Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 259 (2010) 2045–2072
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

A logarithmic Hardy inequality

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Received 2 December 2009; accepted 15 June 2010

Communicated by Cédric Villani

Abstract

We prove a new inequality which improves on the classical Hardy inequality in the sense that a nonlinear integral quantity with super-quadratic growth, which is computed with respect to an inverse square weight, is controlled by the energy. This inequality differs from standard logarithmic Sobolev inequalities in the sense that the measure is neither Lebesgue's measure nor a probability measure. All terms are scale invariant. After an Emden–Fowler transformation, the inequality can be rewritten as an optimal inequality of logarithmic Sobolev type on the cylinder. Explicit expressions of the sharp constant, as well as minimizers, are established in the radial case. However, when no symmetry is imposed, the sharp constants are not achieved by radial functions, in some range of the parameters.

Keywords: Hardy inequality; Sobolev inequality; Interpolation; Logarithmic Sobolev inequality; Hardy–Sobolev inequalities; Caffarelli–Kohn–Nirenberg inequalities; Scale invariance; Emden–Fowler transformation; Radial symmetry; Symmetry breaking

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2010.06.005

1. Introduction and main results

The classical Hardy inequality in $\mathbb{R}^d$, $d \geqslant 3$, states that for any smooth, compactly supported function $u \in \mathcal{D}(\mathbb{R}^d)$, the following inequality holds:

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2}\,dx \leqslant \frac{4}{(d-2)^2} \int_{\mathbb{R}^d} |\nabla u|^2\,dx. \tag{1}$$

The constant $4/(d-2)^2$ is the best possible one. Many studies have been devoted to extensions and improvements of Hardy's inequality in bounded domains containing zero. In this direction, the first result is due to Brezis and Vázquez; see [24]. In [56], nonlinear improvements have been established, whereas in [42,2] linear and Sobolev type improvements are given. In [3], the best constant in the correction term of Sobolev type is computed. We also refer to [30] for improvements involving nonstandard correction terms. A recent trend seems to be oriented towards weights involving a distance to a manifold rather than a distance to a point singularity; see for instance [11,32,5,54]. In particular, when taking distance to the boundary, the dependence of the correction term on the geometry of the domain has been established in [46,39,10]. In the special case of the half-space in three space dimensions, the best constant of the Sobolev term in the improvement of Hardy's inequality has been found in [18] and it turns out to be the best Sobolev constant.

On the other hand a subject of particular interest has been the analysis of the link between Hardy's inequality (1) and Sobolev's inequality. A family of inequalities that interpolate between Hardy and Sobolev inequalities is given by the *Hardy–Sobolev inequality*,

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{d-\frac{d-2}{2}p}}\,dx\right)^{\frac{2}{p}} \leqslant \mathsf{C}_{\mathrm{HS}}(p) \int_{\mathbb{R}^d} |\nabla u|^2\,dx \tag{2}$$

for any $u \in \mathcal{D}(\mathbb{R}^d)$, where $2 \leqslant p \leqslant 2d/(d-2)$, $d \geqslant 3$, for a certain $\mathsf{C}_{\mathrm{HS}}(p) > 0$. Extremals for (2) are radially symmetric and the best constant $\mathsf{C}_{\mathrm{HS}}(p)$ can be explicitly computed: see [44,49,29, 47,28,35]. We shall recover the expression of $\mathsf{C}_{\mathrm{HS}}(p)$ at the end of Section 3.1. Extensions and improvements of the Hardy–Sobolev inequalities, and more generally of the Caffarelli–Kohn–Nirenberg inequalities established in [26], have been the object of many papers. We refer the reader for instance to [11,59,1,5,54,3] for various contributions to this topic.

The purpose of this paper is to investigate the connection between (1) and another classical Sobolev type inequality: the optimal *logarithmic Sobolev inequality* in $\mathbb{R}^d$ established in [45] which, expressed in a scale invariant form due to Weissler in [60], reads as

$$\int_{\mathbb{R}^d} |u|^2 \log |u|^2\,dx \leqslant \frac{d}{2} \log\left(\frac{2}{\pi d e} \int_{\mathbb{R}^d} |\nabla u|^2\,dx\right) \tag{3}$$

for any $u \in H^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} u^2\,dx = 1$. We point out a parallel between these inequalities: just like (1) is an endpoint of the family (2), that connects with Sobolev's inequality, inequality (3) can be viewed as an endpoint of a family of optimal Gagliardo–Nirenberg inequalities that also connects to Sobolev's inequality; see [33,34] for more details.

We emphasize that Hardy's inequality (1) in $\mathbb{R}^d$ cannot be improved in the usual sense, that is, there are no nontrivial potential $V \geqslant 0$ and no exponent $q > 0$ such that, for any function u,

$$\mathsf{C}\bigg(\int_{\mathbb{R}^d} V(x)|u|^q\,dx\bigg)^{2/q} \leqslant \frac{4}{(d-2)^2}\int_{\mathbb{R}^d} |\nabla u|^2\,dx - \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2}\,dx$$

for some positive constant C, as one can easily see by testing the above inequality with $u_\epsilon(x) = |x|^{-\frac{d-2}{2}+\epsilon}$, $|x| \leqslant 1$, and $u_\epsilon(x) = |x|^{-\frac{d-2}{2}-\epsilon}$, $|x| > 1$, and sending ϵ to zero.

Instead of improving on the potential, we study here the possibility of improving on the control of u. The weight is fixed to be $1/|x|^2$ and we try to get a control on $|u|^2 \log |u|^2$ instead of a control on $|u|^2$ only, as can sometimes be done for inequalities which appear as endpoints of a family, like (3). As a result, we obtain inequalities of logarithmic Sobolev type, with weight $1/|x|^2$ in the term involving the logarithm. Such an inequality is somewhat unusual, because in most of the cases, logarithmic Sobolev inequalities involve bounded positive measures. The euclidean case with Lebesgue's measure is an exception and can actually be reinterpreted in terms of the gaussian measure, see for instance [27,15] for some recent contributions in this direction. In the case of bounded measures, there is a huge literature: one can refer to [51,22,13] for a few key contributions.

The logarithmic Sobolev and Hardy inequalities play an important role in a number of instances. The first one is a very natural tool for obtaining intermediate asymptotics for the heat equation, see [12,55,8,7,37,14] with natural extensions to nonlinear diffusions (see for instance [21,23] and references therein). These inequalities are also useful in obtaining heat kernel estimates (see for instance [45,31]). A related logarithmic Sobolev inequality recently appeared in [40,41], where it was used for obtaining upper bounds for the heat kernel of a degenerate equation.

We shall denote by $\mathcal{D}^{1,2}(\mathbb{R}^d)$ the completion of $\mathcal{D}(\mathbb{R}^d)$ under the $L^2(\mathbb{R}^d)$ norm of the gradient of u. Let

$$\mathsf{S} = \frac{1}{\pi d(d-2)}\left[\frac{\Gamma(d)}{\Gamma(\frac{d}{2})}\right]^{\frac{2}{d}} = \mathsf{C}_{\mathrm{HS}}\left(\frac{2d}{d-2}\right)$$

be the optimal constant in Sobolev's inequality, according to [9,53]. Our first result states the validity of the following *logarithmic Hardy inequality*.

Theorem A. *Let $d \geqslant 3$. There exists a constant $\mathsf{C}_{\mathrm{LH}} \in (0, \mathsf{S}]$ such that, for all $u \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2}\,dx = 1$, we have*

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \log\big(|x|^{d-2}|u|^2\big)\,dx \leqslant \frac{d}{2}\log\bigg[\mathsf{C}_{\mathrm{LH}}\int_{\mathbb{R}^d} |\nabla u|^2\,dx\bigg]. \tag{4}$$

Inequality (4) can be viewed as an infinitesimal form of the Hardy–Sobolev inequality at $p = 2$: we observe that its left hand side is nothing but the derivative in p at $p = 2$ of the left hand side of (2), up to a factor 2. Compared to an entropy term with respect to the measure $|x|^{-2}\,dx$, there is however a $\log(|x|^d)$ term. Such a term is easily explained by scaling considerations and compensates for the presence of a super-quadratic nonlinearity $|u|^2 \log |u|^2$. The quantities involved in (4) give a precise account of the fact that, to exert control by the Dirichlet integral of a power of u larger than two, the singularity has to be at the same time milder.

It is natural to search for the optimal constant and extremals for inequality (4). Our second result answers this question in the class of radially symmetric functions, depending only on $|x|$, $x \in \mathbb{R}^d$.

Theorem B. *Let $d \geqslant 3$. If $u = u(|x|) \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ is radially symmetric, and $\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2}\,dx = 1$, then*

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \log\bigl(|x|^{d-2}|u|^2\bigr)\,dx \leqslant \frac{d}{2} \log\Bigl[\mathsf{C}^*_{\mathrm{LH}} \int_{\mathbb{R}^d} |\nabla u|^2\,dx\Bigr],$$

where

$$\mathsf{C}^*_{\mathrm{LH}} := \frac{4}{d} \frac{[\Gamma(\frac{d}{2})]^{\frac{2}{d}}}{\pi(8\pi e)^{\frac{1}{d}}} \left[\frac{d-1}{(d-2)^2}\right]^{1-\frac{1}{d}}.$$

Equality in the above inequality is achieved by the function

$$u = \frac{\tilde{u}}{\int_{\mathbb{R}^d} \frac{|\tilde{u}|^2}{|x|^2}\,dx} \quad \text{where } \tilde{u}(x) = |x|^{-\frac{d-2}{2}} \exp\biggl(-\frac{(d-2)^2}{4(d-1)}\bigl[\log|x|\bigr]^2\biggr).$$

For $d \geqslant 2$ and $a < (d-2)/2$, by starting from a more general weighted Hardy inequality,

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}}\,dx \leqslant \frac{4}{(d-2-2a)^2} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\,dx, \tag{5}$$

we prove the validity of a whole class of *weighted logarithmic Hardy inequalities*. If we denote by $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$ the completion with respect to the norm defined by the right hand side of (5) of $\mathcal{D}(\mathbb{R}^d \setminus \{0\})$ if $d \geqslant 2$ and of $\{u \in \mathcal{D}(\mathbb{R})\colon\ u'(0) = 0\}$ if $d = 1$, our result reads as follows:

Theorem A′. *Let $d \geqslant 1$. Suppose that $a < (d-2)/2$, $\gamma \geqslant d/4$ and $\gamma > 1/2$ if $d = 2$. Then there exists a positive constant $\mathsf{C}_{\mathrm{GLH}}$ such that, for any $u \in \mathcal{D}^{1,2}_a(\mathbb{R}^d)$ normalized by $\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}}\,dx = 1$, we have*

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}} \log\bigl(|x|^{d-2-2a}|u|^2\bigr)\,dx \leqslant 2\gamma \log\Bigl[\mathsf{C}_{\mathrm{GLH}} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\,dx\Bigr]. \tag{6}$$

On the other hand, in the radial case, we have a more general family of sharp inequalities:

Theorem B′. *Let $d \geqslant 1$, $a < (d-2)/2$ and $\gamma \geqslant 1/4$. If $u = u(|x|) \in \mathcal{D}_a^{1,2}(\mathbb{R}^d)$ is radially symmetric, and $\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}}\,dx = 1$, then*

