The Euclidean Onofri Inequality in Higher Dimensions

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The classical Onofri inequality in the two-dimensional sphere assumes a natural form in the plane when transformed via stereographic projection. We establish an optimal version of a generalization of this inequality in the d-dimensional Euclidean space for any $d \geq 2$, by considering the endpoint of a family of optimal Gagliardo–Nirenberg interpolation inequalities. Unlike the two-dimensional case, this extension involves a rather unexpected Sobolev–Orlicz norm, as well as a probability measure no longer related to stereographic projection.

1 Introduction and main result

The *Onofri inequality* as stated in [18] asserts that

$$\log \left(\int_{\mathbb{S}^2} e^v \, d\sigma_2 \right) - \int_{\mathbb{S}^2} v \, d\sigma_2 \leq \frac{1}{4} \|\nabla v\|^2_{L^2(\mathbb{S}^2, d\sigma_2)} \tag{1}$$

for any function $v \in H^1(\mathbb{S}^2, d\sigma_2)$. Here, $d\sigma_2$ denotes the standard surface measure on the two-dimensional unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, up to a normalization factor $\frac{1}{4\pi}$ so that $\int_{\mathbb{S}^2} 1 \, d\sigma_2 = 1$.

Using stereographic projection from $\mathbb{S}^2$ onto $\mathbb{R}^2$, that is, defining u by

$$u(x) = v(y) \quad \text{with } y = (y_1, y_2, y_3), \quad y_1 = \frac{2x_1}{1 + |x|^2}, \quad y_2 = \frac{2x_2}{1 + |x|^2}, \quad y_3 = \frac{1 - |x|^2}{1 + |x|^2}$$

for any $x = (x_1, x_2) \in \mathbb{R}^2$, then (1) can be reformulated into the *Euclidean Onofri inequality*, namely

$$\log\left(\int_{\mathbb{R}^2} e^u \, d\mu_2\right) - \int_{\mathbb{R}^2} u \, d\mu_2 \leq \frac{1}{16\pi} \|\nabla u\|_{L^2(\mathbb{R}^2, dx)}^2 \tag{2}$$

for any $u \in L^1(\mathbb{R}^2, d\mu_2)$ such that $\nabla u \in L^2(\mathbb{R}^2, dx)$, where

$$d\mu_2(x) := \frac{dx}{\pi(1 + |x|^2)^2}$$

is again a probability measure.

The purpose of this note is to obtain an (optimal) extension of inequality (2) to any space dimension. There is a vast literature on Onofri's inequality, and we shall only mention a few works relevant to our main result below. Onofri's inequality with a nonoptimal constant was first established by Moser [17], a work prior to that of Onofri [18]. For this reason, the inequality is sometimes called the *Moser–Onofri* inequality. We also point out that Onofri's paper is based on an earlier result of Aubin [2]. We refer the interested reader to [14] for a recent account on the Moser–Onofri inequality. The inequality has an interesting version in the cylinder $\mathbb{R} \times \mathbb{S}^1$, see [12], which is however out of the scope of the present work.

In this note, we will establish that the Euclidean version of Onofri's inequality (2) can be extended to an arbitrary dimension $d \geq 3$ in the following manner. Let us consider the probability measure

$$d\mu_d(x) := \frac{d}{|\mathbb{S}^{d-1}|} \frac{dx}{(1 + |x|^{\frac{d}{d-1}})^d}.$$

Let us denote

$$\mathsf{R}_d(X, Y) := |X + Y|^d - |X|^d - d|X|^{d-2} X \cdot Y, \quad (X, Y) \in \mathbb{R}^d \times \mathbb{R}^d,$$

which is a polynomial if d is even. We define

$$\mathsf{H}_d(x, p) := \mathsf{R}_d\left(-\frac{d|x|^{-\frac{d-2}{d-1}}}{1 + |x|^{\frac{d}{d-1}}} x, \frac{d-1}{d} p\right), \quad (x, p) \in \mathbb{R}^d \times \mathbb{R}^d$$

and

$$Q_d[u] := \frac{\int_{\mathbb{R}^d} \mathsf{H}_d(x, \nabla u)\, \mathrm{d}x}{\log(\int_{\mathbb{R}^d} \mathsf{e}^u \, \mathrm{d}\mu_d) - \int_{\mathbb{R}^d} u \, \mathrm{d}\mu_d}.$$

The following is our main result.

