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# Boundary singularities for weak solutions of semilinear elliptic problems

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**Abstract**

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geqslant 2$, with smooth boundary $\partial\Omega$. We construct positive weak solutions of the problem $\Delta u + u^p = 0$ in $\Omega$, which vanish in a suitable trace sense on $\partial\Omega$, but which are singular at prescribed isolated points if $p$ is equal or slightly above $\frac{N+1}{N-1}$. Similar constructions are carried out for solutions which are singular at any given embedded submanifold of $\partial\Omega$ of dimension $k \in [0, N-2]$, if $p$ equals or it is slightly above $\frac{N-k+1}{N-k-1}$, and even on countable families of these objects, dense on a given closed set. The role of the exponent $\frac{N+1}{N-1}$ (first discovered by Brezis and Turner [H. Brezis, R. Turner, On a class of superlinear elliptic problems, Comm. Partial Differential Equations 2 (1977) 601–614]) for boundary regularity, parallels that of $\frac{N}{N-2}$ for interior singularities.


*Keywords:* Prescribed boundary singularities; Very weak solution; Critical exponents

## 1. Introduction and statement of main results

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geqslant 2$ with smooth boundary $\partial\Omega$. A model of nonlinear elliptic boundary value problem is the classical Lane–Emden–Fowler equation,

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$$\begin{cases} \Delta u + u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $p > 1$. We are interested in finding solutions to this problem which are smooth in $\Omega$ and equal to 0 almost everywhere on $\partial\Omega$ with respect to the $(N-1)$-dimensional measure. More precisely, we want to study solutions to problem (1.1) which satisfy the boundary condition in a suitable trace sense, while not necessarily in a continuous fashion.

Following Brezis and Turner [3] and Quittner and Souplet [9], we will say that a positive function $u \in \mathcal{C}^\infty(\Omega)$ is a *very weak solution* of problem (1.1) if

$$u \quad \text{and} \quad \operatorname{dist}(x, \partial\Omega) u^p \in L^1(\Omega)$$

and if

$$\int_\Omega \big(u \Delta v + u^p v\big)\, dx = 0 \quad \text{for all } v \in \mathcal{C}^2(\bar{\Omega}) \text{ with } v = 0 \text{ on } \partial\Omega.$$

From the results in [3,9], it follows that if $p$ satisfies the constraint

$$1 < p < \frac{N+1}{N-1} \tag{1.2}$$

then a very weak solution $u$ is actually in $H_0^1(\Omega)$, and it is a weak solution in the usual variational sense:

$$u \in H_0^1(\Omega) \quad \text{and} \quad \int_\Omega \big(\nabla u \nabla v - u^p v\big)\, dx = 0 \quad \text{for all } v \in H_0^1(\Omega).$$

Elliptic regularity then yields $u \in \mathcal{C}^2(\bar{\Omega})$, so that $u$ solves (1.1) in the classical sense. As it is well known, a constrained minimization procedure involving Sobolev's embedding implies the existence of a weak-variational solution to (1.1) for $1 < p < \frac{N+2}{N-2}$. A natural question is then whether very weak solutions of (1.1) are classical within a broader range of exponents than (1.2). Partially answering this question negatively, Souplet [10] constructed an example of a positive function $a \in L^\infty(\Omega)$ such that problem (1.1), with $u^p$ replaced by $a(x)u^p$ for $p > \frac{N+1}{N-1}$, has a very weak solution which is *unbounded*, developing a point singularity on the boundary. Thus, as far as boundary regularity of very weak solutions is concerned, the exponent $p = \frac{N+1}{N-1}$ is critical. In the same spirit, we would also like to mention the recent paper by McKenna and Reichel [7] where very weak solutions on Lipschitz domain are considered and where critical exponents depending on the local behavior of the boundary are defined, see also Beresticky, Capuzzo-Dolcetta and Nirenberg [1].

The aim of this paper is to construct solutions to problem (1.1) with prescribed singularities on the boundary. To state an important special case of our main results we need a definition:

**Definition 1.1.** Let $u(x)$ be a function defined in $\Omega$ and $y \in \partial\Omega$. We say that

$$u(x) \to \ell \quad \text{as } x \to y \text{ nontangentially}$$

if

$$\lim_{\Gamma_\alpha(y)\ni x\to y} u(x)=\ell \quad \text{for all } \alpha\in\left[0,\frac{\pi}{2}\right),$$

where $\Gamma_\alpha(y)$ denotes the cone with vertex $y$, and angle $\alpha$ with respect to its axis, the inner normal to $\partial\Omega$ at $y$.

Our main result reads:

**Theorem 1.1.** *There exists a number $p_N > \frac{N+1}{N-1}$ such that, given $p\in[\frac{N+1}{N-1}, p_N)$ and given points $y_1, y_2, \dots, y_k\in\partial\Omega$, there exist very weak solutions $u$ to problem* (1.1) *such that $u\in C^2(\bar\Omega\setminus\{y_1,\dots,y_k\})$ and*

$$u(x)\to+\infty \quad \text{as } x\to y_i \text{ nontangentially, for all } i=1,\dots,k.$$

Before proceeding, let us comment briefly on the result. First of all, the solutions we obtain are not unique and in fact, it will be clear from their construction that they belong to a smooth $k$-dimensional family of solutions sharing the same properties, where $k$ is the number of punctures of the boundary. Our result holds for all exponents slightly larger than or equal to $\frac{N+1}{N-1}$ but we conjecture that the result should hold for $p\in[\frac{N+1}{N-1},\frac{N+2}{N-2})$. Finally, let us mention that the study of the behavior near an isolated boundary singularity of any positive solution of (1.1) when the exponent $p\geqslant\frac{N+1}{N-1}$ was recently achieved by Bidaut-Véron–Ponce–Véron in [2] and this result is in agreement with our result.

### 1.1. *The parallel with $p=\frac{N}{N-2}$ and interior singularities*

The role of the exponent $p=\frac{N+1}{N-1}$ for solutions with boundary singularities parallels that of $p=\frac{N}{N-2}$ for solutions to problem (1.1) with *interior singularities*. Let us recall that if $u\in L^p(\Omega)$ is a positive distributional solution of (1.1) and $1<p<\frac{N}{N-2}$, then $u$ is smooth in $\Omega$. On the other hand, for $p\geqslant\frac{N}{N-2}$, distributional solutions of (1.1) with prescribed interior singularities are studied and built for example in [4–6,8].

Basic cells in those constructions are radially symmetric singular solutions $u(x)=u(|x|)$ for the equation

$$\Delta u+u^p=0. \tag{1.3}$$

Whenever $p>\frac{N}{N-2}$, the function

$$u_0(x)=c_{p,N}|x|^{-\frac{2}{p-1}},\quad c_{p,N}=\left(\frac{2}{p-1}\left(N-2-\frac{2}{p-1}\right)\right)^{\frac{1}{p-1}}, \tag{1.4}$$

is an explicit singular solution of (1.3) in $\mathbb{R}^N\setminus\{0\}$. If, in addition, $\frac{N}{N-2}<p<\frac{N+2}{N-2}$, the phase plane analysis for the ODE corresponding to radial solutions of (1.3), yields the existence of a singular positive solution $u_1$ which shares the behavior of $u_0$ near the origin

$$u_1(x) = c_{p,N}|x|^{-\frac{2}{p-1}}\bigl(1 + o(1)\bigr) \quad \text{as } x \to 0, \tag{1.5}$$

and has a *fast decay behavior at infinity*

$$u_1(x) = |x|^{-(N-2)}\bigl(1 + o(1)\bigr) \quad \text{as } |x| \to +\infty \tag{1.6}$$

(note that $N - 2 > \frac{2}{p-1}$). The scalings $u_\lambda(x) = \lambda^{\frac{2}{p-1}} u_1(\lambda x)$ with $\lambda > 0$ are then solutions of (1.3) that all have the same behavior near the origin but which converge uniformly to 0 on any compact subset of $\mathbb{R}^N \setminus \{0\}$, as $\lambda \to \infty$. Thus, given points

$$y_1, y_2, \ldots, y_k \in \Omega,$$

the function

$$u_*(x) = \sum_{i=1}^{k} u_\lambda(x - y_i)$$

constitutes, for large $\lambda > 0$, a "good approximation" to a singular solution of problem (1.1). Linear theory and perturbation arguments lead to establish the presence of an actual solution to (1.1) near $u_*$, see [6]. When $p = \frac{N}{N-2}$ a similar construction can be carried out, see [8]. The basic cell $u_1$ corresponds in this case to a positive radial solution $u_1$ of Eq. (1.3) in $B(0,1)$ with

$$u_1(x) = c_N|x|^{-(N-2)} \log\bigl(1/|x|\bigr)^{-\frac{N-2}{2}}\bigl(1 + o(1)\bigr) \quad \text{as } x \to 0, \tag{1.7}$$

where $c_N > 0$ only depends on $N \geqslant 3$. In this case the scalings $u_\lambda(x) = \lambda^{\frac{N-2}{2}} u_1(\lambda x)$ all have the same behavior as $u_1$ at the origin, and they approach zero uniformly on compact subsets of $\mathbb{R}^N \setminus \{0\}$, as $\lambda \to 0^+$.

### *1.2. The basic cells: singular solutions on a half-space*

In the construction of the solutions predicted by Theorem 1.1 we will follow a scheme similar to that described above for interior singularities. Basic cells will now be positive solutions of Eq. (1.3) defined on the half-space,

$$\mathbb{R}^N_+ := \bigl\{x = (x_1, \ldots, x_N)\colon\, x_N > 0\bigr\}$$

which vanish on its boundary and have a singularity at the origin. Such solutions are of course not radial, and ODE analysis does not apply anymore. Thus, we consider the following two problems:

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \mathbb{R}^N_+ \setminus \{0\}, \\ u > 0 & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{on } \partial\mathbb{R}^N_+ \setminus \{0\}, \end{cases} \tag{1.8}$$

for $p > \frac{N+1}{N-1}$, and

$$\begin{cases} \Delta u + u^{\frac{N+1}{N-1}} = 0 & \text{in } B_+, \\ u > 0 & \text{in } B_+, \\ u = 0 & \text{on } \partial\mathbb{R}^N_+ \cap \bar{B}_+ \setminus \{0\}, \end{cases} \tag{1.9}$$

where $B_+ = \mathbb{R}^N_+ \cap B(0,1)$.

Our purpose is to find families of solutions $u_\lambda$ of the above problems with analogous behavior to the radial singular ones previously described. Let us consider first the case $p > \frac{N+1}{N-1}$. The role of the explicit radial solution $u_0$ in (1.4) is now played by one found by separation of variables: Let us denote by $S^{N-1}_+$ the half sphere

$$S^{N-1}_+ := \{z = (z_1, \ldots, z_N) \in S^{N-1}\colon z_N > 0\}.$$

Looking for a solution of problem (1.8) of the form

$$u_0(x) = |x|^{-\frac{2}{p-1}} \phi_p(z_N), \quad \text{with } z = \frac{x}{|x|}, \tag{1.10}$$

we arrive at the problem on the half sphere,

$$\begin{cases} (\Delta_{S^{N-1}} + N - 1)\phi_p - \frac{p+1}{p-1}\left(N - \frac{p+1}{p-1}\right)\phi_p + \phi_p^p = 0 & \text{in } S^{N-1}_+, \\ \phi_p > 0 & \text{in } S^{N-1}_+, \\ \phi_p = 0 & \text{on } \partial S^{N-1}_+. \end{cases} \tag{1.11}$$

Here $\Delta_{S^{N-1}}$ designates the Laplace–Beltrami operator in $S^{N-1}_+$. Observe that $N-1$ is the first eigenvalue of $-\Delta_{S^{N-1}}$ on the half sphere and under Dirichlet boundary conditions. The corresponding eigenfunction is given by

$$\varphi_1(z) = \left( \int_{S^{N-1}_+} z_N^2 \, d\sigma \right)^{-\frac{1}{2}} z_N.$$

Solvability of (1.11) can be understood from two different complementary points of view. In the considered range of exponents, $N - \frac{p+1}{p-1} > 0$, and the application of the mountain pass lemma yields the existence of a solution to this problem, provided that $p$ is subcritical in dimension $N-1$, namely $p < \frac{N+1}{N-3}$ when $N \geqslant 4$. When $p$ tends from above to $\frac{N+1}{N-1}$, this solution converges uniformly to 0. Alternatively, in this regime, a standard application of Crandall–Rabinowitz local bifurcation theorem yields that this solution defines a continuous branch in $p$ with asymptotic behavior

$$\phi_p(z) = \bar{c}_{p,N} z_N \bigl(1 + o(1)\bigr), \quad \text{as } p \downarrow \frac{N+1}{N-1}, \tag{1.12}$$

where $\bar{c}_{p,N} > 0$ tends to 0 as $p$ tends to $\frac{N+1}{N-1}$. Even though $\phi_p$ is well defined for all $p \in (\frac{N+1}{N-1}, \frac{N+1}{N-3})$, the function $u_0$ does not suffice for the construction of approximate solutions to prove Theorem 1.1 for all $p$ in this range since, when $p$ is not close to $\frac{N+1}{N-1}$, the solution $u_0$ associated to $\phi_p$ does not decay fast enough at infinity. Therefore, in order to be able to prove the result of Theorem 1.1 for all value of $p$, we need an analogue of the radial function $u_1$ in (1.5)–(1.6), namely a solution which behaves like $u_0$ near the origin but which has a *fast decay* at infinity. We are able to prove that this solution, which interpolates between $u_0$ near 0 and Poisson's kernel $x \mapsto |x|^{1-N} z_N$ near infinity, does indeed exist provided that $p$ is sufficiently close to $\frac{N+1}{N-1}$.

