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MULTIPLE SOLUTIONS OF NEUMANN ELLIPTIC PROBLEMS WITH CRITICAL NONLINEARITY

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Abstract. The paper is concerned with a class of Neumann elliptic problems, in bounded domains, involving the critical Sobolev exponent. Some conditions on the lower order term are given, sufficient to guarantee existence and multiplicity of positive solutions without any geometrical assumption on the boundary of the domain.

1. Introduction. Let us consider the following problem

$$(P)\quad \begin{cases} -\Delta u + a(x)u = u^{2^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, with $N \geq 3$, $2^* = \frac{2N}{N-2}$, $a \in L^{\frac{N}{2}}(\Omega)$ is a nonnegative function and ν denotes the outward normal to $\partial\Omega$.

It is easy to verify that the solutions of (P) correspond to the positive functions in $W^{1,2}(\Omega)$, which are critical points for the energy functional $E_a : W^{1,2}(\Omega) \to \mathbb{R}$, defined by

$$E_a(u) = \int_\Omega \left[|Du|^2 + a(x)u^2 \right] dx, \tag{1.1}$$

constrained on the manifold

$$V = \{u \in W^{1,2}(\Omega) \, : \, \int_\Omega |u|^{2^*} dx = 1\}. \tag{1.2}$$

Because of the presence of the critical Sobolev exponent 2^*, the usual compactness conditions are not satisfied, so that the classical variational methods cannot be applied in a standard way.

In the last few years several researches have been devoted to study existence and multiplicity of solutions for problems like (P). In [22] and [1] the problem is solved by minimization methods, arguing as in [10], [15], under the assumption that a geometrical condition (condition $(*)$ of Theorem 2.1) is satisfied.

Afterwards, arguing as in [20], [4], [6], other results have been stated, relating the number of solutions to the geometrical properties of $\partial\Omega$ (see [23], [2], [14], [3]). All these solutions correspond to critical points of E_a, constrained on V, in the sublevel $\{u \in V \, : \, E_a(u) < 2^{-\frac{2}{N}} S\}$, where S denotes the best Sobolev constant (see (2.1)); their existence is strictly related to the fact that the well known Palais-Smale condition holds below $2^{-\frac{2}{N}} S$.

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