Theorem 1.1.   With the above notation, for any smooth compactly supported function u, we have

$$\log \left(\int_{\mathbb{R}^d} \mathsf{e}^u \, \mathrm{d}\mu_d \right) - \int_{\mathbb{R}^d} u \, \mathrm{d}\mu_d \le \alpha_d \int_{\mathbb{R}^d} \mathsf{H}_d(x, \nabla u) \, \mathrm{d}x. \tag{3}$$

The optimal constant α_d is explicit and given by

$$\alpha_d = \frac{d^{1-d} \Gamma(d/2)}{2(d-1)\pi^{d/2}}.$$

Small multiples of the function

$$v(x) = -d \frac{x \cdot \mathsf{e}}{|x|^{\frac{d-2}{d-1}} (1 + |x|^{\frac{d}{d-1}})} \tag{4}$$

for a unit vector e are approximate extremals of (3) in the sense that

$$\lim_{\varepsilon \to 0} Q_d[\varepsilon\, v] = \frac{1}{\alpha_d}. \qquad \qquad \square$$

A rather unexpected feature of inequality (3) when compared with Onofri's inequality (2) is that it involves an inhomogeneous Sobolev–Orlicz-type norm. As we will see below, as a by-product of the proof we obtain a new Poincaré inequality in entire space, (7), of which the function v defined by (4) is an extremal.

Example 1.2.   If $d = 2$, $\int_{\mathbb{R}^d} \mathsf{H}_2(x, \nabla u) \, \mathrm{d}x = \frac{1}{4} \int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x$ and we recover Onofri's inequality (3) as in [11], with optimal constant $1/\alpha_2 = 4\pi$. On the other hand, if, for instance, $d = 4$, we find that $H_4(x, \nabla u)$ is a fourth-order polynomial in the partial derivatives of u, since $\mathsf{R}_4(X, Y) = 4(X \cdot Y)^2 + |Y|^2(|Y|^2 + 4X \cdot Y + 2|X|^2)$. $\qquad \square$

Extensions of inequality (2) to higher dimensions were already obtained long ago. Inequality (1) was generalized to the d-dimensional sphere in [3, 5], where natural conformally invariant, nonlocal generalizations of the Laplacian were used. Those

operators are of different nature than those in Theorem 1.1. Indeed, no clear connection through, for instance, stereographic projection is present. See also [15, 16] in which bounded domains are considered.

Inequality (3) determines a natural Sobolev space in which it holds. Indeed, a classical completion argument with respect to a norm corresponding to the integrals defined on both sides of the inequality determines a space on which the inequality still holds. This space can be identified with the set of all functions $u \in L^1(\mathbb{R}^d, d\mu_d)$ such that the distribution ∇u is a square integrable function. To avoid technicalities, computations will only be done for smooth, compactly supported functions.

Our strategy is to consider the Euclidean inequality of Theorem 1.1 as the endpoint of a family of optimal interpolation inequalities discovered in [7] and then extended in [8]. These inequalities can be stated as follows.

Theorem 1.3. Let $p \in (1, d]$, $a > 1$ such that $a \leq \frac{p(d-1)}{d-p}$ if $p < d$, and $b = p\frac{a-1}{p-1}$. There exists a positive constant $C_{p,a}$ such that, for any function, $f \in L^a(\mathbb{R}^d, dx)$ with $\nabla f \in L^p(\mathbb{R}^d, dx)$, we have

$$\|f\|_{L^b(\mathbb{R}^d)} \leq C_{p,a} \|\nabla f\|_{L^p(\mathbb{R}^d)}^{\theta} \|\nabla f\|_{L^a(\mathbb{R}^d)}^{1-\theta} \quad \text{with } \theta = \frac{(a-p)d}{(a-1)(dp-(d-p)a)} \tag{5}$$

if $a > p$. A similar inequality also holds if $a < p$, namely

$$\|\nabla f\|_{L^a(\mathbb{R}^d)} \leq C_{p,a} \|\nabla f\|_{L^p(\mathbb{R}^d)}^{\theta} \|f\|_{L^b(\mathbb{R}^d)}^{1-\theta} \quad \text{with } \theta = \frac{(p-a)d}{a(d(p-a)+p(a-1))}.$$

In both cases, equality holds for any function taking the form

$$f(x) = A(1 + B|x - x_0|^{\frac{p}{p-1}})_+^{-\frac{p-1}{a-p}} \quad \forall \, x \in \mathbb{R}^d$$

for some $(A, B, x_0) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$, where B has the sign of $a - p$. □

While in [8], only the case $p < d$ was considered, the proof there actually applies to also cover the case $p = d$, for any $a \in (1, \infty)$.

