SIAM J. Math. Anal.
Vol. 38, No. 5, pp. 1542–1564

RESONANCE AND INTERIOR LAYERS IN AN INHOMOGENEOUS PHASE TRANSITION MODEL*

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Abstract. We consider the problem $\varepsilon^2\Delta u + (u - a(x))(1 - u^2) = 0$ in Ω, $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, where Ω is a smooth and bounded domain in $\mathbb{R}^2$, $-1 < a(x) < 1$. Assume that $\Gamma = \{x \in \Omega, a(x) = 0\}$ is a closed, smooth curve contained in Ω in such a way that $\Omega = \Omega_+ \cup \Gamma \cup \Omega_-$ and $\frac{\partial a}{\partial n} > 0$ on Γ, where n is the outer normal to Ω_+. Fife and Greenlee [*Russian Math. Surveys*, 29 (1974), pp. 103–131] proved the existence of an interior transition layer solution u_ε which approaches -1 in Ω_- and $+1$ in Ω_+, for *all* ε sufficiently small. A question open for many years has been whether an interior transition layer solution approaching 1 in Ω_- and -1 in Ω_+ exists. In this paper, we answer this question affirmatively when $n = 2$, provided that ε is small and away from certain critical numbers. A main difficulty is a resonance phenomenon induced by a large number of small critical eigenvalues of the linearized operator.

Key words. interior transition layer, Fife–Greenlee problem, infinite-dimensional reduction, spectral gap

AMS subject classifications. 35J25, 35J20, 35B33, 35B40

DOI. 10.1137/060649574

1. Introduction and statement of main result.

Let Ω be a bounded, smooth domain in $\mathbb{R}^2$. In the gradient theory of phase transitions it is common to seek critical points in $H^1(\Omega)$ of energy of the form

$$J_\varepsilon(u) = \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2 + \varepsilon^{-1}\int_\Omega W(x,u),$$

where $W(x,\cdot)$ is a double-well potential with exactly two strict local minimizers at $u = +1$ and $u = -1$, which also correspond to trivial local minimizers of J_ε in $H^1(\Omega)$. For simplicity of exposition we shall restrict ourselves to a potential of the form

$$W(x,u) = \int_{-1}^{u} (s^2 - 1)(s - a(x))ds, \tag{1.1}$$

for a smooth function $a(x)$ with

$$-1 < a(x) < 1 \text{ for all } x \in \Omega.$$

Critical points of J_ε correspond to solutions of the problem

$$\begin{cases} \varepsilon^2\Delta u + (u - a(x))(1 - u^2) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

*Received by the editors January 10, 2006; accepted for publication (in revised form) September 13, 2006; published electronically January 12, 2007.
http://www.siam.org/journals/sima/38-5/64957.html

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where $\varepsilon > 0$ is a small parameter and ν denotes unit outer normal to $\partial\Omega$. The function $u(x)$ represents a continuous realization of the phase present in a material confined to the region Ω at the point x which, except for a narrow region, is expected to take values close to $+1$ or -1. Of interest are, of course, nontrivial steady state configurations in which the two phases coexist.

The case $a \equiv 0$ corresponds to the standard Allen–Cahn equation [6]

$$\begin{cases} \varepsilon^2 \Delta u + u(1-u^2) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

for which extensive literature on transition layer solutions is available; see, for instance, [4, 17, 18, 24] and the references therein. We observe that in this case, $+1$ and -1 are both global minimizers of the potential (1.1). We are interested in an inhomogeneous situation in which $+1$ is the absolute minimizer of $W(x,\cdot)$ in one region of the domain, while -1 is such a minimizer in its complement. More precisely, we shall assume that the set

$$\Gamma = \{x \in \Omega \ / \ a(x) = 0\}$$

is a smooth, simple, closed curve in Ω which separates the domain into two disjoint components,

$$\Omega = \Omega_- \cup \Omega_+ \cup \Gamma \tag{1.4}$$

such that

$$a(x) < 0 \ \text{ in } \Omega_+, \quad a(x) > 0 \ \text{ in } \Omega_-, \quad \frac{\partial a}{\partial n} > 0 \ \text{ on } \Gamma\,. \tag{1.5}$$

Observe in particular that for the potential (1.1) we have

$$W(x,-1) < W(x,+1) \quad \text{ in } \Omega_-, \quad W(x,+1) < W(x,-1) \quad \text{ in } \Omega_+\,.$$

Thus, if one considers a global minimizer u_ε for J_ε, which exists by standard arguments, then u_ε should minimize $W(x,u)$; namely, u_ε should intuitively have the following asymptotic behavior as $\varepsilon \to 0$:

$$u_\varepsilon \to -1 \text{ in } \Omega_-, \quad u_\varepsilon \to +1 \text{ in } \Omega_+. \tag{1.6}$$

A solution u_ε to problem (1.2) with these characteristics was constructed, and precisely described, by Fife and Greenlee [15] in 1974 via matching asymptotic and bifurcation arguments.

Supersubsolutions were later used by Angenent, Mallet-Paret, and Peletier in the one-dimensional case [7] for construction and classification of stable solutions. Radial solutions were found variationally by Alikakos and Simpson [5]. The construction of the Fife–Greenlee solution allowing Γ to be any closed subset of Ω in any dimension was given by the first author in [10]. Further constructions have been found recently by Dancer and Yan [9] and Do Nascimento [13]. In particular, it was found in [9] that this solution is precisely a minimizer of J_ε. Related results can be found in [1, 3].

On the other hand, a solution exhibiting a transition layer in the *opposite direction*, namely,

$$u_\varepsilon \to +1 \text{ in } \Omega_-, \quad u_\varepsilon \to -1 \text{ on } \Omega_+, \tag{1.7}$$

has been believed to exist for many years. Hale and Sakamoto [19] established the existence of this solution in the one-dimensional case, while this was done in the radial case in a ball in [11]; see also [8]. The opposite direction layer (1.7) in this scalar problem is meaningful in finding transition layer solutions in pattern-formation–reaction-diffusion systems such as the Gierer–Meinhardt system with saturation; see [11, 14, 25, 28, 27] and the references therein. While the singular perturbation methods used in these one-dimensional or radial equations and systems do not see a substantial difference between the stable and unstable layers, except for the sign of the principal $O(\varepsilon)$ eigenvalue of the linearization, one faces a dramatically different situation in higher-dimensional, nonsymmetric situations. This is clearly seen when linearizing around a spherically symmetric solution like (1.7), as bifurcations of nonradial solutions along certain infinite discrete sets of values for $\varepsilon \to 0$ take place, as established in [27]. In particular, the radial solution has a large ε-dependent Morse index. This poses an important difficulty for a general construction. A phenomenon of this type was previously observed in the one-dimensional case by Alikakos, Bates, and Fusco [2] in a construction of solutions with any prescribed Morse index.

In this paper we are able to prove that the opposite-layer solution (1.7) exists as long as ε remains properly away from a set of critical values. More precisely, there is an explicit number $\lambda_* > 0$ such that given $c > 0$, if ε is sufficiently small and satisfies the gap condition

$$|k^2\varepsilon - \lambda_*| \geq c\sqrt{\varepsilon} \quad \text{for all } k \in \mathbb{N}, \tag{1.8}$$

then a solution u_ε with the required concentration property indeed exists. In other words, this will be the case whenever ε is small and away from the critical numbers $\frac{\lambda_*}{k^2}$, in the sense that for fixed and arbitrarily small $c < \lambda_*$,

$$\varepsilon \notin \left[\frac{\lambda_*}{k^2} - \frac{c}{k^3}\,,\, \frac{\lambda_*}{k^2} + \frac{c}{k^3}\right] \quad \text{for all } k \in \mathbb{N}.$$

Here λ_* is defined by

$$\lambda_* = \frac{1}{3\pi^2 \int_{\mathbb{R}} H_x^2 dx} \left(\int_\Gamma \sqrt{\frac{\partial a}{\partial \nu}} \right)^2, \tag{1.9}$$

where $H(y)$ is the unique heteroclinic solution of

$$H^{''} + H - H^3 = 0,\ H(0) = 0,\ H(\pm\infty) = \pm 1. \tag{1.10}$$

We can now state our main result.

THEOREM 1. *Given $c > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ satisfying the gap condition* (1.8)*, problem* (1.2) *has a solution u_ϵ satisfying*

$$u_\varepsilon(x) \to +1 \quad in \quad \Omega_-, \quad u_\varepsilon(x) \to -1 \quad in \quad \Omega_+$$

as $\varepsilon \to 0$.

Much more accurate information on the solution will be provided by its construction; in particular, its shape near Γ is governed by the heteroclinic solution H, in the sense that

$$u_\varepsilon(x) \sim H\left(\frac{t - \varepsilon f(\theta)}{\varepsilon}\right),$$

where f is a bounded function of θ, a choice of arclength coordinate of Γ, and t is the (signed) normal coordinate along the outer normal to Ω_+ on Γ.

The main difficulty in the construction of the interior layer solution in the opposite direction is the appearance of a large number of small *critical eigenvalues*, or *resonance*. This kind of phenomenon has been dealt with in various problems, for example, in the study of periodic orbits for strongly attractive potentials [21, 29] and in boundary concentrations for singularly perturbed Neumann problems [22, 23]. It also arises in our previous work [12] on the construction of a concentrating solution on weighted geodesics for nonlinear Schrodinger equations. The scheme employed here follows the general lines set in [12].

