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Nonlinear Elliptic Problems Above Criticality

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Abstract. We consider the elliptic problem $\Delta u + u^p = 0$, $u > 0$ in an exterior domain, $\Omega = \mathbb{R}^N \setminus \mathcal{D}$ under zero Dirichlet and vanishing conditions, where $\mathcal{D}$ is smooth and bounded, and p is *supercritical*, namely $p > \frac{N+2}{N-2}$. We prove that this problem has infinitely many solutions with *slow decay* $O(|x|^{-\frac{2}{p-1}})$ at infinity. In addition, a fast decay solution exists if p is close enough to the critical exponent. If p differs from certain sequence of resonant values which tends to infinity, then the Dirichlet problem is also solvabe in a bounded domain Ω with a sufficiently small spherical hole.

Keywords. Critical Sobolev exponent, supercritical elliptic problems, exterior domains.

1. Introduction and statement of the main results

A basic model of nonlinear elliptic boundary problem is the Lane-Emden-Fowler equation,

$$\Delta u + u^p = 0\,,\ u > 0 \quad \text{in } \Omega\,, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega\,, \tag{1.2}$$

where Ω is a domain with smooth boundary in $\mathbb{R}^N$ and $p > 1$. Discovered by Lane, an astrophysicist, in the mid 19th century, the role of this and related equations has been broad outside and inside mathematics. While simple looking, the structure of the solution set of this problem may be surprisingly complex. Much has been learned over the last decades, particularly thanks to the development of techniques from the calculus of variations, see [23], but many basic issues remain far from understood. Among those, *solvability above criticality* is a paradigm of the difficulties arising in solving nonlinear elliptic PDEs. A central, intriguing characteristic of this problem is the role played by the critical exponent $p = \frac{N+2}{N-2}$ in the solvability question. When Ω is *bounded* and $1 < p < \frac{N+2}{N-2}$, compactness of Sobolev's embedding yields a solution as a minimizer of the variational problem

$$\inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{p+1}\right)^{\frac{2}{p+1}}}.$$

When $p \geq \frac{N+2}{N-2}$, compactness is lost, and this minimization procedure fails, as existence does in general: Pohozaev [21] discovered in 1965 that no solution exists if the domain is strictly star-shaped. In 1975, Kazdan and Warner [15] observed that in strong contrast, if Ω is an annulus, $\Omega = \{a < |x| < b\}$, compactness holds for any $p > 1$ within the class of radial functions, and a solution can again be found variationally, regardless the value of p. On the other hand, the question of solvability in a non-symmetric annular domain for powers above critical remains notoriously open until today.

The critical case $p = \frac{N+2}{N-2}$ can still be handled by variational arguments, since the loss of compactness of Sobolev's embedding is well-understood. In the classical paper [3], Brezis and Nirenberg considered the critical case $p = \frac{N+2}{N-2}$ and proved that compactness, and hence solvability, is restored by the addition of a suitable linear term. Coron [4] used a variational approach to prove that (1.1)–(1.2) is solvable for $p = \frac{N+2}{N-2}$ if Ω exhibits *a small hole*. Rey [22] established existence of multiple solutions if Ω exhibits several small holes. Bahri and Coron [1] established, by deep topological analysis, that solvability holds for $p = \frac{N+2}{N-2}$ whenever Ω has a non-trivial topology. The question by Rabinowitz, stated by Brezis in [2], whether the presence of non-trivial topology in the domain suffices for solvability in the supercritical case $p > \frac{N+2}{N-2}$, was answered negatively by Passaseo [19] by means of an example for $N \geq 4$ and $p > \frac{N+1}{N-3}$. If Ω is a Coron's type domain, namely one with a sufficiently small hole, then solvability persists slightly above the critical exponent, say $p = \frac{N+2}{N-2} + \varepsilon$ for all small $\varepsilon > 0$. In such a case, a solution with a two-bubble pattern which blows-up as $\varepsilon \to 0^+$ is present, see [8, 9].

Except for results in domains involving symmetries or exponents close to critical, e.g. [8, 9, 13, 18, 20], solvability of (1.1)–(1.2) in the supercritical case has been a widely open matter, particularly since variational machinery no longer applies, at least in its naturally adapted way for subcritical or critical problems.