For $a = p$, inequality (5) degenerates into an equality. By subtracting it from the inequality, dividing by $a - p$ and taking the limit as $a \to p_+$, we obtain an *optimal Euclidean L^p-Sobolev logarithmic inequality* which goes as follows. Assume that

$1 < p \leq d$. Then, for any $u \in W^{1,p}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} |u|^p \, dx = 1$, we have

$$\int_{\mathbb{R}^d} |u|^p \log |u|^p \, dx \leq \frac{d}{p} \log \left[\beta_{p,d} \int_{\mathbb{R}^d} |\nabla u|^p \, dx \right],$$

where

$$\beta_{p,d} := \frac{p}{d} \left(\frac{p-1}{e} \right)^{p-1} \frac{1}{\pi^{\frac{p}{2}}} \left[\frac{\Gamma(\frac{d}{2}+1)}{\Gamma(d\frac{p-1}{p}+1)} \right]^{\frac{p}{d}}$$

is the optimal constant. Equality holds if and only if for some $\sigma > 0$ and $x_0 \in \mathbb{R}^d$

$$u(x) = \left[\frac{1}{2\pi^{\frac{d}{2}}} \frac{p}{p-1} \frac{\Gamma(\frac{d}{2})}{\Gamma(d\frac{p-1}{p})} \left(\frac{p}{\sigma} \right)^{d\frac{p-1}{p}} \right]^{\frac{1}{p}} e^{-\frac{1}{\sigma} |x-x_0|^{\frac{p}{p-1}}} \quad \forall x \in \mathbb{R}^d.$$

This inequality has been established in [9] when $p < d$ and, in general, in [13]; see also [6, 10].

When $p < d$, the endpoint $a = \frac{p(d-1)}{d-p}$ corresponds to the usual optimal Sobolev inequality, for which the extremal functions were already known from the celebrated papers by Aubin and Talenti [1, 20]. See also [4, 19] for earlier related computations, which provided the value of some of the best constants.

When $p = d$, Theorem 1.1 will also be obtained by passing to a limit, namely as $a \to +\infty$. In this way, the d-dimensional Onofri inequality corresponds to nothing but a natural extension of the optimal Sobolev's inequality. In dimension $d = 2$, with $p = 2$, $a = q + 1 > 2$ and $b = 2q$, it has recently been observed in [11] that

$$1 \leq \lim_{q \to \infty} \mathsf{C}_{2,q+1} \frac{\|\nabla f_q\|_{L^2(\mathbb{R}^2)}^{\frac{q-1}{2q}} \|f_q\|_{L^{q+1}(\mathbb{R}^2)}^{\frac{q+1}{2q}}}{\|f_q\|_{L^{2q}(\mathbb{R}^2)}} = \frac{e^{\frac{1}{16\pi} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx}}{\int_{\mathbb{R}^2} e^u \, d\mu_2}$$

if $f_q = (1 + |x|^2)^{-\frac{1}{q-1}} (1 + \frac{u}{2q})$ and $\int_{\mathbb{R}^2} u \, d\mu_2 = 0$. In that sense, Onofri's inequality in dimension $d = 2$ replaces Sobolev's inequality in higher dimensions as an endpoint of the family of Gagliardo–Nirenberg inequalities

$$\|f\|_{L^{2q}(\mathbb{R}^d)} \leq \mathsf{C}_{2,q+1} \|\nabla f\|_{L^2(\mathbb{R}^d)}^{\theta} \|f\|_{L^{q+1}(\mathbb{R}^d)}^{1-\theta}$$

with $\theta = \frac{q-1}{q} \frac{d}{d+2-q(d-2)}$. In dimension $d \geq 3$, we will see below that (3) can also be seen as an endpoint of (5).