**Proposition 1.1.** *There exists a number $p_N > \frac{N+1}{N-1}$, such that for all $p \in (\frac{N+1}{N-1}, p_N)$, there exists a solution $u_1$ to problem* (1.8) *such that*

$$u_1(x) = |x|^{-\frac{2}{p-1}} \phi_p(z_N)\big(1 + o(1)\big) \quad \text{as } x \to 0,$$

*where $\phi_p$ solves* (1.11)*, and*

$$u_1(x) = |x|^{1-N} z_N\big(1 + o(1)\big) \quad \text{as } |x| \to +\infty\ .$$

*In addition, we have the pointwise estimate*

$$|u_1| \leqslant c|x|^{-\frac{2}{p-1}} \|\phi_p\|_{\mathcal{C}^2(S_+^{N-1})} \tag{1.13}$$

*for some constant $c > 0$ which does not depend on $p$.*

This solution has indeed "fast decay" at infinity since $N - 1 > \frac{2}{p-1}$ when $p > \frac{N+1}{N-1}$. Observe that the scalings $u_\lambda(x) = \lambda^{\frac{2}{p-1}} u_1(\lambda x)$ define a family of solutions to problem (1.8) which have a common, $\lambda$-independent behavior at the origin, but which converge uniformly to 0 on compact subsets of $\mathbb{R}_+^N \setminus \{0\}$, as $\lambda \to \infty$.

The result of Propositions 1.1 and the parallel with the radial case and $p$ close to $\frac{N}{N-2}$, to which the ODE phase plane analysis applies, lead us naturally to several questions concerning the existence of solutions of $\Delta u + u^p = 0$ on the punctured half space $\mathbb{R}_+^N - \{0\}$ with Dirichlet boundary data.

**Open problem 1.** We believe that the solution $u_1$ which has been obtained in Proposition 1.1 for $p$ close to $\frac{N+1}{N-1}$ should actually exist for all $p \in (\frac{N+1}{N-1}, \frac{N+2}{N-2})$. This would be important since it would allow one to extend the result of Theorem 1.1 to the full range $p \in (\frac{N+1}{N-1}, \frac{N+2}{N-2})$.

**Open problem 2.** When $p = \frac{N+2}{N-2}$, we believe that there exists a one parameter family of solutions of (1.8) of the form

$$u(x) = |x|^{\frac{2-N}{2}} v\big(-\log|x|, z\big)$$

where $t \mapsto v(t, \cdot)$ is periodic, not constant. This one-parameter family of solutions corresponds to the well-know periodic solutions for the singular Yamabe problem (corresponding to singular radially symmetric solutions of (1.8)).

**Open problem 3.** When $p > \frac{N+2}{N-2}$, $N \geqslant 3$, we believe that there exists a *smooth* solution of $\Delta u + u^p = 0$ defined on $\mathbb{R}^N_+$ which is equal to 0 on $\partial\mathbb{R}^N_+$ and which is asymptotic to $u_0$ in (1.10) at $\infty$. This solution should correspond to the smooth radially symmetric solution of the same equation which is defined on the whole space and decays like $|x|^{-\frac{2}{p-1}}$ at infinity, when $p > \frac{N+2}{N-2}$.

**Open problem 4.** Are there singular solutions of (1.8) when $p \geqslant \frac{N+1}{N-3}$, $N \geqslant 4$? In this regime separation of variables in general fails. Some partial answer to this question is given in [2] where it is proven that (1.8) has no positive solution of the form $u(x) = |x|^{-\frac{2}{p-1}} w(z)$.

When $p = \frac{N+1}{N-1}$ there is no solution to problem (1.11) and thus separation of variables fails. On the other hand, we have an exact analogue of the radial solution of (1.7), as described by the following result.

**Proposition 1.2.** *There exists a solution $u_1$ of problem* (1.9) *such that*

$$u_1(x) = \bar{c}_N |x|^{1-N} \log\bigl(1/|x|\bigr)^{\frac{1-N}{2}} z_N\bigl(1 + o(1)\bigr) \quad \text{as } x \to 0,$$

*where $\bar{c}_N > 0$ only depends on $N \geqslant 2$.*

We observe that in this case the functions $u_\lambda(x) = \lambda^{N-1} u_1(\lambda x)$ satisfy that $u_\lambda(x) \to 0$ uniformly on compact subsets of $\mathbb{R}^N_+ \setminus \{0\}$, as $\lambda \to 0^+$.

### *1.3. Solutions with prescribed singular set: general statements*

In reality, the profiles given by the above results can also be used to approximate solutions to problem (1.1) whose singular set is a smooth $k$-dimensional submanifold of $\partial\Omega$ with $1 \leqslant k \leqslant N-2$. For instance, for $p$ close from above to $\frac{N-k+1}{N-k-1}$, if $x' \mapsto u_1(x')$, $x' \in \mathbb{R}^{N-k}_+$ is the solution of (1.8) given by Proposition 1.1, then $\tilde{u}(x) = u_1(x')$ solves the same problem in $\mathbb{R}^N_+$, but this time with a singular set given by a $k$-dimensional subspace. In the same spirit, we have the following result whose analogue for interior singularities can be found in [6,8].

**Theorem 1.2.** *Let $0 \leqslant k \leqslant N-2$ and let $p_{N-k}$ be the number given by Proposition* 1.1 *with $N$ replaced by $N-k$. Given $p$ such that*

$$\frac{N-k+1}{N-k-1} \leqslant p < p_{N-k},$$

*and given a $k$-dimensional submanifold $S$ embedded in $\partial\Omega$, there exist infinitely many* (*very*) *weak solutions to problem* (1.1) *such that $u \in \mathcal{C}^2(\bar{\Omega} \setminus S)$, and*

$$u(x) \to +\infty \quad \text{as } x \to y \text{ nontangentially, for all } y \in S.$$

When $k = 0$, we agree that $S$ is a finite set of isolated points, so that Theorem 1.1 becomes a particular case of the result of Theorem 1.2. As already mentioned, the solutions found in Theorem 1.1 arise in *continua* and depend on as many real parameters as the number of punctures. By contrast, when $k \geqslant 1$, the solutions we construct in Theorem 1.2 belong to infinite-dimensional families. Since this is not essential to the paper, we shall not prove this point here. The construction actually allows much more: For instance, when $p = \frac{N+1}{N-1}$ or slightly larger than this value, the number of points of the singular set $S \subset \Omega$ can be taken to infinite (countable), to total a dense subset of any given closed set $\mathcal{A}$ of $\partial\Omega$, and $\mathcal{A}$ can be properly called the *singular set* of the solution. In fact, since the solutions we are interested in are smooth in $\Omega$, it is natural to define the singular set of a very weak solution $u$ of (1.1) as the complement in $\partial\Omega$ of the set of points $y \in \partial\Omega$ in a neighborhood of which $u$ is smooth. Observe that, by definition, the singular set of $u$ is a closed subset of $\partial\Omega$. We have the validity of the following general result.

**Theorem 1.3.** *Let* $0 \leqslant k \leqslant N-2$, *and* $\frac{N-k+1}{N-k-1} \leqslant p < p_{N-k}$. *Assume that* $\mathcal{A}$ *of* $\partial\Omega$ *is a nonempty closed subset which contains a sequence of* $k$*-dimensional embedded submanifolds* $S_i$, $i \in \mathbb{N}$, *which are mutually disjoint and for which* $S := \bigcup_i S_i$ *is dense in* $\mathcal{A}$. *Then, there exist positive very weak solutions of problem* (1.1) *whose singular set is exactly* $\mathcal{A}$, *and such that*

$$u(x) \to +\infty \quad \text{as } x \to y \text{ nontangentially, for all } y \in S.$$

*In addition* $u \in W_0^{1,q}(\Omega)$ *for any* $1 < q < N\frac{p-1}{p+1}$.

Let us emphasize that according to this last result, when $p$ is larger than but close enough to $\frac{N+1}{N-1}$, there are infinitely many very weak solutions of (1.1) whose singular set is any prescribed closed subset of $\partial\Omega$ and which belong to $W_0^{1,q}(\Omega)$ for any $1 < q < N\frac{p-1}{p+1}$. Therefore, even though these solutions are not identically equal to 0 at each point of $\partial\Omega$, we can say that they are equal to 0 on $\partial\Omega$ in an appropriate sense of traces.

The proof of these results relies on two ingredients: one is the construction of the basic cells of Propositions 1.1 and 1.2, which we carry out in Section 2. The other ingredient is the analysis of invertibility of Laplace's operator for right-hand sides that involve singular behavior near a point or an embedded manifold of the boundary. After this analysis, which is carried out in Section 3, the proof of Theorem 1.2 then follows from a fixed point argument. The result of Theorem 1.3 is a consequence of an inductive construction taken to the limit under suitable control.

## 2. The half-space case: proofs of Propositions 1.1 and 1.2

It is natural and convenient to look for solutions of (1.8) or (1.9) of the form

$$u(x) = |x|^{-\frac{2}{p-1}}\phi\big(-\log|x|, z\big),$$

where we recall that $z = \frac{x}{|x|}$, so that the equation $\Delta u + u^p = 0$ reads in terms of the function $\phi$ defined for $t \in \mathbb{R}$ and $z \in S_+^{N-1}$, as

$$\partial_t^2\phi - \left(N - 2\frac{p+1}{p-1}\right)\partial_t\phi - \frac{p+1}{p-1}\left(N - \frac{p+1}{p-1}\right)\phi + (\Delta_{S^{N-1}} + N - 1)\phi + \phi^p = 0. \tag{2.1}$$

### 2.1. Proof of Proposition 1.2

When $p = \frac{N+1}{N-1}$, under the change of functions performed above, problem (1.9) becomes:

$$\begin{cases} \partial_t^2\phi + N\partial_t\phi + (\Delta_{S^{N-1}} + N - 1)\phi + \phi^{\frac{N+1}{N-1}} = 0 & \text{in } (t_*, \infty) \times S_+^{N-1}, \\ \phi > 0 & \text{in } (t_*, \infty) \times S_+^{N-1}, \\ \phi = 0 & \text{on } (t_*, \infty) \times \partial S_+^{N-1}. \end{cases} \tag{2.2}$$

We allow here $t_* > 0$ to be a parameter, which we will choose later to be large. To get a solution of problem (1.9) we actually need $t_* = 0$, but (2.2) being autonomous, this can be subsequently achieved by applying a suitable translation in the $t$-variable.