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}} \log\bigl(|x|^{d-2-2a}|u|^2\bigr)\,dx \leqslant 2\gamma \log\left[\mathsf{C}^*_{\mathrm{GLH}} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\,dx\right], \tag{7}$$

where

$$\mathsf{C}^*_{\mathrm{GLH}} = \frac{1}{\gamma} \frac{[\Gamma(\frac{d}{2})]^{\frac{1}{2\gamma}}}{(8\pi^{d+1}e)^{\frac{1}{4\gamma}}} \left(\frac{4\gamma-1}{(d-2-2a)^2}\right)^{\frac{4\gamma-1}{4\gamma}} \quad \text{if } \gamma > \frac{1}{4} \quad \text{and}$$

$$\mathsf{C}^*_{\mathrm{GLH}} = \frac{[\Gamma(\frac{d}{2})]^2}{2\pi^{d+1}e} \quad \text{if } \gamma = \frac{1}{4}. \tag{8}$$

If $\gamma > \frac{1}{4}$, equality in (7) *is achieved by the function*

$$u = \frac{\tilde{u}}{\int_{\mathbb{R}^d} \frac{|\tilde{u}|^2}{|x|^2}\,dx} \quad \text{where } \tilde{u}(x) = |x|^{-\frac{d-2-2a}{2}} \exp\left(-\frac{(d-2-2a)^2}{4(4\gamma-1)}\bigl[\log|x|\bigr]^2\right).$$

Theorems A and B are special cases of Theorems A′ and B′ corresponding to $a = 0$, $\gamma = d/4$, $d \geqslant 3$. The family of inequalities of Theorem B′ implies on the one hand the logarithmic Sobolev inequality, and on the other hand the Hardy inequality, with optimal constants, as we shall see in Section 4. In dimension $d = 1$, radial symmetry simply means that functions are even.

We notice that inequalities (6) and (7) are both homogeneous and scale invariant. Actually, all integrals are individually scale invariant, in the sense that their values are unchanged if we replace $u(x)$ by $u_\lambda(x) = \lambda^{(d-2-2a)/2} u(\lambda x)$. This is of course consistent with the fact that the inequalities behave well under the Emden–Fowler transformation

$$u(x) = |x|^{-\frac{d-2-2a}{2}} w(y) \quad \text{with } y = (s,\omega) := \left(-\log|x|, \frac{x}{|x|}\right) \in \mathcal{C} := \mathbb{R} \times \mathbb{S}^{d-1} \tag{9}$$

and have an equivalent formulation on the cylinder $\mathcal{C}$, which goes as follows.

Theorem A″. *Let $d \geqslant 1$, $a < (d-2)/2$, $\gamma \geqslant d/4$ and $\gamma > 1/2$ if $d = 2$. Then, for any $w \in H^1(\mathcal{C})$ normalized by $\int_{\mathcal{C}} w^2\,dy = 1$, we have*

$$\int_{\mathcal{C}} |w|^2 \log|w|^2\,dy \leqslant 2\gamma \log\left(\mathsf{C}_{\mathrm{GLH}}\left[\int_{\mathcal{C}} |\nabla w|^2\,dy + \frac{1}{4}(d-2-2a)^2\right]\right). \tag{10}$$

The optimal constant $\mathsf{C}_{\mathrm{GLH}}$ is the same in Theorems A′ and A″. Similarly, the case of radial functions depending only on $|x|$ corresponds to the case of functions depending only on $s = -\log|x|$.

Theorem B″. *Let $d \geqslant 1$, $a < (d-2)/2$ and $\gamma \geqslant 1/4$. If $w \in H^1(\mathcal{C})$ depends only on $s \in \mathbb{R}$ and is normalized by $\int_{\mathcal{C}} w^2\,dy = 1$, then*

$$\int_{\mathcal{C}} |w|^2 \log |w|^2\,dy \leqslant 2\gamma \log\biggl(\mathsf{C}^*_{\mathrm{GLH}}\biggl[\int_{\mathcal{C}} |\nabla w|^2\,dy + \frac{1}{4}(d-2-2a)^2\biggr]\biggr). \tag{11}$$

The value of the optimal constant $\mathsf{C}^*_{\mathrm{GLH}}$ is given by (8). If $\gamma > \frac{1}{4}$, equality in (11) is achieved by the function

$$w(s) = \frac{\tilde{w}(s)}{\int_{\mathcal{C}} \tilde{w}^2\,dy} \quad \text{where } \tilde{w}(s) = \exp\biggl(-\frac{(d-2-2a)^2}{4(4\gamma-1)}s^2\biggr).$$

If $d = 1$, $\mathcal{C}$ is equal to $\mathbb{R}$. For any $d \geqslant 1$, one may suspect that the optimal constant for (6) (resp. (10)) is achieved in the class of radially symmetric functions (resp. functions depending only on $s \in \mathbb{R}$) and therefore $\mathsf{C}_{\mathrm{GLH}} = \mathsf{C}^*_{\mathrm{GLH}}$. Using the method developed in [28,38,36], it turns out that there is a range of the parameters a and γ for which this is not the case.

Theorem C. *Let $d \geqslant 2$ and $a < -1/2$. Assume that $\gamma > 1/2$ if $d = 2$. If, in addition,*

$$\frac{d}{4} \leqslant \gamma < \frac{1}{4} + \frac{(d-2a-2)^2}{4(d-1)},$$

then the optimal constant $\mathsf{C}_{\mathrm{GLH}}$ in inequality (6) is not achieved by a radial function and $\mathsf{C}_{\mathrm{GLH}} > \mathsf{C}^*_{\mathrm{GLH}}$.

This paper is organized as follows. In Section 2, we derive Theorems A, A′ and A″ as a consequence of a Caffarelli–Kohn–Nirenberg interpolation inequality. In Section 3, we present a complete study of the radial case and in particular we prove Theorems B, B′ and B″. This study is based on a sharp one-dimensional interpolation inequality. In Section 4, we show that Theorem B′ implies both the logarithmic Sobolev inequality (3) and the Hardy inequality (1). In the final section, we study the symmetry breaking of the interpolation inequalities as well as of the logarithmic Hardy inequality, thus establishing Theorem C.

2. Interpolation inequalities. Proof of Theorems A, A′ and A″

In this section, we will give the proofs of Theorems A, A′ and A″ with the help of a general inequality of Caffarelli–Kohn–Nirenberg type and a differentiation procedure with respect to some of the parameters of the inequality. Our starting point is the following inequality, which has been established in [26]:

$$\biggl(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{bp}}\,dx\biggr)^{\frac{2}{p}} \leqslant \mathsf{C}_{\mathrm{CKN}}(p,a)\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\,dx \quad \forall u \in \mathcal{D}\bigl(\mathbb{R}^d\bigr). \tag{12}$$

Restrictions on the exponents are given by the conditions: $b \in (a+1/2, a+1]$ in case $d=1$, $b \in (a, a+1]$ when $d=2$ and $b \in [a, a+1]$ when $d \geqslant 3$. In addition, for any $d \geqslant 1$, we assume that

$$a < \frac{d-2}{2} \quad \text{and} \quad p = \frac{2d}{d-2+2(b-a)}. \tag{13}$$

See for instance [28] for a review of various known results like existence of optimal functions. In the limit case $b=a+1$, $p=2$, (12) is equivalent to (5) and the optimal constant is then $\mathsf{C}_{\mathrm{CKN}}(2,a) = 4/(d-2-2a)^2$.

The range $a > (d-2)/2$ can also be covered with functions in the space $\mathcal{D}(\mathbb{R}^d \setminus \{0\})$. Inequalities are not restricted to spaces of smooth functions and can be extended to the space $\mathcal{D}_{a,b}(\mathbb{R}^d)$ obtained by completion of $\mathcal{D}(\mathbb{R}^d \setminus \{0\})$ with respect to the norm defined by

$$\|u\|^2 = \big\| |x|^{-b} u \big\|^2_{L^p(\mathbb{R}^d)} + \big\| |x|^{-a} \nabla u \big\|^2_{L^2(\mathbb{R}^d)},$$

but some care is required. For instance, if $a > (d-2)/2$, it turns out that, for any $u \in \mathcal{D}_{a,b}(\mathbb{R}^d)$,

$$\lim_{r \to 0_+} r^{-d} \|u\|_{L^2(B(0,r))} = 0.$$

See [36] for more details.

A key issue for (12) is to determine whether equality is achieved among radial solutions when $\mathsf{C}_{\mathrm{CKN}}$ is the optimal constant, or, alternatively, if symmetry breaking occurs. See [28,38,50,36,35] for conditions for which the answer is known. Here are some cases for which radial symmetry holds:

(i) The dimension is $d=1$.
(ii) If $d \geqslant 3$, we assume either $a \geqslant 0$ or, for any $p \in (2, 2^*)$, $a < 0$ and $|a|$ is small enough, or for any $a < 0$, $p-2 > 0$ is small enough.
(iii) If $d=2$, we assume either $a<0$ with $|a|$ small enough and $|a|p < 2$, or, for any $a < 0$, $p-2 > 0$ is small enough.

In such cases, optimal functions are known and $\mathsf{C}_{\mathrm{CKN}}(p,a)$ is explicit (see Section 3.3). Alternatively, it is known that for $d \geqslant 2$, if

$$a < b < 1 + a - \frac{d}{2}\left(1 - \frac{d-2-2a}{\sqrt{(d-2-2a)^2 + 4(d-1)}}\right), \tag{14}$$

minimizers are not radially symmetric. In such a case the explicit expression of $\mathsf{C}_{\mathrm{CKN}}$ is not known. More details will be given in Section 5.

Let $2^* = \infty$ if $d=1$ or $d=2$, $2^* = 2d/(d-2)$ if $d \geqslant 3$ and define

$$\vartheta(p,d) := \frac{d(p-2)}{2p}.$$

We have a slightly more general family of interpolation inequalities than (12), which has also been established in [26] and goes as follows.

Theorem 1. *(According to [26].) Let $d \geqslant 1$. For any $\theta \in [\vartheta(p,d), 1]$, there exists a positive constant $\mathsf{C}(\theta, p, a)$ such that*

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{bp}}\,dx\right)^{\frac{2}{p}} \leqslant \mathsf{C}(\theta,p,a)\left(\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\,dx\right)^{\theta}\left(\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}}\,dx\right)^{1-\theta} \quad \forall u \in \mathcal{D}^{1,2}_a\big(\mathbb{R}^d\big). \tag{15}$$

Inequality (15) coincides with (12) if $\theta = 1$. We will establish the expression of $\mathsf{C}(\theta, p, a)$ when minimizers are radially symmetric and extend the symmetry breaking results of Felli and Schneider to the case $\theta < 1$ in Sections 3 and 5 respectively. Before, we give an elementary proof of (15), whose purpose is to give a bound on $\mathsf{C}(\theta, p, a)$ in terms of the best constant in (12), and to justify the limiting case that is obtained by passing to the limit $\theta \to 0_+$ and $p \to 2_+$ simultaneously.

