\,dx\\&\le\lim_{n\to\infty}\int_\Omega\frac{a^{i,j}_{\varepsilon_n}(x)}{\varepsilon_n}\partial_{x_i}\overline{u}_n\partial_{x_j}\overline{u}_n\,dx=\overline{\mu},\end{aligned}$$

contrary to (5.9). □

We are now ready to prove Theorem (5.1).

Proof of Theorem (5.1). The solutions of problem $P_\varepsilon(\Omega,g)$ are the critical points of the functional

$$F_\varepsilon(u)=\frac12\int_\Omega\frac{a^{i,j}_\varepsilon(x)}{\varepsilon}\partial_{x_i}u\partial_{x_j}u\,dx-\frac1\varepsilon\int_\Omega G(x,u(x))\,dx$$

constrained to lie upon the manifold

$$M_\varepsilon=\left\{u\in H_0^{1,2}(\Omega):u\neq0\text{ and }\int_\Omega a^{i,j}_\varepsilon(x)\partial_{x_i}u\partial_{x_j}u\,dx=\int_\Omega g(x,u(x))u(x)\,dx\right\}$$

(see Lemma (5.8)). According to Corollary (5.11) and Lemma (5.12) there exists $\delta\in\,]0,1[$ such that, for all $s=1,\dots,k$,

$$\liminf_{\varepsilon\to0^+}\frac{\inf\{F_\varepsilon(u):u\in M_\varepsilon,\ \int_{\Omega_s'}\lambda(x)\frac{|u(x)|^p}{\|u\|^p_{(\lambda,p)}}\,dx=1-\delta\}}{\varepsilon^{2/(p-2)}}>\left(\frac12-\frac1p\right)\overline{\mu}_{0,s}^{p/(p-2)}\tag{5.10}$$

and

$$\limsup_{\varepsilon\to0^+}\frac{\inf\{F_\varepsilon(u):u\in M_\varepsilon,\ \int_{\Omega_s'}\lambda(x)\frac{|u(x)|^p}{\|u\|^p_{(\lambda,p)}}\,dx>1-\delta\}}{\varepsilon^{2/(p-2)}}\le\left(\frac12-\frac1p\right)\overline{\mu}_{0,s}^{p/(p-2)}.\tag{5.11}$$

From (5.10) and (5.11) we can infer that there exists $\overline{\varepsilon} > 0$ such that for all $\varepsilon \in]0, \overline{\varepsilon}[$ and for all $s = 1, \ldots, k$, there exists a function $u_{\varepsilon,s}$ which is a minimum point for the functional F_ε in the set

$$\biggl\{u \in M_\varepsilon : \int_{\Omega'_s} \lambda(x) \frac{|u(x)|^p}{\|u\|^p_{(\lambda,p)}}\, dx > 1 - \delta\biggr\}.$$

Moreover, it is evident that

$$\limsup_{\varepsilon \to 0^+} \frac{F_\varepsilon(u_{\varepsilon,s})}{\varepsilon^{2/(p-2)}} \leq \biggl(\frac{1}{2} - \frac{1}{p}\biggr) \overline{\mu}_{0,s}^{\,p/(p-2)}. \tag{5.12}$$

We now prove that there exists another critical point $u_{\varepsilon,k+1}$ for F_ε on M_ε. For $\varepsilon > 0$ and $s = 1, \ldots, k$ we have already defined

$$\overline{\mu}_{\varepsilon,s} := \inf\{F_\varepsilon(u) : u \in H_0^{1,2}(\Omega'_s) \cap M_\varepsilon\};$$

let $\omega_{\varepsilon,s} \in H_0^{1,2}(\Omega'_s) \cap M_\varepsilon$ be a function that realizes $\overline{\mu}_{\varepsilon,s}$. Let $\overline{\mu}_{\varepsilon,\mathrm{M}} := \max_{1 \leq t \leq k} \overline{\mu}_{\varepsilon,t}$ for $\varepsilon \in]0, \overline{\varepsilon}[$. Suppose $\overline{\mu}_{\varepsilon,\mathrm{M}} = \overline{\mu}_{\varepsilon,1}$; let $\gamma_\varepsilon : [0,1] \to M_\varepsilon$ be a continuous path joining the functions $\omega_{\varepsilon,1}$ and $\omega_{\varepsilon,2}$,