2   Proof of Theorem 1.1

Assume that $u \in \mathcal{D}(\mathbb{R}^d)$ is such that $\int_{\mathbb{R}^d} u \, d\mu_d = 0$ and let

$$f_a := F_a \left(1 + \frac{d-1}{da} u \right),$$

where F_a is defined by

$$F_a(x) = (1 + |x|^{\frac{d}{d-1}})^{-\frac{d-1}{a-d}} \quad \forall x \in \mathbb{R}^d. \tag{6}$$

From Theorem 1.3, Inequality (5), we know that

$$1 \leq \lim_{a \to +\infty} \mathsf{C}_{d,a} \frac{\|\nabla f_a\|_{\mathrm{L}^d(\mathbb{R}^d)}^{\theta} \|f_a\|_{\mathrm{L}^a(\mathbb{R}^d)}^{1-\theta}}{\|f_a\|_{\mathrm{L}^b(\mathbb{R}^d)}}$$

if $p = d$. Our goal is to identify the right-hand side in terms of u. We recall that $b = \frac{d(a-1)}{d-1}$ and $\theta = \frac{a-d}{d(a-1)}$. Using the fact that F_a is an optimal function, we can then rewrite (5) with $f = f_a$ as

$$\frac{\int_{\mathbb{R}^d} |f_a|^{\frac{d(a-1)}{d-1}} \, dx}{\int_{\mathbb{R}^d} |F_a|^{\frac{d(a-1)}{d-1}} \, dx} \leq \left(\frac{\int_{\mathbb{R}^d} |\nabla f_a|^d \, dx}{\int_{\mathbb{R}^d} |\nabla F_a|^d \, dx} \right)^{\frac{a-d}{d(a-1)}} \frac{\int_{\mathbb{R}^d} |f_a|^a \, dx}{\int_{\mathbb{R}^d} |F_a|^a \, dx}$$

and observe that:

(i) $\lim_{a \to +\infty} \int_{\mathbb{R}^d} |F_a|^{\frac{d(a-1)}{d-1}} \, dx = \int_{\mathbb{R}^d} (1 + |x|^{\frac{d}{d-1}})^{-d} \, dx = \frac{1}{d} |\mathbb{S}^{d-1}|$ and

$$\lim_{a \to +\infty} \int_{\mathbb{R}^d} |f_a|^{\frac{d(a-1)}{d-1}} \, dx = \lim_{a \to +\infty} \int_{\mathbb{R}^d} F_a^{\frac{d(a-1)}{d-1}} \left(1 + \frac{d-1}{da} u \right)^{\frac{d(a-1)}{d-1}} \, dx = \int_{\mathbb{R}^d} \frac{e^u}{(1 + |x|^{\frac{d}{d-1}})^d} \, dx,$$

so that

$$\lim_{a \to +\infty} \frac{\int_{\mathbb{R}^d} |f_a|^{\frac{d(a-1)}{d-1}} \, dx}{\int_{\mathbb{R}^d} |F_a|^{\frac{d(a-1)}{d-1}} \, dx} = \int_{\mathbb{R}^d} e^u \, d\mu_d.$$

(ii)   As $a \to +\infty$,

$$\int_{\mathbb{R}^d} |F_a|^a \, dx \approx \frac{2a\pi^{d/2}}{d^2 \Gamma(d/2)}, \quad \lim_{a \to +\infty} \int_{\mathbb{R}^d} |f_a|^a \, dx = \infty,$$

and

$$\lim_{a \to +\infty} \frac{\int_{\mathbb{R}^d} |f_a|^a \, dx}{\int_{\mathbb{R}^d} |F_a|^a \, dx} = 1.$$

(iii)   Finally, as $a \to +\infty$, we also find that

$$\left(\frac{\int_{\mathbb{R}^d} |\nabla f_a|^d \, \mathrm{d}x}{\int_{\mathbb{R}^d} |\nabla F_a|^d \, \mathrm{d}x} \right)^{\frac{a-d}{d(d-1)}} \approx \left(1 + \frac{d(d-1)}{a} \, \alpha_d \int_{\mathbb{R}^d} \mathsf{H}_d(x, \nabla u) \, \mathrm{d}x \right)^{\frac{a-d}{d(d-1)}}$$

$$\approx \exp \left(\alpha_d \int_{\mathbb{R}^d} \mathsf{H}_d(x, \nabla u) \, \mathrm{d}x \right).$$

Here and above $\ell_1(a) \approx \ell_2(a)$ means that $\lim_{a \to +\infty} \ell_1(a)/\ell_2(a) = 1$. Fact (iii) requires some computations which we make explicit next. First of all, we have

$$\int_{\mathbb{R}^d} |\nabla F_a|^d \, \mathrm{d}x = \frac{2 d^{d-2} \pi^{d/2}}{\Gamma(d/2)} a^{1-d}.$$