For notational convenience, we set

$$\mathfrak{N}(\phi) = \partial_t^2\phi + N\partial_t\phi + (\Delta_{S^{N-1}} + N - 1)\phi + |\phi|^{\frac{N+1}{N-1}}.$$

The idea is to look for a solution of (2.2) as a perturbation of an approximate solution. Therefore, we set

$$\phi(t, z) = \phi_0(t, z) + \psi(t, z), \tag{2.3}$$

where the "approximate solution" $\phi_0$ is defined by

$$\phi_0(t, z) = at^{-\frac{N-1}{2}}\varphi_1(z).$$

Here $a > 0$ is parameter which has to be determined so that the function $\mathfrak{N}(\phi_0)$ decays fast enough as $t$ tends to $\infty$ (in a sense to be made precise later on). As in the introduction, $\varphi_1$ denotes the first eigenfunction of $-\Delta_{S^{N-1}}$ on the half sphere, which is associated to the eigenvalue $N - 1$ and which is normalized so that its $L^2$-norm is equal to 1. Explicitly,

$$\varphi_1(z) = \left(\int_{S_+^{N-1}} z_N^2\, d\sigma\right)^{-1/2} z_N.$$

We now explain how to choose the parameter $a$. We compute

$$\mathfrak{N}(\phi_0) = \left(a^{\frac{N+1}{N-1}}\varphi_1^{\frac{N+1}{N-1}} - \frac{N(N-1)}{2}a\varphi_1\right)t^{-\frac{N+1}{2}} + a\frac{N^2-1}{2}t^{-\frac{N+3}{2}}\varphi_1.$$

We choose the constant $a > 0$ such that the function $a^{\frac{N+1}{N-1}}\varphi_1^{\frac{N+1}{N-1}} - \frac{N(N-1)}{2}a\varphi_1$ is $L^2$-orthogonal to the function $\varphi_1$. Namely

$$a^{\frac{2}{N-1}} \int_{S_+^{N-1}} \varphi^{\frac{2N}{N-1}}\, d\sigma = \frac{N(N-1)}{2}.$$

If we insert $\phi = \phi_0 + \psi$ in (2.2), we find that we still have to solve the equation

$$\mathfrak{N}(\phi_0) + \partial_t^2\psi + N\partial_t\psi + (\Delta_{S^{N-1}} + N - 1)\psi + \frac{N+1}{N-1}\phi_0^{\frac{2}{N-1}}\psi + \mathcal{Q}(\psi) = 0 \tag{2.4}$$

on $(t_*, \infty) \times S_+^{N-1}$, with

$$\psi = 0 \quad \text{on } (t_*, \infty) \times \partial S_+^{N-1}.$$

Here, we have defined

$$\mathcal{Q}(\psi) = |\phi_0 + \psi|^{\frac{N+1}{N-1}} - \phi_0^{\frac{N+1}{N-1}} - \frac{N+1}{N-1}\phi_0^{\frac{2}{N-1}}\psi.$$

We further decompose

$$\psi(t,z) = \psi_1(t,z) + f_2(t)\varphi_1(z), \tag{2.5}$$

where $(t,z) \mapsto \psi_1(t,z)$ is $L^2$-orthogonal to $\varphi_1$ for each $t > t_*$.

Let $\Pi^\perp$ denote the $L^2$-orthogonal projection over the orthogonal complement to $\varphi_1$, namely

$$\Pi^\perp(h)(t,z) = h(t,z) - \left(\int_{S_+^{N-1}} h(t,\cdot)\varphi_1\,d\sigma\right)\varphi_1(z).$$

Projecting (2.4) over the $L^2$-orthogonal complement of $\varphi_1$ and over the space spanned by $\varphi_1$, we find out that the equation we have to solve reduces to the coupled system in $(\psi_1, f_2)$ given by

$$\begin{cases} \left(\partial_t^2 + N\partial_t + (\Delta_{S^{N-1}} + N - 1)\right)\psi_1 = N_1(\psi_1, f_2), \\ \left(\partial_t^2 + N\partial_t + \dfrac{N(N+1)}{2}\dfrac{1}{t}\right) f_2 = N_2(\psi_1, f_2), \\ \psi_1 = 0 \quad \text{on } (t_*, \infty) \times \partial S_+^{N-1}, \end{cases} \tag{2.6}$$

where

$$\begin{aligned} N_1(\psi_1, f_2) &= -\Pi^\perp\left(\mathfrak{N}(\phi_0) + \frac{N+1}{N-1}\phi_0^{\frac{2}{N-1}}(\psi_1 + f_2\varphi_1) + \mathcal{Q}(\psi_1 + f_2\varphi_1)\right), \\ N_2(\psi_1, f_2) &= -\int_{S_+^{N-1}} \left(\mathfrak{N}(\phi_0) + \frac{N+1}{N-1}\phi_0^{\frac{2}{N-1}}\psi_1 + \mathcal{Q}(\psi_1 + f_2\varphi_1)\right)\varphi_1\,d\sigma. \end{aligned} \tag{2.7}$$

To obtain this, we have used the fact that

$$\frac{N+1}{N-1}\int_{S_+^{N-1}} \phi_0^{\frac{2}{N-1}}\varphi_1^2\,d\sigma = \frac{N(N+1)}{2}\frac{1}{t},$$

by definition of $a$.

The rational in the resolution of problem (2.6) is simple: we look for a solution $\psi = \psi_1 + f_2\varphi_1$ which is small compared with $\phi_0$. To do so, we will construct right-inverses for the linear operators defined by the left-hand sides of the equations in (2.6). Then, for sufficiently large $t_*$, we will obtain the resolution of the system *via* a contraction mapping principle. Observe that so far we have not imposed boundary conditions at $t = t_*$. We will invert the linear operator in $\psi_1$, for the right-hand sides $L^2$-orthogonal to $\varphi_1$ for all $t$, imposing Dirichlet boundary condition at $t = t_*$. The choice of inverse for the ODE operator in $f_2$ will be basically explicit, and will not require imposing boundary conditions. In the next two lemmas we construct these inverses. It turns out that the natural environment to carry out these inversions is $L^\infty$-weighted spaces.

Thus we now consider the linear problem

$$\begin{cases} \big(\partial_t^2 + N\partial_t + (\Delta_{S^{N-1}} + N - 1)\big)\psi = h & \text{in } (t_*, \infty) \times S_+^{N-1}, \\ \psi = 0 & \text{on } \partial\big((t_*, \infty) \times S_+^{N-1}\big), \\ \displaystyle\int_{S_+^{N-1}} \psi(t, \cdot)\varphi_1 \, d\sigma = 0 & \text{for all } t > t_*, \end{cases} \tag{2.8}$$

for $h$ such that

$$\int_{S_+^{N-1}} h(t, \cdot)\varphi_1 \, d\sigma = 0 \quad \text{for all } t > t_*. \tag{2.9}$$

Let $d : S_+^{N-1} \to (0, \infty)$ denote the distance to $\partial S_+^{N-1}$. We have

**Lemma 2.1.** *Given $\sigma \in \mathbb{R}$, there exists $t_\sigma \geqslant 1$ and for all $t_* \geqslant t_\sigma$, there exists a continuous linear operator*

$$T_1 : t^{-\sigma} L^\infty\big((t_*, \infty) \times S_+^{N-1}\big) \to t^{-\sigma} L^\infty\big((t_*, \infty) \times S_+^{N-1}\big)$$

*such that, if $t^\sigma h \in L^\infty((t_*, \infty) \times S_+^{N-1})$ satisfies* (2.9)*, then $T_1(h)$ is a solution of* (2.8)*. In addition,*

$$\big\| t^\sigma d^{-1} T_1(h) \big\|_{L^\infty} \leqslant c_\sigma \big\| t^\sigma h \big\|_{L^\infty} \tag{2.10}$$

*for some constant $c_\sigma > 0$ which does not depend on $t_* > t_\sigma$.*

Observe that (2.10) implies the pointwise estimate

$$\big| T_1(h)(t, z) \big| \leqslant \tilde{c}_\sigma \big\| t^\sigma h \big\|_{L^\infty} t^{-\sigma} \varphi_1(z).$$

Next, we consider the linear problem

$$\left(\partial_t^2 + N\partial_t + \frac{N(N+1)}{2}\frac{1}{t}\right) f = g \quad \text{in } (t_*, \infty). \tag{2.11}$$

We have the

**Lemma 2.2.** *Given $\sigma \neq \frac{N+1}{2}$, there exists $t_\sigma$ and, for all $t_* > t_\sigma$, there exists a continuous linear operator*

$$T_2 : t^{-\sigma-1} L^\infty\big((t_*, \infty)\big) \to t^{-\sigma} L^\infty_\sigma\big((t_*, \infty)\big)$$

*such that, if $t^{\sigma+1} g \in L^\infty((t_*, \infty))$, then $T_2(g)$ is a solution of* (2.11). *In addition,*

$$\big\| t^\sigma T_2(g) \big\|_{L^\infty} \leqslant c_\sigma \big\| t^{\sigma+1} g \big\|_{L^\infty}$$

*for some constant $c_\sigma > 0$ which does not depend on $t_* > t_\sigma$.*

Before proceeding into the proofs of these lemmas, let us conclude the result.

**Conclusion of the proof of Proposition 1.2.** Let us fix in the above lemmas any number $\sigma$ such that

$$\frac{N-1}{2} < \sigma < \frac{N+1}{2}$$

and $t_*$ larger than both $t_\sigma$ appearing in the above statements. We obtain a solution of problem (2.6) as a solution of a fixed point problem

$$(\psi_1, f_2) = M(\psi_1, f_2)$$

where we have defined

$$M(\psi_1, f_2) = \big(T_1\big(N_1(\psi_1, f_2)\big), T_2\big(N_2(\psi_1, f_2)\big)\big). \tag{2.12}$$

We consider the space of functions

$$(\psi, f) \in L^\infty\big([t_*, \infty) \times S^{N-1}_+\big) \times L^\infty\big([t_*, \infty)\big)$$

for which the norm

$$\big\|(\psi, f)\big\|_\mu = \big\| t^\sigma d^{-1} \psi \big\|_{L^\infty} + \mu \big\| t^\sigma f \big\|_{L^\infty}$$

is finite. Here $\mu \leqslant 1$ is a positive number which we will fix later on and we recall that $d = S^{N-1}_+ \to (0, \infty)$ denotes the distance to $\partial S^{N-1}_+$.

Provided

$$|f_2| \ll t^{-\frac{N-1}{2}} \quad \text{and} \quad |\psi_1| \ll t^{-\frac{N-1}{2}} \varphi_1$$

the following pointwise estimates hold

$$\begin{aligned}
\big|N_1(\psi_1, f_2)\big| &\leqslant c\big(t^{-\frac{N+1}{2}} + t^{-1}|\psi_1| + t^{-1}|f_2|\big),\\
\big|N_2(\psi_1, f_2)\big| &\leqslant c\big(t^{-\frac{N+3}{2}} + t^{-1}|\psi_1| + t^{\frac{N-3}{2}}|f_2|^2\big),
\end{aligned} \tag{2.13}$$

where the constant $c > 0$ only depends on $N$. The first estimate follows at once from Taylor's expansion of the nonlinearity at first-order. To obtain the second estimate, we have used Taylor's expansion of the nonlinearity up to second-order. Observe that $|\psi_1| + |f_2\varphi_1| \ll t^{-\frac{N-1}{2}}\varphi_1$ for $t$ large enough and hence, we are entitled to use Taylor's expansion to estimate the nonlinearity.

We assume that $\|(\psi_1, f_2)\|_\mu \leqslant \mu \leqslant 1$. It follows from these pointwise estimates that

$$\begin{aligned} \big\|t^\sigma N_1(\psi_1, f_2)\big\|_{L^\infty} &\leqslant c t_*^{\sigma-\frac{N+1}{2}}, \\ \big\|t^{1+\sigma} N_2(\psi_1, f_2)\big\|_{L^\infty} &\leqslant c\big(t_*^{\sigma-\frac{N+1}{2}} + \mu + t_*^{\frac{N-1}{2}-\sigma}\big), \end{aligned} \tag{2.14}$$

for some constant $c > 0$ only depending on $N$, provided $t_*$ is chosen large enough. These pointwise estimates, together with Lemmas 2.1 and 2.2, yield

$$\big\|M(\psi_1, f_2)\big\|_\mu \leqslant \tilde{c}\big(t_*^{\sigma-\frac{N+1}{2}} + \mu t_*^{\frac{N-1}{2}-\sigma} + \mu^2\big),$$

for some constant $\tilde{c} > 0$ only depending on $N$ and $\sigma$, provided $\|(\psi_1, f_2)\|_\mu \leqslant \mu \leqslant 1$ and $t_*$ is chosen large enough.