More precisely, the solution to the full problem is roughly decomposed into the form

$$u_\varepsilon(x) = H\left(s - f(\varepsilon z)\right) + \phi_1\left(s - f(\varepsilon z)\right) + \tilde{\phi}(s, z), \tag{1.11}$$

where $x = (t, \theta) = (\varepsilon s, \varepsilon z)$, $t = \varepsilon s$ is the signed distance to Γ, $\theta = \varepsilon z$ is the arclength coordinate of Γ whose length is l, f is an l-periodic function left as a parameter, and ϕ_1 is the correction term to be defined, while $\tilde{\phi}(s, z)$ is $L^2(ds)$-orthogonal for each z to $H_s(s - f(\varepsilon z))$. Solving first in $\tilde{\phi}$ a natural projected problem, where the linear operator is uniformly invertible, the resolution of the full problem becomes reduced to a nonlinear, nonlocal second order system of differential equations in f which turns out to be directly solvable thanks to the assumptions made. This approach is familiar when the parameter f lies in a finite-dimensional space (as in the papers [5, 9, 13, 19]), corresponding this time to adjusting infinitely many parameters. To stress the difference in the radial case, we note that the parameter f is just a single number. The analysis we make takes special advantage through Fourier analysis of the fact that the objects to be adjusted are one-variable functions, while we still believe that the current approach may be modified to the higher-dimensional case. We also believe that the gap condition may be improved to size ε^q, any $q > \frac{1}{2}$.

Additionally we point out the following:

1. The results of Theorem 1 remain true when Ω is an unbounded domain, for instance, $\Omega = \mathbb{R}^2$. Indeed, our proofs, and in particular the matching argument below, can easily be adapted to handle this case.

2. The method and results presented in Theorem 1 can be generalized to more general bistable equations of the form

$$\varepsilon^2 \Delta u - h(x, u, \varepsilon) = 0 \text{ in } \Omega, \quad \varepsilon \frac{\partial u}{\partial \nu} - \sigma(x, \epsilon) u = f(x, \epsilon) \text{ on } \partial\Omega,$$

as treated originally by Fife and Greenlee [15].

3. Our general approach seems also to work when $N = 3$. It will be an interesting problem to consider $N \geq 4$. Note that there is no restriction of dimension in the construction of Fife–Greenlee solutions (1.6); see [10].

The organization of this paper is as follows. In section 2, we set up the local coordinates near Γ and transform (1.2) into a new equation in the stretched variable (s, z). We then introduce the first correction term ϕ_1 and estimate the errors. In section 3, we use a gluing procedure to reduce the nonlinear problem to one on the infinite cylinder and another one away from the interface. Then we solve the inner problem modulo the projections in section 4 and the full problem modulo projections in section 5. In section 6 and section 7, we derive a nonlinear ODE for f, which will be solved in section 8 using the gap condition.

2. The setup near the curve. Let $\Gamma = \{x \in \Omega, a(x) = 0\}$ be a simple, closed, smooth curve in $\Omega \subset \mathbb{R}^2$ and let $\ell = |\Gamma|$ denote its total length. We consider the natural parameterization $\gamma(\theta)$ of Γ with positive orientation, where θ denotes the arclength parameter measured from a fixed point of Γ. Let $\nu(\theta)$ denote outer unit normal to Γ. Points y which are δ_0-close Γ for sufficiently small δ_0 can be represented in the form

$$y = \gamma(\theta) + t\,\nu(\theta), \quad |t| < \delta_0,\ \theta \in [0, \ell)\,, \tag{2.1}$$

where map $y \mapsto (t, \theta)$ is a local diffeomorphism. By slight abuse of notation we denote $a(t, \theta)$ to actually mean $a(y)$ for y in (2.1). Let $k(\theta)$ denote the curvature of Γ.

Stretching variables, absorbing ε from Laplace's operator, and replacing $u(y)$ with $u(\varepsilon y)$, (1.2) becomes

$$\Delta u + (u - a(\varepsilon y))(1 - u^2) = 0 \text{ in } \Omega_\varepsilon, \qquad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_\varepsilon, \tag{2.2}$$

where $\Omega_\varepsilon = \frac{\Omega}{\varepsilon}$.

Let $(s, z) = \varepsilon^{-1}(t, \theta)$ be the natural stretched coordinates associated with the curve $\Gamma_\varepsilon = \varepsilon^{-1}\Gamma$, now defined for

$$z \in [0, \varepsilon^{-1}\ell), \quad s \in (-\varepsilon^{-1}\delta_0, \varepsilon^{-1}\delta_0). \tag{2.3}$$

Equation (2.2) for u expressed in these coordinates becomes

$$u_{zz} + u_{ss} + B_1(u) + B_2(u) + u - u^3 = 0, \tag{2.4}$$

in the region (2.3), where

$$B_1(u) = -u_{zz}\left[1 - \frac{1}{(1 + \varepsilon k(\varepsilon z)s)^2}\right] + \frac{\varepsilon k(\varepsilon z)u_s}{1 + \varepsilon k(\varepsilon z)s} - \frac{\varepsilon^2 s\, k'(\varepsilon z)u_z}{(1 + \varepsilon k(\varepsilon z)s)^3},$$

$$B_2(u) = -a(\varepsilon s, \varepsilon z)(1 - u^2).$$

For further reference, it is convenient to expand B_1 in the form

$$B_1(u) \;=\; (\varepsilon k(\varepsilon z) - \varepsilon^2 s k^2(\varepsilon z))u_s \;+\; B_0(u), \tag{2.5}$$

where

$$B_0(u) \;=\; \varepsilon^2 s a_1(\varepsilon s, \varepsilon z)u_z + \varepsilon s a_2(\varepsilon s, \varepsilon z)u_{zz} + \varepsilon^3 s^2 a_3(\varepsilon s, \varepsilon z)u_s, \tag{2.6}$$

for certain smooth functions $a_j(t, \theta)$, $j = 1, 2, 3$. Observe that all terms in the operator B_1 have ε as a common factor.

We consider now a further change of variables in (2.4). Let $f(\theta)$ be a twice differentiable, ℓ-periodic function whose exact form is to be specified later (see (2.25)). We define $v(x, z)$ by the relation

$$u(s, z) = v(\,x\,,\,z), \quad x = s - f(\varepsilon z). \tag{2.7}$$

We want to express (2.4) in terms of these new coordinates. We compute

$$u_s = v_x, \quad u_{ss} = v_{xx}, \tag{2.8}$$

$$u_z = v_x(-f)_z + v_z, \tag{2.9}$$

$$u_{zz} = v_{xx}|f_z|^2 + 2v_{xz}(-f)_z + v_x\,(-f)_{zz} + v_{zz}. \tag{2.10}$$

In order to write down the equation it is also convenient to expand

$$a(\varepsilon s, \varepsilon z) = a(0, \varepsilon z) + a_t(0, \varepsilon z)\varepsilon s + \frac{1}{2}a_{tt}(0, \varepsilon z)\varepsilon^2 s^2 + a_4(\varepsilon s, \varepsilon z)\varepsilon^3 s^3 \tag{2.11}$$

for a smooth function $a_4(t, \theta)$. It turns out that u solves (2.4) if and only if v defined by (2.7) solves

$$S(v) \equiv v_{zz} + v_{xx} + B_3(v) + B_4(v) + v - v^3 = 0, \tag{2.12}$$

where $B_3(v)$ is a linear differential operator defined by

$$\begin{aligned} B_3(v) = {} & \left[\varepsilon k - \varepsilon^2 k^2\,(x + f\,)\right]\, v_x \\ & + \left[\varepsilon^2\,|f'|^2\; v_{xx} - 2\varepsilon f' v_{xz} - \varepsilon^2 f'' v_x\right] \\ & + B_5(v), \end{aligned}$$

with

$$B_5(v) = B_0(u) - a_4(\varepsilon s, \varepsilon z)\varepsilon^3 s^3 (1 - v^2) \tag{2.13}$$

and

$$B_4(v) = -\left[\varepsilon a_t\,(x + f) + \frac{\varepsilon^2}{2}a_{tt}\,(x + f)^2\right](1 - v^2). \tag{2.14}$$

$B_0(u)$ is the operator in (2.6), where the derivatives are expressed in terms of the formulas (2.8)–(2.10), a_4 is given by (2.11), and s is replaced with $x + f$.

Let $H(x)$ denote the unique positive solution of (1.10). Then, taking $H(x)$ as a first approximation, the error produced is of ε times a function with exponential decay. Let us be more precise. We need to identify both the terms of order ε and those of order ε^2:

$$\begin{aligned} S(H) = B_3(H) + B_4(H) = {} & \left[\varepsilon k - \varepsilon^2 k^2\,(x + f\,)\right]\, H_x \\ & + \left[\,\varepsilon^2 |f'|^2\, H_{xx} - \varepsilon^2 f'' H_x\,\right] \\ & - \left[\varepsilon a_t\,(x + f) + \frac{\varepsilon^2}{2}a_{tt}\,(x + f)^2\right](1 - H^2) + B_5(H), \end{aligned}$$

where

$$B_5(H) = B_0(H) - \varepsilon^3 s^3 a_4(\varepsilon s, \varepsilon z)(1 - H^2).$$

Gathering terms of order ε and ε^2 we get

$$\begin{aligned} S(H) = & -\varepsilon a_t x(1-H^2) \\ & +\varepsilon\left[kH_x - a_t f(1-H^2)\right] \\ & -\varepsilon^2\left[k^2(xH_x) - |f^{'}|^2 H_{xx} + a_{tt} f x(1-H^2)\right] \\ & -\varepsilon^2\left[k^2 f H_x + f^{''} H_x + \frac{a_{tt}}{2}[x^2+f^2](1-H^2)\right] \\ & \qquad + B_6(H) \\ = & \ \varepsilon S_1 + \varepsilon S_2 + \varepsilon^2 S_3 + \varepsilon^2 S_4 + B_6(H)\,. \end{aligned}$$

