On De Giorgi's conjecture and beyond

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We consider the problem of existence of entire solutions to the Allen–Cahn equation $\Delta u + u - u^3 = 0$ in $\mathbb{R}^N$, usually regarded as a prototype for the modeling of phase transition phenomena. In particular, exploiting the link between the Allen–Cahn equation and minimal surface theory in dimensions $N \ge 9$, we find a solution, u, with $\partial_{x_N} u > 0$, such that its level sets are close to a nonplanar, minimal, entire graph. This counterexample provides a negative answer to a celebrated question by Ennio de Giorgi [De Giorgi E (1979) *Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978)*, 131–188, Pitagora, Bologna]. Our results suggest parallels of De Giorgi's conjecture for finite Morse index solutions in two and three dimensions and suggest a possible program of classification of all entire solutions.

The Allen–Cahn equation in $\mathbb{R}^N$ is the semilinear elliptic problem

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N. \qquad [1]$$

Originally formulated in the description of biphase separation in fluids and ordering in binary alloys (1), Eq. **1** has received extensive mathematical study. It is a prototype for the modeling of phase transition phenomena in a variety of contexts.

Introducing a small positive parameter ε and writing $v(x) := u(\varepsilon^{-1}x)$, we get the scaled version of [**1**],

$$\varepsilon^2 \Delta v + v - v^3 = 0 \quad \text{in } \mathbb{R}^N. \qquad [2]$$

On every bounded domain $\Omega \subset \mathbb{R}^N$, [**1**] is the Euler–Lagrange equation for the action functional

$$J_\varepsilon(v) = \int_\Omega \frac{\varepsilon}{2}|\nabla v|^2 + \frac{1}{4\varepsilon}(1 - v^2)^2.$$

Observe that the constant functions $v = \pm 1$ minimize J_ε. They are idealized as two stable phases of a material in Ω. It is of interest to analyze configurations in which the two phases coexist. These states are represented by stationary points of J_ε, or solutions v_ε of Eq. **2**, that aside from a small set take values close to $+1$ in a subregion of Ω and -1 in its complement. Modica and Mortola (2) and Modica (3), established that a family of local minimizers v_ε of J_ε for which

$$\sup_{\varepsilon > 0} J_\varepsilon(v_\varepsilon) < +\infty \qquad [3]$$

must satisfy, after passing to a subsequence,

$$v_\varepsilon \to \chi\Lambda - \chi\Omega \setminus \Lambda \quad \text{in } L^1_{\text{loc}}(\Omega), \qquad [4]$$

as $\varepsilon \to 0$. Here Λ is an open subset of Ω with $\Gamma = \partial\Lambda \cap \Omega$ having minimal perimeter. Therefore, Λ is a (generalized) minimal surface. Moreover, as $\varepsilon \to 0$

$$J_\varepsilon(v_\varepsilon) \to \frac{2}{3}\sqrt{2}H^{n-1}(\Gamma). \qquad [5]$$

The hypersurface Γ is close to the nodal set of v_ε [or more generally, for a given $\lambda \in (-1, 1)$, to any level set $[v_\varepsilon = \lambda]$ for small ε]. Scaling back into Eq. **1**, it is then plausible to conjecture that a relation between the level sets of u and the minimal surface $\varepsilon^{-1}\Gamma$ should exist, at least when u corresponds to a local minimizer of the energy on each given compact set.

What condition guarantees that u is a locally minimizing (or stable) solution to the Allen–Cahn equation? For a solution u of [**1**], this condition is implied by the fact that the linearized operator $\Delta + (1 - 3u^2)$ satisfies the maximum principle. Because the directional derivatives $e \cdot \nabla u$ lie in the kernel of this operator, the assumption that the solution is monotone in some direction, say $u_{x_N} > 0$ is sufficient for this condition. De Giorgi's conjecture, which we state below, is partly motivated by the above facts.

For $N = 1$ the function

$$w(t) := \tanh\left(\frac{t}{\sqrt{2}}\right)$$

connects the stable values -1 and $+1$ in a monotone fashion and solves [**1**]:

$$w'' + w - w^3 = 0, \qquad w(\pm\infty) = \pm 1, \qquad w' > 0.$$

This solution generates a class of solutions to [Allen–Cahn equation (AC)] in the following manner: For any $p, \nu \in \mathbb{R}^N$, $|\nu| = 1$, the functions

$$u(x) := w(z), \qquad z = (x - p)\cdot\nu$$

solve Eq. **1**. Here, the variable $z =$ represents the normal coordinate to the hyperplane through p in the direction of its unit normal ν. A question is whether or not there exist solutions connecting the values -1 and 1 monotonically along some direction, which are different from these trivial ones.

In 1978, De Giorgi (4) made the following celebrated conjecture.

De Giorgi's Conjecture. Let u be a bounded solution of equation

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N,$$

which is monotone in one direction, say $u_{x_N} > 0$. Then, at least when $N \le 8$, there exist p, ν such that

$$u(x) = w[(x - p)\cdot\nu].$$

This conjecture is equivalent to the following:

At least when $N \le 8$, all level sets of u, $[u = \lambda]$ must be hyperplanes.