Methods other than variational analysis are therefore required. In this paper we survey recent progress in the resolution of supercritical problems. We shall concentrate next in Problem (1.1)–(1.2) for exponents p above critical in a *exterior domain*. Let $\mathcal{D}$ be a bounded domain with smooth boundary. We consider the problem of finding classical solutions of the problem

$$\Delta u + u^p = 0\,,\ \ u > 0 \quad \text{in } \mathbb{R}^N \setminus \bar{\mathcal{D}}\,, \tag{1.3}$$

$$u = 0 \ \text{ on } \partial\mathcal{D}\,, \qquad \lim_{|x|\to+\infty} u(x) = 0 \tag{1.4}$$

where $p > \frac{N+2}{N-2}$. The supercritical case is meaningful in this problem since Pohozaev's identity does not pose obstructions for its solvability. To fix ideas, let us consider case the simple case of $\mathcal{D} = B(0,1)$ and look for radially symmetric solutions to the problem $u = u(r)$, $r = |x|$. The equation

$$\Delta u + u^p = 0 \tag{1.5}$$

then corresponds to the ODE

$$u'' + \frac{N-1}{r}\,u' + u^p = 0\,. \tag{1.6}$$

This equation can be analyzed through phase plane analysis after a transformation introduced by Fowler [12] in 1931: $v(s) = r^{\frac{2}{p-1}}u(r)$, $r = e^s$, which transforms equation (1.6) into the autonomous ODE

$$v'' + \alpha v' - \beta v + v^p = 0 \tag{1.7}$$

where

$$\alpha = N - 2 - \frac{4}{p-1}, \quad \beta = \frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right). \tag{1.8}$$

Since α and β are positive for $p > \frac{N+2}{N-2}$, the Hamiltonian energy

$$E(v) = \frac{1}{2}\dot{v}^2 + \frac{1}{p+1}v^{p+1} - \frac{\beta}{2}v^2$$

strictly decreases along trajectories. Using this it is easy to see the existence of a heteroclinic orbit which connects the equilibria $(0,0)$ and $(0, \beta^{\frac{1}{p-1}})$ in the phase plane (v, v'). These equilibria correspond respectively to a saddle point and an attractor. A solution $v(s)$ of (1.7) corresponding this orbit satisfies $v(-\infty) = 0$, $v(+\infty) = \beta^{\frac{1}{p-1}}$ and $w(r) = r^{-\frac{2}{p-1}}v(\log r)$ solves (1.6) and is bounded at $r = 0$. Then all radial solutions of (1.5) defined in all $\mathbb{R}^N$ have the form

$$w_\lambda(x) := \lambda^{\frac{2}{p-1}} w(\lambda |x|), \quad \lambda > 0. \tag{1.9}$$

We denote in what follows by $w(x)$ the unique positive radial solution

$$\Delta w + w^p = 0 \quad \text{in } \mathbb{R}^N, \quad w(0) = 1. \tag{1.10}$$

Coming back to the analysis for (1.7), we see in phase plane (v, v') the presence of a continuum of orbits that begin on the axis $v = 0$ as close to the equilibrium $(0,0)$ as we please, which eventually end in the attractor $(0, \beta^{\frac{1}{p-1}})$. If $v(s)$ is a solution associated to one of these orbits, then a suitable translation makes it defined in $[0, \infty)$ with $v(0) = 0$. Its associated $u(r)$ then satisfies $u(1) = 0$ and represents a positive solution of problem (1.3)–(1.4) with $\mathcal{D} = B(0,1)$. The closer the starting point of the orbit is taken from $(0,0)$, the smaller the associated $v(s)$ gets on compact subsets of $(0, \infty)$, at the same time getting close to the heteroclinic, more precisely the solution $u(|x|)$ is close to some w_λ for small $\lambda > 0$. The solutions u built this way are small in their entire domain and all have the uniform *slow decay*

$$u(x) = \beta^{\frac{1}{p-1}} |x|^{-\frac{2}{p-1}} (1 + o(1)) \quad \text{as } |x| \to \infty,$$

with β given by (1.8). This analysis establishes the existence of a one-parameter, asymptotically vanishing continuum of radial solutions of problem (1.3)–(1.4) with $\mathcal{D} = B(0,1)$ with slow decay.

We establish in Theorem 1.1 below that the above mentioned phenomenon is *very robust.* In fact, we have, for arbitrary domain $\mathcal{D}$ the existence of this continuum of slow decay solutions, in particular proving the striking fact that the supercritical exterior problem (1.3)–(1.4) is *always solvable.*

Theorem 1.1 ([6, 7])**.** *For any $p > \frac{N+2}{N-2}$ there is a continuum of solutions u_λ, $\lambda > 0$, to Problem* (1.3)–(1.4), *such that*

$$u_\lambda(x) = \beta^{\frac{1}{p-1}} |x|^{-\frac{2}{p-1}} (1 + o(1)) \quad \textit{as } |x| \to \infty \tag{1.11}$$

and $u_\lambda(x) \to 0$ as $\lambda \to 0$, uniformly in $\mathbb{R}^N \setminus \mathcal{D}$.