Let us observe that, grouped this way, the quantities S_1, S_3 are odd functions of x while S_2, S_4 are even. In addition, $B_6(H)$ is a term of order ε^3 times an exponentially decaying function. We want now to construct a further approximation to a solution which eliminates the terms of order ε in the error. If ϕ represents such an approximation, then we see that

$$S(H+\phi) = S(H) + L_0(\phi) + B_7(\phi) + N_0(\phi),$$

where

$$L_0(\phi) = \phi_{zz} + \phi_{xx} + (1-3H^2)\phi, \tag{2.15}$$

$$B_7(\phi) = B_3(H+\phi) + B_4(H+\phi) - B_3(H) - B_4(H), \tag{2.16}$$

and

$$N_0(\phi) = -3H\phi^2 - \phi^3. \tag{2.17}$$

We write

$$\begin{aligned} S(H+\phi) = & \ [\varepsilon(S_1+S_2) + \phi_{xx} + (1-3H^2)\phi] \\ & + \varepsilon^2 S_3 + \varepsilon^2 S_4 + B_6(H) + \phi_{zz} + B_7(\phi) + N_0(\phi). \end{aligned} \tag{2.18}$$

We choose $\phi = \phi_1$ in order to eliminate the term between brackets in the above expression. Namely, for fixed z, we need a solution of

$$-\phi_{xx} + (3H^2-1)\phi = \varepsilon(S_1+S_2), \quad \phi(\pm\infty) = 0.$$

As it is well known, this problem is solvable provided that

$$\int_{-\infty}^{\infty} (S_1+S_2) H_x \, dx = 0. \tag{2.19}$$

Furthermore, the solution is unique under the constraint

$$\int_{-\infty}^{\infty} \phi H_x \, dx = 0. \tag{2.20}$$

We compute

$$\int_{-\infty}^{\infty} (S_1+S_2) H_x \, dx = \int_{-\infty}^{\infty} S_2 H_x \, dx = k\int_{-\infty}^{\infty} H_x^2 - a_t f \int_{-\infty}^{\infty} (1-H^2) H_x,$$

where

$$\int_{-\infty}^{\infty}(1-H^2)H_x = \frac{4}{3}.$$

Since $a_t(0,\theta) \neq 0$, we have that the first approximation of f should be

$$f_0(\theta) = c_0 \frac{k(\theta)}{a_t(0,\theta)}, \quad \text{where } c_0 = \frac{3\int_{\mathbb{R}} H_x^2}{4}.$$

The solution has the form

$$\phi_1 = \phi_{11} + \phi_{12}, \tag{2.21}$$

where

$$\phi_{11} = \varepsilon a_{11}(\varepsilon z)H_1(x), \quad \phi_{12} = \varepsilon f_0(\varepsilon z) a_{12}(\varepsilon z) H_2(x), \tag{2.22}$$

$$a_{11} = a_t(0,\theta), \quad a_{12} = k(\theta),$$

H_1 is the unique odd function satisfying

$$-H_{1,xx} - H_1 + 3H^2 H_1 = x(1-H^2), \tag{2.23}$$

and H_2 is the unique even solution satisfying

$$-H_{2,xx} - H_2 + 3H^2 H_2 = H_x - c_0(1-H^2), \quad \int_{\mathbb{R}} H_2 H_x dx = 0. \tag{2.24}$$

Let us now choose f:

$$f(\theta) = f_0(\theta) + \mathbf{f}(\theta). \tag{2.25}$$

In all what follows, we will assume the validity of the following constraints on the parameter $\mathbf{f}$:

$$\|\mathbf{f}\| \equiv \varepsilon \|\mathbf{f}''\|_{L^2(0,\ell)} + \sqrt{\varepsilon}\|\mathbf{f}'\|_{L^2(0,\ell)} + \|\mathbf{f}\|_{L^\infty(0,\ell)} \leq \varepsilon, \tag{2.26}$$

so that

$$\|\mathbf{f}\|_{L^\infty(0,\ell)} \leq \varepsilon, \quad \|\mathbf{f}'\|_{L^2(0,\ell)} \leq \sqrt{\varepsilon}, \quad \|\mathbf{f}''\|_{L^2(0,\ell)} \leq 1. \tag{2.27}$$

By interpolation, it also holds that

$$\|\mathbf{f}'\|_{L^\infty(0,\ell)} \leq \sqrt{\varepsilon}. \tag{2.28}$$

We now take our basic approximation to a solution to the problem near the curve Γ_ε to be

$$f(\theta) = f_0(\theta) + \mathbf{f}(\theta), \qquad \mathtt{H} = H + \phi_1. \tag{2.29}$$

Substituting $\phi = \phi_1$ in (2.18), we can compute the new error:

$$\begin{aligned} E_1 &= S(\mathtt{H}) = S(H+\phi_1) \\ &= \varepsilon(S_1+S_2) + \phi_{1,xx} + (1-3H^2)\phi_1 \\ &\quad + \varepsilon^2 S_3 + \varepsilon^2 S_4 + B_6(H) + \phi_{1,zz} + B_7(\phi_1) + N_0(\phi_1) \\ &= -\varepsilon a_t \mathbf{f}(1-H^2) + \varepsilon^2 S_3 + \varepsilon^2 S_4 + B_6(H) + \phi_{1,zz} + B_7(\phi_1) + N_0(\phi_1). \end{aligned} \tag{2.30}$$

Observe that since ϕ_1 and $\mathbf{f}$ are of size $O(\varepsilon)$, all terms above carry ε^2 in front. Observe also that all functions involved are expressed in (x, z) variables, and the natural domain for those variables is the infinite strip

$$\mathcal{S} = \{-\infty < x < \infty, \quad 0 < z < \ell/\varepsilon\}.$$

We now want to measure the size of the error in the $L^2(\mathcal{S})$ norm.

Note that

$$\| - \varepsilon a_t \mathbf{f}(1 - H^2) + \varepsilon^2 S_3 + \varepsilon^2 S_4\|_{L^2(\mathcal{S})} \leq C\varepsilon^{\frac{3}{2}}. \tag{2.31}$$

A rather delicate term in the cubic remainder $B_6(H)$ is the one carrying $\mathbf{f}''$ since in reality we shall only assume a uniform bound on $\|\mathbf{f}''\|_{L^2(0,\ell)}$. For instance, one term arising from $B_6(H)$ can be written as

$$R = \varepsilon^3(x + f)f''(\varepsilon z)a_2(\varepsilon(x + f), \varepsilon z)H_x(x), \quad f = f_0 + \mathbf{f},$$

with a_2 smooth (see (2.6)). Observe that

$$\int_{\mathcal{S}} |R|^2 \leq C\varepsilon^6 \int_0^{\frac{\ell}{\varepsilon}} |f''(\varepsilon z)|^2 dz = \varepsilon^5 \|f''\|^2_{L^2(0,\ell)}.$$

Hence

$$\|R\|_{L^2(\mathcal{S})} \leq C\varepsilon^{\frac{5}{2}} \|f''\|_{L^2(0,\ell)}.$$

Since ϕ_1 can be bounded by $C\varepsilon|x|^2 e^{-c|x|}$ for large $|x|$, we obtain that

$$\|B_7(\phi_1)\|_{L^2(\mathcal{S})} \leq C\varepsilon^{\frac{3}{2}} \ .$$

A similar bound holds for the term $N_0(\phi_1)$:

$$\|N_0(\phi_1)\|_{L^2(\mathcal{S})} \leq C\varepsilon^{\frac{3}{2}}. \tag{2.32}$$

In summary, we have

$$\|S(H + \phi_1)\|_{L^2(\mathcal{S})} \leq C\varepsilon^{\frac{3}{2}}. \tag{2.33}$$

We set up the full problem in the form $S(\mathtt{H} + \phi) = 0$, which takes the form near the curve,

$$S(\mathtt{H} + \phi) = L_0(\phi) + B_8(\phi) + E_1 + N_1(\phi) = 0, \tag{2.34}$$

where $E_1 = S(\mathtt{H})$ and

$$L_0(\phi) = \phi_{xx} + \phi_{zz} + (1 - 3\mathtt{H}^2)\phi \ , \tag{2.35}$$

$$B_8(\phi) = B_7(\phi + \phi_1) - B_7(\phi_1), \tag{2.36}$$

$$N_1(\phi) = N_0(\phi + \phi_1) - N_0(\phi_1). \tag{2.37}$$

We recall that the description made here is only local. However, we will be able to reduce the problem to one qualitatively similar to that of the above form in the infinite strip.

3. The matching procedure. We follow [12] to perform a procedure that we refer to as an infinite-dimensional Liapunov–Schmidt reduction (see the explanations at the end of this section). Since it is quite similar to that of [12], we shall only sketch the proofs.

First, we need to match solutions near and outside Γ. The idea is to solve the problem outside a tubular neighborhood of Γ and then to reduce the problem to an infinite strip.

