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The two-dimensional Lazer–McKenna conjecture for an exponential nonlinearity

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Abstract

We consider the problem of Ambrosetti–Prodi type

$$\begin{cases} \Delta u + e^u = s\phi_1 + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded, smooth domain in $\mathbb{R}^2$, ϕ_1 is a positive first eigenfunction of the Laplacian under Dirichlet boundary conditions and $h \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$. We prove that given $k \geqslant 1$ this problem has at least k solutions for all sufficiently large $s > 0$, which answers affirmatively a conjecture by Lazer and McKenna [A.C. Lazer, P.J. McKenna, On the number of solutions of a nonlinear Dirichlet problem, J. Math. Anal. Appl. 84 (1981) 282–294] for this case. The solutions found exhibit multiple concentration behavior around maxima of ϕ_1 as $s \to +\infty$.

1. Introduction and statement of main results

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded and smooth domain. This paper deals with the boundary value problem

$$\begin{cases} \Delta u + e^u = s\phi_1 + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $h \in C^{0,\alpha}(\overline{\Omega})$ is given, s is a large, positive parameter and ϕ_1 is a positive first eigenfunction of the problem $-\Delta\phi = \lambda\phi$ under Dirichlet boundary condition in Ω. We denote its eigenvalues as

$$0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots.$$

The *Ambrosetti–Prodi problem* is the equation

$$\begin{cases} \Delta u + g(u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is bounded and smooth, $f \in C^{0,\alpha}(\overline{\Omega})$, and the limits

$$\nu \equiv \lim_{t\to-\infty} \frac{g(t)}{t} < \mu \equiv \lim_{t\to+\infty} \frac{g(t)}{t}$$

are assumed to exist. Problem (1.1) corresponds to a case in which $\nu = 0$ and $\mu = +\infty$. In 1973, Ambrosetti and Prodi [2] assumed that

$$0 < \nu < \lambda_1 < \mu < \lambda_2$$

and additionally that $g'' > 0$. They proved the existence of a C^1 manifold $\mathcal{M}$ of codimension 1 which separates $C^{0,\alpha}(\overline{\Omega})$ into two disjoint open regions,

$$C^{0,\alpha}(\overline{\Omega}) = \mathcal{O}_0 \cup \mathcal{M} \cup \mathcal{O}_2,$$

such that problem (1.2) has no solutions for $f \in \mathcal{O}_0$, exactly two solutions if $f \in \mathcal{O}_2$, and exactly one solution if $f \in \mathcal{M}$.

In 1975, Berger and Podolak [4] obtained a more explicit representation for the result in [2] by decomposing

$$f = s\phi_1 + h, \qquad \int_\Omega h\phi_1 = 0,$$

and proving that for each such an h there is a number $\alpha(h)$ such that the problem

$$\begin{cases} \Delta u + g(u) = s\phi_1 + h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

has no solution if $s < \alpha(h)$ and exactly two solutions if $s > \alpha(h)$. Written in this form, letting s be a parameter and h fixed, is what is commonly referred to as the Ambrosetti–Prodi problem.

The convexity assumption in the multiplicity result for large and positive s was relaxed subsequently in [1,9,21]. In [22], Lazer and McKenna obtained a third solution of (1.3) under the further assumption

$$\nu < \lambda_1 < \lambda_2 < \mu < \lambda_3,$$

while a fourth solution under this circumstance was found by Hofer [20] and by Solimini [29]. In [22] it was further conjectured that the number of solutions for very large $s > 0$ grows as the interval (ν, μ) contains more and more eigenvalues, in particular, they conjectured that if

$$\nu < \lambda_1 < \mu = +\infty \tag{1.4}$$

and g does not grow "too fast" at infinity, then for all $k \geqslant 1$ there is a number s_k such that for all $s > s_k$, problem (1.3) has at least k solutions.

Surprisingly enough, Dancer [10] was able to disprove the conjecture in the asymptotically linear case in which ν and μ are finite, exhibiting an example in $N \geqslant 2$ in which the interval (ν, μ) contains a large number of eigenvalues but *no more than four solutions* for large s exist. The conjecture, for both μ finite and infinite actually holds true in one-dimensional and radial cases under various situations, see [8,16,19,23,28] for these and related results. See also [5,13, 14,30] for other results in the PDE case.

How fast should "too fast" be in the growth of g under the situation (1.4)? The authors of the conjecture had probably in mind a growth not beyond critical for the nonlinearity. This constraint was indeed used in [8] in the radial case.

Recently Dancer and Yan [11,12] proved that the Lazer–McKenna conjecture *holds true* when $N \geqslant 3$ and

$$g(t) = \lambda t + t_+^p, \quad 1 < p < \frac{N+2}{N-2},\ \lambda < \lambda_1,$$

by constructing and describing asymptotic behavior of the solutions found as $s \to +\infty$. In this case $\nu = \lambda$ and $\mu = +\infty$. This has also been done in the critical case $p = \frac{N+2}{N-2}$ if, in addition, $0 < \lambda$ and $N \geqslant 7$, by Li, Yan and Yang in [24].

Problem (1.1) is also a problem involving criticality in $\mathbb{R}^2$. While, strictly speaking, the nonlinearity stays below the threshold of compactness given by Trudinger–Moser embedding, for which e^{u^2} is critical, two-dimensional equations involving e^u exhibit *bubbling phenomena*, similar to that found at the critical exponent in higher dimensions. This has been a subject broadly treated in the literature, in what regards to construction and classification of unbounded families of solutions for this type of exponential nonlinearities.

The main result of this paper is a *positive* answer to the Lazer–McKenna conjecture for problem (1.1). Given any $m \geqslant 1$, there are at least m solutions for all $s > 0$ sufficiently large. These solutions can be explicitly described: they exhibit multiple bubbling behavior around maximum points of ϕ_1.

Theorem 1. *Given any $m \geqslant 1$ and any s sufficiently large, there exists a solution u_s of problem* (1.1) *such that*

$$\lim_{s \to +\infty} \int_\Omega e^{u_s} = 8\pi m.$$

More precisely, given any subset Λ of Ω for which

$$\sup_{\partial\Lambda} \phi_1 < \sup_{\Lambda} \phi_1$$

and a sequence $s \to +\infty$, there is a subsequence and m points $\xi_i \in \Lambda$ with

$$\phi_1(\xi_i) = \sup_{\Lambda} \phi_1$$

such that as $s \to +\infty$

$$e^{u_s} \rightharpoonup 8\pi \sum_{i=1}^{m} \delta_{\xi_i}.$$

In particular, we observe that associated to any isolated local maximum point of ξ_0 of ϕ_1 one has the phenomenon of multiple bubbling at a single point, namely, $e^{u_s} \rightharpoonup 8\pi m \delta_{\xi_0}$.

The construction gives much more accurate information on the asymptotic profile of these solutions, in particular we have the expansion

$$u_s = -\frac{s}{\lambda_1}\phi_1 - \rho + \sum_{i=1}^{m} G(\xi_i, x) + o(1)$$

uniformly on compact subsets of $\overline{\Omega} \setminus \{\xi_1, \ldots, \xi_m\}$, where $\rho = (-\Delta)^{-1}h$ in $H_0^1(\Omega)$ and $G(x,\xi)$ denotes the Green function of the problem

$$\begin{cases} -\Delta_x G(x,\xi) = 8\pi\delta_\xi(x), & x \in \Omega, \\ G(x,\xi) = 0, & x \in \partial\Omega. \end{cases} \tag{1.5}$$

In order to restate the problem in perhaps more familiar terms, let us substitute u in Eq. (1.1) by $u - \frac{s}{\lambda_1}\phi_1 - \rho$. Replacing further the parameter s by $\lambda_1 s$ and setting $k(x) = e^{-\rho}$, (1.1) becomes equivalent to

$$\begin{cases} \Delta u + k(x)e^{-s\phi_1}e^u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

and thus what one typically expects are solutions of u_s (1.6) that resemble

$$u_s(x) \sim \sum_{j=1}^{k} m_j G(\xi_i, x),$$

with $m_j > 1$, where ξ_i's are maxima of ϕ_1. This multiple bubbling phenomenon is in strong opposition to the seemingly similar, well studied problem

$$\begin{cases} \Delta u + \varepsilon^2 k(x)e^u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

with $k \in \mathcal{C}^2(\overline{\Omega})$, $\inf_\Omega k > 0$ and $\varepsilon \to 0$, where bubbling of solutions with

$$\int_\Omega \varepsilon^2 k(x)e^u = O(1)$$

is forced to be simple, namely with all m_j's equal to one, as it follows from the results in [6,25–27]. Blowing up families of solutions to this problem have been constructed in [3,7,15,17].

For instance, it is found in [15] the presence of solutions with arbitrary number of bubbling points whenever Ω is not simply connected, see also [18] for a similar phenomenon for large exponents in a power nonlinearity. Multiple bubbling has been built recently, in [32], for the anisotropic problem

$$\begin{cases} \operatorname{div}(a(x)\nabla u) + \varepsilon^2 k(x) e^u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

around isolated local maxima of the (uniformly positive) coefficient a. The moral of our result is that multiple bubbling in the isotropic case may be triggered by the fact that the coefficient in front of e^u does not go to zero in uniform way. Multiple bubbling "wants to take place" where the coefficient vanishes faster in s. This should be somehow connected with phenomena associated to (1.7), where $k(x)$ is replaced by $|x|^\alpha k(x)$, weight resulting for Liouville type equations with singular sources. Important advances in understanding of blowing-up solutions for that problem have been obtained, see, for instance, [31] and references therein.