In the radial case, the analysis explained above makes it natural to seek for a solution u_λ in the form of a small perturbation of w_λ. This naturally leads to construct an inverse of the linearized operator $\Delta + p w_\lambda^{p-1}$. in $\mathbb{R}^N \setminus \mathcal{D}$ under Dirichlet boundary conditions. Since w_λ itself is small on bounded sets for small λ, such an inverse can be found as a small perturbation of an inverse of this operator in entire $\mathbb{R}^N$. By scaling, it suffices to carry out that analysis for $\lambda = 1$. This inverse indeed exists for $p \geq \frac{N+1}{N-3}$, however if $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ the linearized operator is not surjective, having a range orthogonal to the generators of translations. This suggests that a further adjustment of the location of the origin may produce a family of solutions as in Theorem 1.1. As we shall see, this is indeed the case.

The invertibility analysis for $p \geq \frac{N+1}{N-3}$ is in strong analogy with one carried out in [17] in the construction of singular solutions with prescribed singularities for $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ in bounded domains. At the radial level, supercritical and subcritical in this range are completely dual: In equation (1.7) β remains positive but α becomes negative. The effect of this is basically to make the phase portraits equivalent, just with arrows inverted in the orbits, with obvious dual consequences. For instance, inner-subcritical in a ball has a classical solution, which in the phase diagram is represented by the unstable manifold of $(0, 0)$. Correspondingly, in the supercritical case, to the orbit representing the stable manifold of $(0, 0)$, it corresponds to the unique solution w_* to the exterior problem with *fast decay*, namely w_* satisfies

$$\Delta w_* + w_*^p = 0\,,\ \ w_* > 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}_1(0)\,, \tag{1.12}$$

$$w_* = 0 \ \text{ on } \partial B_1(0)\,, \quad \limsup_{|x|\to+\infty} |x|^{2-N} w_*(x) < +\infty\,, \tag{1.13}$$

We do not know if for arbitrary $\mathcal{D}$ a solution with this property in $\mathbb{R}^N \setminus \mathcal{D}$ actually exists. We are able to establish that this is the case for supercritical powers sufficiently close to critical.

Theorem 1.2 **([7]).** *There exists a number $p_0 > \frac{N+2}{N-2}$ such that for any $\frac{N+2}{N-2} < p < p_0$, problem* (1.3)–(1.4) *has a fast decay solution.*

Given that we are finding solutions in exterior domains which decay at infinity, it is reasonable to ask whether we can also find solutions in bounded domains with small holes. We consider next Problem (1.1)–(1.2) for exponents p above critical in a Coron's type domain: one exhibiting a small hole. Thus we assume in what follows that the domain Ω has the form

$$\Omega = \mathcal{D} \setminus B_\delta(Q) \tag{1.14}$$

where $\mathcal{D}$ is a bounded domain with smooth boundary, $B_\delta(Q) \subset \mathcal{D}$ and $\delta > 0$ is to be taken small. Thus we consider the problem of finding classical solutions of

$$\Delta u + u^p = 0\,,\ \ u > 0 \quad \text{in } \mathcal{D} \setminus B_\delta(Q)\,, \tag{1.15}$$

$$u = 0 \quad \text{on } \partial\mathcal{D} \cup \partial B_\delta(Q)\,. \tag{1.16}$$

Our main result states that there is a sequence of *resonant exponents*,

$$\frac{N+2}{N-2} < p_1 < p_2 < p_3 < \cdots\,, \quad \text{with} \lim_{k\to+\infty} p_k \;=\; +\infty \tag{1.17}$$

such that if p is supercritical and differs from all elements of this sequence then Problem (1.3)–(1.4) is solvable whenever δ is sufficiently small.

Theorem 1.3 ([11]). *There exists a sequence of the form* (1.17) *such that if* $p > \frac{N+2}{N-2}$ *and* $p \neq p_j$ *for all* j*, then there is a* $\delta_0 > 0$ *such that for any* $\delta < \delta_0$*, Problem* (1.15)–(1.16) *possesses at least one solution.*

In the background of our result is problem (1.12). The solutions we find have a profile similar to w suitably rescaled. More precisely, Let us observe that

$$w_\delta(x) = \delta^{-\frac{2}{p-1}} w(\delta^{-1}|x-Q|) \tag{1.18}$$

solves uniquely the same problem with $B_1(0)$ replaced with $B_\delta(Q)$. The idea is to consider w_δ as a first approximation for a solution of Problem (1.1)–(1.2), provided that $\delta > 0$ is chosen small enough. What we shall prove is that an actual solution of the problem, which differs little from w_δ does exist. To this end, it is necessary to understand the linearized operator around w_δ.

The rest of this paper presents the main elements involved in the proofs of the above results. Full details are provided in the articles [6, 7, 11].