Let $\mathtt{H}(y)$ denote the first approximation constructed near the curve in the coordinate $y = (y_1, y_2)$ in $\mathbb{R}^2$. Let $\delta < \delta_0/100$ be a fixed number. We consider a smooth cut-off function $\eta_\delta(t)$ such that $\eta_\delta(t) = 1$ if $|t| < \delta$ and $= 0$ if $|t| > 2\delta$. Denote as well $\eta_\delta^\varepsilon(s) = \eta_\delta(\varepsilon|s|)$, where s is the normal coordinate to Γ_ε. We define our first global approximation to be simply

$$\mathbf{H}(y) = \begin{cases} \eta_{3\delta}^\varepsilon(s)(\mathtt{H}+1) - 1 \text{ if } y \in \Omega_+, \\ \eta_{3\delta}^\varepsilon(s)(\mathtt{H}-1) + 1 \text{ if } y \in \mathbb{R}^2 \setminus \Omega_+. \end{cases}$$

Denote $S(u) = \Delta u + (u - a(\varepsilon s, \varepsilon z))(1-u^2)$ for $u = \mathbf{H} + \tilde{\phi}$. Then $S(\mathbf{H} + \tilde{\phi}) = 0$ if and only if

$$\tilde{L}(\tilde{\phi}) = \tilde{E} + \tilde{N}(\tilde{\phi}), \tag{3.1}$$

where

$$\tilde{E} = -S(\mathbf{H}),$$
$$\tilde{L}(\tilde{\phi}) = \Delta\tilde{\phi} + [1 - 3\mathbf{H}^2 + 2a(\varepsilon y)\mathbf{H}]\tilde{\phi},$$

and

$$\tilde{N}(\tilde{\phi}) = -3\mathbf{H}(\tilde{\phi})^2 - (\tilde{\phi})^3 + a(\varepsilon y)(\tilde{\phi})^2.$$

We further separate $\tilde{\phi}$ in the following form:

$$\tilde{\phi} = \eta_{3\delta}^\varepsilon \phi + \psi,$$

where, in coordinates (x, z), we assume that ϕ is defined in the whole strip $\mathcal{S}$. We want

$$\tilde{L}(\eta_{3\delta}^\varepsilon \phi) + \tilde{L}(\psi) = \tilde{E} + \tilde{N}(\eta_{3\delta}^\varepsilon \phi + \psi).$$

We achieve this if the pair (ψ, ϕ) satisfies the following nonlinear coupled system:

$$\eta_{3\delta}^\varepsilon \tilde{L}(\phi) = \eta_\delta^\varepsilon \tilde{E} + \eta_\delta^\varepsilon \tilde{N}(\eta_{3\delta}^\varepsilon \phi + \psi) - 3\eta_\delta^\varepsilon (1 - \mathbf{H}^2)\psi, \tag{3.2}$$

$$\begin{aligned} &\Delta\psi - 2(1 - a\mathbf{H})\psi + 3(1 - \eta_\delta^\varepsilon)(1 - \mathbf{H}^2)\,\psi = (1 - \eta_\delta^\varepsilon)\,\tilde{E} - 2\varepsilon\nabla\eta_{3\delta}^\varepsilon\,\nabla\phi \\ &\qquad - 2\varepsilon^2(\Delta\eta_{3\delta}^\varepsilon)\,\phi + (1 - \eta_\delta^\varepsilon)\,\tilde{N}(\,\eta_{3\delta}^\varepsilon\,\phi + \psi\,), \end{aligned} \tag{3.3}$$

where ϕ is defined globally on $\mathcal{S}$ and ψ is defined in Ω_ε and is required to satisfy the Neumann boundary condition.

Notice that the operator $\tilde{L}$ in the strip $\mathcal{S}$ may be taken as any compatible extension outside the $6\delta/\varepsilon$-neighborhood of the curve.

What we want to do next is to reduce the problem to one in the strip. To do this, we solve, given a small ϕ, problem (3.3) for ψ. This can be done in an elementary way: Let us observe first that since $|a(x)| < 1$, we have

$$\gamma_0^2 = \min_{x\in\bar\Omega} 2(1-|a(x)|) > 0. \tag{3.4}$$

Since $1-\mathbf{H}^2$ is exponentially small for $|s| > \delta\varepsilon^{-1}$, where s is the normal coordinate to Γ_ε, then the problem

$$\Delta\psi - 2(1-a(\varepsilon y)\mathbf{H})\psi + 3(1-\eta_\delta^\varepsilon)(1-\mathbf{H}^2)\psi = h, \text{ in } \Omega, \quad \frac{\partial\psi}{\partial\nu} = 0 \text{ on } \partial\Omega_\varepsilon, \tag{3.5}$$

has a unique bounded solution ψ whenever $\|h\|_\infty < +\infty$. Moreover,

$$\|\psi\|_\infty \le C\|h\|_\infty.$$

Assume now that ϕ satisfies the following decay condition:

$$|\nabla\phi(y)| + |\phi(y)| \le e^{-\frac{\gamma}{\varepsilon}} \quad \text{ for } |s| > \frac{\delta}{\varepsilon}. \tag{3.6}$$

Since $\tilde N$ has a power-like behavior with power greater than one, a direct application of the contraction mapping principle yields that problem (3.3) has a unique (small) solution $\psi = \psi(\phi)$ with

$$\|\psi(\phi)\|_\infty \le Ce^{-\delta/\varepsilon} + C\varepsilon[\,\|\phi\|_{L^\infty(|s|>\delta\varepsilon^{-1})} + \|\nabla\phi\|_{L^\infty(|s|>\delta\varepsilon^{-1})}\,],$$

where with some abuse of notation by $\{|s| > \delta/\varepsilon\}$ we denote the complement of the δ/ε-neighborhood of Γ_ε. The nonlinear operator ψ satisfies a Lipschitz condition of the form

$$\|\psi(\phi_1) - \psi(\phi_2)\|_\infty \le C\varepsilon[\,\|\phi_1-\phi_2\|_{L^\infty(|s|>\delta\varepsilon^{-1})} + \|\nabla(\phi_1-\phi_2)\|_{L^\infty(|s|>\delta\varepsilon^{-1})}\,].$$

The full problem has been reduced to solving the (nonlocal) problem in the infinite strip $\mathcal{S}$,

$$L_2(\phi) = \eta_\delta^\varepsilon \tilde E + \eta_\delta^\varepsilon \tilde N(\eta_{3\delta}^\varepsilon\phi + \psi(\phi)) - 3\eta_\delta^\varepsilon(1-\mathbf{H}^2)\psi(\phi), \tag{3.7}$$

for a $\phi \in H^2(\mathcal{S})$ satisfying condition (3.6). Here L_2 denotes a linear operator that coincides with $\tilde L$ on the region $|s| < 10\delta/\varepsilon$.

We shall define this operator next. The operator $\tilde L$ for $|s| > 20\delta/\varepsilon$ is given in coordinates (x,z) by

$$L_1(\phi) = \phi_{xx} + \phi_{zz} + (1-3\mathtt{H}^2)\phi.$$

We extend it for functions ϕ defined in the entire strip $\mathcal{S}$, in terms of (x,z), as follows:

$$L_2(\phi) = L_1(\phi) + 2\chi(\varepsilon|x|)a(\varepsilon s,\varepsilon z)\mathtt{H}\phi + \chi(\varepsilon|x|)B_1(\phi), \tag{3.8}$$

where $\chi(r)$ is a smooth cut-off function which equals 1 for $r < 10\delta$ and vanishes identically for $r > 20\delta$.

Rather than solving problem (3.1) directly, we shall do it in steps. We consider the following projected problem in $H^2(\mathcal{S})$: Given $f = f_0 + \mathbf{f}$, with $\mathbf{f}$ satisfying bounds (2.26), find functions $\phi \in H^2(\mathcal{S})$, c such that

$$L_2(\phi) = \eta_\delta^\varepsilon \tilde{E} + N_2(\phi) + c(\varepsilon z)\chi_\delta^\varepsilon H_x \quad \text{in } \mathcal{S}, \tag{3.9}$$

$$\phi(x,0) = \phi(x,\ell/\varepsilon), \quad \phi_z(x,0) = \phi_z(x,\ell/\varepsilon), \quad -\infty < x < +\infty, \tag{3.10}$$

$$\int_{-\infty}^{\infty} \phi(x,z)\, H_x(x)\, dx = 0, \quad 0 < z < \frac{\ell}{\varepsilon}. \tag{3.11}$$

Here $N_2(\phi) = \eta_\delta^\varepsilon \tilde{N}(\eta_{3\delta}^\varepsilon \phi + \psi(\phi)) - 3\eta_\delta^\varepsilon(1 - \mathtt{H}^2)\psi(\phi)$ and $\chi_\delta^\varepsilon(x) = \chi_1(\varepsilon|x|/\delta)$, where $\chi_1(t)$ is a cut-off function equal to 1 for $|t| < 1/2$ and equal to 0 for $|t| > 1$.

We will prove that this problem has a unique solution whose norm is controlled by the L^2 norm of $\eta_\delta^\varepsilon \tilde{E} = E_1 = S(\mathtt{H})$. The main step here is to show bounded invertibility of a suitable perturbation of the operator L_2. The proof of this fact is a combination of an a priori estimate (Lemma 4.1) with an application of the Fredholm alternative (Lemma 4.2). After this first step, our task is to adjust the parameter $\mathbf{f}$ in such a way that c is identically zero. As we will see, this turns out to be equivalent to solving a nonlocal, nonlinear, second order differential equation for $\mathbf{f}$ under periodic boundary conditions. This system is solvable in a region where the bound (2.26) holds.

We call the entire procedure described above as *infinite-dimensional Lyapunov–Schmidt reduction* because of its analogy to a method devised by Floer and Weinstein [16] in a finite-dimensional context for a related problem. In a finite-dimensional setting, the main step in this method, which corresponds to adjustment of a parameter to make $c = 0$, is also known as *quasi-invariant manifold reduction*. The whole scheme has been refined and widely used in singular perturbation elliptic problems.

We will carry out the outlined program in the next sections. To solve (3.9)–(3.11) we need to investigate invertibility of L_2 in the L^2-H^2 setting under periodic boundary conditions and orthogonality conditions.