The rest of this paper will devoted to the proof of Theorem 1. We will actually give to it a precise version in terms of problem (1.6) in Theorem 2 below.

As we have mentioned, we do not intend to express our results in their most general forms. For instance, the choice of ϕ_1 as the positive function in the right-hand side of (1.1) is made for historical reasons but it is certainly not essential. We could in principle replace it, for instance, by any positive function ϕ, where now concentration will take place around local maxima of the function $(-\Delta)^{-1}\phi$ in $H_0^1(\Omega)$.

On the other hand, we also remark that a similar result to Theorem 1 is valid for the problem

$$\begin{cases} \Delta u + \lambda u + e^u = s\phi_1 + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

provided that $\lambda < \lambda_1$. Note that $\nu = \lambda$, $\mu = +\infty$ in this case. The basic fact is that $\Delta + \lambda$ satisfies maximum principle. Green's function should consistently be replaced by the one associated to this operator.

2. Preliminaries and ansatz for the solution

In what remains of this paper we fix a set Λ as in the statement of Theorem 1. For notational simplicity we assume

$$\max_{x \in \overline{\Lambda}} \phi_1(x) = 1.$$

What we will do next is to construct a reasonably good approximation U to a solution of (1.6) which will have as parameters yet to be adjusted, points ξ_i where the spikes are meant to take place. As we will see, a convenient set to select $\xi = (\xi_1, \dots, \xi_m)$ is

$$\mathcal{O}_s \equiv \left\{ \xi \in \overline{\Lambda}^m\colon\ 1 - \phi_1(\xi_j) \leqslant \frac{1}{\sqrt{s}},\ \forall j = 1, \dots, m,\ \text{and}\ \min_{i \neq j} |\xi_i - \xi_j| \geqslant \frac{1}{s^\beta} \right\}, \tag{2.1}$$

where the number $\beta > 1$ will be specified later. We thus fix $\xi \in \mathcal{O}_s$.

For numbers $\mu_j > 0$, $j = 1, \dots, m$, yet to be chosen, we define

$$u_j(x) = u_{j,s}(x) = \log \frac{8\mu_j^2\delta_j^2}{\left(\mu_j^2\delta_j^2 + |x - \xi_j|^2\right)^2} + s\phi_1(\xi_j) - \log k(\xi_j), \tag{2.2}$$

so that u_j solves

$$\Delta u + k(\xi_j)\delta_j^2 e^u = 0 \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} k(\xi_j)\delta_j^2 e^u = 8\pi, \tag{2.3}$$

where, since we are approximating a solution to (1.6), we naturally choose

$$\delta_j = \delta_j(s) \equiv \exp\left\{-\frac{s}{2}\phi_1(\xi_j)\right\}. \tag{2.4}$$

Note that u_j is not zero on the boundary of Ω, so that we add to it a harmonic correction so that boundary condition is satisfied. Let $H_j(x)$ be the solution of

$$\begin{cases} \Delta H_j = 0 & \text{in } \Omega, \\ H_j = -u_j & \text{on } \partial\Omega. \end{cases}$$

We define our first approximation $U(\xi)$ as

$$U(\xi) \equiv \sum_{j=1}^m U_j, \quad U_j \equiv u_j + H_j. \tag{2.5}$$

As we will see precisely below, $u_j + H_j \sim G(x, \xi_j)$, where $G(x, \xi)$ is the Green function defined in (1.5). Let us consider $H(x, \xi)$, its *regular part*, namely, the solution of

$$\begin{cases} -\Delta_x H(x, \xi) = 0, & x \in \Omega, \\ H(x, y) = \Gamma(x - y) = -4\log \frac{1}{|x-y|}, & x \in \partial\Omega, \end{cases} \tag{2.6}$$

so that

$$G(x, y) = H(x, y) - \Gamma(x - y).$$

While u_j is a good approximation to a solution of (1.6) near ξ_j, it is not so much the case for U, namely,

$$U = u_j + \left(H_j + \sum_{k \neq j} u_k\right),$$

unless the remainder vanishes at main order near ξ_j. This is achieved through the following precise choice of the parameters μ_k:

$$\log 8\mu_k^2 = \log k(\xi_j) + H(\xi_k, \xi_k) + \sum_{i \neq k} G(\xi_i, \xi_k). \tag{2.7}$$

Let us observe, in particular, that since $\xi \in \mathcal{O}_s$,

$$\frac{1}{C} \leqslant \mu_k \leqslant C s^{2\beta} \quad \text{for all } k = 1, \ldots, m \text{ and some } C > 0. \tag{2.8}$$

The following lemma expands U_j in Ω.

Lemma 2.1. *Assume $\xi \in \mathcal{O}_s$. Then we have*

$$H_j(x) = H(x, \xi_j) - \log 8\mu_j^2 + \log k(\xi_j) + O\big(\mu_j^2 \delta_j^2\big), \tag{2.9}$$

uniformly in Ω, and

$$u_j(x) = \log 8\mu_j^2 - \log k(\xi_j) - \Gamma(x - \xi_j) + O\big(\mu_j^2 s^{2\beta} \delta_j^2\big), \tag{2.10}$$

uniformly in the region $|x - \xi_j| \geqslant \frac{1}{2s^\beta}$, so that there,

$$U_j(x) = G(x, \xi_j) + O\big(\mu_j^2 s^{2\beta} \delta_j^2\big). \tag{2.11}$$

Proof. Let us prove (2.9). Define $z(x) = H_j(x) + \log 8\mu_j^2 - \log k(\xi_j) - H(x, \xi_j)$. Since z is harmonic we have

$$\begin{aligned} \max_{\overline{\Omega}} |z| &= \max_{\partial\Omega} \big| -u_j + \log 8\mu_j^2 - \log k(\xi_j) - \Gamma(\cdot - \xi_j) \big| \\ &= \max_{x \in \partial\Omega} \left| \log \frac{1}{|x - \xi_j|^4} - \log \frac{1}{(\mu_j^2 \delta_j^2 + |x - \xi_j|^2)^2} \right| = O\big(\mu_j^2 \delta_j^2\big), \end{aligned}$$

uniformly in Ω, as $s \to \infty$. Expansion (2.10) is directly obtained by definition of u_j and μ_j. □

Now, let us write

$$\delta = \delta(s) = e^{-s/2}, \qquad \Omega_s = \delta^{-1}\Omega, \qquad \xi_j = \delta \xi_j'. \tag{2.12}$$

Then u solves (1.6) if and only if $v(y) \equiv u(\delta y) - 2s$ satisfies

$$\begin{cases} \Delta v + q(y, s) e^v = 0 & \text{in } \Omega_s, \\ v(y) = -2s, & y \in \partial\Omega_s, \end{cases} \tag{2.13}$$

where

$$q(y, s) \equiv k(\delta y) \exp\big\{-s\big(\phi_1(\delta y) - 1\big)\big\}.$$

Let us define $V(y) = U(\delta y) - 2s$, with U our approximate solution (2.5). We want to measure the size of the error of approximation

$$R \equiv \Delta V + q(y, s) e^V. \tag{2.14}$$

It is convenient to do so in terms of the following norm:

$$\|v\|_* = \sup_{y\in\Omega_s}\left|\left[\sum_{j=1}^{m}\frac{\gamma_j}{(\gamma_j^2+|y-\xi_j'|^2)^{3/2}}+\delta^2\right]^{-1}v(y)\right|, \tag{2.15}$$

where

$$\gamma_j = \mu_j\delta_j\delta^{-1} = \mu_j\exp\left\{\frac{s}{2}\big(1-\phi_1(\xi_j)\big)\right\}. \tag{2.16}$$

Important facts in the analysis below are the estimates

$$\frac{1}{C}\leqslant\gamma_j\leqslant Cs^{2\beta}e^{\sqrt{s}/2},\quad (\delta\gamma_j)\leqslant Cs^{2\beta}e^{-s/4}. \tag{2.17}$$

Here and in what follows, C denotes a generic constant independent of s or $\xi\in\mathcal{O}_s$.