$$\gamma_\varepsilon = \alpha_\varepsilon[\tau \omega_{\varepsilon,2} + (1-\tau)\omega_{\varepsilon,1}]$$

with α_ε a positive constant depending on ε and on $\tau \in [0,1]$ (see Proposition (5.7)). Define $m_\varepsilon := \max\{F_\varepsilon \circ \gamma_\varepsilon(\tau) : \tau \in [0,1]\}$. It is clear that $\overline{\mu}_{\varepsilon,\mathrm{M}} = \overline{\mu}_{\varepsilon,1} \leq m_\varepsilon$. Let $\mu'_\varepsilon > \overline{\mu}_{\varepsilon,\mathrm{M}}$ be such that

$$\mu'_\varepsilon < \inf \biggl\{F_\varepsilon(u) : u \in M_\varepsilon,\ \int_{\Omega'_1} \lambda(x) \frac{|u(x)|^p}{\|u\|_{(\lambda,p)}}\, dx = 1 - \delta\biggr\}.$$

For $\varepsilon \in]0, \overline{\varepsilon}[$, $\omega_{\varepsilon,1}$ and $\omega_{\varepsilon,2}$ belong to $\{u \in M_\varepsilon : F_\varepsilon(u) \leq \mu'_\varepsilon\}$ because of Lemma (5.10) and they are not connected in that sublevel, which does not meet the set

$$\biggl\{u \in M_\varepsilon : \int_{\Omega'_1} \lambda(x) \frac{|u(x)|^p}{\|u\|^p_{(\lambda,p)}}\, dx = 1 - \delta\biggr\};$$

while the two functions $\omega_{\varepsilon,1}$ and $\omega_{\varepsilon,2}$ are connected in the sublevel

$$\{u \in M_\varepsilon : F_\varepsilon(u) \leq m_\varepsilon\}$$

to which the curve γ_ε belongs.

Moreover, as $\liminf_{\varepsilon \to 0^+}(\inf_{x \in \Omega} \Lambda_1(\varepsilon, x)/\varepsilon) > 0$ and $p \in]2, 2N/(N-2)[$ (see (g.2)), the functional F_ε constrained on M_ε satisfies the well-known Palais–Smale condition. Hence, by the Mountain Pass Theorem of Ambrosetti–Rabinowitz, there is a critical value for F_ε on M_ε in the interval $]\mu'_\varepsilon, m_\varepsilon]$. Let $u_{\varepsilon,k+1}$ be the corresponding critical point: it is a solution for $P_\varepsilon(\Omega, g)$ and it is distinct from the previous ones because it corresponds to a greater critical level. □

The next proposition provides some qualitative information on the behaviour of the solutions $u_{\varepsilon,1}, \ldots, u_{\varepsilon,k+1}$ of problem $P_\varepsilon(\Omega, g)$ when ε goes to 0.

(5.13) Proposition. *For $\varepsilon > 0$ small enough and $s = 1, \dots, k+1$, let $u_{\varepsilon,s} \in H_0^{1,2}(\Omega)$ be the solutions of problem $P_\varepsilon(\Omega, g)$ given by Theorem (5.1). They have the following properties:*

(I) *for all $s = 1, \dots, k+1$,*

$$\lim_{\varepsilon\to 0^+} \|u_{\varepsilon,s}\|^2 = 0;$$

(II) *for all $s = 1, \dots, k$,*

$$\lim_{\varepsilon\to 0^+} \int_{\Omega'_s} \lambda(x) \frac{|u_{\varepsilon,s}(x)|^p}{\|u_{\varepsilon,s}\|^p_{(\lambda,p)}}\, dx = 1;$$

(III) *for all $s = 1, \dots, k$, if we set $\overline{u}_{\varepsilon,s}(x) = u_{\varepsilon,s}(x)/\|u_{\varepsilon,s}\|_{(\lambda,p)}$, we have*

$$\limsup_{\varepsilon\to 0^+} \|\overline{u}_{\varepsilon,s}\| < \infty;$$

moreover, if for a sequence $(\varepsilon_n)_n \to 0$, the sequence $(\overline{u}_{\varepsilon_n,s})_n \to \overline{u}_s$ in $L^p(\Omega)$, then $\overline{u}_s \in H_0^{1,2}(\Omega'_s)$ and it is a function which realizes the minimum $\overline{\mu}_{0,s}$;