4. Invertibility of L_2.

Let L_2 be the operator defined in $H^2(\mathcal{S})$ by (3.8). In this section we study the linear problem

$$L_2(\phi) = h + c(\varepsilon z)\chi_\delta^\varepsilon H_x \quad \text{in } \mathcal{S}, \tag{4.1}$$

$$\phi(x,0) = \phi(x,\ell/\varepsilon), \quad \phi_z(x,0) = \phi_z(x,\ell/\varepsilon), \quad -\infty < x < +\infty, \tag{4.2}$$

$$\int_{-\infty}^{\infty} \phi(x,z)\, H_x(x)\, dx = 0, \quad 0 < z < \frac{\ell}{\varepsilon} \tag{4.3}$$

for a given $h \in L^2(\mathcal{S})$. Our main result in this section is the following.

PROPOSITION 4.1. *If δ in the definition of L_2 is chosen sufficiently small, then there exists a constant $C > 0$, independent of ε, such that for all small ε, problem* (4.1)–(4.3) *has a unique solution $\phi = T(h)$, which satisfies the estimate*

$$\|\phi\|_{H^2(\mathcal{S})} \le C\|h\|_{L^2(\mathcal{S})}.$$

For the proof of this result we need the validity of the corresponding assertion for a simpler operator which does not depend on δ. Let us consider the problem

$$\mathtt{L}(\phi) = \phi_{ss} + \phi_{zz} + (1 - 3H^2)\phi = h \quad \text{in } \mathcal{S}, \tag{4.4}$$

$$\phi(x,0) = \phi(x,\ell/\varepsilon), \quad \phi_z(x,0) = \phi_z(x,\ell/\varepsilon), \quad -\infty < x < +\infty, \tag{4.5}$$

$$\int_{-\infty}^{\infty} \phi(x,z)\, H_x(x)\, dx = 0, \quad 0 < z < \frac{\ell}{\varepsilon}. \tag{4.6}$$

LEMMA 4.1. *There exists a constant $C > 0$, independent of ε such that the solutions of* (4.4)–(4.6) *satisfy the a priori estimate*

$$\|\phi\|_{H^2(\mathcal{S})} \leq C\|h\|_{L^2(\mathcal{S})}.$$

Proof. Let us consider Fourier series decompositions for h and ϕ of the form

$$\phi(x,z) = \sum_{k=0}^{\infty}\left[\phi_{1k}(x)\cos\left(\frac{2\pi k}{\ell}\varepsilon z\right) + \phi_{2k}(x)\sin\left(\frac{2\pi k}{\ell}\varepsilon z\right)\right],$$
$$h(x,z) = \sum_{k=0}^{\infty}\left[h_{1k}(x)\cos\left(\frac{2\pi k}{\ell}\varepsilon z\right) + h_{2k}(x)\sin\left(\frac{2\pi k}{\ell}\varepsilon z\right)\right].$$

Then we have the validity of the equations

$$-\frac{4\pi^2k^2\varepsilon^2}{l^2}\phi_{lk} + \mathtt{L}_0(\phi_{lk}) = h_{lk}, \quad x \in \mathbb{R}, \tag{4.7}$$

with orthogonality conditions

$$\int_{-\infty}^{\infty}\phi_{lk}\, H_x\, dx = 0. \tag{4.8}$$

We have denoted here

$$\mathtt{L}_0(\phi_{lk}) = \phi_{lk,xx} + (1-3H^2)\phi_{lk}.$$

Let us consider the bilinear form in $H^1(\mathbb{R})$ associated with the operator $\mathtt{L}_0$, namely,

$$B(\psi,\psi) = \int_{-\infty}^{\infty}[|\psi_x|^2 + (3H^2-1)|\psi|^2]\, dx\ .$$

Since (4.8) holds, we conclude that

$$C[\|\phi_{lk}\|^2_{L^2(\mathbb{R})} + \|\phi_{lk,x}\|^2_{L^2(\mathbb{R})}] \leq B(\phi_{lk},\phi_{lk}) \tag{4.9}$$

for a constant $C > 0$ independent of l, k. Using this fact and (4.7) we conclude with the estimate

$$(1 + k^4\varepsilon^4)\|\phi_{lk}\|^2_{L^2(\mathbb{R})} + \|\phi_{lk,x}\|^2_{L^2(\mathbb{R})} \leq C\|h_{lk}\|^2_{L^2(\mathbb{R})}.$$

In particular, we see from (4.7) that ϕ_{lk} satisfies an equation of the form

$$-\phi_{lk,xx} + 2\phi_{lk} = \tilde{h}_{lk}, \quad x \in \mathbb{R},$$

where $\|\tilde{h}_{lk}\|_{L^2(\mathbb{R})} \leq C\|h_{lk}\|_{L^2(\mathbb{R})}$. Hence it follows that additionally we have the estimate

$$\|\phi_{lk,xx}\|^2_{L^2(\mathbb{R})} \leq C\|h_{lk}\|^2_{L^2(\mathbb{R})}. \tag{4.10}$$

Adding up estimates (4.9), (4.10) in k and l we conclude that

$$\|D^2\phi\|^2_{L^2(\mathcal{S})} + \|D\phi\|^2_{L^2(\mathcal{S})} + \|\phi\|^2_{L^2(\mathcal{S})} \leq C\|h\|^2_{L^2(\mathcal{S})},$$

which ends the proof. ∎

We consider now the following problem: Given $h \in L^2(\mathcal{S})$, find functions $\phi \in H^2(\mathcal{S})$, $c \in L^2(0,\ell)$ such that

$$\mathtt{L}(\phi) \;=\; h \,+\, c(\varepsilon z)\chi_\delta^\varepsilon H_x \quad \text{in } \mathcal{S}, \tag{4.11}$$

$$\phi(x,0) = \phi(x,\ell/\varepsilon), \quad \phi_z(x,0) = \phi_z(x,\ell/\varepsilon), \quad -\infty < x < +\infty, \tag{4.12}$$

$$\int_{-\infty}^{\infty} \phi(x,z)\, H_x(x)\, dx = 0, \quad 0 < z < \frac{\ell}{\varepsilon}. \tag{4.13}$$

Lemma 4.2. *Problem* (4.11)–(4.13) *possesses a unique solution. Moreover,*

$$\|\phi\|_{H^2(\mathcal{S})} \le C\|h\|_{L^2(\mathcal{S})}.$$

Proof. To establish existence, we assume that

$$h(x,z) = \sum_{k=0}^{\infty} \left[h_{1k}(x)\cos\left(\frac{2\pi k}{\ell}\varepsilon z\right) \,+\, h_{2k}(x)\sin\left(\frac{2\pi k}{\ell}\varepsilon z\right)\right]$$

and consider the problem of finding $\phi_{lk} \in H^1(\mathbb{R})$, and constants c_{lk}, such that

$$-\frac{4\pi^2k^2\varepsilon^2}{l^2}\phi_{lk} + L_0(\phi_{lk}) = h_{lk} + c_{lk}\chi_\delta^\varepsilon H_x, \quad x \in \mathbb{R},$$

and

$$\int_{-\infty}^{\infty} \phi_{lk}\, H_x\, dx = 0.$$

Fredholm's alternative yields that this problem is solvable with the choices

$$c_{lk} = -\frac{\int_{-\infty}^{\infty} h_{lk}H_x dx}{\int_{-\infty}^{\infty} H_x^2\chi_\delta^\varepsilon dx}.$$

Observe in particular that

$$\sum_{k=0}^{\infty} |c_{lk}|^2 \le C\varepsilon\|h\|^2_{L^2(\mathcal{S})}. \tag{4.14}$$

Finally, define

$$\phi(x,z) = \sum_{k=0}^{\infty} \left[\phi_{1k}(x)\cos\left(\frac{2\pi k}{\ell}\varepsilon z\right) \,+\, \phi_{2k}(x)\sin\left(\frac{2\pi k}{\ell}\varepsilon z\right)\right],$$

and correspondingly

$$c(z) = \sum_{k=0}^{\infty} \left[c_{1k}\cos\left(\frac{2\pi k}{\ell} z\right) \,+\, c_{2k}\sin\left(\frac{2\pi k}{\ell} z\right)\right].$$

The estimate (4.14) gives that $c(\varepsilon z)\chi_\delta^\varepsilon H_x$ has the $L^2(\mathcal{S})$ norms controlled by that of h. The a priori estimates of the previous lemma tell us that the series for ϕ is convergent in $H^2(\mathcal{S})$ and defines a unique solution for the problem with the desired bounds. ∎

Proof of Proposition 4.1. Problem (4.1)–(4.3) can be reduced to a small perturbation of a problem of the form (4.11)–(4.13) in which Lemma 4.2 is applicable. In fact, we have

$$L_2(\phi) = \mathtt{L}(\phi) + \tilde{B}(\phi), \tag{4.15}$$

where

$$\tilde{B}(\phi) = 3(\mathtt{H}^2 - 3H^2)\phi + 2\chi(\varepsilon|x|)a(\varepsilon s, \varepsilon z)\phi + \chi(\varepsilon|x|)B_1(\phi).$$

In the operator $B_1(\phi)$, consider for instance the following term involving f'':

$$B_f(\phi) = \varepsilon^2 f''(\varepsilon z)\phi_x.$$

Then we have

$$\|B_f(\phi)\|^2_{L^2(\mathcal{S})} \le \varepsilon^3 \int_0^\ell |f''(\theta)|^2 d\theta \left(\sup_z \int_{-\infty}^\infty |\phi_x(x,z)|^2\, dx\right).$$

Let $\varphi(z) = \int_{-\infty}^\infty |\phi_x(x,z)|^2 dx$. Then

$$\begin{aligned}\sup_z \varphi(z) &\le \varepsilon \int_{\mathcal{S}} |\phi_x|^2 + 2\int_{\mathcal{S}} |\phi_x||\phi_{xz}| \\ &\le \frac{1}{2}\sup_z \varphi(z) + 4\varepsilon^{-1}\int_{\mathcal{S}} |\phi_{xz}|^2 + \varepsilon \int_{\mathcal{S}} |\phi_x|^2.\end{aligned}$$

Hence

$$\varphi(z) \le C\varepsilon^{-1}\|\phi\|^2_{H^2(\mathcal{S})}, \tag{4.16}$$

so that finally

$$\|B_f(\phi)\|_{L^2(\mathcal{S})} \le C\varepsilon\|f''\|_{L^2(0,\ell)}.$$

For other terms the analysis follows in a simpler way. In fact we get

$$\|\tilde{B}(\phi)\|_{L^2(\mathcal{S})} \le C\delta\|\phi\|_{H^2(\mathcal{S})}.$$

This last estimate is a rather straightforward consequence of the fact that $|\varepsilon s| < 20\delta$ wherever the operator $\chi(\varepsilon|x|)B_1$ is supported, and $|a(\varepsilon s, \varepsilon z)| \le C\delta$ in $\mathcal{S}$. Thus, by reducing δ if necessary, we apply the invertibility result of Lemma 4.2. This concludes the proof. ∎

5. Solving the nonlinear intermediate problem. In this section we will solve problem (3.9)–(3.11). For brevity we let $E_2 = \eta^\varepsilon_\delta \tilde{E}$.