Lemma 2.2. *The error R in* (2.14) *satisfies*

$$\|R\|_*\leqslant Cs^{2\beta+1}e^{-s/4}\quad \textit{as } s\to\infty.$$

Proof. We assume first $|y-\xi_k'|\leqslant\frac{1}{2s^\beta\delta}$, for some index k. We have

$$\Delta V(y) = -\delta^2\sum_{j=1}^{m}k(\xi_j)e^{-s\phi_1(\xi_j)}e^{u_j(\delta y)} = -\sum_{j=1}^{m}\frac{8\gamma_j^2}{(\gamma_j^2+|y-\xi_j'|^2)^2}$$
$$= -\frac{8\gamma_k^2}{(\gamma_k^2+|y-\xi_k'|^2)^2}+\sum_{j\neq k}O\big(\mu_j^2s^{4\beta}\delta^2\delta_j^2\big).$$

Let us estimate $q(y,s)e^V(y)$. By (2.9) and the definition of μ_j's,

$$H_k(x) = H(\xi_k,\xi_k)-\log 8\mu_k^2+\log k(\xi_j)+O\big(\mu_k^2\delta_k^2\big)+O\big(|x-\xi_k|\big)$$
$$= -\sum_{j\neq k}G(\xi_j,\xi_k)+O\big(\mu_k^2\delta_k^2\big)+O\big(|x-\xi_k|\big),$$

and if $j\neq k$, by (2.11)

$$U_j(x) = u_j(x)+H_j(x) = G(\xi_j,\xi_k)+O\big(|x-\xi_k|\big)+O\big(\mu_j^2s^{2\beta}\delta_j^2\big).$$

Then

$$H_k(x)+\sum_{j\neq k}U_j(x) = \sum_j O\big(\mu_j^2s^{2\beta}\delta_j^2\big)+O\big(|x-\xi_k|\big). \tag{2.18}$$

Therefore,

$$\begin{aligned} q(y,s)e^{V(y)} &= q(y,s)\delta^4 \exp\Big\{u_k(\delta y) + H_k(\delta y) + \sum_{j\neq k} U_j(\delta y)\Big\} \\ &= \frac{8\mu_k^2 q(y,s)}{(\gamma_k^2 + |y-\xi_k'|^2)^2 k(\xi_k)}\Big\{1 + \sum_{j\neq k} O\big(\mu_j^2 s^{2\beta}\delta_j^2\big) + O\big(\delta|y-\xi_k'|\big)\Big\} \\ &= \frac{8\gamma_k^2}{(\gamma_k^2 + |y-\xi_k'|^2)^2}\big\{1 + O\big(s\delta|y-\xi_k'|\big)\big\}. \end{aligned}$$

We can conclude that in this region

$$\big|R(y)\big| \leqslant C(m,\Omega)\frac{s\gamma_k^2\delta|y-\xi_k'|}{(\gamma_k^2 + |y-\xi_k'|^2)^2} + \sum_{j\neq k} O\big(\mu_j^2 s^{4\beta}\delta^2\delta_j^2\big).$$

If $|y-\xi_j'| > \frac{1}{2s^\beta\delta}$ for all j, using (2.9)–(2.11) we obtain

$$\Delta V = \sum_j O\big(\mu_j^2 s^{4\beta}\delta^2\delta_j^2\big) \quad \text{and}$$

$$q(y,s)e^{V(y)} = O\left(\delta^4 \exp\Big\{-\sum_{j=1}^m \Gamma(x-\xi_j)\Big\}\right) = O\big(s^{4m(m-1)\beta}\delta^4\big).$$

Hence,

$$R(y) = \sum_j O\big(s^K\delta^2\delta_j^2\big)$$

for some $K > 0$ so that, finally,

$$\|R\|_* = \sum_k O(s\gamma_k\delta)$$

and by estimate (2.17) the proof is concluded. □

Next consider the energy functional associated with (1.6)

$$J_s[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega k(x)e^{-s\phi_1}e^u. \tag{2.19}$$

We will give an asymptotic estimate of $J_s[U]$, where $U(\xi)$ is the approximation (2.5). The choice of parameters μ_j as in (2.7) and computations essentially contained in [15] show that the following expansion holds.

Lemma 2.3. *With the election of μ_j's given by* (2.7),

$$J_s[U] = 16\pi \sum_{i\neq j} \log|\xi_i - \xi_j| + 8\pi s \sum_{j=1}^{m} \phi_1(\xi_j) + O(1), \tag{2.20}$$

where $O(1)$ is uniform in $\xi \in \mathcal{O}_s$.

In the subsequent analysis we will stay in the expanded variable $y \in \Omega_s$ so that we will look for solutions of problem (2.13) in the form $v = V + \psi$, where ψ will represent a lower order correction. In terms of ψ, problem (2.13) now reads

$$\begin{cases} \mathcal{L}(\psi) \equiv \Delta\psi + W\psi = -[R + N(\psi)] & \text{in } \Omega_s, \\ \psi = 0 & \text{on } \partial\Omega_s, \end{cases} \tag{2.21}$$

where

$$N(\psi) = W\big[e^{\psi} - 1 - \psi\big] \quad \text{and} \quad W = q(y,s)e^{V}.$$

Note that

$$W(y) = \sum_{j=1}^{m} \frac{8\gamma_j^2}{(\gamma_j^2 + |y - \xi_j'|^2)^2}\big(1 + O\big(s\delta\big|y - \xi_j'\big|\big)\big) \quad \text{for } y \in \Omega_s,$$

which can be written in the following way:

Lemma 2.4. *For $y \in \Omega_s$ and $\xi \in \mathcal{O}_s$, $W(y) = O(\delta^2 \sum_{j=1}^{m} \delta_j^2 e^{u_j(\delta y)})$, and then $\|W\|_* = O(1)$.*

3. The linearized problem

In this section we develop a solvability theory for the linear operator $\mathcal{L}$ defined in (2.21) under suitable orthogonality constrains. We consider

$$\mathcal{L}(\psi) \equiv \Delta\psi + W(y)\psi, \tag{3.1}$$

where $W(y)$ was introduced in (2.21). By Lemma 2.4 the operator $\mathcal{L}$ resembles

$$\mathcal{L}_0(\psi) \equiv \Delta\psi + \left(\delta^2 \sum_{j=1}^{m} \delta_j^2 e^{u_j}\right)\psi, \tag{3.2}$$

which is a essentially a superposition of linear operators which, after translations and dilations, approach as $s \to \infty$ the operator in $\mathbb{R}^2$

$$\mathcal{L}_*(\psi) \equiv \Delta\psi + \frac{8}{(1+|z|^2)^2}\psi, \tag{3.3}$$

namely, equation $\Delta v + e^v = 0$ linearized around the radial solution $v(y) = \log \frac{8}{(1+|y|^2)^2}$. The key fact to develop a satisfactory solvability theory for the operator $\mathcal{L}$ is the nondegeneracy of v up to the natural invariances of the equation under translations and dilations. In fact, if we set

$$Z_0(z) = \frac{|z|^2 - 1}{|z|^2 + 1}, \tag{3.4}$$

$$Z_i(z) = \frac{4z_i}{1 + |z|^2}, \quad i = 1, 2, \tag{3.5}$$

the only bounded solutions of $\mathcal{L}_*(\psi) = 0$ in $\mathbb{R}^2$ are linear combinations of Z_i, $i = 0, 1, 2$; see [3] for a proof.

We define for $i = 0, 1, 2$ and $j = 1, \dots, m$,

$$Z_{ij}(y) \equiv \frac{1}{\gamma_j} Z_i\left(\frac{y - \xi'_j}{\gamma_j}\right), \quad i = 0, 1, 2.$$

Additionally, let us consider R_0 a large but fixed number and χ a radial and smooth cut-off function with $\chi \equiv 1$ in $B(0, R_0)$ and $\chi \equiv 0$ in $B(0, R_0 + 1)^c$. Let

$$\chi_j(y) = \chi\left(\gamma_j^{-1}\left|y - \xi'_j\right|\right), \quad j = 1, \dots, m.$$

Given $h \in L^\infty(\Omega_s)$, we consider the problem of finding a function ψ such that for certain scalars c_{ij} one has

$$\begin{cases} \mathcal{L}(\psi) = h + \sum_{i=1}^2 \sum_{j=1}^m c_{ij}\chi_j Z_{ij} & \text{in } \Omega_s, \\ \psi = 0 & \text{on } \partial\Omega_s, \\ \int_{\Omega_s} \chi_j(y) Z_{ij}\psi = 0 & \text{for all } i = 1, 2,\ j = 1, \dots, m. \end{cases} \tag{3.6}$$

Proposition 3.1. *There exist positive constants $s_0 > 0$ and $C > 0$ such that for any $h \in L^\infty(\Omega_s)$ and any $\xi \in \mathcal{O}_s$, there is a unique solution $\psi = T(h)$ to problem* (3.6) *for all $s > s_0$, which defines a linear operator of h. Besides, we have the estimate*

$$\left\|T(h)\right\|_\infty \leqslant Cs\|h\|_*. \tag{3.7}$$

The proof will be split into a series of lemmas which we state and prove next.

Lemma 3.1. *The operator $\mathcal{L}$ satisfies the maximum principle in $\Omega_R \equiv \Omega_s \setminus \bigcup_{j=1}^m B(\xi'_j, R\gamma_j)$, for R large but independent of s. Namely, if $\mathcal{L}(\psi) \leqslant 0$ in Ω_R and $\psi \geqslant 0$ on $\partial\Omega_R$, then $\psi \geqslant 0$ in Ω_R.*

Proof. Notice that for s sufficiently large, $\gamma_j \leqslant \delta^{-1}$, for all j. This ensures that Ω_R is well defined. Now, it is sufficient to find a smooth function $f(y)$ such that $f > 0$ in $\overline{\Omega}_R$ and $\mathcal{L}(f) \leqslant 0$ in $\overline{\Omega}_R$.

For this purpose, we use the following lemma, whose proof is contained in [32].

Lemma 3.2. *There exist constants $R_1 > 0$, $C > 0$ such that for any $s > 0$ large enough, there exists $f : \Omega_{R_1} \to [1, \infty)$ smooth and positive verifying*

$$\mathcal{L}(f) \leqslant -\sum_{j=1}^{m} \frac{\gamma_j}{|y - \xi_j'|^3} - \delta^2$$

in Ω_{R_1}, and $1 < f \leqslant C$ uniformly in Ω_{R_1}.