Now, we choose $\mu$ sufficiently small so that $\tilde{c}\mu < \frac{1}{4}$. For all $t_*$ sufficiently large, the operator $M$ sends the ball $\|(\psi_1, f_2)\|_\mu \leqslant \mu$ into itself. Similar estimates show that (reducing $\mu$ if necessary) $M$ is a contraction mapping with this norm inside this region, for all $t_*$ large enough. We leave the details to the reader. Hence there is a fixed point $(\psi_1, f_2)$ in this ball. The solution obtained this way renders the function

$$\phi = \phi_0 + \psi_1 + f_2\varphi_1$$

positive in $(t_*, +\infty) \times S_+^{N-1}$, provided $t_*$ is chosen large enough. This is then a solution of problem (2.2) and this completes the proof of Proposition 1.2. □

Next we carry out the proofs of the lemmas.

**Proof of Lemma 2.1.** Let us consider first the case $\sigma = 0$, so that $h$ is bounded. Without loss of generality, we can assume that $t_* = 0$. We check that problem (2.8) has at most one bounded solution. This can be shown for instance expanding a bounded solution of the equation with $h = 0$ in eigenfunctions of the Laplace–Beltrami operator with zero boundary conditions on $S_+^{N-1}$. The coefficients in this expansion will be functions of $t$ which correspond to bounded solution of certain homogeneous ODE's which only have the zero solution as a bounded solution. Thus, we only have to prove the existence of the solution. To do so, let us consider, for any given number $\bar{t} > 0$, the problem

$$\begin{cases} \big(\partial_t^2 + N\partial_t + (\Delta_{S^{N-1}} + N - 1)\big)\psi = h & \text{in } (0, \bar{t}) \times S_+^{N-1}, \\ \psi = 0 & \text{on } \partial\big((0, \bar{t}) \times S_+^{N-1}\big). \end{cases} \tag{2.15}$$

This problem is uniquely solvable since it is just a rephrasing of a Dirichlet problem for the Laplacian in a half-annular region. Let us denote by $\psi = \psi_{\bar{t}}$ its unique solution. By assumption, $h(t, \cdot)$ is $L^2$-orthogonal to $\varphi_1$ for all $t \in (0, \bar{t})$, and hence, so is $\psi$.

It suffices to check that there exists a constant $c > 0$ independent of $\bar{t} \geqslant 1$ such that

$$\|\psi\|_{L^\infty([0,\bar{t}]\times S_+^{N-1})} \leqslant c\|h\|_{L^\infty([0,\bar{t}]\times S_+^{N-1})}. \tag{2.16}$$

Indeed, once this estimate is proven, we can use elliptic estimates together with Ascoli's theorem to show that, as $\bar{t}$ tends to $\infty$, the sequence of functions $\psi_{\bar{t}}$ converges uniformly to a function $\psi$ solution of (2.8) which satisfies

$$\|\psi\|_{L^\infty([0,\infty)\times S_+^{N-1})} \leqslant c\|h\|_{L^\infty([0,\infty)\times S_+^{N-1})}.$$

Elliptic estimates then imply that

$$\|\nabla\psi\|_{L^\infty([0,\infty)\times S_+^{N-1})} + \|\psi\|_{L^\infty([0,\infty)\times S_+^{N-1})} \leqslant c\|h\|_{L^\infty([0,\infty)\times S_+^{N-1})}. \tag{2.17}$$

Observe that the bound on the gradient of $\psi(t,\cdot)$ together with the fact that $\psi(t,\cdot)$ vanishes on $\partial S_+^{N-1}$ imply that

$$\left\|d^{-1}\psi\right\|_{L^\infty([0,\infty)\times S_+^{N-1})} \leqslant c\|h\|_{L^\infty([0,\infty)\times S_+^{N-1})}, \tag{2.18}$$

where we recall that $d : S_+^{N-1} \to (0,\infty)$ denotes the distance to $\partial S_+^{N-1}$. The orthogonality conditions on $\psi$ pass certainly to the limit, and existence of a solution with the desired properties thus follows.

It remains to prove the uniform estimate (2.16). We argue by contradiction. Since the result is certainly true when $\bar{t}$ remains bounded, we assume that there exists a sequence $\bar{t} = \bar{t}_i$ tending to $\infty$, functions $h = h_i$ and $\psi_i$ corresponding solutions to problem (2.15) for which

$$\|\psi_i\|_{L^\infty([0,\bar{t}_i]\times S_+^{N-1})} = 1 \quad \text{and} \quad \lim_{i\to\infty} \|h_i\|_{L^\infty([0,\bar{t}_i]\times S_+^{N-1})} = 0.$$

We choose $t_i \in (0, \bar{t}_i)$ where $\|\psi_i\|_{L^\infty([0,\bar{t}_i]\times S_+^{N-1})}$ is achieved and define

$$\tilde{\psi}_i(t,z) = \psi_i(t+t_i, z).$$

It is easy to check that both sequences $(t_i)_i$ and $(\bar{t}_i - t_i)_i$ remain bounded away from 0. Using elliptic estimates together with Ascoli's theorem, we can extract from $(\tilde{\psi}_i)_i$ some subsequence which converges uniformly on compact sets to $\tilde{\psi}$, a bounded solution of

$$\left(\partial_t^2 + N\partial_t + (\Delta_{S^{N-1}} + N - 1)\right)\tilde{\psi} = 0 \tag{2.19}$$

which is either defined on $(t_0,\infty) \times S_+^{N-1}$, on $(-\infty, t_0) \times S_+^{N-1}$ or on $(-\infty,\infty) \times S_+^{N-1}$. Furthermore,

$$\|\tilde{\psi}\|_{L^\infty} = 1, \tag{2.20}$$

with $\tilde{\psi}$ having 0 boundary data whenever a boundary data is needed (i.e. $t_0$ is finite). Furthermore $\tilde{\psi}(t,\cdot)$ is $L^2$-orthogonal to $\varphi_1$, for all $t$. Eigenfunction decomposition of $\tilde{\psi}(t,\cdot)$ for the Laplace–Beltrami operator yields that there is a nontrivial bounded solution of (2.19) and this contradicts (2.20). When $\sigma = 0$, this completes the proof of the uniform estimate, and thus existence of a unique bounded solution of (2.8) with the desired estimate follows. This solution of course defines a linear operator on bounded $h$.

To establish the result for $\sigma \neq 0$ and $t_* > 0$ is sufficiently large, let us write

$$h = t^{-\sigma}\tilde{h} \quad \text{and} \quad \psi = t^{-\sigma}\tilde{\psi},$$

so that $\tilde{h}$ is bounded. Solvability of (2.8) reduces to

$$\big(\partial_t^2 + N\partial_t + (\Delta_{S^{N-1}} + N - 1)\big)\tilde{\psi} + \left(\frac{\sigma(\sigma+1)}{t^2} - \frac{N\sigma}{t}\right)\tilde{\psi} - \frac{2\sigma}{t}\partial_t\tilde{\psi} = \tilde{h}. \tag{2.21}$$

We can estimate

$$\begin{aligned}&\left\|\left(\frac{\sigma(\sigma+1)}{t^2} - \frac{N\sigma}{t}\right)\tilde{\psi} - \frac{2\sigma}{t}\partial_t\tilde{\psi}\right\|_{L^\infty([t_*,\infty)\times S_+^{N-1})}\\ &\quad \leqslant c t_*^{-1}\big(\|\nabla\tilde{\psi}\|_{L^\infty([t_*,\infty)\times S_+^{N-1})} + \|\tilde{\psi}\|_{L^\infty([t_*,\infty)\times S_+^{N-1})}\big),\end{aligned}$$

for some constant $c > 0$ depending on $\sigma$. For all $t_*$ large enough, the resolution of (2.21) with the desired bound then follows from that of (2.8) with $\sigma = 0$ together with a direct linear perturbation argument. This finishes the proof. □

**Proof of Lemma 2.2.** Observe that there exist $w_1, w_2$, two linearly independent solutions of the homogeneous problem

$$\partial_t^2 w + N\partial_t w + \frac{N(N+1)}{2}\frac{w}{t} = 0,$$

whose asymptotic behaviors at $\infty$ are given by

$$w_1(t) = t^{-\frac{N+1}{2}}\big(1 + o(1)\big) \quad \text{and} \quad w_2(t) = t^{\frac{N+1}{2}}e^{-Nt}\big(1 + o(1)\big).$$

The function $t \mapsto e^{Nt}(\partial_t w_1 w_2 - \partial_t w_2 w_1)$ is easily seen to be constant and evaluation at $\infty$ shows that it is equal to $N$. When $\sigma < \frac{N+1}{2}$, a solution of (2.11) is given by

$$G(g)(t) = \frac{1}{N}\left(w_1(t)\int_{t_*}^{t} w_2(s)e^{Ns}g(s)\,ds - w_2(t)\int_{t_*}^{t} w_1(s)e^{Ns}g(s)\,ds\right),$$

and one checks directly that

$$\big\|t^\sigma G(g)\big\|_{L^\infty((t_*,+\infty))} \leqslant c\big\|t^{1+\sigma}g\big\|_{L^\infty((t_*,+\infty))},$$

for some constant $c > 0$, independent of $t_*$, chosen large enough. This follows at once from the computation

$$\int_{t_*}^{t} e^{Ns} s^{-\tau}\, ds = \left[\frac{1}{N} e^{Ns} s^{-\tau}\right]_{t_*}^{t} + \frac{\tau}{N}\int_{t_*}^{t} e^{Ns} s^{-\tau-1}\, ds$$

$$\leqslant \frac{1}{N} e^{Nt} t^{-\tau} + \frac{\tau}{N t_*}\int_{t_*}^{t} e^{Ns} s^{-\tau}\, ds$$

for all $t \geqslant t_* > 0$, and hence,

$$\int_{t_*}^{t} e^{Ns} s^{-\tau}\, ds \leqslant \frac{t_*}{N t_* - \tau} e^{Nt} t^{-\tau}$$

provided $N t_* - \tau > 0$. When $\sigma > \frac{N+1}{2}$, a similar estimate can be obtained starting from the formula

$$G(g)(t) = -\frac{1}{N}\left(w_1(t)\int_{t}^{\infty} w_2(s) e^{Ns} g(s)\, ds + w_2(t)\int_{t_*}^{t} w_1(s) e^{Ns} g(s)\, ds\right).$$

This completes the proof of the result. □

### 2.2. *Proof of Proposition 1.1*

Recall that, when $p \in (\frac{N+1}{N-1}, \frac{N+1}{N-3})$ the Mountain Pass Lemma yields the existence of $\phi_p$, a nontrivial positive solution of (1.11). This solution then induces a solution

$$u_p(x) = |x|^{-\frac{2}{p-1}} \phi_p(z), \tag{2.22}$$

of problem (1.8), for which the emphasize the dependence on $p$. We have to show that there exists a solution of (1.8) which is asymptotic to $u_p$ near 0 and it is asymptotic to

$$u_\infty(x) = |x|^{1-N} z_N,$$

at infinity. Alternatively, we have to find a solution of (2.1) which is close to $\phi_p$ at $+\infty$ and converges to 0 (at a precise rate) at $-\infty$. Again this will be performed by first constructing an approximate solution and then applying some perturbation result. We define

$$\bar{\mathfrak{N}}(\phi) = \partial_t^2 \phi + A\partial_t \phi + (\Delta_{S^{N-1}} + B)\phi + |\phi|^p,$$

where

$$A = -\left(N - 2\frac{p+1}{p-1}\right) \quad \text{and} \quad B = -\frac{2}{p-1}\left(N - \frac{2p}{p-1}\right).$$

In the range of interest, namely $p \in (\frac{N+1}{N-1}, \frac{N+2}{N-2})$, we have $A > 0$ and $B < N - 1$. Moreover, $B = N - 1$ when $p = \frac{N+1}{N-1}$. Since we are interested in the case where $p$ is close to $\frac{N+1}{N-1}$ it will be convenient to define

$$\epsilon = N - 1 - B = \frac{p+1}{p-1}\left(N - \frac{p+1}{p-1}\right).$$

One should keep in mind that $A$, $B$ and $\epsilon$ do depend on $p$ even though this does not appear in the notation.

We first proceed with the construction of the approximate solution. Given $\gamma > 0$ (to be chosen later on), we define $a_\infty$ by

$$\gamma a_\infty^{p-1} = \epsilon.$$

We look for a positive function $a$ which is a solution of

$$\partial_t^2 a + A \partial_t a - \epsilon a + \gamma a^p = 0 \tag{2.23}$$

which converges to 0 as $t$ tends to $-\infty$ and converges to $a_\infty$ as $t$ tends to $+\infty$. Observe that, when $p \in (\frac{N+1}{N-1}, \frac{N+2}{N-2})$ the coefficients $A$ and $\epsilon$ are positive and, therefore, in this range, classical ODE techniques yield the existence of $a$, a positive heteroclinic solution of (2.23) tending to 0 at $-\infty$ and tending to $a_\infty$ as $t$ tends to $+\infty$. The equation being autonomous the function $a$ is not unique and $a$ can be normalized so that $a(0) = \frac{1}{2} a_\infty$. The informations we will need on the function $a$ are collected in the following results.