2. The Proof of Theorem 1.1 for $p \geq \frac{N+1}{N-3}$

2.1. The fixed point argument

We look for a solution of Problem (1.3)–(1.4) of the form $u = \eta w_\lambda + \phi$, where η is a smooth cut-off function with $\eta(x) = 0$ for $|x| \leq R$, $\eta(x) = 1$ for $|x| \geq R+1$ and $\mathcal{D} \subset B(0,R)$. This u solves (1.3)–(1.4) if ϕ satisfies

$$\begin{cases} \Delta\phi + p w_\lambda^{p-1}\phi = N(\phi) + E & \text{in } \mathbb{R}^N \setminus \bar{\mathcal{D}} \\ \phi = 0 & \text{on } \partial\mathcal{D} \\ \phi(x) \to 0 & \text{as } |x| \to +\infty \end{cases} \tag{2.1}$$

where

$$N(\phi) = -(\eta w_\lambda + \phi)^p + (\eta w_\lambda)^p + p(\eta w_\lambda)^{p-1}\phi + p(1-\eta^{p-1}) w_\lambda^{p-1}\phi,$$

and

$$E = -\Delta(\eta w_\lambda) - (\eta w_\lambda)^p.$$

We write the above problem in fixed point form on the basis of the existence of a right inverse for the linear operator $\Delta + pw_\lambda^{p-1}$ in suitable weighted L^∞-spaces. Thus we consider the linear problem

$$\begin{cases} \Delta\phi + pw_\lambda^{p-1}\phi = h & \text{in } \mathbb{R}^N \setminus \bar{\mathcal{D}} \\ \phi = 0 & \text{on } \partial\mathcal{D} \\ \phi(x) \to 0 & |x| \to +\infty, \end{cases} \tag{2.2}$$

and the norms

$$\|\phi\|_{*,\lambda} = \lambda^\sigma \sup_{|x|\le\frac{1}{\lambda}} |x|^\sigma|\phi(x)| + \lambda^{\frac{2}{p-1}} \sup_{|x|\ge\frac{1}{\lambda}} |x|^{\frac{2}{p-1}}|\phi(x)|$$

$$\|h\|_{**,\lambda} = \lambda^\sigma \sup_{|x|\le\frac{1}{\lambda}} |x|^{2+\sigma}|h(x)| + \lambda^{\frac{2}{p-1}} \sup_{|x|\ge\frac{1}{\lambda}} |x|^{2+\frac{2}{p-1}}|h(x)|.$$

We have the validity of the following result.

Lemma 2.1. *Assume that $N \ge 4$ and $p \ge \frac{N+1}{N-3}$. Then there exists a constant $C > 0$ such that for all sufficiently small $\lambda > 0$ and all h with $\|h\|_{**,\lambda} < +\infty$, Problem* (2.2) *has a solution $\phi = \mathcal{T}_\lambda(h)$ such that $\mathcal{T}_\lambda$ is a linear map and*

$$\|\mathcal{T}_\lambda(h)\|_{*,\lambda} \le C\|h\|_{**,\lambda}.$$

By this result, we have a solution to (2.1) if ϕ solves the fixed point problem

$$\phi = \mathcal{T}_\lambda(N(\phi) + E). \tag{2.3}$$

We can check the estimates

$$\|N(\phi)\|_{**,\lambda} \le C(\lambda^2\|\phi\|_{*,\lambda} + \lambda^{-\frac{2}{p-1}}\|\phi\|^2_{*,\lambda} + \lambda^{-2}\|\phi\|^p_{*,\lambda}), \tag{2.4}$$

and

$$\|E\|_{**,\lambda} \le C\lambda^{\frac{2}{p-1}+\sigma}. \tag{2.5}$$

Let $\phi_0 = \mathcal{T}_\lambda(E)$. From Lemma 2.1, and (2.5), we get $\|\phi_0\|_{*,\lambda} \le C\lambda^{\frac{2}{p-1}+\sigma}$. Let us write $\phi = \phi_0 + \phi_1$. Then solving equation (2.3) is equivalent to solving the fixed point problem $\phi = \mathcal{T}_\lambda(N(\phi_0 + \phi))$. We consider the set

$$\mathcal{F} = \{\phi \in L^\infty(\mathbb{R}^N \setminus \mathcal{D}) \ / \ \|\phi\|_{*,\lambda} \le \rho\lambda^{\frac{2}{p-1}}\}$$

where $\rho > 0$ is going to be fixed independently of λ, and the operator

$$\mathcal{A}(\phi) = \mathcal{T}_\lambda(N(\phi_0 + \phi)).$$

Next we show that $\mathcal{A}$ has a fixed point in $\mathcal{F}$. For $\phi \in \mathcal{F}$ we have

$$\|\mathcal{A}(\phi)\|_{*,\lambda} \le C\|N(\phi_0 + \phi)\|_{**,\lambda} \tag{2.6}$$

$$\le C(\lambda^2\|\phi_0 + \phi\|_{*,\lambda} + \lambda^{-\frac{2}{p-1}}\|\phi_0 + \phi\|^2_{*,\lambda} + \lambda^{-2}\|\phi_0 + \phi\|^p_{*,\lambda}). \tag{2.7}$$

Thus for a fixed sufficiently small ρ and all small λ we get

$$\|\mathcal{A}(\phi)\|_{*,\lambda} \le C\lambda^{\frac{2}{p-1}}(\rho\lambda^2 + \lambda^{2\sigma} + \lambda^{p\sigma} + \rho^2 + \rho^p) \le \rho\lambda^{\frac{2}{p-1}}. \tag{2.8}$$

Hence $\mathcal{A}(\mathcal{F}) \subset \mathcal{F}$ for all small λ.