An intriguing feature of this statement is its presumed space dependence. Because $u_{x_N} > 0$, the level sets $[u = \lambda]$ are graphs of

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functions of the first $N-1$ variables. The rationale behind De Giorgi's statement is that these graphs should behave like minimal hypersurfaces that are graphs of entire functions. Indeed, De Giorgi's conjecture is intimately related to Bernstein's Problem for entire minimal graphs, which are surfaces in $\mathbb{R}^N$ of the form

$$\Gamma = \{[x', F(x')] \in \mathbb{R}^{N-1} \times \mathbb{R} / x' \in \mathbb{R}^{N-1}\},$$

where F solves the minimal surface equation

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^{N-1}. \qquad [6]$$

Note that any affine function is an obvious solution of this equation, representing a hyperplane.

Bernstein's Problem. Is it true that all entire minimal graphs are hyperplanes?

In 1917, Bernstein (5) proved the validity of this fact for $N=2$. In 1962, Fleming (6) provided a proof for $N=3$ and conjectured its validity in all dimensions. In 1965, De Giorgi (7) proved it for $N=4$, in 1966 Almgren (8) proved it for $N=5$, and in 1968 Simons (9), did so for $N=6,7,8$. Strikingly, in 1969, Bombieri, De Giorgi, and Giusti (BDG) (10) found that Fleming's conjecture was false for $N \geq 9$ exhibiting a counterexample (the BDG surface).

*The construction in ref. 10 begins with an example of a minimal and local area minimizing cone in dimension $N=8$ found by Simons (9). The Simons cone in $\mathbb{R}^8$ is a surface of the form $|\mathbf{u}| = |\mathbf{v}|$, $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^4 \times \mathbb{R}^4$ and the solution in ref. 10 depends of two radial variables $(|\mathbf{u}|, |\mathbf{v}|)$ only and is a function of the form $F(|\mathbf{u}|, |\mathbf{v}|)$ for $F: \mathbb{R}^2 \to \mathbb{R}$. Moreover, it is assumed a priori that $F(|\mathbf{u}|, |\mathbf{v}|) > 0$ for $|\mathbf{v}| > |\mathbf{u}|$ and $F(|\mathbf{u}|, |\mathbf{v}|) = -F(|\mathbf{v}|, |\mathbf{u}|)$. In ref. 10 ingenious explicit super and subsolutions for Eq. **6** written in the radial variables are found and they lead to the existence result.*

The BDG surface plays a crucial role in the construction of a counterexample to the De Giorgi conjecture and in ref. 11 we need to improve the result of ref. 10 to find very precise information about the asymptotic behavior of the BDG graph at infinity. Introducing polar coordinates

$$|\mathbf{u}| = r\cos\theta, \qquad |\mathbf{v}| = r\sin\theta, \qquad \theta \in (0, \frac{\pi}{2}),$$

the barriers in ref. 10 can be refined to yield quite accurate asymptotics for F for large r. We established in ref. 11 that there exists a function $g(\theta)$ such that $g > 0$ in $(\frac{\pi}{4}, \frac{\pi}{2})$ and with $g(\frac{\pi}{4}) = 0 = g'(\frac{\pi}{2})$, such that for $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$ we have, for $0 < \sigma < 1$,

$$r^3 g(\theta) \leq F(r,\theta) \leq r^3 g(\theta) + A r^{-\sigma} \quad \text{as } r \to +\infty. \qquad [7]$$

*The function g is a solution of the second order ODE obtained when formally substituting $F = r^3 g(\theta)$ in Eq. **6** and letting $r \to +\infty$. Although proving that $r^3 g(\theta)$ is a subsolution is relatively straightforward, finding the supersolution with the right asymptotic behavior is nontrivial.*

For De Giorgi's conjecture, many contributions have been made since it was formulated. In particular the conjecture was proven to be true for $N=2$ by Ghoussoub and Gui (12) in 1998, and by Ambrosio and Cabré (13) for $N=3$ in 1999. In 2009, Savin (14) proved that De Giorgi's conjecture is true for $4 \leq N \leq 8$ under the additional assumption

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1 \quad \text{for all } x' \in \mathbb{R}^{N-1}.$$

The latter assumption is indeed a posteriori satisfied by the solution. If the limits above are assumed to exist uniformly in x', then the claim that $u = w(x_N)$ is known as Gibbons' conjecture, and it has been proven in all dimensions and without the monotonicity hypothesis. In fact, different approaches have been given by Barlow et al. (15), Berestycki et al. (16), Caffarelli and Córdoba (17), and Farina (18). In refs. 15 and 17, it is proven that the conjecture is true for any solution that has one level set, which is globally a Lipschitz graph. Without monotonicity or uniformity in limits, the one-dimensional symmetry of the solution is not true. This fact is, for instance, clearly reflected in the entire planar solutions built in ref. 19 with any given finite number of nearly parallel nodal lines.