Notice that

$$\|E_2\|_{L^2(\mathcal{S})} \le C\varepsilon^{\frac{3}{2}}.$$

For further reference, it is useful to point out the Lipschitz dependence of the term of error E_2 on the parameters $\mathbf{f}$ for the norms defined in (2.26). We have the validity of the estimate

$$\|E_2(\mathbf{f}_1) - E_2(\mathbf{f}_2)\|_{L^2(\mathcal{S})} \le C\varepsilon^{\frac{1}{2}}[\|\mathbf{f}_1 - \mathbf{f}_2\|]. \tag{5.1}$$

Let T be the operator defined by Proposition 4.1. Then the equation is equivalent to the fixed point problem

$$\phi = T(E_2 + N_2(\phi)) \equiv \mathcal{A}(\phi). \tag{5.2}$$

The operator T has a useful property: Assume h has support contained in $|x| \leq \frac{\delta}{\varepsilon}$. Then by elliptic estimates, $\phi = T(h)$ satisfies the estimate

$$|\phi(x,z)| + |\nabla\phi(x,z)| \leq \|\phi\|_\infty e^{-\frac{\gamma_0\delta}{\varepsilon}} \quad \text{for } |x| > \frac{\delta}{\varepsilon}. \tag{5.3}$$

Now we recall that the operator $\psi(\phi)$ satisfies, as seen directly from its definition,

$$\|\psi(\phi)\|_{L^\infty} \ \leq\ C\left[\,\|\,|\nabla\phi| + |\phi|\,\|_{L^\infty(|s|>\frac{\delta}{\varepsilon})} + e^{-\frac{\gamma_0\delta}{\varepsilon}}\right], \tag{5.4}$$

and also the Lipschitz condition

$$\|\psi(\phi_1) - \psi(\phi_2)\|_{L^\infty} \ \leq\ C\left[\,\|\,|\nabla(\phi_1-\phi_2)| \ +\ |\phi_1-\phi_2|\,\|_{L^\infty(|s|>\frac{\delta}{\varepsilon})}\right]; \tag{5.5}$$

here $s = x + f$. These facts will allow us to construct a region where the contraction mapping principle applies. As we have said,

$$\|E_2\|_{L^2(\mathcal{S})} \leq C_*\varepsilon^{\frac{3}{2}}$$

for certain constant $C_* > 0$. We consider the following closed, bounded subset of $H^2(\mathcal{S})$:

$$\mathcal{B} = \left\{ \phi \in H^2(\mathcal{S}) \ \middle|\ \begin{array}{l} \|\phi\|_{H^2(\mathcal{S})} \leq D\varepsilon^{\frac{3}{2}}, \\ \|\,|\phi| + |\nabla\phi|\,\|_{L^\infty(|s|>\frac{\delta}{\varepsilon})} \leq \|\phi\|_{H^2(\mathcal{S})} e^{-\frac{\gamma_0\delta}{2\varepsilon}} \end{array} \right\}.$$

We claim that if the constant D is fixed sufficiently large, then the map $\mathcal{A}$ defined in (5.2) is a contraction from B into itself.

Let us analyze the Lipschitz character of the nonlinear operator $N_2(\phi)$, involved in $\mathcal{A}$ for functions in $\mathcal{B}$. Arguing as in [12], we have the following Lipschitz estimates for $N_2(\phi)$:

$$\|N_2(\phi_1) - N_2(\phi_2)\|_{L^2(\mathcal{S})} \ \leq\ C\varepsilon^{\frac{3}{2}}\|\phi_1 - \phi_2\|_{H^2(\mathcal{S})}. \tag{5.6}$$

Now let $\phi \in \mathcal{B}$; then $\varphi = \mathcal{A}(\phi)$ satisfies

$$\|\varphi\|_{H^2(\mathcal{S})} \leq C_*\varepsilon^{\frac{3}{2}}\|T\|.$$

Choosing any number $D > C_*\|T\|$ we get that for small ε

$$\|\varphi\|_{H^2(\mathcal{S})} \leq D\varepsilon^{\frac{3}{2}}.$$

On the other hand we have

$$\|\varphi\|_{L^\infty(\mathcal{S})} \leq C\|\varphi\|_{H^2(\mathcal{S})}.$$

But φ satisfies an equation of the form $L_2(\varphi) = h$ with h compactly supported. Hence φ belongs to $\mathcal{B}$ thanks to the discussion above. $\mathcal{A}$ is clearly a contraction mapping thanks to (5.6). We conclude that $\mathcal{A}$ has a unique fixed point in $\mathcal{B}$.

We recall that the error E_2 and the operator T themselves carry the function $\mathbf{f}$ as a parameter. A tedious but straightforward analysis of all terms involved in the differential operator and in the error yield that this dependence is indeed Lipschitz with respect to the H^2 norm (for each fixed ε). Indeed, emphasizing now the dependence of L_2 on $\mathbf{f}$ we can write

$$L_{2,\mathbf{f}_1}(\phi(\mathbf{f}_1)) - L_{2,\mathbf{f}_2}(\phi(\mathbf{f}_2)) = L_{2,\mathbf{f}_1}[\phi(\mathbf{f}_1) - \phi(\mathbf{f}_2)] + [L_{2,\mathbf{f}_1} - L_{2,\mathbf{f}_2}](\phi(\mathbf{f}_1))$$

and use the theory just developed to estimate $\phi(\mathbf{f}_1) - \phi(\mathbf{f}_2)$. Taking advantage of the Lipschitz character of the error term $E_2(\mathbf{f})$, we can show the Lipschitz character of T, and we find

$$\|T_{\mathbf{f}_1} - T_{\mathbf{f}_2}\| \le C\varepsilon\|\mathbf{f}_1 - \mathbf{f}_2\|.$$

Hence

$$\|\phi(\mathbf{f}_1) - \phi(\mathbf{f}_2)\|_{H^2(\mathcal{S})} \le C\varepsilon\|\mathbf{f}_1 - \mathbf{f}_2\|. \tag{5.7}$$

We summarize the result we have obtained in the following.

PROPOSITION 5.1. *There is a number $D > 0$ such that for all sufficiently small ε and all $\mathbf{f}$ satisfying* (2.26)*, problem* (3.9)–(3.11) *has a unique solution $\phi = \phi(\mathbf{f})$ which satisfies*

$$\|\phi\|_{H^2(\mathcal{S})} \le D\varepsilon^{\frac{3}{2}},$$
$$\|\,|\phi| + |\nabla\phi|\,\|_{L^\infty(|s|>\frac{\delta}{\varepsilon})} \le \|\phi\|_{H^2(\mathcal{S})} e^{-\frac{\gamma_0\delta}{2\varepsilon}}.$$

Besides, ϕ depends Lipschitz continuously on $\mathbf{f}$ in the sense of estimate (5.7).

Next we carry out the second part of the program, which is to set up an equation for $\mathbf{f}$, which is equivalent to making c identically zero. The equation is obtained by simply integrating the equation (only in x) against H_x. It is therefore of crucial importance to carry out computations of the terms $\int_{\mathbb{R}} E_2 H_x\,dx$. We do that in the next section.

6. Estimates for projections of the error.

In this section we carry out some estimates for the terms $\int_{\mathbb{R}} E_2 H_x\,dx$, where $E_2 = \eta_\delta^\varepsilon E_1$ and E_1 was defined as in (2.30). Observe that it suffices to evaluate $\int_{\mathbb{R}} E_1 H_x\,dx$ instead since the difference $E_2 - E_1$ is exponentially small in ε. Notice that the odd terms in x in E_1 do not contribute to the value of the integral since H_x is an even function.