We briefly recall the argument: we consider numbers R_1, s large enough and define

$$\frac{1}{C}\alpha^2 f(y) = f_0(\delta y) - \sum_{j=1}^{m} \frac{\gamma_j^\alpha}{|y - \xi_j'|^\alpha},$$

with f_0 the solution of $-\Delta f_0 = 1$ in Ω, $f_0 = 2$ on $\partial\Omega$ and $\alpha \in (0, 1)$. It is directly checked that f verifies the required conditions. □

Let us consider now the *inner norm*

$$\|\psi\|_i \equiv \sup_{\Omega_R^c} |\psi|,$$

where we understand $\Omega_R^c \equiv \Omega_s \backslash \Omega_R = \bigcup_{j=1}^m B(\xi_j', R\gamma_j)$.

Lemma 3.3. *There exists a constant $C = C(R, m) > 0$ such that if $\mathcal{L}(\psi) = h$ in Ω_s, $\psi = 0$ on $\partial\Omega_s$, $h \in L^\infty(\Omega_s)$, and s is sufficiently large, we have*

$$\|\psi\|_\infty \leqslant C\big\{\|\psi\|_i + \|h\|_*\big\}. \tag{3.8}$$

Proof. We will establish this estimate with the aid of Lemmas 3.1 and 3.2. We let f be the function defined in the latter result. We consider the function

$$\hat{\psi} = \big(\|\psi\|_i + \|h\|_*\big) f,$$

and claim that $\hat{\psi} \geqslant |\psi|$ on $\partial\Omega_R$ if R is sufficiently large. In fact, if $y \in \partial\Omega_s$, by the positivity of f, we have

$$\hat{\psi}(y) \geqslant 0 = \big|\psi(y)\big|.$$

On the other hand, if $|y - \xi_k'| = R\gamma_k$ for some $k = 1, \dots, m$,

$$\hat{\psi}(y) \geqslant \|\psi\|_i f \geqslant \|\psi\|_i \geqslant \big|\psi(y)\big| \quad \text{for } \big|y - \xi_k'\big| = R\gamma_k,\ k = 1, \dots, m.$$

Finally, using that

$$\left|h(y)\right| \leqslant \left(\sum_{j=1}^{m} \frac{\gamma_j}{(\gamma_j^2 + |y - \xi_j'|^2)^{3/2}} + \delta^2\right) \|h\|_*,$$

we have for $y \in \Omega_R$,

$$\begin{aligned}\mathcal{L}(\hat{\psi})(y) &\leqslant \left(\|\psi\|_i + \|h\|_*\right)\mathcal{L}(f) \leqslant -\|h\|_* \left\{\sum_{j=1}^{m} \frac{\gamma_j}{|y - \xi_j'|^3} + \delta^2\right\} \\ &\leqslant -\|h\|_* \left\{\sum_{j=1}^{m} \frac{\gamma_j}{(\gamma_j^2 + |y - \xi_j'|^2)^{3/2}} + \delta^2\right\} \leqslant -\left|h(y)\right| \leqslant -\left|\mathcal{L}(\psi)(y)\right|\end{aligned}$$

provided R large. In particular, we have $\mathcal{L}(\hat{\psi}) \leqslant -\mathcal{L}(\psi)$ and $\mathcal{L}(\hat{\psi}) \leqslant \mathcal{L}(\psi)$, in Ω_R. Hence, by Maximum Principle in Lemma 3.1 we have $|\psi(y)| \leqslant \hat{\psi}(y)$, for $y \in \Omega_R$. From this we obtain

$$\|\psi\|_\infty \leqslant \|\hat{\psi}\|_\infty \leqslant C\left\{\|\psi\|_i + \|h\|_*\right\}$$

as desired. □

The next step is to obtain a priori estimates for the problem

$$\begin{cases} \mathcal{L}(\psi) = h & \text{in } \Omega_s, \\ \psi = 0 & \text{on } \partial\Omega_s, \\ \int_{\Omega_s} \chi_j Z_{ij} \psi = 0 & \text{for all } i = 0, 1, 2, \ \ j = 1, \ldots, m. \end{cases} \tag{3.9}$$

which involves more orthogonality conditions than those in (3.6). We have the following estimate.

Lemma 3.4. *Let ψ be a solution of problem* (3.9) *with $\xi \in \mathcal{O}_s$. Then, there exists a $C > 0$ such that*

$$\|\psi\|_\infty \leqslant C\|h\|_* \tag{3.10}$$

for all $s > 0$ sufficiently large.

Proof. We carry out the proof by a contradiction argument. If the result was false, then, there would exist a sequence $s_n \to \infty$, points $\xi^n \in \mathcal{O}_{s_n}$, functions h_n with $\|h_n\|_* \to 0$ and associated solutions ψ_n with $\|\psi_n\|_\infty = 1$ such that

$$\begin{cases} \mathcal{L}(\psi_n) = h_n & \text{in } \Omega_{s_n}, \\ \psi_n = 0 & \text{on } \partial\Omega_{s_n}, \\ \int_{\Omega_{s_n}} \chi_j Z_{ij} \psi_n = 0 & \text{for all } i = 0, 1, 2, \ \ j = 1, \ldots, m. \end{cases} \tag{3.11}$$

By virtue of Lemma 3.3 and $\|\psi_n\|_\infty = 1$ we have $\liminf_{n\to\infty} \|\psi_n\|_i \geqslant \alpha > 0$. Let us set $\hat{\psi}_n(z) = \psi_n((\xi_j')^n + \gamma_j^n z)$, where the index $j = j(n)$ is such that $\sup_{B(\xi_j'^n, R\gamma_j)} |\psi_n| \geqslant \alpha$, and can be assumed to be the same for all n. We notice that $\hat{\psi}_n$ satisfies

$$\Delta\hat{\psi}_n + \left(\gamma_j^n\right)^2 W\hat{\psi}_n = \left(\gamma_j^n\right)^2 h_n \quad \text{in } \Omega_n \equiv \gamma_j^{-1}\left(\Omega_s - \left(\xi_j'\right)^n\right).$$

Elliptic estimates allow us to assume that $\hat{\psi}_n$ converges uniformly over compact subsets of $\mathbb{R}^2$ to a bounded, nonzero solution $\hat{\psi}$ of

$$\Delta\psi + \frac{8}{(1+|z|^2)^2}\psi = 0.$$

This implies that $\hat{\psi}$ is a linear combination of the functions Z_i, $i = 0, 1, 2$, namely, $\hat{\psi} = \sum_{k=0}^{2} \alpha_k Z_k$. But orthogonality conditions over $\hat{\psi}_n$ pass to the limit thanks to $\|\hat{\psi}_n\|_\infty \leqslant 1$. Dominated convergence then yields

$$0 = \int_{\Omega_{s_n}} \chi_j Z_{ij}\psi_n = \int_{\mathbb{R}^2} \chi Z_i \hat{\psi}_n + o(1) = \sum_{k=0}^{2} \alpha_k \int_{\mathbb{R}^2} \chi Z_i Z_k + o(1), \quad i = 0, 1, 2.$$

But $\int_{\mathbb{R}^2} \chi Z_i Z_k = 0$ for $i \neq k$ and $\int_{\mathbb{R}^2} \chi Z_i^2 > 0$. Then $\alpha_k = 0$ for all $k = 0, 1, 2$ and hence $\hat{\psi} \equiv 0$, a contradiction with $\liminf_{n\to\infty} \|\psi_n\|_i > 0$. □

Now we will deal with problem (3.9) lifting the orthogonality constraints $\int_{\Omega_s} \chi_j Z_{0j}\psi = 0$, $j = 1, \ldots, m$, namely,

$$\begin{cases} \mathcal{L}(\psi) = h & \text{in } \Omega_s, \\ \psi = 0 & \text{on } \partial\Omega_s, \\ \int_{\Omega_s} \chi_j Z_{ij}\psi = 0 & \text{for all } i = 1, 2, \ j = 1, \ldots, m. \end{cases} \tag{3.12}$$

We have the following a priori estimates for this problem.

Lemma 3.5. *Let ψ be a solution of* (3.12) *with $\xi \in \mathcal{O}_s$. Then, there exists a $C > 0$ such that*

$$\|\psi\|_\infty \leqslant Cs\|h\|_* \tag{3.13}$$

for all s sufficiently large.