It is illustrative to review the proof of De Giorgi's conjecture for $N=2$ in ref. 12. Let us set $\phi = \frac{u_{x_1}}{u_{x_2}}$, which is well-defined because $u_{x_2} > 0$. Then ϕ satisfies the equation

$$\nabla \cdot (u_{x_2}^2 \nabla \phi) = 0.$$

Let $\eta(s)$ be a smooth cut-off function with $\eta(s) = 1$ for $s < 1$ and $=0$ for $s > 2$, and set $\eta_R(x) = \eta(|x|/R)$ for $R > 0$. Testing this equation against $\phi \eta_R^2$ and integrating we find that

$$\int_{\mathbb{R}^2} |\nabla\phi|^2 \eta_R^2 u_{x_N}^2 = -2 \int_{\mathbb{R}^2} \eta_R \nabla\eta_R \nabla\phi \phi u_{x_N}^2 \leq C \left(\int_{\{R<|x|<2R\}} |\nabla\phi|^2 \eta_R^2 u_{x_N}^2 \right)^{\frac{1}{2}},$$

where C is a constant dependent on uniform bounds for u and ∇u (which exist by the boundedness assumption and standard elliptic estimates). Letting $R \to \infty$, the above formula clearly implies that $\int_{\mathbb{R}^2} |\nabla\phi|^2 u_{x_N}^2 < +\infty$. Applying the formula a second time with $R \to \infty$ we find that this integral actually equals zero. Hence $\phi = \alpha = \text{constant}$ and $\nabla u \cdot (1, -\alpha) = 0$. This result implies that all level sets must be parallel lines as desired. The higher dimensional cases are more difficult to handle and the full result for dimensions $4 \leq N \leq 8$ remains open.

A counterexample to De Giorgi's conjecture in dimension $N \geq 9$ was believed to exist for a long time, possibly by De Giorgi himself. Partial progress in this direction was made by Jerison and Monneau (20) and by Cabré and Terra (21). See also the survey article by Farina and Valdinoci (22). The following result disproves De Giorgi's conjecture in dimension 9 (hence in any dimension higher than 9).

Theorem 1. *Let Γ be a BDG minimal graph in $\mathbb{R}^9$ and let ν be its unit normal. Set $\Gamma_\varepsilon := \varepsilon^{-1}\Gamma$. There exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists a bounded solution u_ε of (AC), monotone in the x_9-direction, with*

$$u_\varepsilon(x) = w(\zeta) + O(\varepsilon), \qquad x = y + \zeta\nu(\varepsilon y), \qquad y \in \Gamma_\varepsilon, |\zeta| < \frac{\delta}{\varepsilon},$$

$$\text{and} \quad \lim_{x_9 \to \pm\infty} u(x', x_9) = \pm 1 \quad \text{for all } x' \in \mathbb{R}^8.$$

Note that our result provides not just one example of a solution that violates De Giorgi's conjecture in dimensions $N \geq 9$, but a one parameter family parameterized by ε. This construction is possible because the dilated minimal graphs Γ_ε are themselves minimal graphs. In fact, the key idea of our work is that a connection between the minimal surface theory in $\mathbb{R}^N$ and the entire solutions of the Allen–Cahn equation can be made in the limit $\varepsilon \to 0$. One can speculate that the family of solutions $\{u_\varepsilon\}$ can be continued for values of $\varepsilon > \varepsilon_0$. Then, the nodal sets of such solutions will no longer be close to minimal surfaces.

The main ingredients in the proof of this above result will be described next. Details can be found in ref. 11.

The Proof of Theorem 1: Let Γ be a hypersurface embedded in $\mathbb{R}^N$ and let ν be the unit normal chosen so that $\nu_9 > 0$. Points of space, which are near Γ can be described by the local system of coordinates

$$x = y + z\nu(y), \qquad y \in \Gamma, |z| < \delta.$$

The following expression for the Laplacian in these coordinates holds,

$$\Delta_x = \partial_{zz} + \Delta_{\Gamma^z} - H_{\Gamma^z}(y)\partial_z. \qquad [8]$$

Here

$$\Gamma^z := \{y + z\nu(y)/y \in \Gamma\},$$

Δ_{Γ^z} is the Laplace–Beltrami operator on Γ^z and $H_{\Gamma^z}(y)$ its mean curvature. Let $k_1, \ldots, k_{N-1}$ be the principal curvatures of Γ. Then, it is also known that

$$H_{\Gamma^z} = \sum_{i=1}^{N-1} \frac{k_i}{1 - zk_i}. \qquad [9]$$

Now, similar relations hold if we consider the dilated surfaces Γ_ε instead of Γ, for instance,

$$x = y + \zeta\nu(\varepsilon y), \qquad y \in \Gamma_\varepsilon, |\zeta| < \delta/\varepsilon,$$