With $X_a := \left(1 + \frac{d-1}{da} u \right) \nabla F_a$ and $Y_a := \frac{d-1}{da} F_a \nabla u$, we can write, using the definition of R_d, that

$$|\nabla f_a|^d = |\nabla F_a|^d \left(1 + \frac{d-1}{da} u \right)^d + F_a |\nabla F_a|^{d-2} \nabla F_a \cdot \nabla \left(1 + \frac{d-1}{da} u \right)^d + \mathsf{R}_d(X_a, Y_a).$$

Consider the second term on the right-hand side and integrate by parts. A straightforward computation shows that

$$\int_{\mathbb{R}^d} F_a |\nabla F_a|^{d-2} \nabla F_a \cdot \nabla \left(1 + \frac{d-1}{da} u \right)^d \, \mathrm{d}x = - \int_{\mathbb{R}^d} |\nabla F_a|^d \left(1 + \frac{d-1}{da} u \right)^d \, \mathrm{d}x$$

$$- \int_{\mathbb{R}^d} F_a \Delta_d F_a \left(1 + \frac{d-1}{da} u \right)^d \, \mathrm{d}x,$$

where $\Delta_p F_a = \nabla \cdot (|\nabla F_a|^{p-2} \nabla F_a)$ is computed for $p = d$. Collecting terms, we obtain

$$\int_{\mathbb{R}^d} |\nabla f_a|^d \, \mathrm{d}x = - \int_{\mathbb{R}^d} F_a \Delta_d F_a \left(1 + \frac{d-1}{da} u \right)^d \, \mathrm{d}x + \int_{\mathbb{R}^d} \mathsf{R}_d(X_a, Y_a) \, \mathrm{d}x.$$

We may next observe that

$$a \nabla F_a(x) = - \frac{da}{a-d} |x|^{-\frac{d-2}{d-1}} x (1 + |x|^{\frac{d}{d-1}})_+^{-\frac{a-1}{a-d}} \to -d \frac{|x|^{-\frac{d-2}{d-1}} x}{1 + |x|^{\frac{d}{d-1}}} \quad \text{a.e. as } a \to +\infty,$$

while $a\nabla(1 + \frac{d-1}{da}u) = \frac{d-1}{d}\nabla u$, so that both $X_a = (1 + \frac{d-1}{da}u)\nabla F_a$ and $Y_a = \frac{d-1}{da}F_a\nabla u$ in $\mathsf{R}_d(X_a, Y_a)$ are of the order of $1/a$. By homogeneity, it follows that

$$a^d\mathsf{R}_d(X_a, Y_a) \to \mathsf{R}_d\left(-\frac{d|x|^{-\frac{d-2}{d-1}}}{1 + |x|^{\frac{d}{d-1}}}x, \frac{d-1}{d}\nabla u\right) = \mathsf{H}_d(x, \nabla u) \quad \text{as } a \to +\infty,$$

by definition of H_d. Hence we have established the fact that

$$\int_{\mathbb{R}^d} |\nabla f_a|^d \, dx = -\int_{\mathbb{R}^d} F_a\Delta_d F_a\left(1 + \frac{d-1}{da}u\right)^d dx + \int_{\mathbb{R}^d} \mathsf{R}_d(X_a, Y_a) \, dx$$

$$= -\int_{\mathbb{R}^d} F_a\Delta_d F_a\left(1 + d\frac{d-1}{da}u + o(a^{-1})\right) dx + a^{-d}\int_{\mathbb{R}^d} \mathsf{H}_d(x, \nabla u) \, dx.$$

Next we can observe that $-\int_{\mathbb{R}^d} F_a\Delta_d F_a \, dx = \int_{\mathbb{R}^d} |\nabla F_a|^d \, dx$, while $-\lim_{a\to+\infty} a^{d-1}F_a\Delta_d F_a = d^{d-1}|\mathbb{S}^{d-1}|\mu_d$, so that

$$-\int_{\mathbb{R}^d} F_a\Delta_d F_a u \, dx = a^{1-d}d^{d-1}|\mathbb{S}^{d-1}|\int_{\mathbb{R}^d} u \, d\mu_d + o(a^{1-d}) = o(a^{1-d}) \quad \text{as } a \to +\infty$$

by the assumption that $\int_{\mathbb{R}^d} u \, d\mu_d = 0$. Altogether, this means that