(IV) *for all $s = 1, \dots, k$,*

$$\lim_{\varepsilon\to 0^+} \frac{F_\varepsilon(u_{\varepsilon,s})}{\varepsilon^{2/(p-2)}} = \left(\frac{1}{2} - \frac{1}{p}\right) \overline{\mu}_{0,s}^{p/(p-2)}.$$

Proof. (I) Since $\liminf_{\varepsilon\to 0^+} (\inf_{x\in\Omega} \Lambda_1(\varepsilon, x)/\varepsilon) > 0$, we can choose $\lambda_1 > 0$ such that $\inf_{x\in\Omega} \Lambda_1(\varepsilon, x)/\varepsilon \ge \lambda_1 > 0$. Then, from (g.4), for $\varepsilon > 0$ small enough and for all $s = 1, \dots, k$, we have

$$\begin{aligned}
&\|u_{\varepsilon,s}\|^2 \\
&\quad = \int_\Omega \frac{\Lambda_1(\varepsilon,x)}{\varepsilon} \cdot \frac{\varepsilon}{\Lambda_1(\varepsilon,x)} |Du_{\varepsilon,s}(x)|^2\, dx \le \frac{1}{\lambda_1} \int_\Omega \frac{\Lambda_1(\varepsilon,x)}{\varepsilon} |Du_{\varepsilon,s}(x)|^2\, dx \\
&\quad \le \frac{1/2 - \theta}{\lambda_1(1/2-\theta)} \int_\Omega \frac{a_\varepsilon^{i,j}(x)}{\varepsilon} \partial_{x_i} u_{\varepsilon,s} \partial_{x_j} u_{\varepsilon,s}\, dx \\
&\quad = \frac{1}{\lambda_1(1/2-\theta)} \left[\frac{1}{2} \int_\Omega \frac{a_\varepsilon^{i,j}(x)}{\varepsilon} \partial_{x_i} u_{\varepsilon,s} \partial_{x_j} u_{\varepsilon,s}\, dx - \frac{\theta}{\varepsilon} \int_\Omega g(x, u_{\varepsilon,s}(x)) u_{\varepsilon,s}(x)\, dx\right] \\
&\quad \le \frac{1}{\lambda_1(1/2-\theta)} \left[\frac{1}{2} \int_\Omega \frac{a_\varepsilon^{i,j}(x)}{\varepsilon} \partial_{x_i} u_{\varepsilon,s} \partial_{x_j} u_{\varepsilon,s}\, dx - \frac{1}{\varepsilon} \int_\Omega G(x, u_{\varepsilon,s}(x))\, dx\right] \\
&\quad = \frac{1}{\lambda_1(1/2-\theta)} F_\varepsilon(u_{\varepsilon,s})
\end{aligned}$$

with $\lim_{\varepsilon\to 0^+} F_\varepsilon(u_{\varepsilon,s}) = 0$, because of (5.12).

If $s = k+1$ the same conclusion holds. In fact, the proof of Theorem (5.1) shows that

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_{\varepsilon,k+1})/\overline{\mu}_{\varepsilon,\mathrm{M}} < \infty$$

and consequently, by Lemma (5.10),

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_{\varepsilon,k+1})/\varepsilon^{2/(p-2)} < \infty.$$

(II) If we define $\overline{u}_{\varepsilon,s}(x) := u_{\varepsilon,s}(x)/\|u_{\varepsilon,s}\|_{(\lambda,p)}$ for $s = 1,\dots,k$, it is evident that $\int_{\Omega'_s} \lambda(x)|\overline{u}_{\varepsilon,s}(x)|^p\,dx \le 1$. By contradiction, assume that there exists an infinitesimal sequence $(\varepsilon_n)_n$ of positive numbers such that

$$\lim_{n\to\infty}\int_{\Omega'_s} \lambda(x)|\overline{u}_{\varepsilon_n,s}(x)|^p\,dx < 1. \tag{5.13}$$