$k_{\varepsilon,i}(y) = \varepsilon k_\varepsilon(\varepsilon y)$, etc. The change of variables described above is a diffeomorphism, ϕ_ε, of a neighborhood of Γ_ε onto a set $\Gamma_\varepsilon \times (-\delta/\varepsilon, \delta/\varepsilon)$. In what follows we will abuse the notation and denote functions of the variable $x \in \mathbb{R}^9$ and of the local variables $(y, \zeta) = \phi_\varepsilon(x)$ by the same symbol, for instance given $u : \mathbb{R}^9 \to \mathbb{R}$ we write $u(y, \zeta)$ when x is close to Γ_ε, instead of $u \circ \phi_\varepsilon^{-1}(y, \zeta)$. Thus, letting $f(u) = u - u^3$ and $S(u) = \Delta u + f(u)$ the Allen–Cahn equation near Γ_ε becomes

$$S(u) = \Delta_{\Gamma_\varepsilon^\zeta} u - \varepsilon H_{\Gamma_\varepsilon^\zeta}(\varepsilon y)\partial_\zeta u + \partial_\zeta^2 u + f(u) = 0.$$

The solution we seek, at least near Γ_ε, should be of the following form:

$$u_\varepsilon(x) = w[\zeta - \varepsilon h(\varepsilon y)] + \phi, \qquad x = y + \zeta\nu(\varepsilon y),$$

where the function, h, defined on Γ, is left as a parameter to be adjusted and the function, ϕ, which should be small for ε. Set $r(y', y_9) = |y'|$ and $\omega_r = \sqrt{1 + r^2}$. We assume a priori that

$$\|\omega_r^3 D_\Gamma^2 h\|_{C^\sigma(\Gamma)} + \|\omega_r^2 D_\Gamma h\|_{L^\infty(\Gamma)} + \|\omega_r h\|_{L^\infty(\Gamma)} \le M$$

for some large, fixed number M. Let us change variables to $t = \zeta - \varepsilon h(\varepsilon y)$, and write, again abusing notation,

$$u(y, t) := u(x) \quad x = y + [t + \varepsilon h(\varepsilon y)]\nu(\varepsilon y).$$

The equation becomes

$$\begin{aligned} S(u) &= \partial_{tt} u + \Delta_{\Gamma_\varepsilon^\zeta} u - \varepsilon H_{\Gamma^{\varepsilon\zeta}}(\varepsilon y)\partial_t u \\ &\quad + \varepsilon^4 |\nabla_{\Gamma^{\varepsilon\zeta}} h(\varepsilon y)|^2 \partial_{tt} u - 2\varepsilon^3 \langle \nabla_{\Gamma^{\varepsilon\zeta}} h(\varepsilon y), \partial_t \nabla_{\Gamma^{\varepsilon\zeta}} u\rangle \\ &\quad - \varepsilon^3 \Delta_{\Gamma^{\varepsilon\zeta}} h(\varepsilon y)\partial_t u + f(u) = 0. \end{aligned}$$

Consequently, we look for solution, u_ε, of the form

$$u_\varepsilon(t, y) = w(t) + \phi(t, y)$$

for a small function ϕ. The equation in terms of ϕ becomes

$$\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + B\phi + f'[w(t)]\phi + N(\phi) + E = 0, \qquad [10]$$

where B is a small linear second order operator, and

$$E = S[w(t)], \quad N(\phi) = f(w + \phi) - f(w) - f'(w)\phi \approx f''(w)\phi^2.$$

The error of approximation is then given by the quantity

$$E = \varepsilon^4 |\nabla_{\Gamma^{\varepsilon\zeta}} h(\varepsilon y)|^2 w''(t) - [\varepsilon^3 \Delta_{\Gamma^{\varepsilon\zeta}} h(\varepsilon y) + \varepsilon H_{\Gamma^{\varepsilon\zeta}}(\varepsilon y)]w'(t),$$

where

$$\begin{aligned} \varepsilon H_{\Gamma^{\varepsilon\zeta}}(\varepsilon y) &= \varepsilon^2 [t + \varepsilon h(\varepsilon y)] |A_\Gamma(\varepsilon y)|^2 \\ &\quad + \varepsilon^3 [t + \varepsilon h(\varepsilon y)]^2 \sum_{i=1}^{8} k_i^3(\varepsilon y) + \cdots. \end{aligned}$$

A crucial fact for estimating the size of this error is the following result of L. Simon (23): $k_i = O(r^{-1})$ as $r \to +\infty$. In particular

$$|E(y, t)| \le C\varepsilon^2 r(\varepsilon y)^{-2}.$$

So far we have reduced our original problem to the Eq. **10** only near Γ_ε, namely for $|t| < \delta\varepsilon^{-1}$. To address this problem, we introduce a gluing procedure, which reduces the full problem to

$$\partial_{tt}\phi + \Delta\Gamma_\varepsilon\phi + B\phi + f'(w)\phi + N(\phi) + E = 0 \quad \text{in } \mathbb{R} \times \Gamma\varepsilon, \qquad [11]$$

where E and B are the same as before, but cutoff for $|t| > \delta/\varepsilon$, and N is accordingly modified by the addition of a small nonlocal operator of ϕ.