**Lemma 2.3.** *The following pointwise estimates hold*

$$a^{p+1} \leqslant \frac{p+1}{2\gamma} \epsilon a^2 \quad \text{and} \quad (\partial_t a)^2 \leqslant \epsilon a^2.$$

**Proof.** The estimates follow at once from the fact that

$$t \mapsto (\partial_t a)^2 - \epsilon a^2 + \frac{2\gamma}{p+1} a^{p+1}$$

decreases with $t$ if $a$ is a solution of (2.23). The solution we are interested in tends to 0 at $-\infty$ therefore,

$$(\partial_t a)^2 + \frac{2\gamma}{p+1} a^{p+1} \leqslant \epsilon a^2$$

for this solution. This completes the proof. □

For the next results, we need to distinguish between the solutions of (2.23) corresponding to different values of $p$. Therefore, we set $a = a_p$ for the solution of (2.23) normalized as above.

**Lemma 2.4.** *Given a sequence $t_i \in \mathbb{R}$ and a sequence $p_i$ tending to $\frac{N+1}{N-1}$, the sequence of functions $a_{p_i}(t_i)^{-1} a_{p_i}(t_i + \cdot)$ converges uniformly on compacts to the constant function* 1.

**Proof.** The claim follows at once from the result of the previous lemma which implies that $(\partial_t \log a)^2 \leqslant \epsilon$. $\square$

We define

$$\delta^- = \frac{1}{2}\left(\sqrt{A^2 + 4\epsilon} - A\right) \quad \text{and} \quad \bar{\delta}^+ = \frac{1}{2}\left(\sqrt{A^2 - 4(p-1)\epsilon} - A\right)$$

and

$$\tilde{a}_p = \frac{a_p}{a_\infty}.$$

Precise estimates concerning the behavior of $a_p$ as $p$ tends to $\frac{N+1}{N-1}$ which will be needed are included in the following lemma whose proof is rather technical and postponed to Appendix A.

**Lemma 2.5.** *There exists $c > 1$, $\bar{t} > 0$ and $p_N > \frac{N+1}{N-1}$ such that, for $p \in (\frac{N+1}{N-1}, p_N)$ we have*

$$\frac{1}{2} e^{\delta^- t} \leqslant \tilde{a}_p \leqslant e^{\delta^- t} \tag{2.24}$$

*if $\epsilon t \leqslant -\bar{t}$, and*

$$\frac{1}{2} e^{\bar{\delta}^+ t} \leqslant \tilde{1} - \tilde{a}_p \leqslant 2 e^{\bar{\delta}^+ t} \tag{2.25}$$

*for $\epsilon t \geqslant \bar{t}$. Finally,*

$$\frac{1}{c}\epsilon \tilde{a}_p (1 - \tilde{a}_p) \leqslant \partial_t \tilde{a}_p \leqslant c \epsilon \tilde{a}_p (1 - \tilde{a}_p), \tag{2.26}$$

*for all $t \in \mathbb{R}$.*

The approximate solution $\bar{\phi}_0$ to our problem is defined by

$$\bar{\phi}_0(t, z) = a(t)\varphi_1(z),$$

where the function $a$ is the solution of (2.23) described above when parameter $\gamma > 0$ is chosen to be

$$\gamma = \int\limits_{S_+^{N-1}} \varphi_1^{p+1}\, d\sigma.$$

Observe that, with these choices,

$$\bar{\mathfrak{N}}(\bar{\phi}_0) = a^p\big(\varphi_1^p - \gamma\varphi_1\big) \tag{2.27}$$

is $L^2$-orthogonal to $\varphi_1$ for each $t$.

We now turn to the study of the operator

$$\bar{L}_p = \partial_t^2 + A\partial_t + (\Delta_{S^{N-1}} + N - 1 - \epsilon) + p\bar{\phi}_0^{p-1}$$

which is the nonlinear operator $\bar{\mathfrak{N}}$ linearized about $\bar{\phi}_0$. We have the validity of the following result.

**Lemma 2.6.** *There exists a* (*unique*) *continuous operator*

$$\bar{G}_p : a^p L^\infty\big(\mathbb{R} \times S_+^{N-1}\big) \to a^p L^\infty\big(\mathbb{R} \times S_+^{N-1}\big),$$

*such that for each $a^{-p} f \in L^\infty(\mathbb{R} \times S_+^{N-1})$, the function $\psi = \bar{G}_p(f)$ is the unique solution of $\bar{L}_p\psi = f$ with* 0 *Dirichlet boundary data which satisfies*

$$\big\|a^{-p} d^{-1} \bar{G}_p(f)\big\|_{L^\infty} \leqslant c\epsilon^{-1}\big\|a^{-p} f\big\|_{L^\infty}.$$

*If in addition $f(t,\cdot)$ is $L^2$-orthogonal to $\varphi_1$ for a.e. $t$, then we have*

$$\big\|a^{-p} d^{-1} \bar{G}_p(f)\big\|_{L^\infty} \leqslant c\big\|a^{-p} f\big\|_{L^\infty}$$

*where we recall that $d : S_+^{N-1} \to (0,\infty)$ denotes the distance to $\partial S_+^{N-1}$.*

**Proof.** Let us observe that $\delta(\delta - N) < 0$ if $\delta \in (0, N)$ is fixed. Therefore, we can define $\varphi_*$ to be the unique, positive solution of

$$\begin{cases} -\big(\Delta_{S^{N-1}} + N - 1 + \delta(\delta - N)\big)\varphi_* = 1 & \text{in } S_+^{N-1}, \\ \varphi_* = 0 & \text{on } \partial S_+^{N-1}. \end{cases}$$

A direct computation shows that

$$\bar{L}_p\big(e^{-\delta t}\varphi_*\big) = -\big(1 + \mathcal{O}(\epsilon)\big)e^{-\delta t}, \tag{2.28}$$

as $p$ tends to $\frac{N+1}{N-1}$. This implies that, provided $\epsilon$ is small enough, the function $(t,z) \mapsto e^{-\delta t}\varphi_*(z)$ can be used as a barrier to show that, given a function $f$ such that $a^{-p} f \in L^\infty(\mathbb{R} \times S_+^{N-1})$ and given $t_1 < -1 < 1 < t_2$, we can solve the equation

$$\bar{L}_p\psi = f$$

in $(t_1, t_2) \times S_+^{N-1}$, with 0 boundary conditions.

Let us first restrict our attention to the case where the function $f$ satisfies

$$\int_{S_+^{N-1}} f(t,\cdot)\varphi_1\, d\sigma = 0, \tag{2.29}$$

for a.e. $t \in \mathbb{R}$. In this case, the proof follows very closely the proof of Lemma 2.1, the only difference being that the exponent $p$ is now larger than, but close to $\frac{N+1}{N-1}$.

We claim that, there exists a constant $c > 0$ (independent of $f$ and $t_1 < -1 < 1 < t_2$) such that, for $p$ close enough to $\frac{N+1}{N-1}$ and for we have

$$\left\|a^{-p}\psi\right\|_{L^\infty} \leqslant c\left\|a^{-p} f\right\|_{L^\infty}. \tag{2.30}$$

As in the proof of (2.16), we argue by contradiction and we assume that, for a sequence $p_i$ tending to $\frac{N+1}{N-1}$, there exists $t_{1,i} < -1 < 1 < t_{2,i}$, a sequence of functions $f_i$ satisfying (2.29) and a sequence of solutions $\psi_i$ of $L_{p_i}\psi_i = f_i$ satisfying

$$\left\|a^{-p_i}\psi_i\right\|_{L^\infty} = 1,$$

while

$$\left\|a^{-p_i} f_i\right\|_{L^\infty} = 0.$$

We denote by $\bar{t}_i$ a point where $\|a^{-p_i}\psi_i\|_{L^\infty(\{\bar{t}_i\}\times S_+^{N-1})} = 1$ and we set

$$\bar{\psi}_i(t,z) = a^{-p_i}(\bar{t}_i)\psi_i(t+\bar{t}_i, z).$$

Using elliptic estimates together with Ascoli's theorem, we can extract subsequences so that the sequence of functions $\bar{\psi}_i$ converges on compacts to $\bar{\psi}$ solution of

$$\partial_t^2\bar{\psi} + N\partial_t\bar{\psi} + (\Delta_{S^{N-1}} + N - 1)\bar{\psi} = 0,$$

with 0 Dirichlet boundary data. In addition using Lemma 2.4 we see that

$$\|\bar{\psi}\|_{L^\infty} = 1$$

and also that $\bar{u}$ satisfies (2.29). Depending on the behavior of the sequences $t_{1,i} - \bar{t}_i$ and $t_{2,i} - \bar{t}_i$ the function $\bar{\psi}$ is defined on $(\bar{t}_1, \bar{t}_2) \times S_+^{N-1}$ where $-\infty \leqslant \bar{t}_1 < -1 < 1 < \bar{t}_2 \leqslant +\infty$. A contradiction follows at once from the eigenfunction expansion of $\bar{\psi}$ in the $z$ variables.

Now that (2.30) has been proven, we may use elliptic estimates together with Ascoli's Theorem to pass to the limit as $t_1$ tends to $-\infty$ and $t_2$ tends to $+\infty$ and get the existence of a solution of $\bar{L}_p\psi = f$ which is defined in $\mathbb{R} \times S_+^{N-1}$ and satisfies

$$\left\|a^{-p}\psi\right\|_{L^\infty} \leqslant c\left\|a^{-p} f\right\|_{L^\infty}$$

provided $f$ satisfies (2.29). Elliptic regularity then also implies that $\|a^{-p}\nabla\psi\|_{L^\infty} \leqslant c\|a^{-p} f\|_{L^\infty}$ which immediately yields

$$\left\|a^{-p} d^{-1}\psi\right\|_{L^\infty} \leqslant c\left\|a^{-p} f\right\|_{L^\infty}$$

since $\psi$ has 0 boundary data. This completes the proof of the result in the case where the function $f(t,\cdot)$ is $L^2$-orthogonal to $\varphi_1$, for a.e. $t$.

We now turn to the general case, namely, we do not assume anymore that $f$ satisfies (2.29). We look for a solution of $\bar{L}_p u = f$ of the form

$$u(t,z) = u^{\perp}(t,z) + h(t)\varphi_1(z)$$

where $u^{\perp}$ solves

$$\bar{L}_p u^{\perp} = f - \bar{L}_p(h\varphi_1) \tag{2.31}$$

and $h$ solves the ordinary differential equation

$$\partial_t^2 h + A\partial_t h - \epsilon h + \gamma p a^{p-1} h = \int\limits_{S_+^{N-1}} f(t,\cdot)\varphi_1\,d\sigma$$

so that the right-hand side of (2.31) satisfies (2.29). The solution of this problem is explicitly given by

$$h(t) = w(t)\int\limits_{-\infty}^{t} w^{-2}(s)e^{-As}\left(\int\limits_{-\infty}^{s} w(\zeta)e^{A\zeta}\tilde{f}(\zeta)\,d\zeta\right)ds$$

where $w = \partial_t a$. Using the estimates of Lemma 2.5, we find that

$$\left\|a^{-p}h\right\|_{L^\infty} \leqslant c\epsilon^{-1}\left\|a^{-p}f\right\|_{L^\infty}.$$

The proof of the result in the general case then follows at once from the collection of these results. □

**Conclusion of the proof of Proposition 1.1.** To find a solution of problem (1.8), we write

$$\phi = \bar{\phi}_0 + \psi$$

and we let $\bar{G}_p$ be the operator defined in Lemma 2.6. To conclude the proof, it is enough to find a function $\psi$ solution of the fixed point problem

$$\psi = -\bar{G}_p\big(\bar{\mathfrak{N}}(\bar{\phi}_0) + \bar{\mathcal{Q}}(\psi)\big), \tag{2.32}$$

in the space $a^p d L^\infty(\mathbb{R}\times S_+^{N-1})$, where

$$\bar{\mathcal{Q}}(\psi) = |\bar{\phi}_0 + \psi|^p - \bar{\phi}_0^p - p\bar{\phi}_0^{p-1}\psi.$$

Also, we need to check that $\phi > 0$, but we will see that the solution we obtain is much smaller than $\bar{\phi}_0$ and this will immediately guarantee that $\phi > 0$.