Proposition 2. *Let $b \in (a+1/2, a+1]$ when $d = 1$, $b \in (a, a+1]$ when $d = 2$ and $b \in [a, a+1]$ when $d \geqslant 3$. In addition, for any $d \geqslant 1$, we assume that* (13) *holds. Then we have*

(i) *Let $\mathfrak{K} := \{k \in (0,2)\colon\ k \leqslant d - (d-2)p/2 \text{ if } d \geqslant 3\}$. For any $\theta \in [\vartheta(p,d), 1] \cap (1 - 2/p, 1]$, we have*

$$\mathsf{C}(\theta,p,a) \leqslant \inf_{k\in\mathfrak{K}}\left[\mathsf{C}_{\mathrm{CKN}}\left(\frac{2(p-k)}{2-k}, a\right)\right]^{1-\frac{k}{p}}\left(\frac{2}{d-2-2a}\right)^{2(\frac{k}{p}+\theta-1)}.$$

(ii) *Let $d \geqslant 2$. Suppose that $a < (d-2)/2$, $\gamma \geqslant d/4$ and $\gamma > 1/2$ if $d = 2$. We have*

$$\mathsf{C}_{\mathrm{GLH}} \leqslant \mathsf{C}_{\mathrm{CKN}}\left(\frac{4\gamma}{2\gamma-1}, a\right).$$

Proof. If $d \geqslant 3$ and $p = 2^*$, that is for $b = a$, then $\vartheta(p,d) = 1 = \theta$ and (15) is reduced to a special case of (12). Assume that $p < 2^*$. Let $u \in \mathcal{D}(\mathbb{R}^d)$. For any $k \in (0,2)$, we have:

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{bp}}\,dx &= \int_{\mathbb{R}^d} \left(\frac{|u|}{|x|^{1+a}}\right)^k \left(\frac{|u|^{p-k}}{|x|^{bp-k(1+a)}}\right) dx \\ &\leqslant \left(\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}}\,dx\right)^{\frac{k}{2}} \left(\int_{\mathbb{R}^d} \frac{|u|^{2\frac{p-k}{2-k}}}{|x|^{2\frac{bp-k(1+a)}{2-k}}}\,dx\right)^{\frac{2-k}{2}}.\end{aligned} \tag{16}$$

We observe that (12) holds for some a, b and p if, due to the scaling invariance, these parameters are related by the relation

$$b = a + 1 + d\left(\frac{1}{p} - \frac{1}{2}\right) = a + 1 - \vartheta(p,1).$$

For any $k \in (0,2)$, we also have the relation $B = A + 1 + d(\frac{1}{P} - \frac{1}{2})$ if

$$A = a, \qquad P = \frac{2(p-k)}{2-k} \quad \text{and} \quad BP = \frac{2(bp - k(1+a))}{2-k}.$$

Hence, using (12), we have that

$$\left(\int_{\mathbb{R}^d} \frac{|u|^{2\frac{p-k}{2-k}}}{|x|^{2\frac{bp-k(1+a)}{2-k}}}\,dx\right)^{\frac{2-k}{p-k}} \leqslant \mathsf{C}_{\mathrm{CKN}}\left(2\frac{p-k}{2-k}, a\right)\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\,dx \tag{17}$$

provided that $2 < 2(p-k)/(2-k) \leqslant 2^*$ if $d \geqslant 3$, which is equivalent to $k \leqslant d-(d-2)p/2$ using the fact that $k<2$. On the other hand, we may estimate the first integral in the right hand side of (16) by (5) and get

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}}\,dx \leqslant \left(\frac{4}{(d-2-2a)^2}\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\,dx\right)^{1-\alpha}\left(\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}}\,dx\right)^{\alpha} \tag{18}$$

for any $\alpha \in [0,1]$. Combining (16), (17) and (18) we get

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{bp}}\,dx\right)^{\frac{2}{p}} \leqslant \mathsf{C}(\theta,p,a)\left(\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\,dx\right)^{\theta}\left(\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}}\,dx\right)^{1-\theta}$$

with $\theta = 1-\alpha k/p \in [\vartheta(p,d),1]$ and this proves (15). Notice that for $d \geqslant 3$, the restriction $\theta \geqslant \vartheta(p,d)$ comes from the condition $k \leqslant d-(d-2)p/2$. If $d=2$, $\theta > \vartheta(p,2) = 1-2/p$ is due to the restriction $k<2$. However, if $\theta = \vartheta(p,2)$, inequality (15) still holds true. For this case, we refer to [26]. If $d=1$, one knows that the inequality (15) holds true under the condition $\theta > 1-2/p$, but the inequality still holds under the weaker condition $\theta \geqslant \vartheta(p,1) = 1/2-1/p$. See Section 3.1 for further details in the one-dimensional case.

Let $P \in (2,2^*]$ if $d \geqslant 3$, $P>2$ if $d=1$ or 2. For any $p \in [2,P)$ we choose $k = 2\frac{P-p}{P-2} \in (0,2]$, which also satisfies $k \leqslant d-(d-2)p/2$ so that $P = 2\frac{p-k}{2-k} \leqslant 2^*$, if $d \geqslant 3$ and $k<2$. We also set $B := a+1-d(\frac{1}{P}-\frac{1}{2})$, so that $BP = 2\frac{bp-k(1+a)}{2-k}$. Then (16) can be written as

$$\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{bp}}\,dx \leqslant \left(\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}}\,dx\right)^{\frac{P-p}{P-2}}\left(\int_{\mathbb{R}^d} \frac{|u|^P}{|x|^{BP}}\,dx\right)^{\frac{p-2}{P-2}}. \tag{19}$$

Here we assume that $b = a+1+d(\frac{1}{p}-\frac{1}{2})$. If $d \geqslant 3$, estimate (19) is valid for any $2<p<P \leqslant 2^*$, and it is an equality for $p=2$ and any $P \in (2,2^*]$. By differentiating (19) with respect to p at $p=2$, we get that for any $P \in (2,2^*]$,

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}}\log\left(\frac{|x|^{d-2-2a}|u|^2}{\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}}\,dx}\right)dx \leqslant \frac{P}{P-2}\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}}\,dx \log\left[\frac{(\int_{\mathbb{R}^d} \frac{|u|^P}{|x|^{BP}}\,dx)^{\frac{2}{P}}}{\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}}\,dx}\right].$$

For $d \geqslant 3$, let $2\gamma := \frac{P}{P-2} \in [\frac{d}{2},\infty)$. For $d=1,2$, let $2\gamma := \frac{P}{P-2} \in (1,\infty)$. Then, for any $\gamma \geqslant d/4$ if $d \geqslant 3$ and any $\gamma > 1/2$ if $d=1,2$ and any $a<(d-2)/2$, using once more (12), we have shown (ii). □

Proof of Theorems A, A′ and A″. The existence of $\mathsf{C}_{\mathrm{GLH}}$ is a straightforward consequence of Proposition 2 if $d \geqslant 2$. This proves Theorem A′, except for the case $d = 1$ which will be considered in Section 3.4. Theorem A follows with $\gamma = d/4$, $d \geqslant 3$ and $a = 0$. In particular we get an upper bound for the optimal constant: $\mathsf{C}_{\mathrm{LH}} \leqslant \mathsf{C}_{\mathrm{CKN}}(2^*, 0) = \mathsf{S}$.

By the Emden–Fowler transformation (9), the inequalities of Proposition 2, on $\mathbb{R}^d$, are transformed into equivalent ones on the cylinder $\mathcal{C} = \mathbb{R} \times \mathbb{S}^{d-1}$. More precisely (15) can be reformulated as

$$\left(\int_{\mathcal{C}} |w|^p\,dy\right)^{\frac{2}{p}} \leqslant \mathsf{C}(\theta, p, a)\left(\int_{\mathcal{C}} |\nabla w|^2\,dy + \frac{1}{4}(d-2-2a)^2 \int_{\mathcal{C}} |w|^2\,dy\right)^{\theta}\left(\int_{\mathcal{C}} |w|^2\,dy\right)^{1-\theta}. \tag{20}$$

Notice that by standard arguments, the sharp constant in (20) is achieved in $H^1(\mathcal{C})$ when the parameters are in the range corresponding to the assumptions of Proposition 2, provided $p > 2$ and $\theta > \vartheta(p, d)$.

By (9), the logarithmic Hardy inequality (6) of Theorem A′ takes the form (10) of Theorem A″ if $\gamma \geqslant d/4$, $d \geqslant 3$ and $a < (d-2)/2$, or $\gamma > 1/2$, $d = 2$ and $a < 0$. The one-dimensional case, for which $\mathcal{C} = \mathbb{R}$, will be directly investigated in the next section. □

3. The one-dimensional and the radial cases. Proof of Theorems B, B′ and B″

In this section we will study the Caffarelli–Kohn–Nirenberg interpolation inequality (20) as well as the logarithmic Hardy inequality (10) in the one-dimensional case and under the restriction to the set of radial functions. As a consequence, we shall also establish Theorems B, B′ and B″.

3.1. The sharp interpolation inequality in the one-dimensional cylindric case

If w depends only on $s = -\log|x|$, inequality (20) can be reduced to its one-dimensional version,

$$\left(\int_{\mathbb{R}} |w|^p\,ds\right)^{\frac{2}{p}} \leqslant \mathsf{K}(\theta, p, \sigma)\left(\int_{\mathbb{R}} |w'|^2\,ds + \sigma^2 \int_{\mathbb{R}} |w|^2\,ds\right)^{\theta}\left(\int_{\mathbb{R}} |w|^2\,ds\right)^{1-\theta}, \tag{21}$$

for any $w \in H^1(\mathbb{R})$, with $\sigma = (d-2-2a)/2$, provided $\mathsf{C}(\theta, p, a)|\mathbb{S}^{d-1}|^{1-2/p} = \mathsf{K}(\theta, p, \sigma)$. Inequality (21) is however of interest by itself and can be considered as depending on the parameters θ, p and σ, independently of a and d.

If $\theta > \vartheta(p, 1)$, the proof of the existence of an optimal function is standard. After optimizing the inequality under scalings, (21) reduces to a Gagliardo–Nirenberg inequality, whose optimal function is defined up to a scaling and a multiplication by a constant. Let us give some details.