Although it is not apparent in the way **[11]** is written, we have two unknown functions ϕ and h to determine and we find them in two steps, which constitute an infinite dimensional Lyapunov–Schmidt reduction. This procedure resembles in principle the approach in ref. 24, and also has common features with ref. 25. However, the difference and the major difficulty comes from the fact that neither the manifold $\mathbb{R} \times \Gamma_\varepsilon$, nor its minimal submanifold $\{0\} \times \Gamma_\varepsilon$ are compact. More specifically, the steps of the Lyapunov–Schmidt reduction are the following:

Step 1: Given the parameter function h, find a function ϕ in $\mathbb{R} \times \Gamma_\varepsilon$, which is a solution to the problem

$$\begin{aligned} &\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + B\phi + f'[w(t)]\phi + N(\phi) + E = c(y)w'(t) \\ &\int_{\mathbb{R}} \phi(t, y)w'(t)dt = 0 \quad \text{for all } y \in \Gamma_\varepsilon. \end{aligned} \qquad [12]$$

Note that the map $h \mapsto \phi$ defines a nonlinear and nonlocal operator $\phi = \phi(h)$.

Step 2: Find a function h such that for all $y \in \Gamma_\varepsilon$,

$$c(y) := \frac{1}{\int_{\mathbb{R}} w'^2 dt} \int_{\mathbb{R}} \{E + B\phi(h) + N[\phi(h)]\}w' dt = 0.$$

To carry out Step 1 we solve first the linear problem

$$\begin{aligned} &\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + f'[w(t)]\phi = g(t, y) - c(y)w'(t) \quad \text{in } \mathbb{R} \times \Gamma_\varepsilon \\ &\int_{\mathbb{R}} \phi(y, t)w'(t)dt = 0 \quad \text{in } \Gamma_\varepsilon, c(y) := \frac{\int_{\mathbb{R}} g(y, t)w'(t)dt}{\int_{\mathbb{R}} w'^2 dt}. \end{aligned} \qquad [13]$$

Our claim is that there is a unique bounded solution $\phi := A(g)$ if g is bounded. Moreover, for any $\nu \ge 0$ we have

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$$\|[1+r(\varepsilon y)]^{\nu}\phi\|_{\infty} \le C\|[1+r(\varepsilon y)]^{\nu}g\|_{\infty}.$$

The proof of this claim is quite simple when Γ_ε is replaced by $\mathbb{R}$. Because $\Gamma_\varepsilon \approx \mathbb{R}^8$, locally uniformly as $\varepsilon \to 0$, the claim will follow from the analogous statement for the linear model problem: The equation

$$\partial_{tt}\phi + \Delta_y\phi + f'[w(t)]\phi = g(t,y) - c(y)w'(t) \quad \text{in } \mathbb{R}^9$$
$$\int_{\mathbb{R}} \phi(y,t)w'(t)dt = 0 \quad \text{in } \mathbb{R}^8, \qquad c(y) := \frac{\int_{\mathbb{R}} g(y,t)w'(t)dt}{\int_{\mathbb{R}} w'^2 dt} \qquad \textbf{[14]}$$

has a unique bounded solution ϕ if g is bounded, and

$$\|\phi\|_{\infty} \le C\|g\|_{\infty}. \qquad \textbf{[15]}$$

Let us first prove **[15]**. If the estimate is not true, there exist sequences $\{\phi_n\}$, $\{g_n\}$ such that

$$\partial_{tt}\phi_n + \Delta_y\phi_n + f'[w(t)]\phi_n = g_n(t,y), \qquad \int_{\mathbb{R}} \phi_n(y,t)w'(t)dt = 0,$$

while at the same time $\|\phi_n\|_\infty = 1$, $\|g_n\|_\infty \to 0$.

Using maximum principle and local elliptic estimates, we may assume that $\phi_n \to \phi_*$ uniformly over compact sets, where

$$\partial_{tt}\phi_* + \Delta_y\phi_* + f'[w(t)]\phi_* = 0, \qquad \int_{\mathbb{R}} \phi_*(y,t)w'(t)dt = 0.$$

Now, we claim that the above $\phi_* = 0$, which is a contradiction with the normalization $\|\phi_n\|_\infty = 1$.