Proof. Let $R > R_0 + 1$ be a large and fixed number. Let us consider the function

$$\widehat{Z}_{0j} = Z_{0j}(y) - \frac{1}{\gamma_j} + a_{0j} G(\delta y, \xi_j), \tag{3.14}$$

where

$$a_{0j} \equiv \frac{1}{\gamma_j\{H(\xi_j, \xi_j) - 4\log(\delta\gamma_j R)\}}. \tag{3.15}$$

From estimate (2.8), we have

$$C_1|\log\delta_j| \leqslant \log(\delta\gamma_j R) \leqslant C_2|\log\delta_j| \quad \text{and}$$
$$\widehat{Z}_{0j}(y) = O\left(\frac{G(\delta y, \xi_j)}{\gamma_j|\log\delta_j|}\right). \tag{3.16}$$

Next we consider radial smooth cut-off functions η_1 and η_2 with the following properties:

$$0 \leqslant \eta_1 \leqslant 1, \quad \eta_1 \equiv 1 \text{ in } B(0,R), \quad \eta_1 \equiv 0 \text{ in } B(0,R+1)^c; \quad \text{and}$$
$$0 \leqslant \eta_2 \leqslant 1, \quad \eta_2 \equiv 1 \text{ in } B(0,1), \quad \eta_2 \equiv 0 \text{ in } B\left(0,\frac{4}{3}\right)^c.$$

With no loss of generality we assume that $B(0,\frac{4}{3}) \subseteq \Omega$. Then we set

$$\eta_{1j}(y) = \eta_1\left(\frac{|y-\xi_j'|}{\gamma_j}\right), \qquad \eta_{2j}(y) = \eta_2\big(4\delta|y-\xi_j'|\big), \tag{3.17}$$

and define the test function

$$\widetilde{Z}_{0j} = \eta_{1j}Z_{0j} + (1-\eta_{1j})\eta_{2j}\widehat{Z}_{0j}.$$

Let ψ be a solution to problem (3.12). We will modify ψ so that the extra orthogonality conditions with respect to Z_{0j}'s hold. We set

$$\tilde{\psi} = \psi + \sum_{j=1}^m d_j\widetilde{Z}_{0j} + \sum_{i=1}^2\sum_{j=1}^m e_{ij}\chi_j Z_{ij}. \tag{3.18}$$

We adjust $\tilde{\psi}$ to satisfy the orthogonality condition

$$\int_{\Omega_s} \chi_j Z_{ij}\tilde{\psi} = 0 \quad \text{for all } i=0,1,2;\ j=1,\dots,m. \tag{3.19}$$

Then,

$$\mathcal{L}(\tilde{\psi}) = h + \sum_{j=1}^m d_j\mathcal{L}\big(\widetilde{Z}_{0j}\big) + \sum_{i=1}^2\sum_{j=1}^m e_{ij}\mathcal{L}(\chi_j Z_{ij}). \tag{3.20}$$

If (3.19) holds, the previous lemma allows us to conclude

$$\|\tilde{\psi}\|_\infty \leqslant C\left\{\|h\|_* + \sum_{j=1}^m |d_j|\big\|\mathcal{L}\big(\widetilde{Z}_{0j}\big)\big\|_* + \sum_{i=1}^2\sum_{j=1}^m |e_{ij}|\big\|\mathcal{L}\big(\chi_j\widetilde{Z}_{ij}\big)\big\|_*\right\}. \tag{3.21}$$

Estimate (3.13) is a direct consequence of the following two claims:

Claim 1. *The constants d_j and e_{ij} are well defined and*

$$\left\|\mathcal{L}(\chi_j Z_{ij})\right\|_* \leqslant \frac{C}{\gamma_j}, \qquad \left\|\mathcal{L}\big(\widetilde{Z}_{0j}\big)\right\|_* \leqslant \frac{C\log s}{\gamma_j|\log\delta_j|}, \quad i=1,2;\ j=1,\dots,m. \tag{3.22}$$

Claim 2. *The following bounds hold*:

$$|d_j| \leqslant C\gamma_j|\log\delta_j|\|h\|_*, \qquad |e_{ij}| \leqslant C\gamma_j \log s\|h\|_*, \quad i=1,2;\ j=1,\dots,m. \tag{3.23}$$

After these facts have been established, using that

$$\|\widetilde{Z}_{0j}\|_\infty \leqslant \frac{C}{\gamma_j} \quad \text{and} \quad \|\chi_j Z_{ij}\|_\infty \leqslant \frac{C}{\gamma_j}$$

we obtain (3.13), as desired.

Let us prove now Claim 1. First we find d_j and e_{ij}. From definition (3.18), orthogonality conditions (3.19) and the fact that $\operatorname{supp}\chi_j\chi_k=\emptyset$ if $j\neq k$, we can write

$$e_{ij} = -\frac{\sum_{k=1}^m d_k\int_{\Omega_s}\chi_j Z_{ij}\widetilde{Z}_{0k}}{\int_{\Omega_s}\chi_j Z_{ij}^2}, \quad i=1,2;\ j=1,\dots,m. \tag{3.24}$$

Notice that $\int_{\Omega_s} Z_{ij}^2\chi_j^2 = c>0$, for all i, j, and

$$\int_{\Omega_s}\chi_j Z_{ij}\widetilde{Z}_{0l} = O\left(\frac{\gamma_j\log s}{\gamma_l|\log\delta_l|}\right), \quad j\neq l.$$

Then, from (3.24)

$$|e_{ij}| \leqslant C\sum_{l\neq j}|d_l|\frac{\gamma_j\log s}{\gamma_l|\log\delta_l|}. \tag{3.25}$$

We need to show that d_j is well defined. In fact, multiplying definition (3.18) by $Z_{0k}\chi_k$, integrating and using the orthogonality condition (3.19) for $i=0$, we get

$$\sum_{j=1}^m d_j\int_{\Omega_s}\chi_k Z_{0k}\widetilde{Z}_{0j} = -\int_{\Omega_s}\chi_k Z_{0k}\psi, \quad \forall k=1,\dots,m. \tag{3.26}$$

But $\int_{\Omega_s}\chi_k Z_{0k}\widetilde{Z}_{0k} = \int_{\Omega_s}\chi_k Z_{0k}^2 = C$ for all k, and

$$\int_{\Omega_s}\chi_k Z_{0k}\widetilde{Z}_{0j} = O\left(\frac{\gamma_k\log s}{\gamma_j|\log\delta_j|}\right),$$

if $k \neq j$. Then, if we define

$$m_{kj} = \int_{\Omega_s} \chi_k Z_{0k} \widetilde{Z}_{0j} \quad \text{and} \quad f_k = -\int_{\Omega_s} \chi_k Z_{0k} \psi,$$

system (3.26) can be written as

$$\sum_{j=1}^{m} m_{kj} d_j = f_k, \quad k = 1, \ldots, m.$$

But the matrix with coefficients $\gamma_j m_{kj} \gamma_k^{-1}$ is clearly diagonal-dominant, thus invertible, so the matrix m_{kj} is also invertible. Thus d_k is well defined.

Let us prove inequalities (3.22). We note that in the region $|y - \xi_j'| \leqslant (R+1)\gamma_j$,

$$\begin{aligned}\mathcal{L}(\chi_j Z_{ij}) &= \chi_j \mathcal{L}(Z_{ij}) + \Delta\chi_j Z_{ij} + 2\nabla\chi_j \cdot \nabla Z_{ij} \\ &= O\left(\frac{s\delta_j \gamma_j^2}{(\gamma_j^2 + |y - \xi_j'|^2)^2}\right) + O\left(\frac{\gamma_j^{-1}}{(\gamma_j^2 + |y - \xi_j'|^2)^{1/2}}\right) + O\left(\frac{1}{(\gamma_j^2 + |y - \xi_j'|^2)^{3/2}}\right),\end{aligned}$$

and then $\|\mathcal{L}(\chi_j Z_{ij})\|_* = O(\gamma_j^{-1})$. We prove now the second inequality in (3.22). In fact,

$$\begin{aligned}\mathcal{L}(\widetilde{Z}_{0j}) &= \Delta\eta_{1j}(Z_{0j} - \widehat{Z}_{0j}) + 2\nabla\eta_{1j} \cdot \nabla(Z_{0j} - \widehat{Z}_{0j}) + 2\nabla\eta_{2j} \cdot \nabla\widehat{Z}_{0j} + \Delta\eta_{2j}\widehat{Z}_{0j} \\ &\quad + \eta_{1j}\{\mathcal{L}(Z_{0j}) - \mathcal{L}(\widehat{Z}_{0j})\} + \eta_{2j}\mathcal{L}(\widehat{Z}_{0j}).\end{aligned}$$

Now we consider the four regions

$$\Omega_1 \equiv \{|y - \xi_j'| \leqslant \gamma_j R\}, \qquad \Omega_2 \equiv \{\gamma_j R < |y - \xi_j'| \leqslant \gamma_j(R+1)\},$$
$$\Omega_3 \equiv \left\{\gamma_j(R+1) < |y - \xi_j'| \leqslant \frac{1}{4\delta}\right\} \quad \text{and} \quad \Omega_4 \equiv \left\{\frac{1}{4\delta} < |y - \xi_j'| \leqslant \frac{1}{3\delta}\right\}.$$

Notice that (2.19) and (3.3) imply

$$\eta_{1j}\{\mathcal{L}(Z_{0j}) - \mathcal{L}(\widehat{Z}_{0j})\} + \eta_{2j}\mathcal{L}(\widehat{Z}_{0j}) = O\left(\frac{s\delta\gamma_j^2}{(\gamma_j^2 + |y - \xi_j'|^2)^{3/2}}\right) \tag{3.27}$$

for all $y \in \Omega_1 \cup \Omega_2$. Lut us now analyze $\mathcal{L}(\widetilde{Z}_{0j})$ in each Ω_i. In Ω_1,

$$\mathcal{L}(\widetilde{Z}_{0j}) = O\left(\frac{s\delta\gamma_j^2}{(\gamma_j^2 + |y - \xi_j'|^2)^{3/2}}\right). \tag{3.28}$$

In Ω_2,

$$Z_{0j} - \widehat{Z}_{0j} = \frac{1}{\gamma_j} - a_{0j} G(\delta y, \chi_j) = -a_{0j}\left\{4\log\frac{\gamma_j R}{|y - \xi_j'|} + O(\delta\gamma_j)\right\}, \tag{3.29}$$

hence we conclude

$$\left|Z_{0j}-\widehat{Z}_{0j}\right|=O\left(\frac{1}{\gamma_j|\log\delta_j|}\right)\quad\text{and}\quad\left|\nabla\left(Z_{0j}-\widehat{Z}_{0j}\right)\right|=O\left(\frac{1}{\gamma_j^2|\log\delta_j|}\right),\tag{3.30}$$

and then

$$\mathcal{L}\left(\widetilde{Z}_{0j}\right)=O\left(\frac{1}{\gamma_j^2|\log\delta_j|}\right).\tag{3.31}$$