We set

$$\bar{M}(\psi) = -\bar{G}_p\big(\bar{\mathfrak{N}}(\bar{\phi}_0) + \bar{\mathcal{Q}}(\psi)\big).$$

Thanks to the careful choice of $\gamma$, the function $\bar{\mathfrak{N}}(\bar{\phi}_0)$ is $L^2$ orthogonal to $\varphi_1$ for each $t$ and according to Lemma 2.6 we have

$$\big\|a^{-p}d^{-1}\bar{M}(0)\big\|_{L^\infty} \leqslant c\big\|a^{-p}\bar{\mathfrak{N}}(\bar{\phi}_0)\big\|_{L^\infty} \leqslant c_0. \tag{2.33}$$

It is easy to see that there exists $c > 0$ such that, for all $p$ close enough to $\frac{N+1}{N-1}$

$$\big\|a^{-p}d^{-1}\big(\bar{M}(\psi_2) - \bar{M}(\psi_1)\big)\big\|_{L^\infty} \leqslant c\epsilon^2\big\|a^{-p}d^{-1}(\psi_2 - \psi_1)\big\|_{L^\infty} \tag{2.34}$$

for all $\psi_2, \psi_1 \in a^p d L^\infty(\mathbb{R} \times S_+^{N-1})$ satisfying

$$\big\|a^{-p}d^{-1}\psi_i\big\|_{L^\infty} \leqslant 2c_0,$$

where $c_0$ is the constant which appears in (2.33).

Using the above estimates and the result of Lemma 2.6, the existence of a solution to the fixed point problem (2.32) can then be obtained by contraction mapping principle in the ball of radius $2c_0$ in the space $a^p d L^\infty(\mathbb{R} \times S_+^{N-1})$, provided that $p$ is chosen larger than (but close enough to) $\frac{N+1}{N-1}$. We will denote by $\psi_p$ this fixed point.

Observe that $|\psi_p| \ll a\varphi_1$ is $p$ is close enough to $\frac{N+1}{N-1}$. Therefore, $\phi = \bar{\phi}_0 + \psi_p$ is positive. This completes the proof of Proposition 1.1. $\square$

## 3. The bounded domain case: proofs of Theorems 1.2 and 1.3

The proof of our main results relies on two basic ingredients: One is the, already established, existence of the "basic cells" given by Propositions 1.1 and 1.2 which we will use to construct approximations to singular solutions. Another important ingredient, on which we elaborate in the next two subsections, is the analysis of invertibility of Laplace's operator, for the right-hand sides exhibiting a controlled singular behavior on a given embedded submanifold of $\partial\Omega$, in the same spirit to that of Lemma 2.6. Then we will use a fixed point scheme analogous to that in the proof of Proposition 1.1 to perturb the approximate solutions.

Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^N$ and $S$ be a smooth embedded submanifold of $\partial\Omega \subset \mathbb{R}^N$ with dimension $k \leqslant N-2$. We define

$$n = N - k.$$

We start by setting up a suitable description of the space and Laplacian operator in natural coordinates associated to $S$. While the analysis below is done for $k \geqslant 1$, it applies equally well to the point-singularity case $k = 0$, being actually simpler.

### 3.1. Local coordinate system

In a neighborhood of a point $p_0$ of $S$, we choose sections $E_{k+1}, \dots, E_{N-1}$ of $NS$ forming an orthonormal frame the normal bundle of $S$ in $\partial\Omega$. We define Fermi coordinates in some tubular neighborhood of $S$ in $\partial\Omega$ by using the exponential map

$$F(p; x_{k+1}, \dots, x_{N-1}) = \mathrm{Exp}_p^{\partial\Omega}\left(\sum_{j=k+1}^{N-1} x_j E_j\right),$$

for $p \in S$ in a neighborhood of $p_0$ and $\bar{x} = (x_{k+1}, \dots, x_{N-1})$ in some neighborhood of 0 in $\mathbb{R}^{n-1}$. In these coordinates, the induced metric $\mathring{g}$ on $\partial\Omega$ can be expanded as

$$\mathring{g} = g_{\mathbb{R}^{n-1}} + g_S + \mathcal{O}\big(|\bar{x}|\big),$$

where $g_S$ denotes the induced metric on $S$.

Finally, to parameterize a neighborhood of a point of $\partial\Omega$ in $\Omega$, we denote by $E_N$ the normal (inward pointing) vector field about $\partial\Omega$ and define

$$\bar{F}(q; x_N) = q + x_N E_N(q)$$

for $q \in \partial\Omega$ in a neighborhood of a $p_0$ and $x_N \geqslant 0$ in some neighborhood of 0. In these coordinates, the Euclidean metric in $\Omega$ can be expanded as

$$g_{\mathbb{R}^N} = dx_N^2 + \mathring{g} + \mathcal{O}(x_N).$$

Collecting these two expansions, we conclude that the Laplacian can be expanded as

$$\Delta_{\mathbb{R}^N} = \Delta_{NS} + \mathcal{O}\big(|\tilde{x}|\big)\nabla^2 + \mathcal{O}(1)\nabla \tag{3.1}$$

where $\tilde{x} = (x_{k+1}, \dots, x_N)$ and $\Delta_{NS} = \Delta_{\mathbb{R}^n} + \Delta_S$ is the Laplace–Betrami operator on *NS*, the normal bundle of $S$.

*3.2. Analysis of the Laplacian in weighted spaces*

We want to prove, in the current setting, a result in the spirit as that of Lemma 2.6. To do this, we need to define weighted spaces on $\bar{\Omega} \setminus S$, which have a controlled blow up rate as $S$ is approached. Unlike those in Lemma 2.6, we now have to choose Hölder spaces, since they are more suitable to handle with linear perturbations which are second-order operators. Let us define, for $R > 0$, half "balls" and "annuli"

$$B_+(R) := \big\{(p, X) \in NS\colon\ |X| \in (0, R),\ X \cdot E_N > 0\big\}$$

and

$$A_+(R_1, R_2) := \big\{(p, X) \in NS\colon\ |X| \in (R_1, R_2),\ X \cdot E_N > 0\big\}.$$

In other words, $B_+(R)$ is roughly the "half" of a tubular neighborhood of radius $R$ of the manifold $S$, or just a half ball in case that $S$ reduces to a isolated points. We consider the following weighted space of functions defined on $\bar{B}_+(R) \setminus S$.

**Definition 3.1.** The space $\mathcal{C}^{\ell,\alpha}_\delta(\bar{B}_+(R)\setminus S)$ is the space of functions $u\in\mathcal{C}^{\ell,\alpha}_{\mathrm{loc}}(\bar{B}_+(R)\setminus S)$ for which the norm

$$\|u\|_{\mathcal{C}^{\ell,\alpha}_\delta(\bar{B}_+(R)\setminus S)}=\sup_{p\in S}\sup_{r\in(0,R/2]} r^{-\delta}\big\|u\big(\mathrm{Exp}^S_p(r\cdot),r\cdot\big)\big\|_{\mathcal{C}^{\ell,\alpha}(\bar{B}(1)\times(\bar{B}_+(2)-B_+(1)))},$$

is finite.

In other words, if $(\hat{x},\tilde{x})=(x_1,\dots,x_N)$ are local coordinates on $B_+(R)$, $\mathcal{C}^{\ell,\alpha}_\delta(\bar{B}_+(R)\setminus S)=|\tilde{x}|^\delta\tilde{\mathcal{C}}^{\ell,\alpha}(\bar{B}_+(R)\setminus S)$, when, to evaluate the norm in $\tilde{\mathcal{C}}^{\ell,\alpha}(\bar{B}_+(R)\setminus S)$ partial derivatives are computed with respect to the vector fields $|\tilde{x}|\partial_{x_j}$ instead of $\partial_{x_j}$. In particular functions in $\mathcal{C}^{\ell,\alpha}_\delta(\bar{B}_+(R)\setminus S)$ are bounded by a constant times $|\tilde{x}|^\delta$ and have their $\tilde{\ell}$th-order partial derivatives with respect to the vector fields $\partial_{x_j}$, bounded by a constant times $|\tilde{x}|^{\delta-\tilde{\ell}}$, for $\tilde{\ell}\leqslant\ell+\alpha$.

We consider now the linear problem

$$\begin{cases}\Delta_{NS}u=|\tilde{x}|^{-2}f & \text{in } \bar{B}_+(R)\setminus S,\\ u=0 & \text{on } \partial\bar{B}_+(R)\setminus S.\end{cases}\tag{3.2}$$

We have the validity of the following result.

**Lemma 3.1.** *Assume that $\delta\in(1-n,1)$. For all $R>0$, there exists a unique operator*

$$G_{\delta,R}:\mathcal{C}^{0,\alpha}_\delta\big(\bar{B}_+(R)\setminus S\big)\to\mathcal{C}^{2,\alpha}_\delta\big(\bar{B}_+(R)\setminus S\big),$$

*such that, for each $f\in\mathcal{C}^{0,\alpha}_\delta(\bar{B}_+(R)\setminus S)$, the function $G_{\delta,R}(f)$ is a solution of problem* (3.2). *Moreover, the norm of $G_{\delta,R}$ is bounded by a constant $c>0$ which does not depend on $R$.*

**Proof.** First we solve for each $r\in(0,R/2)$ the problem

$$\begin{cases}\Delta_{NS}u=|\tilde{x}|^{-2}f & \text{in } A_+(r,R),\\ u=0 & \text{on } \partial A_+(r,R),\end{cases}\tag{3.3}$$

and call $u_r$ its unique solution.

Since $\delta\in(1-n,1)$, we can define $\varphi_*$ to be the unique, positive solution of

$$\begin{cases}-\big(\Delta_{S^{n-1}}+n-1+(\delta-1)(\delta+n-1)\big)\varphi_*=1 & \text{in } S^{n-1}_+,\\ \varphi_*=0 & \text{on } \partial S^{n-1}_+.\end{cases}$$

A direct computation shows that

$$\Delta_{NS}\big(|\tilde{x}|^\delta\varphi_*\big)=-|\tilde{x}|^{\delta-2},\tag{3.4}$$

and the maximum principle employed as in the proof of Lemma 2.6, yields the a priori bound

$$|u_r|\leqslant c\|f\|_{\mathcal{C}^{0,\alpha}_\delta(\bar{B}_+(R)\setminus S)}|\tilde{x}|^\delta,$$

where $c > 0$. Then, elliptic estimates applied on geodesic balls of radius $r$ centered at distance $2r$ from $S$ give the following bound on the gradient of $u$

$$|\nabla u_r| \leqslant c\|f\|_{\mathcal{C}^{0,\alpha}_\delta(\bar{B}_+(R)\setminus S)}|\tilde{x}|^{\delta-1},$$

for some $c > 0$. Using Arzela's theorem, we conclude that, for a sequence of radii tending to 0, the sequence $u_r$ converges to a function $u$ which satisfies

$$|u| \leqslant c\|f\|_{\mathcal{C}^{0,\alpha}_\delta(\bar{B}_+(R)\setminus S)}|\tilde{x}|^{\delta},$$

and solves (3.2). Again, elliptic estimates applied on geodesic balls of radius $r$ centered at distance $2r$ from $S$ yield the bound

$$\|u\|_{\mathcal{C}^{2,\alpha}_\delta(\bar{B}_+(R)\setminus S)} \leqslant c\|f\|_{\mathcal{C}^{0,\alpha}_\delta(\bar{B}_+(R)\setminus S)},$$

for some constant $c > 0$. Uniqueness of the limit $u$ is easy to get and we leave it to the reader. The proof is concluded. □

Next we will extend the previous result to the entire domain $\bar{\Omega} \setminus S$. To do so, we consider a smooth, positive function

$$\gamma : \bar{\Omega} \setminus S \to (0, \infty),$$

which in the above defined local coordinates coincides with $|\tilde{x}|$ in a neighborhood of $S$ in $\bar{\Omega}$. This function will play the role of the function $|\tilde{x}|$ defined in $\bar{B}_+(R) \setminus S$. For $R$ small enough, we isometrically identify $\bar{B}_+(R) \setminus S$ with its image in $\Omega$ by the exponential map. Accordingly, we define weighted Hölder spaces in $\bar{\Omega} \setminus S$ as follows.