To establish this claim we need the following spectral gap estimate: Let

$$L_0(p) := p'' + f'[w(t)]p.$$

Then there is a $\gamma > 0$ such that if $p \in H^1(\mathbb{R})$ and $\int_{\mathbb{R}} pw'dt = 0$ then

$$-\int_{\mathbb{R}} L_0(p)pdt = \int_{\mathbb{R}} [|p'|^2 - f'(w)p^2]dt \ge \gamma \int_{\mathbb{R}} p^2 dt.$$

Using the maximum principle, we find $|\phi_*(y,t)| \le Ce^{-|t|}$. Set $\varphi(y) = \int_{\mathbb{R}} \phi_*^2(y,t)dt$. Then

$$\begin{aligned}\Delta_y\varphi(y) &= 2\int_{\mathbb{R}} \phi_*\Delta\phi_*(y,t)dt + 2\int_{\mathbb{R}} |\nabla_y\phi_*(y,t)|^2 dt \\ &\ge -2\int_{\mathbb{R}} \phi_*\partial_{tt}\phi_* + f'(w)\phi_*^2 dt \\ &= 2\int_{\mathbb{R}} [|\partial_t\phi_*|^2 - f'(w)\phi_*^2]dt \ge \gamma\varphi(y),\end{aligned}$$

whence

$$-\Delta_y\varphi(y) + \gamma\varphi(y) \le 0$$

and as $\varphi \ge 0$ and is bounded, this inequality implies $\varphi \equiv 0$. Hence $\phi_* = 0$, a contradiction. This result proves **[15]**.

Given **[15]**, the existence of a solution ϕ of the linear model problem **[14]** is now established by a variational scheme. To this end let us initially take g compactly supported and let H be the space of all $\phi \in H^1(\mathbb{R}^9)$ with

$$\int_{\mathbb{R}} \phi(y,t)w'(t)dt = 0 \quad \text{for all } y \in \mathbb{R}^8.$$

Clearly H is a closed subspace of $H^1(\mathbb{R}^N)$. The problem is, as follows: $\phi \in H$ and

$$\partial_{tt}\phi + \Delta_y\phi + f'[w(t)]\phi = g(t,y) - w'(t)\frac{\int_{\mathbb{R}} g(y,\tau)w'(\tau)d\tau}{\int_{\mathbb{R}} w'^2 d\tau}$$

can be written variationally as that of minimizing the energy

$$I(\phi) = \frac{1}{2}\int_{\mathbb{R}^9} |\nabla_y\phi|^2 + |\partial_t\phi|^2 - f'(w)\phi^2 + \int_{\mathbb{R}^9} g\phi, \quad \phi \in H.$$

Thanks to the spectral gap estimate the functional I is coercive in H. Existence in the general case follows by the L^∞-a priori estimate and approximations.

Accepting that we have the above result not only for the linear model problem **[14]** but also for the linear problem **[13]**, we can write the problem **[12]** as a fixed point problem:

$$\phi = A[B\phi + N(\phi) + E].$$

The contraction mapping principle implies the existence of a unique solution $\phi := \phi(h)$ with $\|\omega_r^3\phi\|_\infty = O(\varepsilon^2)$.

Finally, we carry out Step 2. We need to find h such that

$$\int_{\mathbb{R}} \{E + B\phi(h) + N[\phi(h)]\}(\varepsilon^{-1}y,t)w'(t)dt = 0 \ \forall y \in \Gamma.$$

Because

$$\begin{aligned}-E(\varepsilon^{-1}y,t) &= \varepsilon^2 tw'(t)|A_\Gamma(y)|^2 + \varepsilon^3 t^2 w'(t)\sum_{j=1}^{8} k_j(y)^3 \\ &\quad + \varepsilon^3[\Delta_\Gamma h(y) + |A_\Gamma(y)|^2 h(y)]w'(t) + \ldots,\end{aligned}$$

where … represent smaller terms, the problem we need to solve is of the form

$$\Delta_\Gamma h + |A_\Gamma|^2 h = c\sum_{i=1}^{8} k_i^3 + g(y) + N(h) \quad \text{in } \Gamma, \qquad \textbf{[16]}$$

where $\mathcal{N}(h)$ is a small operator and g is a small function. We recognize the operator on the right-hand side as the Jacobi operator of Γ, denoted later by $\mathcal{J}_\Gamma(h)$.

An important ingredient of the analysis is the following claim: Let $0 < \sigma < 1$. Then if $\|(1 + r^{4+\sigma})\tilde{\mathrm{g}}\|_{L^\infty(\Gamma)} < +\infty$ there is a unique solution $h = T(\tilde{\mathrm{g}})$ to the problem

$$\mathcal{J}_\Gamma[h] := \Delta_\Gamma h + |A_\Gamma(y)|^2 h = \tilde{\mathrm{g}}(y) \quad \text{in } \Gamma$$

with

$$\|(1+r)^{2+\sigma}h\|_{L^\infty(\Gamma)} \le C\|(1+r)^{4+\sigma}\tilde{\mathrm{g}}\|_{L^\infty(\Gamma)}.$$