Since

$$\limsup_{\varepsilon\to 0^+}\frac{F_\varepsilon(u_{\varepsilon,s})}{\varepsilon^{2/(p-2)}} \le \left(\frac{1}{2}-\frac{1}{p}\right)\overline{\mu}_{0,s}^{p/(p-2)}$$

and $\liminf_{\varepsilon\to 0^+}(\inf_{x\in\Omega}\Lambda_1(\varepsilon,x)/\varepsilon) > 0$, it can be proved that $(\overline{u}_{\varepsilon_n,s})_n$, or a subsequence, converges in $L^p(\Omega)$ and almost everywhere in Ω to a function $\overline{u}_s \in H_0^{1,2}(\Omega)$, in the same way as we proved it in Lemma (5.12). Moreover, we can assume that, for this subsequence, there exists

$$\lim_{n\to\infty}\int_\Omega \frac{a^{i,j}_{\varepsilon_n}(x)}{\varepsilon_n}\partial_{x_i}\overline{u}_{\varepsilon_n,s}\partial_{x_j}\overline{u}_{\varepsilon_n,s}\,dx = \overline{\mu} < \infty.$$

Since $\lim_{n\to\infty}\|u_{\varepsilon_n,s}\|_{(\lambda,p)} = 0$ (see (I)) and $u_{\varepsilon_n,s}\in M_{\varepsilon_n}$, we can obtain

$$\lim_{n\to\infty}\|u_{\varepsilon_n,s}\|_{(\lambda,p)}/\varepsilon_n^{1/(p-2)} = \overline{\mu}^{1/(p-2)}$$

arguing as in step 1 of the proof of Lemma (5.12). From (5.12) and by using the Lebesgue Theorem we see that

$$\left(\frac{1}{2}-\frac{1}{p}\right)\overline{\mu}_{0,s}^{p/(p-2)} \ge \limsup_{n\to\infty}\frac{F_{\varepsilon_n}(u_{\varepsilon_n,s})}{\varepsilon_n^{2/(p-2)}} \ge \left(\frac{1}{2}-\frac{1}{p}\right)\overline{\mu}^{p/(p-2)}$$

and so $\overline{\mu} \le \overline{\mu}_{0,s}$.

Moreover, since for all $\eta > 0$ we have

$$\liminf_{n\to\infty}\left(\inf\left\{\frac{\Lambda_1(\varepsilon_n,x)}{\varepsilon_n} : x\in\Omega\setminus\bigcup_{t=1}^k\Omega_t(\eta)\right\}\right) = \infty,$$

we conclude that $\int_{\Omega\setminus\bigcup_{t=1}^k\Omega_t}|D\overline{u}_s(x)|^2\,dx = 0$. Since $\overline{u}_s \in H_0^{1,2}(\Omega)$ and every connected component of $\overline{\Omega}\setminus\bigcup_{t=1}^k\Omega'_t$ meets $\partial\Omega$, we have $\overline{u}_s(x) = 0$ for almost all $x\in\Omega\setminus\bigcup_{t=1}^k\Omega'_t$.

Since $\int_\Omega\lambda(x)|\overline{u}_s(x)|^p\,dx = 1$ and, by (5.13), $\int_{\Omega'_s}\lambda(x)|\overline{u}_s(x)|^p\,dx < 1$, there exists $t\in\{1,\dots,k\}, t\ne s$, such that $\int_{\Omega'_t}\lambda(x)|\overline{u}_s(x)|^p\,dx \ne 0$. Moreover, $1-\delta < \int_{\Omega'_s}\lambda(x)|\overline{u}_s(x)|^p\,dx$ with $\delta \in\,]0,(\overline{\mu}_{\mathrm{m}}/\overline{\mu}_{\mathrm{M}})^{p/(p-2)}[$ (see proof of Theorem (5.1)) and $\int_{\Omega_t'\setminus\Omega_t} |D\overline{u}_s(x)|^2\,dx = 0$ for all $t = 1,\dots,k$. Therefore Remark (4.3) assures that

$$\begin{aligned}\overline{\mu}_{0,s} &< \sum_{t=1}^{k}\int_{\Omega_t} a^{i,j}(x)\partial_{x_i}\overline{u}_s(x)\partial_{x_j}\overline{u}_s(x)\,dx \\ &\le \liminf_{n\to\infty}\int_{\Omega}\frac{a^{i,j}_{\varepsilon_n}(x)}{\varepsilon_n}\partial_{x_i}\overline{u}_{\varepsilon_n,s}(x)\partial_{x_j}\overline{u}_{\varepsilon_n,s}(x)\,dx \\ &\le \lim_{n\to\infty}\int_{\Omega}\frac{a^{i,j}_{\varepsilon_n}(x)}{\varepsilon_n}\partial_{x_i}\overline{u}_{\varepsilon_n,s}(x)\partial_{x_j}\overline{u}_{\varepsilon_n,s}(x)\,dx = \overline{\mu}.\end{aligned}$$

This is a contradiction with what was previously proved.