If we optimize inequality (21) under scalings, we find that it is equivalent to the one-dimensional Gagliardo–Nirenberg inequality. Let

$$Q[w] := \left(\int_{\mathbb{R}} |w'|^2\,ds + \sigma^2 \int_{\mathbb{R}} |w|^2\,ds\right)^{\theta}\left(\int_{\mathbb{R}} |w|^2\,ds\right)^{1-\theta}$$

be the functional which appears in the right hand side of inequality (21) and consider $w_\lambda(s) = \lambda^{1/p} w(\lambda s)$, $\lambda > 0$. These scalings leave the left hand side of inequality (21) invariant, while

$$Q[w_\lambda] = \left(\lambda^{2-b} \int_{\mathbb{R}} |w'|^2\, ds + \sigma^2 \lambda^{-b} \int_{\mathbb{R}} |w|^2\, ds\right)^{\theta} \left(\int_{\mathbb{R}} |w|^2\, ds\right)^{1-\theta}$$

with $b = (p-2)/(p\theta)$. We observe that $a = 2 - b$ is positive if and only if $\theta > (p-2)/(2p) = \vartheta(p,1)$. Hence we find that

(i) if $\theta < \vartheta(p,1)$, then $\inf_{\lambda>0} Q[w_\lambda] = \lim_{\lambda\to\infty} Q[w_\lambda] = 0$, and inequality (21) does not hold;
(ii) if $\theta = \vartheta(p,1)$, then

$$\inf_{\lambda>0} Q[w_\lambda] = \lim_{\lambda\to\infty} Q[w_\lambda] = \left(\int_{\mathbb{R}} |w'|^2\, ds\right)^{\vartheta(p,1)} \left(\int_{\mathbb{R}} |w|^2\, ds\right)^{1-\vartheta(p,1)},$$

so that (21) is equivalent to the one-dimensional Gagliardo–Nirenberg inequality

$$\|w\|_{L^p(\mathbb{R})} \leqslant \mathsf{C}_{\mathrm{GN}} \|w'\|_{L^2(\mathbb{R})}^{\vartheta(p,1)} \|w\|_{L^2(\mathbb{R})}^{1-\vartheta(p,1)} \quad \forall w \in H^1(\mathbb{R}). \tag{22}$$

Hence $\mathsf{K}(\vartheta(p,1), p, \sigma) = \mathsf{C}_{\mathrm{GN}}^2$ is independent of $\sigma > 0$, and inequality (21) admits no optimal function if $\theta = \vartheta(p,1)$, $\sigma > 0$. It degenerates into (22) in the limit $\sigma \to 0_+$, for which an optimal function exists;
(iii) if $\theta > \vartheta(p,1)$, then $\inf_{\lambda>0} Q[w_\lambda]$ is achieved for

$$\lambda^2 = \frac{b}{a} \frac{\sigma^2 \int_{\mathbb{R}} |w|^2\, ds}{\int_{\mathbb{R}} |w'|^2\, ds},$$

that is

$$\inf_{\lambda>0} Q[w_\lambda] = \kappa \left(\int_{\mathbb{R}} |w'|^2\, ds\right)^{\vartheta(p,1)} \left(\int_{\mathbb{R}} |w|^2\, ds\right)^{1-\vartheta(p,1)}$$

with $\kappa = (\frac{a}{b\sigma^2})^{\frac{p-2}{2p}} (\frac{a+b}{a}\sigma^2)^{\theta}$, *i.e.*

$$\frac{1}{\kappa} = \left[\frac{(p-2)\sigma^2}{2+(2\theta-1)p}\right]^{\frac{p-2}{2p}} \left[\frac{2+(2\theta-1)p}{2p\theta\sigma^2}\right]^{\theta}.$$

As a consequence, $\mathsf{K}(\theta, p, \sigma) = \kappa^{-1} \mathsf{C}_{\mathrm{GN}}^2$ and optimality is achieved in inequality (21), since (22) admits an optimal function; see for instance [4,52].

The above Gagliardo–Nirenberg inequality (22) is equivalent to the Sobolev inequality corresponding to the embedding $H_0^1(0,1) \hookrightarrow L^q(0,1)$ for some $q = q(p) > 2$; see

[19,57,58] for more details. Also notice that, using the radial symmetry of the minimizers of the optimal functions of the Hardy–Sobolev inequality (2), we recover the expression of $\mathsf{C}_{\mathrm{HS}}(p)=|\mathbb{S}^{d-1}|^{-(p-2)/p}\mathsf{K}(1,p,(d-2)/2)$ that can be found in [29,47,28,35], using $|\mathbb{S}^{d-1}|=2\pi^{d/2}/\Gamma(d/2)$, $\sqrt{\pi}\,\Gamma(d)=2^{d-1}\Gamma(d/2)\Gamma((d+1)/2)$ and Lemma 3 below. Similarly, in all cases for which optimal functions are known to be radially symmetric in inequality (12) (see Section 2), we have $\mathsf{C}_{\mathrm{CKN}}(p,a)=|\mathbb{S}^{d-1}|^{-(p-2)/p}\mathsf{K}(1,p,(d-2-2a)/2)$. The constant $\mathsf{K}(\theta,p,\sigma)$ can be computed as follows.

Lemma 3. *Let $\sigma>0$, $p>2$ and $\theta\in[\vartheta(p,1),1]$. Then the best constant in inequality* (21) *is given by*:

$$\mathsf{K}(\theta,p,\sigma)=\left[\frac{(p-2)^2\sigma^2}{2+(2\theta-1)p}\right]^{\frac{p-2}{2p}}\left[\frac{2+(2\theta-1)p}{2p\theta\sigma^2}\right]^{\theta}\left[\frac{4}{p+2}\right]^{\frac{6-p}{2p}}\left[\frac{\Gamma(\frac{2}{p-2}+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\frac{2}{p-2})}\right]^{\frac{p-2}{p}}. \tag{23}$$

If $\theta>\vartheta(p,1)$, the best constant is achieved by an optimal function $\overline{w}(s)$, which is unique up to multiplication by constants and shifts and is given by

$$\overline{w}(s)=\big(\cosh(\lambda s)\big)^{-\frac{2}{p-2}}\quad \textit{with } \lambda=\frac{1}{2}(p-2)\sigma\left[\frac{p+2}{2+(2\theta-1)p}\right]^{\frac{1}{2}}.$$

Proof. Using the Emden–Fowler transformation, the value of $\mathsf{K}(\theta,p,\sigma)$ can be computed using the equation

$$(p-2)^2w''-4w+2p|w|^{p-2}w=0 \tag{24}$$

such that $w'(0)=0$ and $\lim_{|s|\to\infty}w(s)=0$. A minimizer for (21) is indeed defined up to a translation (a scaling in the original variables) and a multiplication by a constant, which can be adjusted to fix one of the coefficients in the Euler–Lagrange equation as desired. An optimal function can therefore be written as $w(\lambda s)$ for some $\lambda>0$, on which we can optimize. The solution w of (24) is unique if we further assume that it is positive with a maximum at $s=0$. This can be seen as follows. Multiply (24) by w and integrate from s to $+\infty$. Since $\lim_{|s|\to\infty}w'(s)=0$, the function $s\mapsto\frac{1}{2}(p-2)^2w'(s)^2-2w(s)^2+2w(s)^p$ is constant and therefore equal to 0. This determines $w(0)=1$, so that the solution is unique and given by

$$w(s)=(\cosh s)^{-\frac{2}{p-2}}\quad\forall s\in\mathbb{R}.$$

Hence

$$\mathsf{K}(\theta,p,\sigma)=\max_{\lambda>0}\frac{(\lambda^{-1}I_p)^{\frac{2}{p}}}{(\lambda J_2+\sigma^2\lambda^{-1}I_2)^{\theta}(\lambda^{-1}I_2)^{1-\theta}}$$

where

$$I_q:=\int_{\mathbb{R}}\big|w(s)\big|^q\,ds\quad\text{and}\quad J_2:=\int_{\mathbb{R}}\big|w'(s)\big|^2\,ds.$$

With $\lambda = \mu^\theta$, let $g(\mu) := \mu^{\frac{2}{p}+2\theta-1}\frac{J_2}{I_2} + \sigma^2\mu^{\frac{2}{p}-1}$ and observe that

$$\mathsf{K}(\theta, p, \sigma) = \frac{I_p^{\frac{2}{p}}}{(\min_{\mu>0} g(\mu))^\theta I_2}.$$

If $\theta > \vartheta(p, 1)$, the minimum of $g(\mu)$ is achieved at $\lambda^2 = \mu^{2\theta} = \frac{p-2}{2+(2\theta-1)p}\sigma^2\frac{I_2}{J_2}$ and

$$\Big(\min_{\mu>0} g(\mu)\Big)^\theta = h(p, \theta, \sigma)\left(\frac{J_2}{I_2}\right)^{\frac{1}{2}-\frac{1}{p}}$$
$$\text{where } h(p, \theta, \sigma) := \left[\frac{2 + (2\theta - 1)p}{(p - 2)\sigma^2}\right]^{\frac{p-2}{2p}}\left[\frac{2p\theta\sigma^2}{2 + (2\theta - 1)p}\right]^\theta.$$

If $\theta = \vartheta(p, 1)$, then $\inf_{\mu>0} g(\mu) = \frac{J_2}{I_2}$ and we set $h(p, \theta, \sigma) := 1$. For any $\theta \in [\vartheta(p, 1), 1]$, we thus obtain

$$\mathsf{K}(\theta, p, \sigma) = \frac{I_p^{\frac{2}{p}}}{h(p, \theta, \sigma) J_2^{\frac{1}{2}-\frac{1}{p}} I_2^{\frac{1}{2}+\frac{1}{p}}}.$$

Using the formula

$$\int_{\mathbb{R}} \frac{ds}{(\cosh s)^q} = \frac{\sqrt{\pi}\,\Gamma(\frac{q}{2})}{\Gamma(\frac{q+1}{2})} =: f(q),$$

we can compute

$$I_2 = f\left(\frac{4}{p-2}\right), \qquad I_p = f\left(\frac{2p}{p-2}\right) = f\left(\frac{4}{p-2} + 2\right),$$

and get

$$I_2 = \frac{\sqrt{\pi}\,\Gamma(\frac{2}{p-2})}{\Gamma(\frac{p+2}{2(p-2)})}, \qquad I_p = \frac{4I_2}{p+2}, \qquad J_2 := \frac{4}{(p-2)^2}(I_2 - I_p) = \frac{4I_2}{(p+2)(p-2)}.$$

Hence

$$\mathsf{K}(\theta, p, \sigma) = \left[\frac{(p-2)^2\sigma^2}{2 + (2\theta - 1)p}\right]^{\frac{p-2}{2p}}\left[\frac{2 + (2\theta - 1)p}{2p\theta\sigma^2}\right]^\theta\left[\frac{4}{p+2}\right]^{\frac{6-p}{2p}}\left[\frac{1}{I_2}\right]^{\frac{p-2}{p}},$$

which proves (23). □

3.2. *The sharp logarithmic Hardy inequality in the one-dimensional cylindric case*

With $2\gamma = p/(p-2)$ and $\theta = \gamma(p-2)$, we observe that the condition $\theta \in [\vartheta(p,1),1]$ is equivalent to $\gamma \in [1/(2p), 1/(p-2)]$ and that $2+(2\theta-1)p = (2\gamma p-1)(p-2)$ is positive for any $p>2$ since $\gamma > 1/(2p)$. Substituting θ with $\gamma(p-2)$ in the expression of $\mathsf{K}(\theta,p,\sigma)$ given by (23) and taking the logarithm, we get

$$\log \mathsf{K}\big(\gamma(p-2),p,\sigma\big) = \frac{p-2}{2p}\log\left[\frac{2\sigma^2}{2\gamma p-1}\right] + \gamma(p-2)\log\left[\frac{2\gamma p-1}{2p\gamma\sigma^2}\right] + \frac{6-p}{2p}\log\left[\frac{4}{p+2}\right] + \frac{p-2}{p}\log\left[\frac{\Gamma(\frac{2}{p-2}+\frac{1}{2})}{\sqrt{\pi}\sqrt{\frac{2}{p-2}}\,\Gamma(\frac{2}{p-2})}\right].$$

Using Stirling's formula, it is easy to see that $\lim_{t\to\infty}\frac{\Gamma(t+\frac12)}{\sqrt{t}\,\Gamma(t)} = 1$, so that $\lim_{p\to 2}\mathsf{K}(\gamma(p-2), p,\sigma) = 1$. Hence, for any $\gamma > 1/(2p)$, let

$$\mathcal{K}(\gamma,\sigma) := -2\frac{d}{dp}\big[\mathsf{K}\big(\gamma(p-2),p,\sigma\big)\big]\big|_{p=2}$$

and consider the limit as $p\to 2$ in inequality (21). We observe that $1/4 > 1/(2p)$ for any $p>2$ so that $\gamma > 1/4$ guarantees $\gamma > 1/(2p)$ uniformly in the limit $p \to 2_+$. The case $\gamma = 1/4$ is achieved as a limit case.