On the other hand, we also have that $\mathcal{A}$ is a contraction mapping in $\mathcal{F}$. Let us take ϕ_1, ϕ_2 in $\mathcal{F}$. It is straightforward to check that

$$\|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\|_{*,\lambda} \le C\,(\rho + \lambda^2)\,\|\,\phi_1 - \phi_2\,\|_{*,\lambda}\,. \tag{2.9}$$

Thus $\mathcal{A}$ is a contraction mapping in $\mathcal{F}$, and hence a fixed point in this region indeed exists. The solutions u_λ built this way satisfy the requirement of Theorem 1.1. $\square$

2.2. The proof of Lemma 2.1

The proof is based on a similar result valid in entire $\mathbb{R}^N$: Let us consider the problem

$$\Delta\phi + pw_\lambda^{p-1}\phi = h \quad \text{in } \mathbb{R}^N. \tag{2.10}$$

Lemma 2.2. *Assume $N \ge 4$ and $p \ge \frac{N+1}{N-3}$. For $0 < \sigma < N-2$ there exists a constant $C > 0$ such that for any $\lambda > 0$ and h with $\|h\|_{**,\lambda} < +\infty$, equation* (2.10) *has a solution $\phi = T_\lambda(h)$ such that T_λ defines a linear map and*

$$\|T_\lambda(h)\|_{*,\lambda} \le C\|h\|_{**,\lambda}.$$

Before proving this we proceed to the

Proof of Lemma 2.1. We shall solve (2.2) by writing $\phi = \eta\varphi + \psi$ where η is a smooth cut-off function with

$$\eta(x) = 0 \quad \text{for } |x| \le R_0, \quad \eta(x) = 1 \quad \text{for } |x| \ge R_0 + 1$$

and $R_0 > 0$ is fixed so that $\mathcal{D} \subseteq B_{R_0}$. We also set $\zeta(x) = \eta(x/2)$, so that $\eta\zeta = \zeta$.

To find a solution of (2.2) it is sufficient to solve the following system

$$\Delta\varphi + pw_\lambda^{p-1}\varphi = -p\zeta w_\lambda^{p-1}\psi + \zeta h \quad \text{in } \mathbb{R}^N \tag{2.11}$$

$$\begin{cases} \Delta\psi + p(1-\zeta)w_\lambda^{p-1}\psi = -2\nabla\eta\nabla\varphi - \varphi\Delta\eta + (1-\zeta)h & \text{in } \mathbb{R}^N \setminus \bar{\mathcal{D}} \\ \psi = 0 & \text{on } \partial\mathcal{D} \\ \psi(x) \to 0 & |x| \to +\infty. \end{cases} \tag{2.12}$$

We assume $\|h\|_{**,\lambda} < \infty$. Let us consider the Banach space X consisting of functions φ such that $\|\varphi\|_{*,\lambda} < \infty$ and that are Lipschitz on $E = B_{2R_0} \setminus B_{R_0}$ equipped with the norm

$$\|\varphi\|_X = \|\varphi\|_{*,\lambda} + \|\nabla\varphi\|_{L^\infty(E)}.$$

Given $\varphi \in X$ we solve first (2.12) and denote by $\psi(\varphi, h)$ the solution, which is clearly linear in its argument. Then note that $\zeta\psi$ is well defined in $\mathbb{R}^N$ and that $|\psi| \le \frac{C}{|x|^{N-2}}$ for large $|x|$ so hence the right hand side of (2.11) has a finite $\|\ \|_{**,\lambda}$ norm. We obtain a solution to the system, which defines a linear operator in h, if we solve the fixed point problem

$$\varphi \;=\; T_\lambda(\,-p\zeta w_\lambda^{p-1}\,\psi(\varphi, h) \;+\; \zeta h\,) \;\equiv\; F(\varphi)\,.$$

where T_λ is the operator in Propoposition 2.2. Then we have the estimate

$$\|F(\varphi)\|_{*,\lambda} \le C\,\| - p\zeta w_\lambda^{p-1}\psi + \zeta h\|_{**,\lambda} \le C(\|\zeta w_\lambda^{p-1}\psi\|_{**,\lambda} + \|h\|_{**,\lambda}). \tag{2.13}$$