(III) The proof of property (III) can be easily obtained with the same arguments used for (II) and in the proof of Theorem (5.1).

(IV) Because of (5.12), it suffices to prove that

$$\liminf_{\varepsilon\to 0^+}\frac{F_\varepsilon(u_{\varepsilon,s})}{\varepsilon^{2/(p-2)}} \ge \left(\frac{1}{2}-\frac{1}{p}\right)\overline{\mu}_{0,s}^{p/(p-2)}.$$

We argue by contradiction and suppose that there exists a sequence $(\varepsilon_n)_n$ of positive numbers such that $\lim_{n\to\infty}\varepsilon_n = 0$ and

$$\lim_{n\to\infty}\frac{F_{\varepsilon_n}(u_{\varepsilon_n,s})}{\varepsilon_n^{2/(p-2)}} < \left(\frac{1}{2}-\frac{1}{p}\right)\overline{\mu}_{0,s}^{p/(p-2)}. \tag{5.14}$$

Then one can prove that the sequence $\overline{u}_{\varepsilon_n,s} = u_{\varepsilon_n,s}/\|u_{\varepsilon_n,s}\|_{(\lambda,p)}$ or one of its subsequences converges in $L^p(\Omega)$ and almost everywhere in Ω to a function $\overline{u}_s$ which belongs to $H_0^{1,2}(\Omega)$ and is zero in $\Omega\setminus\Omega_s'$ (argue as in the proofs of Lemma (5.12) and of (I)).

Since for all $\eta > 0$,

$$\liminf_{\varepsilon\to 0^+}\left(\inf\left\{\frac{\Lambda_1(\varepsilon,x)}{\varepsilon} : x\in\Omega\setminus\bigcup_{t=1}^{k}\Omega_t(\eta)\right\}\right) = \infty$$

we have $\int_{\Omega_s'\setminus\Omega_s}|D\overline{u}_s|^2\,dx = 0$. Moreover, $\overline{u}_s \in H_0^{1,2}(\Omega_s')$, $\int_{\Omega_s'}\lambda(x)|\overline{u}_s(x)|^p\,dx = 1$ and from the fact that $u_{\varepsilon_n,s}\in M_{\varepsilon_n}$ it follows that

$$\lim_{n\to\infty}\frac{\|u_{\varepsilon_n,s}\|_{(\lambda,p)}}{\varepsilon_n^{1/(p-2)}} = \left(\int_{\Omega_s'} a^{i,j}(x)\partial_{x_i}\overline{u}_s\partial_{x_j}\overline{u}_s\,dx\right)^{1/(p-2)}.$$

From (5.14) we find, by using (a.3) and the Lebesgue Theorem (as in the proof of Lemma (5.12)),

$$\begin{aligned}&\left(\frac{1}{2}-\frac{1}{p}\right)\left(\int_{\Omega_s} a^{i,j}(x)\partial_{x_i}\overline{u}_s\partial_{x_j}\overline{u}_s\,dx\right)^{p/(p-2)} \\ &\qquad\le \lim_{n\to\infty}\frac{F_{\varepsilon_n}(u_{\varepsilon_n,s})}{\varepsilon_n^{2/(p-2)}} < \left(\frac{1}{2}-\frac{1}{p}\right)\overline{\mu}_{0,s}^{p/(p-2)},\end{aligned}$$

i.e. $\int_{\Omega_s} a^{i,j}(x)\partial_{x_i}\overline{u}_s\partial_{x_j}\overline{u}_s\,dx < \overline{\mu}_{0,s}$, contrary to the definition of $\overline{\mu}_{0,s}$. □

(5.14) Remark. In order to obtain the multiplicity results stated in Theorems (4.4) and (5.1), it would be sufficient to require that

$$\limsup_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \sup_{x\in\bigcup_{s=1}^k \Omega_s} \Lambda_2(\varepsilon,x) < \infty$$

instead of condition (a.3). However, in this paper we have assumed (a.3) for the sake of simplicity and also because it has been useful in studying the asymptotic behaviour of the solutions.

References

[1] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[2] A. Bahri and J. M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent. The effect of the topology of the domain*, Comm. Pure Appl. Math. **41** (1988), 253–294.

[3] V. Benci and G. Cerami, *The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems*, Arch. Rational Mech. Anal. **144** (1991), 79–93.