In Ω_4, thanks to (3.16), $|\widehat{Z}_{0j}|=O(\frac{1}{\gamma_j|\log\delta_j|})$, $|\nabla\widehat{Z}_{0j}|=O(\frac{\delta}{\gamma_j|\log\delta_j|})$ and

$$\begin{aligned}\mathcal{L}\left(\widehat{Z}_{0j}\right)&=\Delta Z_{0j}+W\widehat{Z}_{0j}\\&=O\left(\frac{\gamma_j}{(\gamma_j^2+|y-\xi_j'|^2)^2}\right)+\sum_{k\neq j}O\left(\frac{1}{\gamma_j|\log\delta_j|}\frac{\gamma_k^2}{(\gamma_k^2+|y-\xi_k'|^2)^2}\right).\end{aligned}$$

Then, in this region

$$\left\|\mathcal{L}\left(\widetilde{Z}_{0j}\right)\right\|_*=O\left(\frac{1}{\gamma_j|\log\delta_j|}\right).\tag{3.32}$$

Finally, we consider $y\in\Omega_3$. We have

$$\begin{aligned}\mathcal{L}\left(\widetilde{Z}_{0j}\right)&=\mathcal{L}\left(\widehat{Z}_{0j}\right)\\&=\left\{W-\frac{8\gamma_j^2}{(\gamma_j^2+|y-\xi_j'|^2)^2}\right\}Z_{0j}+W\left\{a_{0j}G(\delta y,\xi_j)-\frac{1}{\gamma_j}\right\}\equiv A_1+A_2.\end{aligned}$$

To estimate these two terms, we need to split Ω_3 into several subregions. We let

$$\Omega_{3,j}\equiv\left\{\gamma_j(R+1)<\left|y-\xi_j'\right|\leqslant\frac{1}{2s^\beta\delta}\right\},\qquad\Omega_{3,k}\equiv\left\{y\in\Omega_3\;\middle|\;\left|y-\xi_k'\right|\leqslant\frac{1}{2s^\beta\delta}\right\},\quad k\neq j,$$

$$\text{and}\quad\widetilde{\Omega}_3\equiv\left\{y\in\Omega_3\;\middle|\;\left|y-\xi_l'\right|\geqslant\frac{1}{2s^\beta\delta},\ \forall l\right\}.$$

From Lemma 2.2, $A_1=O(\frac{s\gamma_j\delta}{(\gamma_j^2+|y-\xi_j'|^2)^{3/2}})$ in $\Omega_{3,j}$, and $A_1=O(s^K\delta^2\delta_j^2\gamma_j^{-1})$ in $\widetilde{\Omega}_3$.

If $y\in\Omega_{3,j}$,

$$\begin{aligned}A_2&=O\left(\frac{\gamma_j^2a_{0j}}{(\gamma_j^2+|y-\xi_j'|^2)^2}\left\{-\log\gamma_jR+\log\left|y-\xi_j'\right|+\delta\left|y-\xi_j'\right|\right\}\right)\\&=O\left(\frac{1}{|\log\delta_j|}\frac{1}{(\gamma_j^2+|y-\xi_j'|^2)^{3/2}}\right),\end{aligned}$$

and $A_2 = O(s^K \delta^2 \delta_j^2)$, for some large K. Finally we get, for all $y \in \Omega_{3,j} \cup \widetilde{\Omega}_3$,

$$\left|\mathcal{L}\big(\widetilde{Z}_{0j}\big)\right| = O\left(\frac{1}{|\log\delta_j|}\frac{1}{(\gamma_j^2 + |y-\xi_j'|^2)^{3/2}}\right). \tag{3.33}$$

In $\Omega_{3,k}$, $k \neq j$, we write

$$\begin{aligned}
\mathcal{L}\big(\widetilde{Z}_{0j}\big) = \Delta Z_{0j} + W\widehat{Z}_{0j} &= \frac{-8\gamma_j^2}{(\gamma_j^2 + |y-\xi_j'|^2)^2} Z_{0j} + W\widehat{Z}_{0j} \\
&= O\big(s^K \delta_j \delta^3\big) + O\left(\frac{\gamma_k^2}{(\gamma_k^2 + |y-\xi_k'|^2)^2}\frac{G(\delta y, \xi_j)}{\gamma_j|\log\delta_j|}\right) \\
&= O\left(\frac{\gamma_k^2}{(\gamma_k^2 + |y-\xi_k'|^2)^2}\frac{\log s}{\gamma_j|\log\delta_j|}\right),
\end{aligned}$$

and then, combining (3.27)–(3.33) and the previous estimate, we arrive at

$$\left\|\mathcal{L}\big(\widetilde{Z}_{0j}\big)\right\|_* = O\left(\frac{\log s}{\gamma_j|\log\delta_j|}\right).$$

Finally, we prove Claim 2. Testing Eq. (3.20) against $\widetilde{Z}_{0j}$ and using relations (3.21), (3.22), we get

$$\begin{aligned}
\sum_{k=1}^m d_k \int_{\Omega_s} \mathcal{L}\big(\widetilde{Z}_{0k}\big)\widetilde{Z}_{0j} &= -\int_{\Omega_s} h\widetilde{Z}_{0j} - \int_{\Omega_s} \tilde{\psi}\mathcal{L}\big(\widetilde{Z}_{0j}\big) + \sum_{l=1}^2\sum_{k=1}^m e_{lk}\int_{\Omega_s}\chi_k Z_{lk}\mathcal{L}\big(\widetilde{Z}_{0j}\big) \\
&\leqslant C\frac{\|h\|_*}{\gamma_j} + C\|\tilde{\psi}\|_\infty \left\|\mathcal{L}\big(\widetilde{Z}_{0j}\big)\right\|_* + C\sum_{l=1}^2\sum_{k=1}^m |e_{lk}|\frac{\|\mathcal{L}(\widetilde{Z}_{0j})\|_*}{\gamma_k} \\
&\leqslant C\|h\|_*\left\{\frac{1}{\gamma_j} + \left\|\mathcal{L}\big(\widetilde{Z}_{0j}\big)\right\|_*\right\} + C\sum_{k=1}^m |d_k|\left\|\mathcal{L}\big(\widetilde{Z}_{0k}\big)\right\|_*\left\|\mathcal{L}\big(\widetilde{Z}_{0j}\big)\right\|_* \\
&\quad + C\sum_{l=1}^2\sum_{k=1}^m |e_{lk}|\frac{\|\mathcal{L}(\widetilde{Z}_{0j})\|_*}{\gamma_k},
\end{aligned}$$

where we have used that

$$\int_{\Omega_s} \frac{\gamma_j}{(\gamma_j^2 + |y-\xi_j'|^2)^{3/2}} \leqslant C \quad \text{for all } j.$$

But estimate (3.25) and Claim 1 imply

$$|d_j| \int_{\Omega_s} \mathcal{L}(\widetilde{Z}_{0j})\widetilde{Z}_{0j} \leqslant C\frac{\|h\|_*}{\gamma_j} + C\sum_{k=1}^{m} \frac{|d_k|\log^2 s}{\gamma_j\gamma_k|\log\delta_j||\log\delta_k|} + C\sum_{k\neq j}|d_k|\left|\int_{\Omega_s} \mathcal{L}(\widetilde{Z}_{0j})\widetilde{Z}_{0k}\right|. \tag{3.34}$$

We only need to estimate the terms $\int_{\Omega_s} \mathcal{L}(\widetilde{Z}_{0j})\widetilde{Z}_{0k}$, for all k. We have the following claim.

Claim 3. *If R is sufficiently large,*

$$\int_{\Omega_s} \mathcal{L}(\widetilde{Z}_{0j})\widetilde{Z}_{0j} = \frac{E}{\gamma_j^2|\log\delta_j|}\big(1+o(1)\big), \tag{3.35}$$

where E is a positive constant independent of s and R. Besides, if $k \neq j$

$$\int_{\Omega_s} \mathcal{L}(\widetilde{Z}_{0j})\widetilde{Z}_{0k} = O\left(\frac{\log^2 s}{\gamma_j\gamma_k|\log\delta_j||\log\delta_k|}\right). \tag{3.36}$$

Assuming for the moment the validity of this claim, then replacing (3.35) and (3.36) in (3.34), we get

$$\frac{|d_j|}{\gamma_j} \leqslant C|\log\delta_j|\|h\|_* + C\sum_{k=1}^{m}\frac{|d_k|}{\gamma_k}\frac{\log^2 s}{|\log\delta_k|}, \tag{3.37}$$

and then,

$$|d_j| \leqslant C\gamma_j|\log\delta_j|\|h\|_*.$$

Finally, using estimate (3.25), we conclude

$$|e_{ij}| \leqslant C\gamma_j \log s\|h\|_*$$

and Claim 2 holds. Let us proof Claim 3. Let us try with the first term (3.35). We decompose

$$\begin{aligned}\int_{\Omega_s} \mathcal{L}(\widetilde{Z}_{0j})\widetilde{Z}_{0j} &= O(s\delta_j) + \int_{\Omega_2} \mathcal{L}(\widetilde{Z}_{0j})\widetilde{Z}_{0j} + \int_{\Omega_3} \mathcal{L}(\widetilde{Z}_{0j})\widetilde{Z}_{0j} + \int_{\Omega_4} \mathcal{L}(\widetilde{Z}_{0j})\widetilde{Z}_{0j} \\ &\equiv O(s\delta_j) + I_2 + I_3 + I_4.\end{aligned}$$