Lemma 4. *Let $\sigma > 0$ and $\gamma \geqslant 1/4$. Then for any $w \in H^1(\mathbb{R})$ the following inequality holds true*

$$\int_{\mathbb{R}} |w|^2 \log\left(\frac{|w|^2}{\int_{\mathbb{R}}|w|^2\,ds}\right) ds + \mathcal{K}(\gamma,\sigma)\int_{\mathbb{R}}|w|^2\,ds \leqslant 2\gamma \int_{\mathbb{R}}|w|^2\,ds \log\left[\frac{\int_{\mathbb{R}}|w'|^2\,ds}{\int_{\mathbb{R}}|w|^2\,ds}+\sigma^2\right], \tag{25}$$

with sharp constant given by

$$\mathcal{K}(\gamma,\sigma) = 2\gamma\log\gamma - \frac{4\gamma-1}{2}\log\left(\frac{4\gamma-1}{4\sigma^2}\right) + \frac12\log(2\pi e) \tag{26}$$

if $\gamma > 1/4$, and equality holds in (25) for $w(s) = \exp(-\frac{\sigma^2}{4\gamma-1}s^2)$. If $\gamma = 1/4$, then $\mathcal{K}(\gamma,\sigma) = 2\gamma\log\gamma + \frac12\log(2\pi e)$.

Proof. The result follows by taking the logarithm of (21) and differentiating at $p=2$ for $\gamma > 1/4$. The equality case in (25) can be checked by a direct computation. □

3.3. The sharp inequalities for radial functions

Let $d \geqslant 2$ and consider the interpolation inequality (15) restricted to the subset $\mathcal{D}_a^*(\mathbb{R}^d)$ of radial functions in $\mathcal{D}_a(\mathbb{R}^d)$, *i.e.*

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{bp}}\,dx\right)^{\frac{2}{p}} \leqslant \mathsf{C}^*(\theta,p,a)\left(\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\,dx\right)^{\theta}\left(\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}}\,dx\right)^{1-\theta} \quad \forall u \in \mathcal{D}_a^*(\mathbb{R}^d)$$

where $\mathsf{C}^*(\theta,p,a)$ denotes the best constant. Let $\sigma = (d-2-2a)/2$. By the Emden–Fowler change of coordinates (9), the above inequality is equivalent to

$$\left(\int_{\mathcal{C}} |w|^p\,dy\right)^{\frac{2}{p}} \leqslant \mathsf{C}^*(\theta,p,a)\left(\int_{\mathcal{C}} |\nabla w|^2\,dy + \sigma^2\int_{\mathcal{C}} |w|^2\,dy\right)^{\theta}\left(\int_{\mathcal{C}} |w|^2\,dy\right)^{1-\theta} \tag{27}$$

for all functions $w \in H^1(\mathcal{C})$ depending only on $s = -\log|x|$. Up to a normalization factor depending on $|\mathbb{S}^{d-1}| = 2\pi^{d/2}/\Gamma(d/2)$, (27) is equivalent to the one-dimensional inequality (21) with the best constant $\mathsf{K}(\theta,p,\sigma)$. It is straightforward to check that $\mathsf{C}^*(\theta,p,a) = |\mathbb{S}^{d-1}|^{-(p-2)/p}\mathsf{K}(\theta,p,\sigma)$. We also note that the range of θ is as in Lemma 3, that is, $\theta \in [\vartheta(p,1),1]$.

Similarly, inequality (25) is equivalent to (11) with $[\mathcal{K}(\gamma,\sigma) + \log|\mathbb{S}^{d-1}|] = -2\gamma\log\mathsf{C}^*_{\mathrm{GLH}}$. This proves Theorem B″. Theorem B′ follows by the Emden–Fowler change of coordinates (9). Theorem B corresponds to the special case $a = 0$, $d \geqslant 3$. Notice that, with $\sigma = (d-2-2a)/2$, inequality (7) written in terms of a function f on $\mathbb{R}^+$ such that $u(x) = f(|x|)$ takes the form

$$\int_0^\infty r^{d-3-2a}|f|^2\log\left(r^{d-2-2a}|f|^2\right)dr + \mathcal{K}(\gamma,\sigma) \leqslant 2\gamma\log\left[\int_0^\infty r^{d-1-2a}\left|f'\right|^2 dr\right],$$

under the normalization condition $\int_0^\infty r^{d-3-2a}|f|^2\,dr = 1$.

3.4. The sharp interpolation inequality in the case of the one-dimensional real line

Recall that inequalities written on the euclidean space $\mathbb{R}^d$ are equivalent to one-dimensional inequalities on $\mathcal{C}$ by the Emden–Fowler transformation (9) only in case of radial functions and, for $d = 1$, only for *even* functions. However, in this case, we may notice that the restriction $a < (d-2)/2 = -1/2$ means that the weight $|x|^{-2a}$ corresponds to a positive power, so that we may consider the problem on $\mathbb{R}^+$ and $\mathbb{R}^-$ as two independent problems when dealing with a smooth function u such that $u'(0) = 0$.

Using $X_+\log X_+ + X_-\log X_- \leqslant (X_+ + X_-)\log(X_+ + X_-)$ with $X_\pm = \int_{\mathbb{R}^\pm} \frac{|u|^2}{|x|^{2(a+1)}}\,dx$, we get

$$\int_{\mathbb{R}} \frac{|u|^2}{|x|^{2(a+1)}}\log\left(\frac{|x|^{d-2-2a}|u|^2}{\int_{\mathbb{R}} \frac{|u|^2}{|x|^{2(a+1)}}\,dx}\right)dx$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \frac{|u|^2}{|x|^{2(a+1)}} \log\bigl(|x|^{d-2-2a}|u|^2\bigr)\,dx - (X_+ + X_-)\log(X_+ + X_-) \\
&\leqslant \int_{\mathbb{R}^-} \frac{|u|^2}{|x|^{2(a+1)}} \log\bigl(|x|^{d-2-2a}|u|^2\bigr)\,dx - X_-\log X_- \\
&\quad + \int_{\mathbb{R}^+} \frac{|u|^2}{|x|^{2(a+1)}} \log\bigl(|x|^{d-2-2a}|u|^2\bigr)\,dx - X_+\log X_+ \\
&= \int_{\mathbb{R}^-} \frac{|u|^2}{|x|^{2(a+1)}} \log\biggl(\frac{|x|^{d-2-2a}|u|^2}{\int_{\mathbb{R}^-} \frac{|u|^2}{|x|^{2(a+1)}}\,dx}\biggr)\,dx + \int_{\mathbb{R}^+} \frac{|u|^2}{|x|^{2(a+1)}} \log\biggl(\frac{|x|^{d-2-2a}|u|^2}{\int_{\mathbb{R}^+} \frac{|u|^2}{|x|^{2(a+1)}}\,dx}\biggr)\,dx.
\end{aligned}
$$

By the Emden–Fowler transformation, we know that

$$
\int_{\mathbb{R}^\pm} \frac{|u|^2}{|x|^{2(a+1)}} \log\biggl(\frac{|x|^{d-2-2a}|u|^2}{\int_{\mathbb{R}^\pm} \frac{|u|^2}{|x|^{2(a+1)}}\,dx}\biggr)\,dx \leqslant 2\gamma X_\pm \log\biggl[\mathsf{C}_{\mathrm{GLH}} \frac{Y_\pm}{X_\pm}\biggr]
$$

with $Y_\pm = \int_{\mathbb{R}^\pm} \frac{|\nabla u|^2}{|x|^{2a}}\,dx$. Using $X_-\log[\frac{Y_-}{X_-}] + X_+\log[\frac{Y_+}{X_+}] \leqslant (X_+ + X_-)\log[\frac{Y_+ + Y_-}{X_+ + X_-}]$, we end up with the inequality

$$
\int_{\mathbb{R}} \frac{|u|^2}{|x|^{2(a+1)}} \log\biggl(\frac{|x|^{d-2-2a}|u|^2}{\int_{\mathbb{R}} \frac{|u|^2}{|x|^{2(a+1)}}\,dx}\biggr)\,dx \leqslant 2\gamma \int_{\mathbb{R}} \frac{|u|^2}{|x|^{2(a+1)}}\,dx \log\biggl[\mathsf{C}_{\mathrm{GLH}} \frac{\int_{\mathbb{R}} \frac{|\nabla u|^2}{|x|^{2a}}\,dx}{\int_{\mathbb{R}} \frac{|u|^2}{|x|^{2(a+1)}}\,dx}\biggr],
$$

which completes the proof of Theorem A′ in the one-dimensional case.

4. Connection with logarithmic Sobolev and Hardy inequalities

In this section, we study the connection of the *logarithmic Hardy inequality* (4) and its generalized form (5) with the *euclidean logarithmic Sobolev inequality* (see below), the *Hardy inequality* (1) and its generalized form (2), and the *logarithmic Sobolev inequality on* $\mathcal{C}$ (see below).