[4] ———, *Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology*, Calc. Var. Partial Differential Equations **2** (1994), 29–48.

[5] V. Benci, G. Cerami and D. Passaseo, *On the number of positive solutions of some nonlinear elliptic problems*, Nonlinear Analysis. A Tribute in Honour of G. Prodi (A. Ambrosetti and A. Marino, eds.), Quaderni della Scuola Norm. Sup., Pisa, 1991, pp. 93–107.

[6] H. Brezis, *Elliptic equations with limiting Sobolev exponents—the impact of topology*, Proc. 50th Anniv. Courant Inst., Comm. Pure Appl. Math. **39** (1986), 17–39.

[7] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), 437–477.

[8] G. Cerami and D. Passaseo, *Existence and multiplicity of positive solutions for nonlinear elliptic problems in exterior domains with rich topology*, Nonlinear Anal. **18** (1992), 109–119.

[9] ———, *Existence and multiplicity results for semilinear elliptic Dirichlet problems in exterior domains*, Nonlinear Anal. (to appear).

[10] J. M. Coron, *Topologie et cas limite des injections de Sobolev*, C. R. Acad. Sci. Paris Sér. I **299** (1984), 209–212.

[11] E. N. Dancer, *A note on an equation with critical exponent*, Bull. London Math. Soc. **20** (1988), 600–602.

[12] W. Ding, *Positive solutions of* $\Delta u + u^{\frac{n+2}{n-2}} = 0$ *on contractible domains*, J. Partial Differential Equations **2** (1989), 83–88.

[13] E. Fabes, C. Kenig and R. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), 77–116.

[14] D. Fortunato, E. Jannelli and S. Solimini, (in preparation).

[15] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case*, Rev. Mat. Iberoamericana **1** (1985), 45–121, 145–201.

[16] F. Merle and L. A. Peletier, *Asymptotic behaviour of positive solutions of elliptic equations with critical and supercritical growth. I. The radial case*, Arch. Rational Mech. Anal. **112** (1990), 1–19.

[17] M. K. V. Murthy and G. Stampacchia, *Boundary value problems for some degenerate elliptic operators*, Ann. Mat. Pura Appl. **80** (1968), 1–122.

[18] D. Passaseo, *Multiplicity of positive solutions of nonlinear elliptic equations with critical Sobolev exponent in some contractible domains*, Manuscripta Math. **65** (1989), 147–166.

[19] ______, *The effect of the domain shape on the existence of positive solutions of the equation* $\Delta u + u^{2^\star - 1} = 0$, Topol. Methods Nonlinear Anal. **3** (1994), 27–54.

[20] ______, *Su alcune successioni di soluzioni positive di problemi ellittici con esponente critico*, Rend. Mat. Acc. Lincei (9) **3** (1992), 15–21.

[21] ______, *Nonexistence results for elliptic problems with supercritical nonlinearity in nontrivial domains*, J. Funct. Anal. **114** (1993), 97–105.

[22] ______, *Some sufficient conditions for the existence of positive solutions to the equations* $-\Delta u + a(x)u = u^{2^\star - 1}$ *in bounded domains*, Ann. Inst. H. Poincaré Anal. Non Linéaire (to appear).

[23] ______, *Esistenza e molteplicità di soluzioni positive per equazioni ellittiche con nonlinearitá sopracritica*, preprint no. 619, Dip. Mat. Pisa 1992.

[24] ______, *Some concentration phenomena in degenerate semilinear elliptic problems*, Nonlinear Anal. (to appear).

[25] S. I. Pokhozhaev, *Eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl. **6** (1965), 1408–1411.

[26] O. Rey, *Sur un problème variationnel non compact : l'effet de petits trous dans le domaine*, C. R. Acad. Sci. Paris Sér. I **308** (1989), 349–352.

[27] E. W. Stredulinsky, *Weighted Inequalities and Degenerate Elliptic Partial Differential Equations*, Lecture Notes in Math., vol. 1074, Springer-Verlag, 1984.

[28] M. Struwe, *A global compactness result for elliptic boundary value problems involving limiting nonlinearities*, Math. Z. **187** (1984), 511-517.

Manuscript received March 25, 1995

Monica Musso and Donato Passaseo
Dipartimento di Matematica
Università di Pisa
Via Buonarroti, 2
56127 Pisa, ITALY