We recall

$$S(H+\phi_1) = -\varepsilon a_t\mathbf{f}(1-H^2) + \varepsilon^2 S_3 + \varepsilon^2 S_4 + B_6(H) + \phi_{1,zz} + B_7(\phi_1) + N_0(\phi_1),$$

where S_3 is an odd function, S_4 is an even function, and $B_6(H)$ is of order ε^3. Thus, we see that

$$\begin{aligned}
&\int_{\mathbb{R}} S(H+\phi_1)H_x \\
&\quad = -\varepsilon a_t\mathbf{f}\int_{\mathbb{R}}(1-H^2)H_x \\
&\qquad -\varepsilon^2\left\{\mathbf{f}''\int_{\mathbb{R}} H_x^2 + \mathbf{f}\left[k^2\int_{\mathbb{R}} H_x^2 + a_{tt}f_0\int_{\mathbb{R}}(1-H^2)H_x\right]\right. \\
&\qquad\qquad\qquad\qquad\left. + \frac{\mathbf{f}^2}{2}a_{tt}\int_{\mathbb{R}}(1-H^2)H_x dx\right\} \\
&\qquad + \int_{\mathbb{R}} N_0(\phi_1)H_x + \int_{\mathbb{R}} B_7(\phi_1)H_x + \varepsilon^2\gamma_0(\varepsilon z) + \varepsilon^3 b_{1\varepsilon}\mathbf{f}'' + \varepsilon^3 b_{2\varepsilon}\,.
\end{aligned}$$

Here and below we denote by $b_{l\varepsilon}$, $l = 1, 2$, generic, uniformly bounded continuous functions of the form

$$b_{l\varepsilon} = b_{l\varepsilon}(z, \mathbf{f}(\varepsilon z), \mathbf{f}'(\varepsilon z)),$$

where additionally $b_{1\varepsilon}$ is uniformly Lipschitz in its last two arguments. Here and below, functions $\gamma_j(\theta), j = 0, 1, 2, \ldots$, are C^2 smooth in its argument $\theta \in (0, \ell)$.

Next we estimate $\int_{\mathbb{R}} N_0(\phi_1) H_x$. This term is to main order of the form $\int_{\mathbb{R}} H \phi_1^2 H_x$. Since ϕ_1 doesn't depend on $\mathbf{f}$, we have

$$\int_{\mathbb{R}} N_0(\phi_1) H_x = \varepsilon^2 \gamma_1(\varepsilon z).$$

Now, let us consider $\int_{\mathbb{R}} B_7(\phi_1) H_x$. All terms in this expression, with the exception of the terms of size ε in B_7, carry in the L^2 norm as functions of $\theta = \varepsilon z$ powers 3 or higher. Thus, we find

$$\begin{aligned}\int_{\mathbb{R}} B_7(\phi_1) H_x &= \varepsilon \int_{\mathbb{R}} [k\phi_{1,x} - a_t(0, \varepsilon z)(x + f)\phi_1]\ H_x \ + O(\varepsilon^3)\\ &= -\varepsilon^2 \mathbf{f} a_t(0, \varepsilon z) \int_{\mathbb{R}} \phi_1 H_x\, dx + \varepsilon^2 \gamma_2(\varepsilon z) + \varepsilon^3 b_{3\varepsilon} f^{''} + \varepsilon^3 b_{4\varepsilon},\end{aligned}$$

where $b_{3\varepsilon}$ is uniformly Lipschitz in $\mathbf{f}$ and $\mathbf{f}^{'}$.

In summary, we have established that

$$\begin{aligned}\int_{\mathbb{R}} S(H + \phi_1) H_x\, dx = &- \left[\varepsilon^2 (\mathbf{f}''(\varepsilon z) + \gamma_3(\varepsilon z)\mathbf{f}\ ) + \varepsilon \mathbf{f} \gamma_4(\varepsilon z)\right] \int_{\mathbb{R}} H_x^2\\ &+ \varepsilon^2 \gamma_5(\varepsilon z) + \varepsilon^3 [b_{5\varepsilon} \mathbf{f}'' + b_{6\varepsilon}],\end{aligned} \tag{6.1}$$

where γ_4 is given by

$$\gamma_4(\theta) = \frac{a_t(0, \theta) \int_{\mathbb{R}} (1 - H^2) H_x}{\int_{\mathbb{R}} H_x^2}, \tag{6.2}$$

and $b_{5\varepsilon}$ is uniformly Lipschitz in $\mathbf{f}$ and $\mathbf{f}^{'}$.

7. Projections of terms involving $\boldsymbol{\phi}$. We will estimate next the terms that involve ϕ in (3.9)–(3.11) integrated against H_x. We call the sum of them $\varphi(\phi)$:

$$\begin{aligned}\varphi = -2 &\int_{\mathbb{R}} \chi(\varepsilon|x|) a(\varepsilon s, \varepsilon z) \phi H_x\, dx\\ &- \int_{\mathbb{R}} \chi(\varepsilon|x|) B_8(\phi) H_x\, dx + \int_{\mathbb{R}} N_2(\phi) H_x\, dx\\ &+ 3 \int_{\mathbb{R}} [\mathtt{H}^2 - H^2] \phi H_x\, dx = \sum_{i=1}^{4} \varphi_i.\end{aligned}$$

Let $\varphi_1(\varepsilon z) = -2 \int_{\mathbb{R}} a(\varepsilon s, \varepsilon z) \chi(\varepsilon|x|) \phi H_x$. Then it is easy to see that

$$\int_0^{\ell} |\varphi_1(\theta)|^2 d\theta \le C\varepsilon^3 \|\phi\|^2_{H^2(S)},$$

and hence

$$\|\varphi_1\|_{L^2(0,\ell)} \le C\varepsilon^3.$$

The Lipschitz continuity of φ_1 follows from the Lipschitz continuity of ϕ.

Next we let $\varphi_2(\varepsilon z) = \int_{\mathbb{R}} B_1(\phi)\chi(\varepsilon|x|)H_x$. We make the following observation: All terms in $B_1(\phi)$ carry ε and involve powers of x times derivatives of powers of $0, 1$ or two orders of ϕ. The conclusion is that since H_x has exponential decay, then

$$\int_0^\ell |\varphi_2(\theta)|^2 d\theta \le C\varepsilon^3 \|\phi\|^2_{H^2(S)}.$$

Hence

$$\|\varphi_2\|_{L^2(0,\ell)} \le C\varepsilon^3.$$

To prove the Lipschitz regularity of φ_2, we single out one less regular terms in $B_8(\phi)$. The one whose coefficient depends on $\mathbf{f}''$ explicitly has the form

$$\varphi_{2*} = \varepsilon^2 \mathbf{f}'' \int_{\mathbb{R}} \phi_x H_x = -\varepsilon^2 \mathbf{f}'' \int_{\mathbb{R}} \phi H_{xx}.$$

Since ϕ has Lipschitz dependence on $\mathbf{f}$ in the form (5.7), we see that this is transmitted from Sobolev's embedding into

$$\|\phi(\mathbf{f}_1) - \phi(\mathbf{f}_2)\|_{L^\infty(\mathcal{S})} \le C\varepsilon^{\frac{3}{2}} \|\mathbf{f}_1 - \mathbf{f}_2\|,$$

from where it follows

$$\|\varphi_{2*}(\mathbf{f}_1) - \varphi_{2*}(\mathbf{f}_2)\|_{L^2(0,\ell)} \le C\varepsilon^{1+\alpha} \|\mathbf{f}_1 - \mathbf{f}_2\|.$$

The remainder $\varphi_2 - \varphi_{2*}$ actually defines for fixed ε a compact operator for $\mathbf{f}$ in $L^2(0,\ell)$. This is a consequence of the fact that weak convergence in $H^2(\mathcal{S})$ implies local strong convergence in $H^1(\mathcal{S})$, and the same is the case for $H^2(0,\ell)$ and $C^1[0,\ell]$. If $\mathbf{f}_j$ are weakly convergent sequences in $H^2(0,\ell)$, then clearly the functions $\phi(\mathbf{f}_j)$ constitute a bounded sequence in $H^1(\mathcal{S})$. In the above remainder one can integrate by parts, if necessary, once in x. Averaging against H_x, which decays exponentially, localizes the situation, and the desired fact follows.

We observe also that $\varphi_3(\varepsilon z) = \int_{\mathbb{R}} N_2(\phi)H_x$ can be estimated similarly. Using the definition of $N_2(\phi)$ and the exponential decay of H_x we obtain

$$\|\varphi_3\|_{L^2(0,\ell)} \le C\varepsilon^{\frac{1}{2}} \|\phi\|^2_{H^2(\mathcal{S})} \le C\varepsilon^3.$$

Let us consider now

$$\varphi_4(\varepsilon z) = \int_{\mathbb{R}} 3[\mathtt{H}^2 - H^2]\phi H_x.$$

Since $\mathtt{H} = H + \phi_1$ and ϕ_1 can be estimated as

$$|\phi_1(x,z)| \le C\varepsilon(|x|^2 + 1)\, e^{-c|x|},$$

we easily see that

$$\|\varphi_4\|_{L^2(0,\ell)} \le C\varepsilon^{\frac{3}{2}} \|\phi\|_{H^2(\mathcal{S})} \le C\varepsilon^3 .$$

These terms define compact operators similarly as before.