But

$$\|\zeta w_\lambda^{p-1}\psi\|_{**,\lambda} = \lambda^\sigma \sup_{R_1 \le |x| \le \frac{1}{\lambda}} \left(|x|^{2+\sigma} w_\lambda(x)^{p-1}|\psi(x)|\right) + \lambda^{\frac{2}{p-1}} \sup_{|x| \ge \frac{1}{\lambda}} \left(|x|^{2+\sigma+\frac{2}{p-1}} w_\lambda(x)^{p-1}|\psi(x)|\right).$$

Using equation (2.12) and the fact that $w_\lambda(x) \to 0$ uniformly on compact sets we have

$$|\psi(x)| \le \frac{C}{|x|^{N-2}}(\|\varphi\|_X + \|h\|_{**,\lambda}). \tag{2.14}$$

Using this, and the asymptotic behavior of $w(x)$ we then obtain the estimate

$$\|\zeta w_\lambda^{p-1}\psi\|_{**,\lambda} \le C\lambda^\gamma(\|\varphi\|_X + \|h\|_{**,\lambda}).$$

where $\gamma = \min(2+\sigma, N-2)$. This together with (2.13) yields

$$\|F(\varphi)\|_{*,\lambda} \le C(\lambda^\gamma \|\varphi\|_X + \|h\|_{**,\lambda}). \tag{2.15}$$

While, using elliptic estimates, we get

$$\|\nabla F(\varphi)\|_{L^\infty(E)} \le C(\|F(\varphi)\|_{*,\lambda} + \|h\|_{**,\lambda} + \lambda^\gamma(\|\varphi\|_X + \|h\|_{**,\lambda})).$$

This and (2.15) imply that

$$\|F(\varphi)\|_X \le C(\lambda^\gamma \|\varphi\|_X + \|h\|_{**,\lambda}).$$

It follows that for sufficiently small λ, F defines a contraction mapping of the region

$$\{\varphi \in X \,|\, \|\varphi\|_X \le 2C\|h\|_{**,\lambda}\}.$$

A unique fixed point thus exists in this region, which inherits a solution with the required properties. The proof of Lemma 2.1 is concluded. □

2.3. The proof of Lemma 2.2

By scaling out λ and using the definitions of the norms, we just need to prove the result for $\lambda = 1$. We denote the norms involved simply by $\|\cdot\|_*$ and $\|\cdot\|_{**}$. Let us consider h with $\|h\|_{**} < +\infty$ and decompose it in the form

$$h(x) = \sum_{k=0}^\infty h_k(r)\Theta_k(\theta), \quad r > 0,\ \theta \in S^{N-1} \tag{2.16}$$

where Θ_k, $k \ge 0$ are eigenfunctions of the Laplace-Beltrami operator in S^{N-1}, normalized so that they constitute an orthonormal system in L^2 (S^{N-1}). We take Θ_0 to be a positive constant, associated to the eigenvalue 0 and Θ_i, $1 \le i \le N$ is an appropriate multiple of $\frac{x_i}{|x|}$ which has eigenvalue $\lambda_i = N-1$, $1 \le i \le N$. We recall that the set of eigenvalues is given by $\{j(N-2+j) \,|\, j \ge 0\}$.

We look for a solution ϕ to (2.10) in the form

$$\phi(x) = \sum_{k=0}^\infty \phi_k(r)\Theta_k(\theta)\,. \tag{2.17}$$

Then ϕ satisfies (2.10) if and only if

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \left(pw^{p-1} - \frac{\lambda_k}{r^2}\right)\phi_k = h_k, \quad \text{for all } r>0, \text{ for all } k \ge 0. \tag{2.18}$$

To construct solutions of this ODE we need to consider two linearly independent solutions $z_{1,k}$, $z_{2,k}$ of the homogeneous equation

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \left(pw^{p-1} - \frac{\lambda_k}{r^2}\right)\phi_k = 0, \quad r \in (0,\infty). \tag{2.19}$$

Once these generators are identified, the general solution of the equation can be written through the variation of parameters formula as

$$\phi(r) = z_{1,k}(r)\int z_{2,k}h_k r^{N-1}dr - z_{2,k}(r)\int z_{1,k}h_k r^{N-1}dr$$

where the symbol $\int$ designates arbitrary antiderivatives, which we will specify in the choice of the operators. It is helpful to recall that if one solution $z_{1,k}$ to (2.19) is known, a second, linearly independent solution can be found in any interval where $z_{1,k}$ does not vanish as

$$z_{2,k}(r) = z_{1,k}(r)\int z_{1,k}(r)^{-2}r^{1-N}dr\,. \tag{2.20}$$

One can get the asymptotic behaviors of any solution z as $r \to 0$ and as $r \to +\infty$ by examining the indicial roots of the associated Euler equations. It is known that $r^2 w(r)^{p-1} \to \beta$ as $r \to +\infty$ where