$$\left(\frac{\int_{\mathbb{R}^d} |\nabla f_a|^d \, dx}{\int_{\mathbb{R}^d} |\nabla F_a|^d \, dx}\right)^{\frac{a-d}{d(d-1)}} \approx \left(1 + \frac{\int_{\mathbb{R}^d} \mathsf{H}_d(x, \nabla u) \, dx}{a^d \int_{\mathbb{R}^d} |\nabla F_a|^d \, dx}\right)^{\frac{a-d}{d(d-1)}} \approx \left(1 + \frac{d(d-1)}{a}\alpha_d\int_{\mathbb{R}^d} \mathsf{H}_d(x, \nabla u) \, dx\right)^{\frac{a-d}{d(d-1)}}$$

as $a \to +\infty$, which concludes the proof of (iii).

Before proving the optimality of the constant α_d, let us state an intermediate result which is of interest in itself. Let us assume that $d \geq 2$ and define Q_d as

$$\mathsf{Q}_d(X, Y) := 2\lim_{\varepsilon\to 0} \varepsilon^{-2}\mathsf{R}_d(X, \varepsilon Y) = \frac{d^2}{dt^2}|X + tY|^d_{|t=0} = d|X|^{d-4}[(d-2)(X \cdot Y)^2 + |X|^2 |Y|^2].$$

We also define

$$\mathsf{G}_d(x, p) := \mathsf{Q}_d\left(-\frac{d|x|^{-\frac{d-2}{d-1}}}{1 + |x|^{\frac{d}{d-1}}}x, \frac{d-1}{d}p\right), \quad (x, p) \in \mathbb{R}^d \times \mathbb{R}^d.$$

Corollary 2.1. With α_d as in Theorem 1.1, we have

$$\int_{\mathbb{R}^d} |v - \bar{v}|^2 \, \mathrm{d}\mu_d \le \alpha_d \int_{\mathbb{R}^d} \mathsf{G}_d(x, \nabla v) \, \mathrm{d}x \quad \text{with } \bar{v} = \int_{\mathbb{R}^d} v \, \mathrm{d}\mu_d, \tag{7}$$

for any $v \in \mathrm{L}^1(\mathbb{R}^d, \mathrm{d}\mu_d)$, such that $\nabla v \in \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x)$. $\qquad\qquad\square$

This inequality is a Poincaré inequality, which is remarkable. Indeed, if we prove that the optimal constant in (7) is equal to α_d, then α_d is also optimal in Theorem 1.1, Inequality (3). We will see below that this is the case.

Proof of Corollary 2.1. Inequality (7) is a straightforward consequence of (3), written with u replaced by $\varepsilon\, v$. In the limit $\varepsilon \to 0$, both sides of the inequality are of order ε^2. Details are left to the reader. $\qquad\qquad\blacksquare$

To conclude the proof of Theorem 1.1, let us check that there is a nontrivial function v which achieves equality in (7). Since F_a is optimal for (3), we can write that

$$\log\left(\int_{\mathbb{R}^d} |F_a|^{\frac{d(a-1)}{d-1}} \, \mathrm{d}x\right) = \log \mathsf{C}_{d,a} + \frac{a-d}{d(d-1)} \log\left(\int_{\mathbb{R}^d} |\nabla F_a|^d \, \mathrm{d}x\right) + \log\left(\int_{\mathbb{R}^d} |F_a|^a \, \mathrm{d}x\right).$$

However, equality also holds true if we replace F_a by $F_{a,\varepsilon}$ with $F_{a,\varepsilon}(x) := F_a(x + \varepsilon\, \mathsf{e})$, for an arbitrary given $\mathsf{e} \in \mathbb{S}^{d-1}$, and it is clear that one can differentiate twice with respect to ε at $\varepsilon = 0$. Hence, for any $a > d$, we have

$$\frac{d(a-1)}{d-1}\left(\frac{d(a-1)}{d-1} - 1\right)\frac{\int_{\mathbb{R}^d} |F_a|^{\frac{d(a-1)}{d-1}} |v_a|^2 \, \mathrm{d}x}{\int_{\mathbb{R}^d} |F_a|^{\frac{d(a-1)}{d-1}} \, \mathrm{d}x} = \frac{a-d}{d(d-1)}\frac{\int_{\mathbb{R}^d} \mathsf{Q}_d(X_a, \frac{d-1}{d} Y_a) \, \mathrm{d}x}{\int_{\mathbb{R}^d} |\nabla F_a|^d \, \mathrm{d}x}$$

$$+ a(a-1)\frac{\int_{\mathbb{R}^d} |F_a|^a |v_a|^2 \, \mathrm{d}x}{\int_{\mathbb{R}^d} |F_a|^a \, \mathrm{d}x} \tag{8}$$

with $X_a = \nabla F_a$, $Y_a = \frac{d}{d-1} F_a \nabla v_a$, and $v_a := \mathsf{e} \cdot \nabla \log F_a$, that is,

$$v_a(x) = -\frac{d}{a-d}\frac{x \cdot \mathsf{e}}{|x|^{\frac{d-2}{d-1}}\left(1 + |x|^{\frac{d}{d-1}}\right)}.$$