**Definition 3.2.** We let the space $\mathcal{C}^{\ell,\alpha}_\delta(\bar{\Omega} \setminus S)$ be that of functions $u \in \mathcal{C}^{\ell,\alpha}_{\mathrm{loc}}(\bar{\Omega} \setminus S)$ for which the norm

$$\|u\|_{\mathcal{C}^{\ell,\alpha}_\delta(\bar{\Omega}\setminus S)} = \|u\|_{\mathcal{C}^{\ell,\alpha}_\delta(\bar{B}_+(R)\setminus S)} + \|u\|_{\mathcal{C}^{\ell,\alpha}(\bar{\Omega}\setminus B_+(R/2))},$$

is finite.

We consider now the problem

$$\begin{cases} \Delta_{\mathbb{R}^N} u = \gamma^{-2} f & \text{in } \Omega \setminus S, \\ u = 0 & \text{on } \partial\Omega \setminus S. \end{cases} \tag{3.5}$$

We have the following result, extension of Lemma 3.1.

**Lemma 3.2.** *Assume that $\delta \in (1-n, 1)$. There exists a unique operator*

$$G_\delta : \mathcal{C}^{0,\alpha}_\delta(\bar{\Omega} \setminus S) \to \mathcal{C}^{2,\alpha}_\delta(\bar{\Omega} \setminus S),$$

*such that, for each $f \in \mathcal{C}^{0,\alpha}_\delta(\bar{\Omega} \setminus S)$, the function $G_\delta(f)$ is a solution of problem* (3.5).

**Proof.** The proof follows from Lemma 3.1, expansion (3.1) and a linear perturbation argument. First, we claim that the result of Lemma 3.1 remains true in $\bar{B}_+(R) \setminus S$ if the operator $\Delta_{NS}$ is replaced by $\Delta_{\mathbb{R}^N}$ and if $R$ is chosen small enough. Indeed, we have from (3.1) and Proposition 3.1

$$\left\| \gamma^2(\Delta_{\mathbb{R}^N} - \Delta_{NS}) \circ G_{\delta,R}(f) \right\|_{\mathcal{C}^{0,\alpha}_\delta(\bar{B}_+(R)\setminus S)} \leqslant cR \| f \|_{\mathcal{C}^{0,\alpha}_\delta(\bar{B}_+(R)\setminus S)}.$$

The claim follows at once from a perturbation argument, provided that $R$ is fixed small enough. We denote by $\bar{G}_{\delta,R}$ the right inverse for $\Delta$ in $\bar{B}_+(R) \setminus S$.

We consider a cut-off function $\eta_R$ which is equal to 1 in $\bar{B}_+(R/2) \setminus S$ and equal to 0 in $\bar{\Omega} \setminus \bar{B}_+(R)$. We define

$$\tilde{f} := f - \gamma^2 \Delta_{\mathbb{R}^N}(\eta_R u_1),$$

where $u_1 = \bar{G}_{\delta,R}(f)$. Observe that this function is supported in $\bar{\Omega} \setminus B_+(R/2)$. We have that $\tilde{f} \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$ and

$$\|\tilde{f}\|_{\mathcal{C}^{0,\alpha}(\bar{\Omega})} \leqslant c \| f \|_{\mathcal{C}^{0,\alpha}_\delta(\bar{\Omega}\setminus S)}$$

for some constant $c > 0$.

Finally, we can solve

$$\begin{cases} \Delta_{\mathbb{R}^N} u_2 = \gamma^{-2} \tilde{f} & \text{in } \Omega, \\ u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

We have the bound

$$\|u_2\|_{\mathcal{C}^{2,\alpha}(\bar{\Omega})} \leqslant c \|\tilde{f}\|_{\mathcal{C}^{0,\alpha}(\bar{\Omega}\setminus S)}.$$

The desired result then follows by letting the solution of (3.5) be $u = u_1 + u_2$. □

### 3.3. *Proof of Theorems 1.1, 1.2 and 1.3*

We are now in a position to provide the proof of Theorems 1.2 and 1.3. The argument goes along the same lines as that in the proof of Proposition 1.1, now with Lemma 3.2 playing the role of Lemma 2.6.

We keep the notations of the previous sections as far as local coordinates close to $S$ are concerned.

**Proof of Theorem 1.2 (and Theorem 1.1) in the case where $p = \frac{n+1}{n-1}$.** We assume that $S$ is either a finite number of points of $\partial\Omega$, in which case $k = 0$, or an embedded $k$-dimensional submanifold of $\partial\Omega$, for $k \leqslant N - 2$. For all $\varepsilon > 0$ small enough, we define

$$u_\varepsilon = \eta_R \varepsilon^{n-1} u_1(\varepsilon \tilde{x}),$$

where $u_1$ is the solution provided by Proposition 1.3 and $\eta_R$ is a cut-off function which equals 1 in $\bar{B}_+(R)$ and 0 in $\bar{\Omega} \setminus B_+(2R)$. We assume that $R > 0$ is fixed small enough. Note that, we have $u_\varepsilon = 0$ on $\partial\Omega \setminus S$.

The problem we want to solve then reads

$$\begin{cases} \Delta(u_\varepsilon + v) + |u_\varepsilon + v|^{\frac{n+1}{n-1}} = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \setminus S, \end{cases}$$

where we also require that $u_\varepsilon + v > 0$ in $\Omega$. Let us fix $\delta \in (1-n, 2-n)$. By virtue of Lemma 3.2, we can rewrite this equation as the fixed point problem

$$v = -G_\delta\big(\gamma^2\big(\Delta u_\varepsilon + |u_\varepsilon + v|^{\frac{n+1}{n-1}}\big)\big). \tag{3.6}$$

We have the validity of the following fact: there is a constant $c_0 > 0$ such that

$$\big\|\gamma^2\big(\Delta u_\varepsilon + u_\varepsilon^{\frac{n+1}{n-1}}\big)\big\|_{\mathcal{C}^{0,\alpha}_\delta(\bar{\Omega}\setminus S)} \leqslant c_0\big(\log(1/\varepsilon)\big)^{\frac{1-n}{2}}. \tag{3.7}$$

This result is a consequence of expansion (3.1) and a direct computation using the asymptotic properties of $u_1$ in Proposition 1.2.

Observe that we have chosen $\delta < 2-n$ since $\gamma^2(\Delta u_\varepsilon + u_\varepsilon^{\frac{n+1}{n-1}})$ is bounded by a constant times $|\tilde{x}|^{2-n}$ near $S$, and $\delta < 2-n$ guarantees that this function belongs to $\mathcal{C}^{0,\alpha}_\delta(\bar{\Omega} \setminus S)$.

A second estimate we can directly check is the following: Assume that $\delta \in (1-n, 2-n)$ is fixed. There exists a constant $c > 0$ such that

$$\big\|\gamma^2\big(|u_\varepsilon + v_2|^{\frac{n+1}{n-1}} - |u_\varepsilon + v_1|^{\frac{n+1}{n-1}}\big)\big\|_{\mathcal{C}^{0,\alpha}_\delta(\bar{\Omega}\setminus S)} \leqslant c\big(\log(1/\varepsilon)\big)^{-1}\|v_2 - v_1\|_{\mathcal{C}^{0,\alpha}_\delta(\bar{\Omega}\setminus S)} \tag{3.8}$$

for all $v_2, v_1 \in \mathcal{C}^{2,\alpha}_\delta(\bar{\Omega} \setminus S)$ satisfying

$$\|v_i\|_{\mathcal{C}^{2,\alpha}_\delta(\bar{\Omega}\setminus S)} \leqslant 2c_0\big(\log(1/\varepsilon)\big)^{\frac{1-n}{2}}.$$

The above estimates allow an application of contraction mapping principle in the ball of radius $2c_0(\log(1/\varepsilon))^{\frac{1-n}{2}}$ in $\mathcal{C}^{2,\alpha}_\delta(\bar{\Omega} \setminus S)$ to predict existence of a solution to problem (3.6), which we denote by $v_\varepsilon$.

Since $\delta > 1-n$, we have $|v_\varepsilon| \ll u_\varepsilon$ near $S$ and hence the solution $u = u_\varepsilon + v_\varepsilon$ is singular along $S$ and is positive near $S$. The maximum principle then implies that $u > 0$ in $\Omega$. This completes the proof of Theorem 1.2 in the case $p = \frac{n+1}{n-1}$. $\square$

When $S$ is the union of several connected components, one can choose different concentration parameters $\varepsilon_1, \ldots, \varepsilon_k$ for each connected component as far as they are commensurable, namely

$$\varepsilon_j = a_j \epsilon$$

where $a_j > 0$ are fixed and $\epsilon$ tends to 0. The result holds for any choice of the $a_j$ and this shows that the set of solutions with fixed singular set $S$ is at least $k$-dimensional, if $k$ is the number of connected components of $S$. More can be done when the dimension of $S$ is positive since in this case we can even choose $\varepsilon$ to be a function on $S$. Namely

$$\varepsilon(p) = a(p)\epsilon$$

where $a(p)$ is a smooth positive function on $S$ and $\epsilon$ is small. It is easy to check that the proof goes through in this case and also that two different functions give rise to two different solutions and hence the space of solutions with fixed singular set is now infinite-dimensional.

**The proof of Theorem 1.3 in the case where $p = \frac{n+1}{n-1}$.** This proof uses similar arguments together with an induction process. By assumption, $\mathcal{A}$ is closed and contains a sequence of $k$-dimensional mutually disjoint submanifolds $S_i$, $i \in \mathbb{N}$ such that $\bigcup_i S_i$ is dense in $\mathcal{A}$. We define inductively the sequence of functions $u_i$ which are solutions of

$$\Delta u_i + u_i^{\frac{n+1}{n-1}} = 0 \tag{3.9}$$

in $\Omega$, satisfy $u_i = 0$ on $\partial\Omega - \bigcup_{j=0}^{i} S_j$ and are singular along $\bigcup_{j=0}^{i} S_j$. Assume for example that $u_{i-1}$ has already been constructed, then, we define

$$\tilde{u}_i = u_{i-1} + \eta_{R_{i+1}} \varepsilon_i^{n-1} u_1(\varepsilon_i \tilde{x}_{S_i})$$

where $u_1$ is the solution provided by Proposition 1.3, $R_i$ is fixed small enough less than half the distance from $S_i$ to $\bigcup_{j=0}^{i-1} S_j$ and $\varepsilon_i > 0$ is as small as we want. Here $\tilde{x}_{S_i}$ corresponds to the variable $\tilde{x}$ associated to $S_i$.