As we have seen in the previous section, the weighted logarithmic Hardy inequality of Theorem B′ (radial case) is equivalent to the one-dimensional inequality (25) with sharp constant given by (26). With the choice $2\sigma = 4\gamma - 1$, we observe that $\lim_{\gamma\to 1/4} \mathcal{K}(\gamma, 4\gamma - 1) = \frac{1}{2}\log(\frac{\pi e}{2})$ and recover the one-dimensional logarithmic Sobolev inequality written in the scale invariant form (see [60]) with optimal constant, namely

$$
\int_{\mathbb{R}} |w|^2 \log\biggl(\frac{|w|^2}{\int_{\mathbb{R}} |w|^2\,ds}\biggr)\,ds \leqslant \frac{1}{2}\int_{\mathbb{R}} |w|^2\,ds \log\biggl[\frac{2}{\pi e} \frac{\int_{\mathbb{R}} |w'|^2\,ds}{\int_{\mathbb{R}} |w|^2\,ds}\biggr]. \tag{28}
$$

Actually, inequality (25) can be written in a simpler form in terms of a function $v \in H^1(\mathbb{R})$, such that $w(s) = v((d-2-2a)s/\sqrt{4\gamma - 1})$ as follows. For any $\gamma \geqslant 1/4$ and any $v \in H^1(\mathbb{R})$,

$$\int_{\mathbb{R}} |v|^2 \log\left(\frac{|v|^2}{\int_{\mathbb{R}} |v|^2\, ds}\right) ds + \left[2\gamma \log\gamma + \frac{1}{2}\log(2\pi e)\right] \int_{\mathbb{R}} |v|^2\, ds$$
$$\leqslant 2\gamma \int_{\mathbb{R}} |v|^2\, ds \log\left[\frac{\int_{\mathbb{R}} |v'|^2\, ds}{\int_{\mathbb{R}} |v|^2\, ds} + \gamma - \frac{1}{4}\right].$$

Consider the γ-dependent terms, *i.e.*

$$2\gamma \int_{\mathbb{R}} |v|^2\, ds \log\left[\frac{\int_{\mathbb{R}} |v'|^2\, ds}{\int_{\mathbb{R}} |v|^2\, ds} + \gamma - \frac{1}{4}\right] - 2\gamma\log\gamma \int_{\mathbb{R}} |v|^2\, ds$$
$$= \left[2\int_{\mathbb{R}} |v'|^2\, ds - \frac{1}{2}\int_{\mathbb{R}} |v|^2\, ds\right] f(t),$$

with $f(t) := \frac{1}{t}\log(t+1)$ and $t = \frac{1}{4\gamma}[4\frac{\int_{\mathbb{R}} |v'|^2\, ds}{\int_{\mathbb{R}} |v|^2\, ds} - 1]$. An elementary analysis shows that f is decreasing so that, in terms of γ, the minimum of the right hand side is always achieved at $\gamma = 1/4$. In this case we recover the one-dimensional logarithmic Sobolev inequality written in the scale invariant form (28). On the other hand, if we send γ to ∞ which implies that $t \to 0$ and $f(t) \to 1$, we recover the logarithmic Sobolev inequality in the standard euclidean form (see [45]):

$$\int_{\mathbb{R}} |v|^2 \log\left(\frac{|v|^2}{\int_{\mathbb{R}} |v|^2\, ds}\right) ds + \frac{1}{2}\int_{\mathbb{R}} |v|^2\, ds \log\left[2\pi e^2\right] \leqslant 2\int_{\mathbb{R}} |v'|^2\, ds. \tag{29}$$

We can also recover Hardy's inequality from (25) by taking the limit $\gamma \to +\infty$ and observing that $\lim_{\gamma\to+\infty} \mathcal{K}(\gamma,\sigma)/(2\gamma) = 2\log\sigma$. The radial function $u \in \mathcal{D}_a^{1,2}(\mathbb{R}^d)$ given in terms of w by the inverse of the Emden–Fowler change of coordinates (9) satisfies

$$\frac{1}{4}(d-2-2a)^2 \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}}\, dx \leqslant \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\, dx. \tag{30}$$

This holds true for any $d \in \mathbb{N}$, $d \geqslant 2$ and any $a < 0$. For $d \geqslant 3$, if we define $f(x) := |x|^{-a} u(x)$, $x \in \mathbb{R}^d$, then inequality (30) is equivalent to the usual Hardy inequality (with $a = 0$), namely

$$\frac{1}{4}(d-2)^2 \int_{\mathbb{R}^d} \frac{|f|^2}{|x|^2}\, dx \leqslant \int_{\mathbb{R}^d} |\nabla f|^2\, dx, \quad f \in \mathcal{D}^{1,2}(\mathbb{R}^d). \tag{31}$$

Using Schwarz' symmetrization, it is then straightforward to see that optimality is achieved for radial functions, thus showing that, with $\sigma = (d-2-2a)/2$, inequality (25) implies inequality (30) for any function $u \in \mathcal{D}_a^{1,2}(\mathbb{R}^d)$ (and not only for radial functions), that is Hardy's inequality if $a = 0$ and all Caffarelli–Kohn–Nirenberg inequalities with $b = a+1$, $a < 0$, otherwise.

Summarizing, the family of inequalities (7) of Theorem B′ implies as extreme cases the logarithmic Sobolev inequality (28) at the endpoint $\gamma = 1/4$, and the euclidean logarithmic Sobolev inequality (29) and the Hardy inequality (31) as γ tends to $+\infty$. In both cases, one-dimensional versions of the inequalities are involved. On $\mathcal{C}$, it is possible to recover the optimal logarithmic Sobolev inequality from the logarithmic Hardy inequality as follows.

Let $d\mu$ and $d\nu_\sigma(t) := (2\pi\sigma^2)^{-1/2}\exp(-t^2/(2\sigma^2))\,dt$ be respectively the uniform probability measure on $\mathbb{S}^{d-1}$ induced by Lebesgue's measure on $\mathbb{R}^d$ and the gaussian probability measure on $\mathbb{R}$. Using the tensorization property of the logarithmic Sobolev inequalities (see for instance [6]), we obtain the

Lemma 5. *For any $d \geqslant 2$, the following inequality holds*

$$\int_{\mathcal{C}} |w|^2 \log\left(\frac{|w|^2}{\int_{\mathcal{C}} |w|^2\,dy}\right) dy + \mathcal{K}_d^\sigma \int_{\mathcal{C}} |w|^2\,dy \leqslant \max\left\{\frac{2}{d-1}, 2\sigma^2\right\} \int_{\mathcal{C}} |\nabla w|^2\,dy \quad \forall w \in H^1(\mathcal{C}) \tag{32}$$

with optimal constant

$$\mathcal{K}_d^\sigma = \frac{1}{2}\log\left(2\pi e^2\sigma^2\left|\mathbb{S}^{d-1}\right|^2\right) = 1 + \frac{1}{2}\log\left(\frac{8\pi^{d+1}\sigma^2}{\Gamma(d/2)^2}\right).$$

Proof. The sharp logarithmic Sobolev inequality on the sphere $\mathbb{S}^{d-1}$ is

$$\int_{\mathbb{S}^{d-1}} |w|^2 \log\left(\frac{|w|^2}{\int_{\mathbb{S}^{d-1}} |w|^2\,d\mu}\right) d\mu \leqslant \frac{2}{d-1} \int_{\mathbb{S}^{d-1}} |\nabla w|^2\,d\mu \quad \forall w \in H^1\left(\mathbb{S}^{d-1}\right)$$

where $d\mu$ is the uniform probability measure on $\mathbb{S}^{d-1}$ induced by Lebesgue's measure in $\mathbb{R}^d$. Using a method which can be found for instance in [16], this inequality can be obtained as the limit as $q \to 2_+$ of sharp interpolation inequalities stated in [17], namely

$$\frac{2}{q-2}\left[\left(\int_{\mathbb{S}^{d-1}} |w|^q\,d\mu\right)^{\frac{2}{q}} - \int_{\mathbb{S}^{d-1}} |w|^2\,d\mu\right] \leqslant \frac{2}{d-1} \int_{\mathbb{S}^{d-1}} |\nabla w|^2\,d\mu,$$

and optimality is easily checked by considering the sequence of test functions $w_n = 1 + \varphi_1/n$, where φ_1 is a spherical harmonic function associated to the first non-zero eigenvalue of the Laplace–Beltrami operator on the sphere. On $\mathbb{R}$, the logarithmic Sobolev inequality associated to the gaussian probability measure $d\nu_\sigma$ has been established by L. Gross in [45]:

$$\int_{\mathbb{R}} |w|^2 \log\left(\frac{|w|^2}{\int_{\mathbb{R}} |w|^2\,d\nu_\sigma}\right) d\nu_\sigma \leqslant 2\sigma^2 \int_{\mathbb{R}} \left|w'\right|^2 d\nu_\sigma \quad \forall w \in H^1(\mathbb{R}).$$

Again the constant $2\sigma^2$ is optimal as can be checked considering the sequence of test functions $w_n = 1 + \psi_1/n$, where $\psi_1(t) = t\exp(-t^2/(2\sigma)^2)$ is the first nonconstant Hermite function, up to a scaling.

The tensorization property of the logarithmic Sobolev inequalities shows that

$$\int_{\mathcal{C}} |w|^2 \log\left(\frac{|w|^2}{\int_{\mathcal{C}} |w|^2\, d\mu \otimes d\nu_\sigma}\right) d\mu \otimes d\nu_\sigma \leqslant \max\left\{\frac{2}{d-1}, 2\sigma^2\right\} \int_{\mathcal{C}} |\nabla w|^2\, d\mu \otimes d\nu_\sigma.$$

Taking into account the normalization of $d\mu$ and $d\nu_\sigma$, $|\mathbb{S}^{d-1}| = 2\pi^{d/2}/\Gamma(d/2)$ and rewriting Gross' inequality with respect to Lebesgue's measure, we end up with (32). The constant $\mathcal{K}_d^\sigma$ is optimal as can be shown again by considering a sequence of test functions based either on spherical harmonics or on Hermite functions. □

As a special case, for $\sigma^2 = 1/(d-1)$ and $\mathcal{K}_d := \mathcal{K}_d^\sigma$, we have the following inequality on the cylinder.

Corollary 6. *For any $d \geqslant 2$, with $\mathcal{K}_d = 1 + \frac{1}{2}\log(\frac{8\pi^{d+1}}{(d-1)\Gamma(d/2)^2})$, the following inequality holds*

$$\int_{\mathcal{C}} |w|^2 \log\left(\frac{|w|^2}{\int_{\mathcal{C}} |w|^2\, dy}\right) dy + \mathcal{K}_d \int_{\mathcal{C}} |w|^2\, dy \leqslant \frac{2}{d-1} \int_{\mathcal{C}} |\nabla w|^2\, dy \quad \forall w \in H^1(\mathcal{C}). \tag{33}$$

Using $\log(1+X) \leqslant \alpha - 1 - \log\alpha + \alpha X$ for any $\alpha > 0$, $X > 0$, which amounts to write $\log Y \leqslant Y - 1$ with $Y = \alpha(X+1)$, and applying it in (10) with $X = \sigma^{-2}\int_{\mathcal{C}} |\nabla w|^2\, dy / \int_{\mathcal{C}} |w|^2\, dy$, $\sigma = (d-2-2a)/2$, $\alpha = \sigma^2/(\gamma(d-1))$, we deduce that

$$\int_{\mathcal{C}} |w|^2 \log\left(\frac{|w|^2}{\int_{\mathcal{C}} |w|^2\, dy}\right) dy - 2\gamma\left[\log\big(\gamma(d-1)\mathsf{C}_{\mathrm{GLH}}\big) + \frac{\sigma^2}{\gamma(d-1)} - 1\right] \int_{\mathcal{C}} |w|^2\, dy$$
$$\leqslant \frac{2}{d-1} \int_{\mathcal{C}} |\nabla w|^2\, dy$$

for any $w \in H^1(\mathcal{C})$. We may observe that

$$\mathcal{K}_d + 2\gamma\left[\log\big(\gamma(d-1)\mathsf{C}^*_{\mathrm{GLH}}\big) + \frac{\sigma^2}{\gamma(d-1)} - 1\right] = \frac{4\gamma-1}{2}[Z - 1 - \log Z]$$
$$\text{with } Z = \frac{4\sigma^2}{(4\gamma-1)(d-1)}.$$

Hence, if $4\sigma^2 = (4\gamma-1)(d-1)$ and if (for this specific value) $\mathsf{C}_{\mathrm{GLH}} = \mathsf{C}^*_{\mathrm{GLH}}$, then the optimal logarithmic Sobolev inequality (33) is a consequence of (10).