We want to solve **[16]** using a fixed point formulation for the operator T above. Making suitable assumptions on h and calculating the function g in **[16]** we conclude that $\tilde{\mathrm{g}} = \mathrm{g} + \mathcal{N}(h)$ satisfies the hypothesis of the claim above, namely it is of order $O(r^{-4-\sigma})$ and consequently the function $T[\mathrm{g} + \mathcal{N}(h)]$ is well-defined. However we only have

$$\sum_{i=1}^{8} k_i^3 = O(r^{-3}),$$

and we need some extra arguments to deal with the equation of the form

$$\mathcal{J}_\Gamma[h] = \sum_{i=1}^{8} k_i^3.$$

At this point we take full advantage of the improved asymptotic estimate (Eq. **7**) for the BDG surface. Using this expansion we can perform fairly direct computations for the principal curvatures k_i and conclude the following two key facts:

1. There is a smooth function p, such that $p(\frac{\pi}{2}-\theta) = -p(\theta)$ for all $\theta \in (0, \frac{\pi}{4})$, and
$$\sum_{i=1}^{8} k_i(y)^3 = \frac{p(\theta)}{r^3} + O(r^{-4-\sigma}).$$
2. There exists a smooth function $h_0(r,\theta)$, such that $h_0 = O(r^{-1})$ and for some $\sigma > 0$,
$$\mathcal{J}_\Gamma[h_0] = \frac{p(\theta)}{r^3} + O(r^{-4-\sigma}) \quad \text{as } r \to +\infty.$$

Our [**16**] finally becomes a problem for $h = h_0 + h_1$, where

$$h_1 = T[O(r^{-4-\sigma}) + \mathcal{N}(h_0 + h_1)],$$

which we can solve for $h_1 = O(r^{-2-\sigma})$, using the contraction mapping principle, keeping track of Lipschitz dependence in h of the objects involved in $\mathcal{N}(h)$.

Beyond De Giorgi's Conjecture

Loosely speaking, the method of construction of solutions described so far applies to finding an entire solution u_ε to $\Delta u + u - u^3 = 0$ with a transition set near $\Gamma_\varepsilon = \varepsilon^{-1}\Gamma$, whenever Γ is a minimal hypersurface embedded in $\mathbb{R}^N$, that splits the space into two components, and for which enough control at infinity is present to invert its Jacobi operator globally. Indeed the main difficulty in ref. 11 is the invertibility of the Jacobi operator of the BDG surface. However, in some situations the Jacobi operator is fairly easy to handle and then more can be said about the solutions of the Allen–Cahn equation in the context of their relation with the underlying minimal surface.

Finite Morse Index Solutions. The assumption of monotonicity in one direction for the solution u in De Giorgi's conjecture implies a form of stability, namely locally minimizing character for u when compactly supported perturbations are considered in the energy. Indeed, the linearized operator $L = \Delta + (1 - 3u^2)$, satisfies maximum principle because $L(Z) = 0$ for $Z = \partial_{x_N} u > 0$. This fact implies the stability of u, in the sense that its associated quadratic form, namely the second variation of the corresponding energy,

$$Q(\psi,\psi) \coloneqq \int_{R^3} |\nabla\psi|^2 + (3u^2 - 1)\psi^2 \qquad [\mathbf{17}]$$

satisfies $\mathcal{Q}(\psi,\psi) > 0$ for all $\psi \neq 0$ smooth and compactly supported. Stability of u is not only necessary but indeed sufficient for De Giorgi's statement to hold in dimension $N = 2$, as observed by Dancer (26). This question is open for $3 \le N \le 7$, and so is the corresponding stable Bernstein problem in that range.

Recently, stable solutions with nonplanar level sets in dimensions $N \ge 8$ have been found in ref. 27. This result uses the existence of a foliation by minimal surfaces asymptotic to minimal cones in dimensions $N \ge 8$.

Motivated by this result we would like to consider the problem of existence of entire solutions to the Allen–Cahn equation together with the question of their stability/instability. To be more precise we need the concept of the Morse index $m(u)$, defined as the maximal dimension of a vector space, E, of compactly supported functions such that

$$\mathcal{Q}(\psi,\psi) < 0 \quad \text{for all } \psi \in E \setminus \{0\}.$$

Considering the simplest case of $\mathbb{R}^N$, with $N = 3$, it seems natural to associate complete, embedded minimal surfaces Γ with finite Morse index, and solutions of [**1**]. The Morse index of the minimal surface Γ, $i(\Gamma)$, has a similar definition relative to the quadratic form for its Jacobi operator $\mathcal{J}_\Gamma \coloneqq \Delta_\Gamma + |A_\Gamma|^2$, namely $i(\Gamma)$ is the largest dimension for a vector spaced E of compactly supported smooth functions in Γ with

$$\int_\Gamma |\nabla k|^2 dV - \int_\Gamma |A_\Gamma|^2 k^2 dV < 0 \quad \text{for all } k \in E \setminus \{0\}.$$

We point out that for complete, embedded surfaces, in $\mathbb{R}^3$, finite index is equivalent to finiteness of the total curvature

$$\int_\Gamma |K| dV < +\infty,$$

where K denotes Gauss curvature of the manifold.