$$\beta = \frac{2}{p-1}\left(N-2-\frac{2}{p-1}\right).$$

Thus we get the limiting equation, for $r \to \infty$,

$$r^2\phi'' + (N-1)r\phi' + (p\beta - \lambda_k)\phi = 0, \tag{2.21}$$

while as $r \to 0$,

$$r^2\phi'' + (N-1)r\phi' - \lambda_k\phi = 0. \tag{2.22}$$

In this way the respective behaviors will be ruled by $z(r) \sim r^{-\mu}$ as $r \to +\infty$ where μ solves

$$\mu^2 - (N-2)\mu + (p\beta - \lambda_k) = 0$$

while as $r \to 0$ μ satisfies

$$\mu^2 - (N-2)\mu - \lambda_k = 0.$$

Next we shall construct each of the ϕ_k's in the expansion (2.17), in such a way that they define bounded linear operators of h_k in the norms considered. This method is reminiscent to that in [17], see also [16].

2.3.1. The construction of ϕ_0.

Lemma 2.3. *Let $k = 0$ and $p > \frac{N+2}{N-2}$. Then equation (2.18) has a solution ϕ_0 which depends linearly on h_0 and satisfies*

$$\|\phi_0\|_* \leq C\|h_0\|_{**}. \tag{2.23}$$

Proof. For $k = 0$ the possible behaviors at 0 for a solution $z(r)$ to (2.19) are simply

$$z(r) \sim 1, \quad z(r) \sim r^{2-N}$$

while at $+\infty$ this behavior is more complicated. The indicial roots of (2.22) are given by

$$\mu_{0\pm} = \frac{N-2}{2} \pm \frac{1}{2}\sqrt{(N-2)^2 - 4p\beta}.$$

The situation depends of course on the sign of $D = (N-2)^2 - 4p\beta$. It is observed in [14] that $D > 0$ if and only if $N > 10$ and $p > p_c$ where we set

$$p_c = \begin{cases} \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N > 10 \\ \infty & \text{if } N \leq 10. \end{cases}$$

Thus when $p < p_c$, $\mu_{0\pm}$ are complex with negative real part, and the behavior of a solution $z(r)$ as $r \to +\infty$ is oscillatory and given by

$$Z(r) = O(r^{-\frac{N-2}{2}}).$$

When $p > p_c$, we have $\mu_{0+} > \mu_{0-} > \frac{2}{p-1}$.

Independently of the value of p, we have that the function

$$z_{1,0} = rw' + \frac{2}{p-1}w$$

satisfies equation (2.19) for $k = 0$. Using asymptotic formulae derived for w in [14], we find the estimates

$$\text{if } p < p_c : \qquad |z_{1,0}(r)| \leq Cr^{\frac{N-2}{2}} \tag{2.24}$$

$$\text{if } p = p_c : \qquad z_{1,0}(r) = cr^{-\frac{N-2}{2}} \log r\,(1 + o(1)) \tag{2.25}$$

$$\text{if } p > p_c : \qquad z_{1,0}(r) = cr^{-\mu_{0-}}\,(1 + o(1))\,, \tag{2.26}$$

where $c \neq 0$.

Case $p < p_c$. We define $z_{2,0}(r)$ for small $r > 0$ by

$$z_{2,0}(r) = z_{1,0}(r) \int_{r_0}^{r} z_{1,0}(s)^{-2} s^{1-N}\, ds \tag{2.27}$$

where r_0 is small so that $z_{1,0} > 0$ in $(0, r_0)$ (which is possible because $z_{1,r} \sim 1$ near 0). Then $z_{2,0}$ is extended to $(0, +\infty)$ so that it is a solution to the homogeneous equation (2.19) (with $k = 0$) in this interval. As mentioned earlier $z_{2,0}(r) = O(r^{-\frac{N-2}{2}})$ as $r \to +\infty$.

We define

$$\phi_0(r) = z_{1,0}(r) \int_1^r z_{2,0} h_0 s^{N-1}\, ds - z_{2,0}(r) \int_0^r z_{1,0} h_0 s^{N-1}\, ds,$$

and omit a calculation that shows that this expression satisfies (2.23).

Case $p \geq p_c$. In this case we let

$$\phi_0(r) = -z_{1,0}(r) \int_1^r z_{1,0}(s)^{-2} s^{1-N} \int_0^s z_{1,0}(\tau) h_0(\tau) \tau^{N-1}\, d\tau\, ds,$$

which is justified because when $p \geq p_c$ we have $z_{1,0}(r) > 0$ for all $r > 0$, which follows from the fact that $\lambda \mapsto \lambda^{\frac{2}{p-1}} w(\lambda r)$ is increasing for $\lambda > 0$, see [14]. Again, a calculation using now (2.25) and (2.26) shows that ϕ_0 satisfies the estimate (2.23). $\square$

2.3.2. The construction of ϕ_k, $1 \leq k \leq N$. All these modes are equivalent, so we only consider $k = 1$. We have the following result.