In summary, we have

$$\|\varphi(\phi)\|_{L^2(0,\ell)} \le C\varepsilon^3. \tag{7.1}$$

8. The reduced equation for f: Proof of the theorem. In this section we set up an equation relating $\mathbf{f}$ such that for the solution ϕ of (3.9)–(3.10) obtained via Proposition 5.1 one has that the coefficient $c(\varepsilon z)$ is identically zero. To achieve this we multiply first the equation against H_x and integrate only in x. The equation $c = 0$ is then equivalent to the relation

$$\int_{\mathbb{R}} E_2 H_x dx + \varphi(\phi) = 0.$$

Using the estimates in the previous sections we then find that these relations are equivalent to the following nonlinear, nonlocal, differential equation for $\mathbf{f}$:

$$\mathcal{L}(\mathbf{f}) \equiv \varepsilon \mathbf{f}'' \ + \ (\varepsilon\gamma_3 + \gamma_4)\mathbf{f} \ = \ \varepsilon\gamma_5(\varepsilon z) + \varepsilon^2 M_\varepsilon. \tag{8.1}$$

We further set

$$\mathbf{f} = \varepsilon \frac{\gamma_5}{\gamma_4 + \varepsilon\gamma_3} + \hat{\mathbf{f}}.$$

Then (8.1) becomes a nonlocal equation for $\hat{\mathbf{f}}$,

$$\mathcal{L}(\hat{\mathbf{f}}) \equiv \varepsilon \hat{\mathbf{f}}'' \ + \ (\varepsilon\gamma_3 + \gamma_4)\hat{\mathbf{f}} = \varepsilon^2 M_\varepsilon. \tag{8.2}$$

The operators $M_\varepsilon = M_\varepsilon(\hat{\mathbf{f}})$ can be decomposed into the following form:

$$M_\varepsilon(\hat{\mathbf{f}}) = A_\varepsilon(\hat{\mathbf{f}}) + K_\varepsilon(\hat{\mathbf{f}}),$$

where K_ε is uniformly bounded in $L^2(0,\ell)$ for $\hat{\mathbf{f}}$ satisfying constraints (2.26) and is also compact. The operator A_ε is Lipschitz in this region:

$$\|A_\varepsilon(\hat{\mathbf{f}}_1) - A_\varepsilon(\hat{\mathbf{f}}_2)\|_{L^2(0,\ell)} \le C\varepsilon\|\hat{\mathbf{f}}_1 - \hat{\mathbf{f}}_2\|.$$

The functions $\gamma_i, i = 1, 2$, are smooth. Furthermore, we have

$$\gamma_4 = \frac{4}{3}\left(\int_{\mathbb{R}} H_x^2\right)^{-1} a_t(0,\theta) > 0.$$

We will solve now (8.2). First we need to use assumption (1.8) to deal with the invertibility of $\mathcal{L}$. We have the following lemma.

LEMMA 8.1. *Assume that condition* (1.8) *holds. If $d \in L^2(0,\ell)$, then there is a unique solution $\hat{\mathbf{f}} \in H^2(0,\ell)$ of $\mathcal{L}(\hat{\mathbf{f}}) = d$ which is ℓ-periodic and satisfies*

$$\varepsilon\|\hat{\mathbf{f}}''\|_{L^2(0,\ell)} + \sqrt{\varepsilon}\|\hat{\mathbf{f}}'\|_{L^2(0,\ell)} + \|\hat{\mathbf{f}}\|_{L^\infty(0,\ell)} \le C\varepsilon^{-1/2}\|d\|_{L^2(0,\ell)}.$$

Moreover, if d is in $H^2(0,\ell)$, then

$$\varepsilon\|\hat{\mathbf{f}}''\|_{L^2(0,\ell)} + \|\hat{\mathbf{f}}'\|_{L^2(0,\ell)} + \|\hat{\mathbf{f}}\|_{L^\infty(0,\ell)} \le C[\|d''\|_{L^2(0,\ell)} + \|d'\|_{L^2(0,\ell)}] + C\|d\|_{L^2(0,\ell)}.$$

Let us accept for the moment the validity of this result and let us conclude the proof of the theorem. From the contraction mapping principle, the equation

$$\mathcal{L}\hat{\mathbf{f}} = g$$

is uniquely solvable for $\hat{\mathbf{f}}$ satisfying (2.26) if $\|g\|_2 < \varepsilon^{\frac{3}{2}+\rho}$ for some $\rho > 0$. The desired result for the full problem (8.2) then follows directly from Schauder's fixed point theorem. In fact, refining the fixed point region, we can actually get $\|\hat{\mathbf{f}}\| = O(\varepsilon^{3/2})$ for the solution.

Proof of Lemma 8.1. We consider the boundary value problem

$$\mathcal{L}(\hat{\mathbf{f}}) = d, \quad \hat{\mathbf{f}}(0) = \hat{\mathbf{f}}(\ell), \qquad \hat{\mathbf{f}}'(0) = \hat{\mathbf{f}}'(\ell). \tag{8.3}$$

We notice that it suffices to show Lemma 8.1 with

$$\mathcal{L}_1(\hat{\mathbf{f}}) = \varepsilon\beta^{-2}\hat{\mathbf{f}}'' + \hat{\mathbf{f}},$$

where $\beta = \sqrt{\gamma_4}$. We make the following *Liouville transformation* (cf. [20]):

$$\ell_0 = \int_0^\ell \beta(\theta)d\theta, \quad t = \frac{\int_0^\theta \beta(\theta)d\theta}{\ell_0}\pi, \quad \lambda_0 = \frac{\ell_0^2}{\pi^2},$$
$$\Psi(\theta) = (\beta(\theta))^{-\frac{1}{2}}, \quad y(t) = \Psi^{-1}(\theta)\hat{\mathbf{f}}(\theta), \quad q(t) = \frac{\ell_0^2}{\pi^2}\frac{\Psi_{\theta\theta}}{\beta^2\Psi},$$
$$\tilde{d}(t) = \Psi^{-1}(\theta)d(\theta).$$

Then (8.3) with $\mathcal{L}$ replaced by $\mathcal{L}_1$ is transformed into

$$\tilde{\mathcal{L}}_2(y) = \varepsilon(y'' + q(t)y) + \lambda_0 y = \tilde{d}, \quad y(0) = y(\pi), \qquad y'(0) = y'(\pi), \tag{8.4}$$

and it then suffices to establish the estimates in Lemma 8.1 for the solution of this problem in terms of the corresponding norms of $\tilde{d}$. It is standard that the eigenvalue problem

$$y'' + q(t)y + \lambda y = 0, \quad y(0) = y(\pi), \quad y'(0) = y'(\pi) \tag{8.5}$$

has an infinite sequence of eigenvalues λ_k, $k \geq 0$, with an associated orthonormal basis in $L^2(0,\pi)$, $\{y_k\}$, constituted by eigenfunctions. A result in [20] provides asymptotic expressions as $k \to +\infty$ for these eigenvalues and eigenfunctions, which turn out to correspond to those for $q \equiv 0$. We have

$$\sqrt{\lambda_k} = 2k + O\Big(\frac{1}{k^3}\Big), \quad k \to \infty. \tag{8.6}$$

Problem (8.4) is then solvable if and only if $\lambda_k\varepsilon \neq \lambda_0$ for all $k \geq 1$. In such a case, the solution to (8.3) then can be written as

$$y(t) = \sum_{k=1}^{\infty} \frac{\tilde{d}_k}{\lambda_0 - \lambda_k\varepsilon} y_k(t)$$

with this series convergent in L^2. Hence

$$\|y\|^2_{L^2(0,\pi)} = \sum_{k=0}^{\infty} \frac{|\tilde{d}_k|^2}{(\lambda_0 - \lambda_k\varepsilon)^2}.$$

We then choose ε such that

$$|4k^2\varepsilon - \lambda_0| \;\geq\; c\sqrt{\varepsilon} \tag{8.7}$$

for all k, where c is small. This corresponds precisely to condition (1.8). From (8.6) we then find that $|\lambda_0 - \lambda_k \varepsilon| \geq \frac{c}{2}\sqrt{\varepsilon}$ if ε is also sufficiently small. It follows that $\|y\|_{L^2(0,\pi)} \leq C\varepsilon^{-\frac{1}{2}}\|\tilde{d}\|_{L^2(0,\pi)}$. Next we notice that

$$\begin{aligned} |y(t)| &\leq \sum_{k=1}^{\infty} \left| \frac{\tilde{d}_k y_k(t)}{\lambda_0 - \lambda_k \varepsilon} \right| \\ &\leq \left(\sum_{k=1}^{\infty} \tilde{d}_k^2 y_k^2(t) \right)^{1/2} \left(\sum_{k=1}^{\infty} \frac{1}{(\lambda_0 - \lambda_k \varepsilon)^2} \right)^{1/2} \\ &\leq \frac{C}{\sqrt{\varepsilon}} \|\tilde{d}\|_{L^2(0,\pi)}; \end{aligned}$$

hence the L^∞ estimate for y follows, and thus we get

$$\varepsilon \|y'\|_{L^2(0,\pi)} + \|y\|_{L^\infty(0,\pi)} \leq C\varepsilon^{-\frac{1}{2}} \|\tilde{d}\|_{L^2(0,\pi)}.$$

Observe also that

$$\|y'\|^2_{L^2(0,\pi)} \leq C \sum_{k=0}^{\infty} |\tilde{d}_k|^2 \frac{1 + |\lambda_k|}{(\lambda_0 - \lambda_k \varepsilon)^2} \leq C \sum_{k=0}^{\infty} (1 + k^4) |\tilde{d}_k|^2.$$

Besides, if d is in $H^2(0, \pi)$ with $d(0) = d(\pi)$, $d'(0) = d'(\pi)$, then the sum $\sum_k k^4 d_k^2$ is finite and bounded by the H^2 norm of d. This and the equation automatically imply

$$\varepsilon \|y''\|_{L^2(0,\pi)} + \|y'\|_{L^2(0,\pi)} + \|y\|_{L^\infty(0,\pi)} \leq C \|\tilde{d}\|_{H^2(0,\pi)},$$

and the proof is complete. □

Remark 8.1. In section 3 of [26], an equivalent form of (8.2) was also derived for a system of singularly perturbed elliptic equations on N-dimensional domains ($N \geq 2$). There it was assumed that $\gamma_4(\theta) < 0$ (condition (A7) in [26]). It was also observed that when $\gamma_4 > 0$, there is a resonance of eigenvalues hitting 0.

Acknowledgments. We thank one of the referees for pointing out [26] to us. We also thank the second referee for valuable comments regarding the bibliography and the history of the problem as well as for a careful and insightful revision of the manuscript.

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