Given these definitions, we have the validity of the following result (28):

Theorem 2. *Let Γ be a complete, embedded minimal surface in $\mathbb{R}^3$ with finite total curvature. Assume additionally that Γ is nondegenerate, namely its bounded Jacobi fields originate only from rigid motions, and further let $\Gamma_\varepsilon = \varepsilon^{-1}\Gamma$ be a dilation of this surface. Then for all small $\varepsilon > 0$, there is a solution, u_ε, to [**1**] whose asymptotic behavior near Γ_ε is given by*

$$u_\varepsilon(x) \approx w(t), \quad x = y + t\nu_\varepsilon(y),$$

where ν_ε is the unit normal to Γ_ε and t is the signed distance from Γ_ε.

Moreover the Morse indices of u_ε and Γ_ε are equal: $m(u_\varepsilon) = i(\Gamma_\varepsilon)$.

For example, nondegeneracy and Morse index are known for the catenoid and the Costa–Hoffmann–Meeks surfaces [found in refs. 29 and 30, see Nayatani (31) and Morabito (32)]. In the case of the catenoid, the solution found is radially symmetric in two of its variables and $m(u_\varepsilon) = 1$. For the Costa–Hoffman–Meeks surface with genus $\ell \ge 1$, we have $m(u_\varepsilon) = 2\ell + 3$. We note finally that $i(\Gamma) = i(\Gamma_\varepsilon)$, for all $\varepsilon > 0$.

An Example with Infinite Total Curvature. The condition of finiteness of the total curvature of a minimal surface is by no means necessary for the existence of solutions of [**1**] whose zero level sets are close to this surface. The helicoid is a classical embedded minimal surface whose total curvature is infinite: This surface, dependent on a parameter λ, can be described as

$$H_\lambda = \{(r\cos\theta, r\sin\theta, z) \in \mathbb{R}^3 / z = \frac{\lambda}{\pi}\theta\}.$$

The following result holds ref. 33:

Theorem 3.

1. If $\lambda > \pi$, then there exists a solution to the Allen–Cahn equation in $\mathbb{R}^3$ whose zero level set is exactly H_λ.
2. If $\lambda \le \pi$ then any solution which vanishes on H_λ must be identically zero.

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This theorem, unlike those previously discussed, is not an asymptotic result: λ corresponds precisely to a dilation parameter of a fixed helicoid.

Toward a Classification of Entire Solutions. Complementing the preceding discussion we observe that the relation between the minimal surface theory and the theory of entire solutions of [**1**] in $\mathbb{R}^3$ is more complicated then it seems at first sight. In fact, whereas one can expect that given an embedded minimal surface, it is possible to find solutions to the Allen–Cahn equation whose zero level set is close to a dilation of this surface, there are known examples of solutions to [**1**] whose level set neither is embedded, nor minimal.

Indeed it is shown in ref. 34 that in $\mathbb{R}^2$ there exists the so-called saddle solution to [**1**], whose zero level set coincides with the straight lines $|x| = |y|$. Asymptotically, along these lines, the saddle solution resembles the heteroclinic profile of the one-dimensional solution of the Allen–Cahn equation. In ref. 19, for each sufficiently small $\alpha > 0$ another type of two-dimensional solution is found, these are even functions of the variables (x, y), and their zero level set in the first quadrant is asymptotically a straight line whose angle with the x axis is precisely α. We denote these solutions by u_α and note that the saddle solution mentioned above consequently should be denoted by $u_{\pi/4}$. Moreover in ref. 35 it is established that u_α for α small, and $u_{\pi/4}$ belong to the same connected component $\mathscr{M}$ of the moduli space of solutions of [**1**] in $\mathbb{R}^2$. Clearly every solution in $\mathscr{M}$ can be trivially extended to a solution in $\mathbb{R}^3$, thus giving a family of solutions whose zero level set is neither embedded, nor minimal, as we have anticipated.

All solutions in $\mathscr{M}$ have finite Morse index (it is expected that their Morse index is 1, see refs. 36–38) when considered as solutions in $\mathbb{R}^2$, but the Morse index of their extensions to $\mathbb{R}^3$ is infinite. It looks like the finiteness of the Morse index is then an important criterion from the point of view of classification of the entire solutions of [**1**] and plays a similar role as the condition of the finiteness of the total curvature in the theory of the minimal surfaces (37). Thus, in analogy with De Giorgi's conjecture, it seems plausible that qualitative properties of embedded minimal surfaces with finite Morse index should hold for the level sets of finite Morse index solutions of Eq. **1**, provided that these sets are embedded manifolds outside a compact set. The following result would be a step in the direction of classification of the simplest class of unstable solutions:

A bounded solution, u, of [**1**] in $\mathbb{R}^3$, with $i(u) = 1$, and $\nabla u \neq 0$ outside a bounded set, must be axially symmetric, namely radially symmetric in two variables.

An example of a solution satisfying the above is given in ref. 28 (in Theorem 2, take Γ to be a catenoid). If proven, the above conjecture would correspond to the famous result by Schoen (38), which says, if $i(\Gamma) = 1$ and Γ has embedded ends, then it must be a catenoid.

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