First we estimate I_3. From (3.33),

$$\begin{aligned}I_3 = \int_{\Omega_3} \mathcal{L}(\widehat{Z}_{0j})\widehat{Z}_{0j} &= \int_{\Omega_{3,j}\cup\widetilde{\Omega}_3} \mathcal{L}(\widehat{Z}_{0j})\widehat{Z}_{0j} + \sum_{k\neq j}\int_{\Omega_{3,k}} \mathcal{L}(\widehat{Z}_{0j})\widehat{Z}_{0j} \\ &= O\left(\frac{1}{R\gamma_j^2|\log\delta_j|}\right) + O\left(\frac{\log^2 s}{\gamma_j^2|\log\delta_j|^2}\right).\end{aligned}$$

Now we estimate I_4. From the estimates in Ω_4, $|I_4| = O(\frac{1}{\gamma_j^2|\log\delta_j|^2})$. On the other hand, we have

$$I_2 = \int_{\Omega_2} \{\Delta\eta_{1l}(Z_{0j} - \widehat{Z}_{0j}) + 2\nabla\eta_{1j}\cdot\nabla(Z_{0j} - \widehat{Z}_{0j})\}\widetilde{Z}_{0j} + O\left(\frac{s\delta}{\gamma_j R^2}\right).$$

Thus integrating by parts the first term above we find

$$\begin{aligned} I_2 &= \int_{\Omega_2} \nabla\eta_{1j}\cdot\nabla(Z_{0j} - \widehat{Z}_{0j})\widehat{Z}_{0j} - \int_{\Omega_2} |\nabla\eta_{1l}|^2(Z_{0j} - \widehat{Z}_{0j})^2 \\ &\quad - \int_{\Omega_2} \nabla\eta_{1j}\cdot\nabla\widehat{Z}_{0j}(Z_{0j} - \widehat{Z}_{0j}) + \int_{\Omega_2} \{\eta_{1j}\mathcal{L}(Z_{0j}) + (1-\eta_{1j})\mathcal{L}(\widehat{Z}_{0j})\} \\ &\equiv I_{2,a} + I_{2,b} + I_{2,c} + I_{2,d}. \end{aligned}$$

Using (3.29) and (3.16), we get $|\nabla\widehat{Z}_{0l}| = O(\frac{1}{R^3\gamma_j^2})$ in Ω_2,

$$I_{2,b} = O\left(\frac{R}{\gamma_j^2|\log\delta_j|^2}\right), \qquad I_{2,c} = O\left(\frac{1}{R^2\gamma_j^2|\log\delta_j|}\right) \quad \text{and} \quad I_{2,d} = O\left(\frac{\delta}{R^3\gamma_j^2|\log\delta_j|}\right).$$

Now, as $\widehat{Z}_{0j} = Z_{0j}(1 + O(\frac{\gamma_j\delta R}{|\log\delta_j|}))$, we conclude

$$I_{2,a} = \frac{1}{\gamma_j^2|\log\delta_j|}\int_R^{R+1} r\eta_1'(r)\left(\frac{1-r^2}{1+r^2}\right)(1+o(1))\,dr = \frac{E}{\gamma_j^2|\log\delta_j|}(1+o(1)),$$

where E is a positive constant independent of s and R. Thus, for fixed R large and s small, we obtain (3.35). The second result can be established with similar arguments. □

Now we can now treat the original linear problem (3.6).

Proof of Proposition 3.1. We first establish the validity of the a priori estimate (3.7) for solutions $\psi \in L^\infty(\Omega)$ of problem (3.6), with $h \in L^\infty(\Omega)$. Lemma 3.5 implies

$$\|\psi\|_\infty \leqslant Cs\left\{\|h\|_* + \sum_{i=1}^{2}\sum_{j=1}^{m}|c_{ij}|\|\chi_j Z_{ij}\|_*\right\}, \tag{3.38}$$

but

$$\|\chi_j Z_{ij}\|_* \leqslant C\gamma_j,$$

then, it is sufficient to estimate the values of the constants c_{ij}. To this end, we multiply the first equation in (3.6) by $Z_{ij}\eta_{2j}$, with η_{2j} the cut-off function introduced in (3.17), and integrate by parts to find

$$\int_{\Omega_s} \psi \mathcal{L}(Z_{ij}\eta_{2j}) = \int_{\Omega_s} h Z_{ij}\eta_{2j} + \sum_{k=1}^{2}\sum_{l=1}^{m} c_{kl} \int_{\Omega_s} \eta_{2j} Z_{ij}\chi_l Z_{kl}. \tag{3.39}$$

It is easy to see that $\int_{\Omega_s} h\eta_{2j}Z_{ij} = O(\gamma_j^{-1}\|h\|_*)$. On the other hand, we have

$$\begin{aligned}\mathcal{L}(\eta_{2j}Z_{ij}) &= \Delta\eta_{2j}Z_{ij} + 2\nabla\eta_{2j}\cdot\nabla Z_{ij} + \eta_{2j}\mathcal{L}(Z_{ij})\\ &= O(\delta^3) + \left\{W - \frac{8\gamma_j^2}{(\gamma_j^2 + |y-\xi_j'|^2)^2}\right\}\eta_{2j}Z_{ij} \equiv O(\delta^3) + B_j.\end{aligned}$$

To estimate B_j, we need to split $\operatorname{supp}\eta_{2j}$ into several pieces. We consider the following subdomains. For a fixed j, we let

$$\widehat{\Omega}_{1k} \equiv \left\{|y-\xi_k'| \leqslant \frac{1}{2s^\beta\delta}\right\},$$

for any $k = 1, \dots, m$, and

$$\widehat{\Omega}_2 \equiv \left\{|y-\xi_j'| \leqslant \frac{1}{3\delta},\ |y-\xi_k'| \geqslant \frac{1}{2s^\beta\delta},\ \forall k\right\}.$$

In $\widehat{\Omega}_{1j}$, using Lemma 2.2, $B_j = O(\frac{s\delta\gamma_j}{(\gamma_j^2+|y-\xi_j'|^2)^{3/2}})$. In $\widehat{\Omega}_{1k}$, $k \neq j$,

$$B_j = O\left(\frac{s^\beta\delta\gamma_k^2}{(\gamma_k^2 + |y-\xi_k'|^2)^2}\right).$$

Finally, in $\widehat{\Omega}_2$, $B_j = O(s^K\delta_j^2\delta^3)$, for some constant $K > 0$ large. Then,

$$\left|\int_{\Omega_s} \psi\mathcal{L}(\eta_{2j}Z_{ij})\right| \leqslant Cs^\beta\delta\|\psi\|_\infty.$$

Now,

$$\int_{\Omega_s} \eta_{2j}\chi_j Z_{ij}Z_{kl} = C\delta_{ik}$$

and if $j \neq l$, and s is sufficiently large,

$$\int_{\Omega_s} \eta_{2j}\chi_l Z_{ij}Z_{kl} = O(\gamma_l s^\beta\delta).$$

Using the above estimates in (3.39), we obtain

$$|c_{ij}| \leqslant Cs^{\beta}\delta\|\psi\|_{\infty} + \frac{C}{\gamma_j}\|h\|_{*} + C\sum_{k=1}^{2}\sum_{l\neq j}|c_{kl}|\gamma_l s^{\beta}\delta \tag{3.40}$$

and then

$$|c_{ij}| \leqslant Cs^{\beta}\delta\|\psi\|_{\infty} + \frac{C}{\gamma_j}\|h\|_{*}.$$

Putting this estimate in (3.38), we conclude the validity of (3.13).

Finally, the a priori estimate implies in particular that the homogeneous problem has only the trivial solution. A standard argument involving Fredholm's alternative, see, e.g., [15], gives existence. This concludes the proof. □

Remark 3.1. The operator T is differentiable with respect to the variables ξ'. In fact, computations similar to those used in [15] yield the estimate

$$\left\|\partial_{\xi'}T(h)\right\|_{\infty} \leqslant Cs^2\|h\|_{*} \quad \text{for all } l = 1,2;\ k = 1,\dots,m. \tag{3.41}$$

Important element in this computation is that $\frac{1}{\gamma_j} \leqslant C$, uniformly on s.