Applying a perturbation argument as above, we can perturb $\tilde{u}_i$ into a solution $u_i = \tilde{u}_i + v_i$ of (3.9) for some function $v_i \in \mathcal{C}_\delta^{2,\alpha}(\bar{\Omega} \setminus \bigcup_{j=0}^{i} S_j)$. Taking $\varepsilon_i$ small enough, we can ensure that

$$\|u_i - u_{i-1}\|_{L^1(\Omega)} \leqslant 2^{-i}, \tag{3.10}$$

$$\left\| \operatorname{dist}(\cdot, \partial\Omega)^{\frac{n-1}{n+1}} |u_i - u_{i-1}| \right\|_{L^{\frac{n+1}{n-1}}(\Omega)} \leqslant 2^{-i} \tag{3.11}$$

and

$$\left\| \operatorname{dist}(\cdot, \partial\Omega)^{-\delta} v_i \right\|_{L^\infty(\Omega)} \leqslant 2^{-i} \tag{3.12}$$

where $\delta \in (1-n, 2-n)$ is fixed. Clearly (3.10) ensures that the sequence $(u_i)_i$ converges in $L^1(\Omega)$ to a function $u$. Moreover (3.10) and (3.11) imply that $u$ is a very weak solution of (1.1). Finally, (3.12) implies that the nontangential limit of $u$ at any point of $\bigcup_{j=0}^{\infty} S_j$ is equal to $\infty$. $\square$

**The proof of the Theorems in the case where $p > \frac{n+1}{n-1}$.** Let us briefly comment on the modifications which are necessary to handle the case where $p > \frac{n+1}{n-1}$ is close to this value. As above $\delta \in (1-n, 2-n)$ is fixed and $p$ is close enough to $\frac{n+1}{n-1}$ to ensure that

$$-\frac{2}{p-1} < \delta < \frac{p-3}{p-1}$$

and $n - \frac{p-3}{p-1} \geqslant \frac{p-3}{p-1} - \delta > 0$. This time we define

$$u_\varepsilon = \eta_R \varepsilon^{-\frac{2}{p-1}} u_1(\tilde{x}/\varepsilon)$$

where $\varepsilon$ is close to 0 and $u_1$ is the solution obtained in Proposition 1.1 (instead of $u_1$ being the solution which is defined in Proposition 1.2) and we obtain (provided $p$ is close enough to $\frac{n+1}{n-1}$)

$$\left\| \gamma^2 \left( \Delta u_\varepsilon + u_\varepsilon^p \right) \right\|_{\mathcal{C}^{0,\alpha}_\delta(\bar{\Omega}\setminus S)} \leqslant c\left( \varepsilon^{n-\frac{p-3}{p-1}} + \varepsilon^{\frac{p-3}{p-1}-\delta} \right) \leqslant c_p \varepsilon^{\frac{p-3}{p-1}-\delta},$$

instead of (3.7), where $c_p > 0$ tends to 0 as $p$ tends to $\frac{n+1}{n-1}$. While, using (1.13), we see that (3.8) can be replaced by

$$\left\| \gamma^2 \left( |u_\varepsilon + v_2|^p - |u_\varepsilon + v_1|^p \right) \right\|_{\mathcal{C}^{0,\alpha}_\delta(\bar{\Omega}\setminus S)} \leqslant c \|\phi_p\|_{\mathcal{C}^2(S^{N-1}_+)} \|v_2 - v_1\|_{\mathcal{C}^{0,\alpha}_\delta(\bar{\Omega}\setminus S)}$$

for all $\varepsilon$ small enough and for all $v_2, v_1 \in \mathcal{C}^{2,\alpha}_\delta(\bar{\Omega} \setminus S)$ satisfying

$$\|v_i\|_{\mathcal{C}^{2,\alpha}_\delta(\bar{\Omega}\setminus S)} \leqslant 2c_p \varepsilon^{\frac{p-3}{p-1}-\delta}.$$

Above, the constant $c > 0$ does not depend on $p$ and hence, to obtain a contraction mapping, it is enough to take $p$ close enough to $\frac{n+1}{n-1}$ to ensure that $\|\phi_p\|_{\mathcal{C}^2(S^{N-1}_+)}$ is as small as needed.

The remaining of the analysis is unchanged and we leave the details to the reader. The only substantial difference between the case where $p = \frac{n+1}{n-1}$ and the case where $p$ is larger than this value is that, when $p > \frac{n+1}{n-1}$ is close to this value, in the proof of Theorem 1.3, in addition to the properties (3.10) to (3.12) which ensure the convergence of the sequence of solutions in the appropriate spaces, we may also ask that the sequence converges in $W^{1,q}(\Omega)$, for some $q$ close enough to 1. The proofs are concluded. □

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## Appendix A

**Proof of Lemma 2.5.** Analysis at $-\infty$. Recall that

$$\delta^- = \frac{1}{2}\left( \sqrt{A^2 + 4\epsilon} - A \right).$$

It follows from standard ODE techniques that there is a unique solution of (2.23) which can be written as

$$\hat{a}_p = a_\infty e^{\delta^- t}\left(1 + o(1)\right),$$

and which is defined in some interval $(-\infty, -\hat{t}/\epsilon)$, provided $\hat{t} > 0$ is fixed (independently of $p$) large enough. Observe that $\hat{a}_p$ and $a_p$ only differ by a shift of time, so the study of $\hat{a}_p$ and $a_p$ are equivalent. If we look for a solution of (2.23) of the form $\hat{a}_p(t) = a_\infty(e^{\delta^- t} + w)$ then $w$ is a solution of the fixed point problem

$$w = -\epsilon e^{\delta^- t} \int_{-\infty}^{t} e^{-2\delta^- s - As} \left( \int_{-\infty}^{s} e^{\delta^- \zeta + A\zeta} \big(e^{\delta^- \zeta} + w(\zeta)\big)^p \, d\zeta \right) ds$$

for which it is easy to find a fixed point in the set of functions defined in $(-\infty, -\hat{t}/\epsilon)$ and satisfying $|w| \leqslant \frac{1}{2} e^{\delta^- t}$, provided $\hat{t}$ is fixed large enough (independently of $p$). Using the integral equation satisfied by $w$, one immediately estimates the derivative of $\hat{a}_p$ and check that

$$\partial_t \hat{a}_p = \delta^- a_\infty e^{\delta^- t} \big(1 + o(1)\big)$$

and this, together with the fact that $\delta^- = \epsilon + \mathcal{O}(\epsilon^2)$ implies that (2.24) and (2.26) hold in $(-\infty, -\bar{t}/\epsilon)$, at least for $p$ close enough to $\frac{N+1}{N-1}$, provided $\bar{t}$ is fixed large enough.

Analysis at $+\infty$. We define

$$\bar{\delta}^{\pm} = \frac{1}{2}\big(-A \pm \sqrt{A^2 - 4(p-1)\epsilon}\big)$$

to be the characteristic roots of

$$\big(\partial_t^2 + A\partial_t + (p-1)\epsilon\big) w = 0,$$

the homogeneous equation associated to the linearized ODE at $a_\infty$. Arguing as above, it is easy to check that there exists a unique solution of (2.23) such that

$$\bar{a}_p = a_\infty \big(1 - e^{\bar{\delta}^- t} \big(1 + o(1)\big)\big)$$

which is defined in $(\bar{t}_p, +\infty)$ and satisfies $\bar{a}_p(\bar{t}_p) = 0$. Indeed, we look for a solution of the form $\bar{a}_p(t) = a_\infty(1 - e^{\bar{\delta}^- t} + w)$, we find out that $w$ satisfies

$$\big(\partial_t^2 + A\partial_t + (p-1)\epsilon\big) w + \epsilon Q\big(-e^{\bar{\delta}^- t} + w\big) = 0$$

where $Q(w) = |1 + w|^p - 1 - pw$. A solution of this equation can be obtained using a fixed point argument and the integral formula

$$w = -\epsilon e^{\bar{\delta}^+ t} \int_{t}^{+\infty} e^{-2\bar{\delta}^+ s - As} \left( \int_{s}^{+\infty} e^{\bar{\delta}^+ \zeta + A\zeta} Q\big(-e^{\bar{\delta}^- \zeta} + w(\zeta)\big) \, d\zeta \right) ds.$$

We leave the details to the reader. It is easy to check that, in the plane $(a, \partial_t a)$, the curve described by $t \mapsto \bar{a}_p(t)$ is close to the straight line $\partial_t a = N(a_\infty - a)$, as $p$ tends to $\frac{N+1}{N-1}$, since $\bar{\delta}^-$ tends to $-N$ as $p$ tends to $\frac{N+1}{N-1}$.

Using this information together with the phase plane analysis, we see that the solution $a_p$ is trapped in the region described by $a > 0$, $\partial_t a > 0$ and is below the curve $\{(\bar{a}_p(t), \partial_t \bar{a}_p(t)):\ t \geqslant \tilde{t}_p\}$. In particular, this implies that $a_p < a_\infty$ and also that $\partial_t a_p > 0$, provided $p$ is close enough to $\frac{N+1}{N-1}$. This already implies that (2.25) holds.

We are now in a position to give a precise expansion of the solution $a_p$ when $t$ tends to $\infty$. Assume that $p$ is close enough to $\frac{N+1}{N-1}$ so that $a_p \leqslant a_\infty$ and $\bar{\delta}^+ > \bar{\delta}^-$. Certainly, since $a_p \neq \bar{a}_p$ it can be expanded as

$$a_p = a_\infty - c_p e^{\bar{\delta}^+ t}\big(1 + \mathcal{O}\big(e^{-\xi_p t}\big)\big)$$

for some positive constants $c_p > 0$ and $\xi_p > 0$, as $t$ tends to $+\infty$. Up to a shift of time, we can replace $a_p$ by $\tilde{a}_p$ which this time is normalized so that

$$\tilde{a}_p = a_\infty\big(1 - e^{\bar{\delta}^+ t}\big(1 + \mathcal{O}\big(e^{-\xi_p t}\big)\big)\big)$$

at $+\infty$. We now write $\tilde{a}_p = a_\infty(1 - e^{\bar{\delta}^+ t} + w)$ and choose $\tilde{t}_p$ such that

$$e^{\bar{\delta}^+ \tilde{t}_p} = \theta$$

where $0 < \theta \ll 1$ is fixed independently of $p$. Assume that, for $t \geqslant \tilde{t}_p$,

$$|w| \leqslant \frac{1}{2} e^{\bar{\delta}^+ t}$$

and also that $w$ tends to 0 faster than $e^{\bar{\delta}^+ t}$ as $t \to \infty$. We write $w$ as a solution of a fixed point problem and with little work, we find that

$$w = \epsilon e^{\bar{\delta}^- t} \int_{\tilde{t}_p}^{t} e^{-2\bar{\delta}^- s - As} \left( \int_{s}^{+\infty} e^{\bar{\delta}^- \zeta + A\zeta} Q\big(-e^{\bar{\delta}^+ \zeta} + w(\zeta)\big)\, d\zeta \right) ds + \lambda_p e^{\bar{\delta}^- t}$$

where the constant $\lambda_p$ has to be determined. Using the a priori bound in Lemma 2.3 which implies that

$$(\partial_t \tilde{a}_p)^2 \leqslant \epsilon \tilde{a}_p^2,$$

we can estimate

$$|\lambda_p| e^{\bar{\delta}^- \tilde{t}_p} \leqslant c\epsilon^{1/2}$$

for some constant $c > 0$, independent of $p$. Therefore, we conclude that

$$\tilde{a}_p - a_\infty = -a_\infty\big(e^{\bar{\delta}^+ t} + \mathcal{O}\big(e^{p\bar{\delta}^+ t}\big) + \mathcal{O}\big(\lambda_p e^{\bar{\delta}^- t}\big)\big),$$

and also that

$$\partial_t \tilde{a}_p = -\bar{\delta}^+ a_\infty \left(e^{\bar{\delta}^+ t} + \mathcal{O}\left(e^{p\bar{\delta}^+ t}\right) + \left(\bar{\delta}^+\right)^{-1} \mathcal{O}\left(\lambda_p e^{\bar{\delta}^- t}\right)\right),$$

for all $t \geqslant 2\tilde{t}_p$. Observe that for $t \geqslant 2\tilde{t}_p$ we have

$$\left(\bar{\delta}^+\right)^{-1} \lambda_p e^{\bar{\delta}^- t} \ll e^{\delta^+ t}$$

provided $\epsilon$ is small enough (in other words, $p$ close enough to $\frac{N+1}{N-1}$). The estimates for $a_\infty - a_p$ and $\partial_t a_p$ needed to prove (2.26) when $t \geqslant \bar{t}/\epsilon$ (for some $\bar{t}$ fixed large enough) follow at once from the corresponding estimates which can be derived for $\tilde{a}_p$ and $\partial \tilde{a}_p$, using the above expansions. We leave the details to the reader.

Finally, when $t \in [-\bar{t}/\epsilon, \bar{t}/\epsilon]$, we simply observe that we already know that

$$\frac{1}{c} \epsilon a_\infty \leqslant a_p \leqslant \frac{c}{1+c} \epsilon a_\infty$$

in this range and also that $\frac{1}{\bar{c}} \epsilon a_\infty \leqslant \partial_t a_p(\pm \bar{t}/\epsilon) \leqslant \bar{c} \epsilon a_\infty$ by the previous analysis. We also observe that, if $\partial_t a_p$ achieves a local maximum or minimum in $(-\bar{t}/\epsilon, \bar{t}/\epsilon)$, then $\partial_t^2 a_p = 0$ at this point and hence we obtain from (2.23) that

$$A \partial_t a_p = \epsilon a_p - \gamma a_p^p$$

at this point. Therefore, there exists $\tilde{c} > 0$ independent of $p$ such that

$$\frac{1}{\tilde{c}} \epsilon a_\infty \leqslant \partial_t a_p \leqslant \tilde{c} \epsilon a_\infty,$$

for all $t \in [-\bar{t}/\epsilon, \bar{t}/\epsilon]$, provided $p$ is close to $\frac{N+1}{N-1}$. This completes the proof of (2.26). $\square$

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