We may observe that $4\sigma^2 = (4\gamma-1)(d-1)$ means $Z = 1$ and exactly corresponds to the threshold for the symmetry breaking result of Theorem C. Notice that proving that $\mathsf{C}_{\mathrm{GLH}} = \mathsf{C}^*_{\mathrm{GLH}}$ is an open question.

5. Symmetry breaking. Proof of Theorem C

In this section we study the symmetry breaking of the Caffarelli–Kohn–Nirenberg interpolation inequality as well as of the logarithmic Hardy inequality. To achieve this, we use a technique introduced Catrina and Wang in [28] and later improved by Felli and Schneider in [38]. Also see [25,50,36]. The method amounts to consider a functional made of the difference of the two sides of the inequality, with a constant chosen so that the functional takes value zero in the optimal case, among radially symmetric functions, when the inequality is written for functions on $\mathbb{R}^d$. Equivalently, we can consider functions depending only on one real variable in the case of the cylinder. By linearizing around the optimal radial function, we obtain an explicit linear operator and can study when the eigenvalue corresponding to the subspace generated by the first nontrivial spherical harmonic function becomes negative. It is then clear that the functional can change sign, so that optimality cannot be achieved by radial functions. This proves the *symmetry breaking*. We will apply the method first to the interpolation inequality (20), thus generalizing the results of Felli and Schneider to a more general Caffarelli–Kohn–Nirenberg interpolation inequality than the one they have considered, and then to the logarithmic Hardy inequality (10).

5.1. Symmetry breaking for the interpolation inequality

Based on (27), consider on $H^1(\mathcal{C})$ the functional

$$\mathcal{J}[w] := \int_{\mathcal{C}} \left(|\nabla w|^2 + \frac{1}{4}(d-2-2a)^2|w|^2\right) dy - \left[\mathsf{C}^*(\theta,p,a)\right]^{-\frac{1}{\theta}} \frac{(\int_{\mathcal{C}} |w|^p\,dy)^{\frac{2}{p\theta}}}{(\int_{\mathcal{C}} |w|^2\,dy)^{\frac{1-\theta}{\theta}}}.$$

Among functions $w \in H^1(\mathcal{C})$ which depend only on s, $\mathcal{J}[w]$ is nonnegative, its minimum is zero and it is achieved by

$$\overline{w}(y) := \left[\cosh(\lambda s)\right]^{-\frac{2}{p-2}}, \quad y = (s,\omega) \in \mathbb{R} \times \mathbb{S}^{d-1} = \mathcal{C},$$

with $\lambda := \frac{1}{4}(d-2-2a)(p-2)\sqrt{\frac{p+2}{2p\theta-(p-2)}}$. See the proof of Lemma 3 for more details. We can notice that

$$\left[\mathsf{C}(\theta,p,a)\right]^{-\frac{1}{\theta}} = \int_{\mathcal{C}} \left(|\nabla \overline{w}|^2 + \frac{1}{4}(d-2-2a)^2|\overline{w}|^2\right) dy \frac{(\int_{\mathcal{C}} |\overline{w}|^2\,dy)^{\frac{1-\theta}{\theta}}}{(\int_{\mathcal{C}} |\overline{w}|^p\,dy)^{\frac{2}{p\theta}}}.$$

With a slight abuse of notations, we shall write $\overline{w}$ as a function of s only, which solves the ODE

$$\lambda^2(p-2)^2 w'' - 4w + 2p|w|^{p-2}w = 0$$

and, as in the proof of Lemma 3,

$$\int_{\mathbb{R}} |\overline{w}|^2\,ds = \frac{1}{\lambda} I_2,$$

$$\int_{\mathbb{R}} |\overline{w}'|^2 \, ds = \lambda J_2 = \frac{4\lambda}{p^2-4} I_2,$$

$$\int_{\mathbb{R}} |\overline{w}|^p \, ds = \frac{1}{\lambda} I_p = \frac{4}{p+2} \frac{1}{\lambda} I_2.$$

In terms of p and θ, we investigate the symmetry of optimal functions for (15) or, equivalently, for (20) in the range

$$0 < p - 2 \leqslant \frac{4}{d-2} \quad \text{and} \quad \vartheta(p,d) \leqslant \theta \leqslant 1.$$

Consider now $\mathcal{J}[\overline{w} + \varepsilon\phi]$ and Taylor expand it at order 2 in ε, using the fact that $\overline{w}$ is a critical point and assuming that $\int_{\mathcal{C}} \overline{w}^{p-1} \phi \, dy = 0$:

$$\frac{1}{\varepsilon^2} \mathcal{J}[\overline{w} + \varepsilon\phi] = \int_{\mathcal{C}} |\nabla\phi|^2 \, dy - \kappa \int_{\mathcal{C}} \overline{w}^{p-2} |\phi|^2 \, dy + \mu \int_{\mathcal{C}} |\phi|^2 \, dy - \nu \Bigl(\int_{\mathcal{C}} \overline{w}\phi \, dy \Bigr)^2 + o(1)$$

as $\varepsilon \to 0$, with

$$\kappa := \frac{p-1}{\theta} \frac{1}{I_p} \Bigl(\lambda^2 J_2 + \frac{1}{4}(d-2-2a)^2 I_2 \Bigr),$$

$$\mu := \frac{1}{4}(d-2-2a)^2 + \frac{1-\theta}{\theta} \frac{1}{I_2} \Bigl(\lambda^2 J_2 + \frac{1}{4}(d-2-2a)^2 I_2 \Bigr),$$

$$\nu := \frac{1-\theta}{2\theta^2} \frac{\lambda}{I_2^2} \Bigl(\lambda^2 J_2 + \frac{1}{4}(d-2-2a)^2 I_2 \Bigr).$$

Spectral properties of the operator $\mathcal{L} := -\Delta + \kappa \overline{w}^{p-2} + \mu$ are well known. Eigenfunctions can be characterized in terms of Legendre's polynomials, see for instance [48, p. 74] and [38]. Using spherical coordinates and spherical harmonic functions, see [20], the discrete spectrum is made of the eigenvalues

$$\lambda_{i,j} = \mu + i(d+i-2) - \frac{\lambda^2}{4} \Bigl(\sqrt{1 + \frac{4\kappa}{\lambda^2}} - (1+2j) \Bigr)^2 \quad \forall i, j \in \mathbb{N},$$

as long as $\sqrt{1 + 4\kappa/\lambda^2} \geqslant 2j + 1$. The eigenspace of $\mathcal{L}$ corresponding to $\lambda_{0,0}$ is generated by $\overline{w}$. Next we observe that the eigenfunction $\phi_{(1,0)}$ associated to $\lambda_{1,0}$ is not radially symmetric and such that $\int_{\mathcal{C}} \overline{w} \phi_{(1,0)} \, dy = 0$ and $\int_{\mathcal{C}} \overline{w}^{p-1} \phi_{(1,0)} \, dy = 0$. Hence, *if* $\lambda_{1,0} < 0$, *optimal functions for* (20) *cannot be radially symmetric* and, as a consequence, $\mathsf{C}(\theta, p, a) > \mathsf{C}^*(\theta, p, a)$.

A lengthy computation allows to characterize for which values of p, θ, a and d, the eigenvalue $\lambda_{1,0}$ takes negative values. Using the fact that for any $a < (d-2)/2$ the quantity $2 + p(2\theta - 1)$ is positive, the corresponding condition turns out to be

$$4p(d-1)\bigl(p^2 + 2p + 8\theta - 8\bigr) - \bigl(d^2 + 4a^2 - 4a(d-2)\bigr)(p-2)(p+2)^2 < 0.$$

This is never the case in the admissible range of our parameters if $0 \leqslant a < (d-2)/2$. On the other hand, for $a < 0$ there is always a domain where symmetry breaking occurs. More precisely let

$$\vartheta(p,d) = \frac{d(p-2)}{2p},$$
$$\Theta(a,p,d) := \frac{p-2}{32(d-1)p}\big[(p+2)^2\big(d^2+4a^2-4a(d-2)\big) - 4p(p+4)(d-1)\big].$$

Symmetry breaking occurs if $\theta < \Theta(a,p,d)$. We observe that, for $p \in [2,2^*)$, we have $\vartheta(p,d) < \Theta(a,p,d)$ if and only if $a < a_-(p)$ with

$$a_-(p) := \frac{d-2}{2} - \frac{2(d-1)}{p+2}.$$

We note that the condition $\vartheta(p,d) < 1$ is always satisfied under the assumption $p \in [2,2^*)$. On the other hand the condition $\Theta(a,p,d) \leqslant 1$ is equivalent to

$$a \geqslant \frac{d-2}{2} - \frac{2\sqrt{d-1}}{\sqrt{(p-2)(p+2)}}.$$

Finally, we notice that $d/4 = \partial\vartheta(p,d)/\partial p|_{p=2} < [1+(d-2a-2)^2/(d-1)]/4 = \partial\Theta(a,p,d)/\partial p|_{p=2}$ if $a < -1/2$. See Fig. 1. Summarizing these observations, we arrive at the following result.

Theorem 7. *Let $d \geqslant 2$, $2 < p < 2^*$ and $a < a_-(p)$. Optimality for* (15) (*resp. for* (20)) *is not achieved among radial* (*resp. s-dependent*) *functions, that is,* $\mathsf{C}(\theta,p,a) > \mathsf{C}^*(\theta,p,a)$ *if either*

$$\vartheta(p,d) \leqslant \theta < \Theta(a,p,d) \quad \textit{when } a \geqslant \frac{d-2}{2} - \frac{2\sqrt{d-1}}{\sqrt{(p-2)(p+2)}}$$

or

$$\vartheta(p,d) \leqslant \theta \leqslant 1 \quad \textit{when } a < \frac{d-2}{2} - \frac{2\sqrt{d-1}}{\sqrt{(p-2)(p+2)}}.$$

In other words, symmetry breaking occurs for the optimal functions of (15) *if a, θ and p are in any of the two above regions. Moreover, if $a < -1/2$, there exist $\varepsilon > 0$, $\gamma_1 > d/4$ and $\gamma_2 > \gamma_1$ such that symmetry breaking occurs if $\theta = \gamma(p-2)$ for any $\gamma \in (\gamma_1,\gamma_2)$ and any $p \in (2, 2+\varepsilon)$.*

An elementary computation shows that, $\Theta(a,p,d) > 1$ amounts to (14). This condition is the symmetry breaking condition for Caffarelli–Kohn–Nirenberg inequalities found in [38].