4. The intermediate nonlinear problem

In order to solve problem (2.21) we consider first the intermediate nonlinear problem:

$$\begin{cases} \mathcal{L}(\psi) = -[R + N(\psi)] + \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}\chi_j Z_{ij} & \text{in } \Omega_s,\\ \psi = 0 \quad \text{on } \partial\Omega_s,\\ \int_{\Omega_s}\chi_j Z_{ij}\psi = 0 \quad \text{for all } i = 1,2,\ j = 1,\dots,m. \end{cases} \tag{4.1}$$

For this problem we will prove:

Proposition 4.1. *Let $\xi \in \mathcal{O}_s$. Then, there exists $s_0 > 0$ and $C > 0$ such that for all $s \geqslant s_0$ the nonlinear problem* (4.1) *has a unique solution $\psi \in$ which satisfies*

$$\|\psi\|_{\infty} \leqslant Cs^{2\beta+1}e^{-s/4}. \tag{4.2}$$

Moreover, if we consider the map $\xi' \in \mathcal{O}_s \to \psi \in \mathcal{C}(\overline{\Omega}_s)$, the derivative $D_{\xi'}\psi$ exists and defines a continuous map of ξ'. Besides

$$\|D_{\xi'}\psi\|_{\infty} \leqslant Cs^{2\beta+2}e^{-s/4}. \tag{4.3}$$

Proof. In terms of the operator T defined in Proposition 3.1, problem (4.1) becomes

$$\psi = \mathcal{B}(\psi) \equiv -T\big(N(\psi) + R\big).$$

Let us consider the region

$$\mathcal{F} \equiv \{\psi \in \mathcal{C}(\overline{\Omega}_s) \mid \|\psi\|_\infty \leqslant s^{2\beta+1} e^{-s/4}\}.$$

From Proposition 3.1,

$$\|\mathcal{B}(\psi)\|_\infty \leqslant Cs\{\|N(\psi)\|_* + \|R\|_*\},$$

and Lemma 2.2 implies

$$\|R\|_* \leqslant Cs^{2\beta+1} e^{-s/4}.$$

Also, from the definition of N in (2.21), Mean-value theorem and Lemma 2.4 we obtain

$$\|N(\psi)\|_* \leqslant \|W\|_* \|\psi\|_\infty^2 \leqslant C\|\psi\|_\infty^2.$$

Hence, if $\psi \in \mathcal{F}_\gamma$, $\|\mathcal{B}(\psi)\|_\infty \leqslant Cs^{2\beta+2} e^{-s/4}$. Along the same way we obtain $\|N(\psi_1) - N(\psi_2)\|_* \leqslant C \max_{i=1,2} \|\psi_i\|_\infty \|\psi_1 - \psi_2\|_\infty$, for any $\psi_1, \psi_2 \in \mathcal{F}_\gamma$. Then, we conclude

$$\|\mathcal{B}(\psi_1) - \mathcal{B}(\psi_2)\|_\infty \leqslant Cs\|N(\psi_1) - N(\psi_2)\|_* \leqslant Cs^{2\beta+2} e^{-s/4} \|\psi_1 - \psi_2\|_\infty.$$

It follows that for all s sufficiently large $\mathcal{B}$ is a contraction mapping of $\mathcal{F}_\gamma$, and therefore a unique fixed point of $\mathcal{B}$ exists in this region. The proof of (4.3) is similar to one included in [15] and we thus omit it. □

5. Variational reduction

We have solved the nonlinear problem (4.1). In order to find a solution to the original problem (2.21) we need to find ξ such that

$$c_{ij} = c_{ij}(\xi') = 0, \quad \text{for all } i, j, \tag{5.1}$$

where $c_{ij}(\xi')$ are the constants in (4.1). Problem (5.1) is indeed variational: it is equivalent to finding critical points of a function of ξ'. In fact, we define the functional for $\xi \in \mathcal{O}_s$:

$$\mathcal{F}(\xi) \equiv J_s[U(\xi) + \hat{\psi}_\xi], \tag{5.2}$$

where $U(\xi)$ is our approximate solution from (2.5) and $\hat{\psi}_\xi = \psi(\frac{x}{\delta}, \frac{\xi}{\delta})$, $x \in \Omega$, with $\psi = \psi_{\xi'}$ the unique solution to problem (4.1) given by Proposition 4.1. Then we obtain that critical points of $\mathcal{F}$ correspond to solutions of (5.1) for large s. That is:

Lemma 5.1. *$\mathcal{F} : \mathcal{O}_s \to \mathbb{R}$ is of class $\mathcal{C}^1$. Moreover, for all s sufficiently large, if $D_\xi \mathcal{F}(\xi) = 0$ then ξ satisfies* (5.1).

Proof. The proof of this fact is standard, see [15,17] or [32]. Here the estimate found for $D_{\xi'}\psi$ is used. □

The estimates for the solution $\psi_{\xi'}$ for problem (4.1) in Proposition 4.1 and a Taylor expansion of $\mathcal{F}$ in the expanded domain Ω_s similar to one done in [15] give us the following lemma.

Lemma 5.2. *For points $\xi \in \mathcal{O}_s$ the following expansion holds*

$$\mathcal{F}_s(\xi) = J_s\big[U(\xi)\big] + \theta_s(\xi), \tag{5.3}$$

where $|\theta_s| = O(s^K e^{-s/2})$, for some fixed constant $K > 0$, uniformly on s.

6. Proof of Theorem 1

We consider the set

$$S = \big\{x \in \Lambda\colon\ \phi_1(x) = 1\big\}. \tag{6.1}$$

The result Theorem 1 is a direct consequence of the following more precise result.

Theorem 2. *Given any positive integer m there exists $s_0 > 0$ sufficiently large such that problem* (1.6) *has a solution u_s positive in Ω of the form*

$$u_s(x) = U\big(\xi^s\big) + \tilde{\psi}_s, \tag{6.2}$$

which possesses exactly m local maximum points $\xi_1^s, \dots, \xi_m^s \in \Lambda$, satisfying that as $s \to \infty$

(i) $\operatorname{dist}(\xi_j^s, S) \to 0$ *and* $|\xi_i^s - \xi_j^s| \geqslant \frac{1}{s^{m(m+1)}}$ *if* $i \neq j$;
(ii) $\|\tilde{\psi}_s\|_\infty \to 0$.

The construction actually yields $1 - \phi_1(\xi_j^s) < s^{-1/2}$. Thus if S is just constituted by a non-degenerate maximum point $\bar{x}$ we will have $|\xi_j^s - \bar{x}| \leqslant Cs^{-1/4}$.

Proof of Theorem 2. According to Lemma 5.1, $U(\xi^s) + \hat{\psi}_{\xi^s}$ is a solution of problem (1.6) if $\xi^s \in \mathcal{O}_s$ is a critical point of the functional $\mathcal{F}$ defined in (5.2). We recall in particular that $\|\tilde{\psi}_{\xi^s}\|_\infty \to 0$ as predicted by estimate (4.2). It thus suffices to establish that $\mathcal{F}$ attains its maximum value in $\mathcal{O}_s$ for all sufficiently large s, for which we will see

$$\sup_{\xi \in \partial\mathcal{O}_s} \mathcal{F}(\xi) < \sup_{\xi \in \mathcal{O}_s} \mathcal{F}(\xi). \tag{6.3}$$

First we obtain a lower bound for $\sup_{\xi\in\mathcal{O}_s} \mathcal{F}(\xi)$ Let us fix a point $\bar{x} \in S$ and set

$$\xi_j^0 \equiv \bar{x} + \frac{1}{\sqrt{s}}\hat{\xi}_j,$$

where $\hat{\xi} = (\hat{\xi}_1, \ldots, \hat{\xi}_m)$ is an m-regular polygon in $\mathbb{R}^2$. Clearly $\xi^0 \in \mathcal{O}_s$ because $\phi_1(\xi_j^0) = 1 + O(s^{-1})$. Then

$$\sup_{\xi \in \mathcal{O}_s} \mathcal{F}(\xi) \geqslant J_s\big(U\big(\xi^0\big)\big) + \theta_s\big(\xi^0\big) = 8\pi \left\{ \sum_{i \neq j}^{m} 2 \log\big|\xi_i^0 - \xi_j^0\big| + s \sum_{j=1}^{m} \phi_1\big(\xi_j^0\big) \right\} + O(1)$$
$$\geqslant 8\pi m \big\{-(m-1)\log s + s\big\} + O(1).$$

Then,

$$\sup_{\xi \in \mathcal{O}_s} \mathcal{F}(\xi) \geqslant 8\pi m s - 8\pi m(m-1)\log s + O(1). \tag{6.4}$$

Next we estimate from above $\mathcal{F}(\xi)$ for $\xi \in \partial\mathcal{O}_s$. Then, there are two possibilities: either (1) there exist indices $i_0, j_0, i_0 \neq j_0$ such that $|\xi_{i_0} - \xi_{j_0}| = s^{-\beta}$, or (2) there exists i_0 such that $1 - \phi_1(\xi_{i_0}) = \frac{1}{\sqrt{s}} > 0$.

In the first case, we have the following upper bound

$$\mathcal{F}(\xi) \leqslant 8\pi \left\{ -2\beta \log s + s \sum_{j=1}^{m} \phi_1(\xi_j) \right\} + O(1) \leqslant 8\pi m \left\{ s - \frac{2}{m}\beta \log s \right\} + O(1). \tag{6.5}$$

In the second case, $1 - \phi_1(\xi_{i_0}^s) \leqslant \frac{1}{2\sqrt{s}}$. Then

$$\mathcal{F}(\xi) \leqslant 8\pi \left\{ O(\log s) + s\left(1 - \frac{1}{2\sqrt{s}} + (m-1)\right)\right\} \leqslant 8\pi s\left(m - \frac{1}{2s^{1/2}}\right) + O(\log s). \tag{6.6}$$

At this point we make the election $\beta > m^2 + m$ in the definition of $\mathcal{O}_s$. Relation (6.3) immediately follows from combining estimates (6.4), (6.5), (6.6) and taking s sufficiently large. This finishes the proof. □

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