Hence, if ϕ is a radial function, we may notice that $\int_{\mathbb{R}^d} \phi v_a \, dx = 0$ and

$$\lim_{a \to +\infty} a^2 \int_{\mathbb{R}^d} \phi |v_a|^2 \, dx = d^2 \int_{\mathbb{R}^d} \phi(x) \frac{|x|^{\frac{2}{d-1}-2} (x \cdot e)^2}{(1 + |x|^{\frac{d}{d-1}})^2} \, dx = d \int_{\mathbb{R}^d} \phi(x) \frac{|x|^{\frac{2}{d-1}}}{(1 + |x|^{\frac{d}{d-1}})^2} \, dx.$$

Since $\int_{\mathbb{R}^d} |F_a|^{\frac{d(a-1)}{d-1}} \, dx = o(\int_{\mathbb{R}^d} |F_a|^a \, dx)$, the last term in (8) is negligible compared with the other ones. Passing to the limit as $a \to +\infty$, with $v := \lim_{a \to +\infty} av_a$, we find that v is given by (4) and

$$\left(\frac{d}{d-1} \right)^2 \int_{\mathbb{R}^d} |v|^2 \, d\mu_d = \alpha_d \int_{\mathbb{R}^d} Q_d \left(-\frac{d|x|^{-\frac{d-2}{d-1}}}{1 + |x|^{\frac{d}{d-1}}} x, \frac{d-1}{d} Y \right) dx,$$

where $Y := \frac{d-1}{d} \nabla v$ and where we have used the fact that

$$d(d-1)\alpha_d \lim_{a \to +\infty} a^d \int_{\mathbb{R}^d} |\nabla F_a|^d \, dx = 1.$$

Since the function Q_d is quadratic, we obtain that

$$\left(\frac{d}{d-1} \right)^2 \int_{\mathbb{R}^d} |v|^2 \, d\mu_d = \alpha_d \int_{\mathbb{R}^d} G_d \left(x, \frac{d}{d-1} \nabla v \right) dx = \alpha_d \left(\frac{d}{d-1} \right)^2 \int_{\mathbb{R}^d} G_d(x, \nabla v) \, dx,$$

which corresponds precisely to equality in (7) since v given by (4) is such that $\bar{v} = 0$.

Equality in (3) is achieved by constants. The optimality of α_d amounts to establish that in the inequality

$$\mathcal{Q}_d[u] \geq \frac{1}{\alpha_d},$$

equality can be achieved along a minimizing sequence. Notice that

$$\mathcal{Q}_d[u] = \frac{\int_{\mathbb{R}^d} H_d(x, \nabla u) \, dx}{\log \left(\int_{\mathbb{R}^d} e^u \, d\mu_d \right)} \quad \text{if} \int_{\mathbb{R}^d} u \, d\mu_d = 0.$$

The reader is invited to check that $\lim_{\varepsilon \to 0} \mathcal{Q}_d[\varepsilon v] = \frac{1}{\alpha_d}$. In dimension $d = 2$, v is an eigenfunction associated to the eigenvalue problem: $-\Delta v = \lambda_1 v \mu_2$, corresponding to the lowest positive eigenvalue, λ_1. The generalization to higher dimensions is given by (4). Notice that the function v is an eigenfunction of the linear form associated to G_d, in the space $L^2(\mathbb{R}^d, d\mu_d)$. This concludes the proof of Theorem 1.1.

Whether there are nontrivial optimal functions, that is, whether there exists a nonconstant function u such that $\mathcal{Q}_d[u] = \frac{1}{\alpha_d}$, is an open question. At least the proof of

Theorem 1.1 shows that there is a loss of compactness in the sense that the limit of $\varepsilon\, v$, that is 0, is not an admissible function for $\mathcal{Q}_d$.

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