5.2. Symmetry breaking for the weighted logarithmic Hardy inequality. Proof of Theorem C

We now consider the weighted logarithmic Hardy inequalities of Theorem A′ in the equivalent form of Theorem A″. As we have seen in Section 2, after the Emden–Fowler transformation these inequalities take the equivalent form:

$$\int_{\mathcal{C}} |w|^2 \log\left(\frac{|w|^2}{\int_{\mathcal{C}} |w|^2\,dy}\right) dy + 2\gamma \log \mathsf{C}^*_{\mathrm{GLH}} \int_{\mathcal{C}} |w|^2\,dy \leqslant 2\gamma \int_{\mathcal{C}} |w|^2\,dy \log\left[\frac{\int_{\mathcal{C}} |\nabla w|^2\,dy}{\int_{\mathcal{C}} |w|^2\,dy} + \sigma^2\right]$$

with $\sigma = (d-2-2a)/2$. It is an open question to give sufficient conditions for which optimality is achieved among functions depending on s only, so that $2\gamma \log \mathsf{C}^*_{\mathrm{GLH}} = \mathcal{K}(\gamma,\sigma) + \log|\mathbb{S}^{d-1}|$. We recall that equality among radial functions in (26) is achieved by

$$\widetilde{w}(s,\omega) := \left|\mathbb{S}^{d-1}\right|^{-1/2}\overline{w}(s), \qquad y=(s,\omega)\in\mathbb{R}\times\mathbb{S}^{d-1}=\mathcal{C},$$

where

$$\overline{w}(s) = \left(\frac{4\sigma^2}{2\pi(4\gamma-1)}\right)^{1/4} \exp\left(-\frac{\sigma^2 s^2}{4\gamma-1}\right) \quad \forall s\in\mathbb{R}.$$

We note that $\widetilde{w}(s,\omega)$ is normalized to 1 in $L^2(\mathcal{C})$. As a consequence, it follows that

$$\mathcal{K}(\gamma,\sigma) + \log\left|\mathbb{S}^{d-1}\right| = 2\gamma \log\left[\int_{\mathcal{C}} |\nabla\widetilde{w}|^2\,dy + \sigma^2\right] - \int_{\mathcal{C}} |\widetilde{w}|^2 \log|\widetilde{w}|^2\,dy.$$

After these preliminaries, consider the functional

$$\mathcal{F}[w] := \frac{\int_{\mathcal{C}} |\nabla w|^2\,dy}{\int_{\mathcal{C}} |w|^2\,dy} + \sigma^2 - \left|\mathbb{S}^{d-1}\right|^{\frac{1}{2\gamma}} \exp\left[\frac{\mathcal{K}(\gamma,\sigma)}{2\gamma} + \frac{1}{2\gamma}\int_{\mathcal{C}} \frac{|w|^2}{\int_{\mathcal{C}} |w|^2\,dy} \log\left(\frac{|w|^2}{\int_{\mathcal{C}} |w|^2\,dy}\right) dy\right].$$

We know that $\mathcal{F}[\widetilde{w}] = 0$. Let

$$\mathcal{G}[\phi] := \lim_{\varepsilon\to 0} \frac{\mathcal{F}[\widetilde{w}+\varepsilon\phi]}{2\varepsilon^2}.$$

We have in mind to consider an angle dependent perturbation of $\widetilde{w}$, so we shall assume that

$$\int_{\mathcal{C}} \widetilde{w}\phi\,dy = 0 \quad \text{and} \quad \int_{\mathcal{C}} \log|\widetilde{w}|^2 \phi\,dy = 0.$$

Under this assumption,

$$\mathcal{G}[\phi] = \int_{\mathcal{C}} |\nabla\phi|^2\,dy - \int_{\mathcal{C}} |\nabla\widetilde{w}|^2\,dy \int_{\mathcal{C}} |\phi|^2\,dy - \frac{1}{2\gamma}\left(\int_{\mathcal{C}} |\nabla\widetilde{w}|^2\,dy + \sigma^2\right)$$
$$\times\left[\left(2 - \int_{\mathcal{C}} |\widetilde{w}|^2 \log|\widetilde{w}|^2\,dy\right)\int_{\mathcal{C}} |\phi|^2\,dy + \int_{\mathcal{C}} \log|\widetilde{w}|^2 |\phi|^2\,dy\right].$$

After some elementary but tedious computations, one finds that

$$\mathcal{G}[\phi] = \int_{\mathcal{C}} (\mathcal{L}\phi)\phi\,dy,$$

with $\mathcal{L}\phi := -\Delta\phi + \frac{1}{4}A^2|s|^2\phi - \frac{3}{2}A\phi$ and $A := \frac{4\sigma^2}{4\gamma-1}$. By separation of variables, it is straightforward to check that the spectrum of $\mathcal{L}$ is purely discrete and made of the eigenvalues

$$\lambda_{i,j} = i(d+i-2) + A(j-1) \quad \forall i, j \in \mathbb{N}.$$

It follows that $\lambda_{1,0} < 0$ if

$$\frac{d}{4} \leqslant \gamma < \frac{1}{4}\left(1 + \frac{(d-2a-2)^2}{d-1}\right). \tag{34}$$

Hence symmetry breaking occurs provided that

$$\frac{d}{4} < \frac{1}{4}\left(1 + \frac{(d-2a-2)^2}{d-1}\right),$$

which is equivalent to $a < -1/2$. This concludes the proof of Theorem C.

We recover the limit range for symmetry breaking in the interpolation inequalities studied in Section 5.1. Condition (34) asymptotically defines a cone in which symmetry breaking occurs (see Theorem 7) given by $d/4 = \partial\vartheta(p,d)/\partial p|_{p=2} < \gamma < [1 + (d-2a-2)^2/(d-1)]/4 = \partial\Theta(a,p,d)/\partial p|_{p=2}$. See Figs. 1 and 2.

5.3. Plots

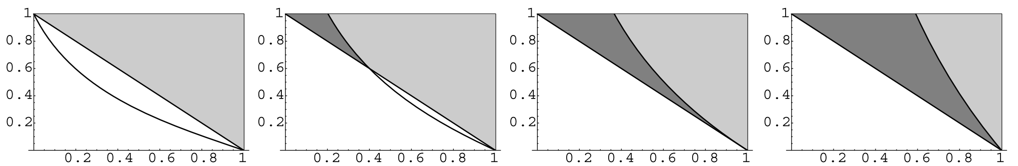

Fig. 1. Plot of the admissible regions (gray areas) with symmetry breaking region established in Theorem 7 (dark grey) in (η,θ) coordinates, with $\eta := b - a$, for various values of a, in dimension $d = 3$: from left to right, $a = 0$, $a = -0.25$, $a = -0.5$ and $a = -1$. The two curves are $\eta \mapsto \vartheta(p,d) = 1 - \eta$ and $\eta \mapsto \Theta(a,p,d)$, for $p = 2d/(d-2+2\eta)$. In the range $a \in (-1/2, 0)$, they intersect for $a = a_-(p)$, *i.e.* $\eta = 2a(1-d)/(d+2a)$. They are tangent at $(\eta,\theta) = (1,0)$ for $a = -1/2$. The symmetry breaking region contains a cone attached to $(\eta,\theta) = (1,0)$ for $a < -1/2$, which determines values of γ for which symmetry breaking occurs in the logarithmic Hardy inequality.

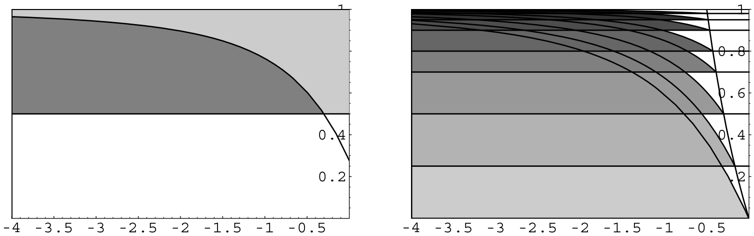

Fig. 2. *Left.* For a given value of $\theta \in (0, 1]$, admissible values of the parameters for which (15) holds are given by (13): in terms of (a, η) with $\eta = b - a$, this simply means $\eta \geqslant 1 - \theta$ (grey areas). According to Theorem 7, symmetry breaking occurs if $\theta < \Theta(a, p, d)$, which determines a region $\eta < g(a, \theta)$ (dark grey). Notice that $\eta < g(a, 1)$ corresponds to condition (14) found by Felli and Schneider. The plot corresponds to $d = 3$ and $\theta = 0.5$. *Right.* Regions of symmetry breaking, *i.e.* $1 - \theta \leqslant \eta < g(a, \theta)$, are shown for $\theta = 1$, 0.75, 0.5, 0.3, 0.2, 0.1, 0.05, 0.02. For each value of θ, the supremum value for which symmetry breaking has been established is $a = a_-(p)$ for $p = 2d/(d - 2\theta)$, which determines a curve $\eta = h(a)$ by requiring that $\theta = 1 - \eta$. The limit case $\eta = 0 = h(0)$ corresponds to the case studied by Felli and Schneider, while $h(-1/2) = 1$ determines the supremum value for which symmetry breaking has been established in the limit case $\eta = 1$, *i.e.* $p = 2$, consistently with Theorem C.

6. Concluding remarks

The purpose of this paper is to establish a new family of inequalities in the euclidean space $\mathbb{R}^d$ and in the cylinder $\mathbb{R} \times \mathbb{S}^{d-1}$. These inequalities are stronger than Hardy's inequality and the logarithmic Sobolev inequalities, but are related to both of them and, for this reason, we have called them *logarithmic Hardy inequalities*. They are invariant term by term under scaling, which distinguishes them from usual logarithmic Sobolev inequalities. On $\mathbb{R}^d$, they are written for unbounded measures and, as far as we know, cannot be easily reduced to inequalities written for probability measures or for Lebesgue's measure. They also appear as an endpoint of a family of Caffarelli–Kohn–Nirenberg inequalities, which is more general than the subfamily studied for instance by Catrina and Wang in [28].

A very natural question is to determine whether optimal functions on $\mathbb{R}^d$ are radially symmetric or not. Using the method introduced by Catrina and Wang in [28] and extended in [38] by Felli and Schneider, we prove that optimal functions in $\mathbb{R}^d$ are not radially symmetric functions in the case of the general Caffarelli–Kohn–Nirenberg inequalities and in the corresponding logarithmic Hardy inequalities, for parameters in a certain range. The method is rather simple. It amounts to linearize the inequality around an optimal function among radial functions and study the sign of the first eigenvalue of an associated operator. A negative eigenvalue then means that optimality is achieved among non-radial functions. The results of *symmetry breaking* that we obtain by this method are fully consistent with previously known results. They allow us to characterize a whole region where the weights are strong enough to break the symmetry that would naturally arise from the nonlinearity in the absence of weights (and can then be proved either by symmetrization techniques or by moving plane methods as in [43]). The symmetry region is by far less understood, although it has recently been established in [35] that both regions are separated by a curve (in the case of the subfamily considered by Catrina and Wang). In the general form of the Caffarelli–Kohn–Nirenberg inequalities, there is an additional term which competes with the nonlinearity to break the symmetry, thus making the analysis more difficult. Hence the main challenge is now to establish the range for symmetry of the optimal functions. This would have some interesting consequences. As mentioned in Section 4, if, for instance, symmetry holds in the complementary region of the one for which symmetry breaking has been established, then we would recover the optimal logarithmic Sobolev inequality on the cylinder as a direct consequence of the logarithmic Hardy inequality.

Acknowledgments

J.D. has been supported by the ECOS contract no. C05E09 and the ANR grants *IFO*, *EVOL* and *CBDif*, and thanks the Department of Mathematics of the University of Crete and the Departamento de Ingeniería Matemática of the University of Chile for their warm hospitality. M.d.P. has also been supported by grants Fondecyt 1070389, Anillo ACT125 CAPDE and Fondo Basal CMM. This work is also part of the MathAmSud *NAPDE* project (M.d.P. & J.D.). Plots have been done with Mathematica™. The authors thank a referee for his careful reading and wise suggestions.

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