Lemma 2.4. *Let $k = 1$ and $p \geq \frac{N+1}{N-3}$. Then equation* (2.18) *has a solution ϕ_1 which defines a linear operator of h_1 and satisfies*

$$\|\phi_1\|_* \leq C \|h_1\|_{**}. \tag{2.28}$$

Proof. In this case the indicial roots that govern the behavior of the solutions $z(r)$ as $r \to +\infty$ of the homogeneous equation (2.19) are given by $\mu_1 = \frac{2}{p-1} + 1$ and $\mu_2 = N - 3 - \frac{2}{p-1}$. Since we are looking for solutions that decay at a rate $r^{-\frac{2}{p-1}}$ as $r \to +\infty$ we will need $N - 3 - \frac{2}{p-1} \geq \frac{2}{p-1}$, which is equivalent to the hypothesis $p \geq \frac{N+1}{N-3}$. On the other hand the behavior near 0 of $z(r)$ can be $z(r) \sim r$ or $z(r) \sim r^{1-N}$.

Similarly as in the case $k = 0$ we have a solution to (2.19), namely $z_1(r) = -w'(r)$ and luckily enough it is positive in all $(0, +\infty)$. With it we can build

$$\phi_1(r) = -z_1(r) \int_1^r z_1(s)^{-2} s^{1-N} \int_0^s z_1(\tau) h_1(\tau) \tau^{N-1}\, d\tau\, ds. \tag{2.29}$$

From this formula and using $p \geq \frac{N+1}{N-3}$ we obtain (2.28). $\square$

2.3.3. The construction of ϕ_k, $k > N$.

Lemma 2.5. *Let $k > N$ and $p > \frac{N+2}{N-2}$. If $\|h_k\|_{**} < \infty$ equation* (2.18) *has a unique solution ϕ_k with $\|\phi_k\|_* < \infty$ and there exists $C_k > 0$ such that*

$$\|\phi_k\|_* \leq C_k \|h_k\|_{**}. \tag{2.30}$$

Proof. Let us write L_k for the operator in (2.18), that is,

$$L_k \phi = \phi'' + \frac{N-1}{r}\phi' + \left(p w^{p-1} - \frac{\lambda_k}{r^2} \right) \phi.$$

This operator satisfies the maximum principle in any interval of the form $(\delta, \frac{1}{\delta})$, $\delta > 0$. Indeed let $z = -w'$, so that $z > 0$ in $(0, +\infty)$ and it is a supersolution, because

$$L_k z = \frac{N-1-\lambda_k}{r^2} z < 0 \quad \text{in } (0, +\infty), \tag{2.31}$$

since $\lambda_k \geq 2N$ for $k \geq 2$. To prove solvability of (2.18) in the appropriate space we construct a supersolution ψ of the form

$$\psi = C_1 z + v, \quad v(r) = \frac{1}{r^\sigma + r^{\frac{2}{p-1}}},$$

Choosing C_1 sufficiently large, we can check that

$$L_k \psi \leq -c \min(r^{-\sigma-2}, r^{-\frac{2}{p-1}-2}) \quad \text{in } (0, +\infty).$$

for some $c > 0$.

Given h_k with $\|h_k\|_{**} < \infty$, by the method of sub and supersolutions, there exists, for any $\delta > 0$ a unique solution ϕ_δ of the two-point boundary value problem

$$L_k \phi_\delta = h_k \qquad \text{in } (\delta, \tfrac{1}{\delta})$$
$$\phi_\delta(\delta) = \phi_\delta(\tfrac{1}{\delta}) = 0.$$

This solution satisfies the bound

$$|\phi_\delta| \leq C\psi \, \|h_k\|_{**} \quad \text{in } (\delta, \tfrac{1}{\delta}).$$

Using standard estimates we have that, up to subsequences, $\phi_\delta \to \phi_k$ as $\delta \to 0$ uniformly on compact subsets of $(0, +\infty)$ where ϕ_k is a solution of (2.18) which satisfies

$$|\phi_k| \leq C\psi \, \|h_k\|_{**} \quad \text{in } (0, \infty).$$

The maximum principle yields that the solution to (2.18) bounded in this way is actually unique, and thus defines the desired linear operator. $\square$

2.3.4. Conclusion of the construction. Let $m > 0$ be an integer. By Lemmas 2.3, 2.4 and 2.5 we see that if $\|h\|_{**} < \infty$ and its Fourier series (2.16) has $h_k \equiv 0\ \forall k \geq m$ there exists a solution ϕ to (2.10) that depends linearly with respect to h and moreover

$$\|\phi\|_* \leq C_m \|h\|_{**}.$$

We can prove that the constant C_m may actually be taken uniform in m. Indeed, an indirect argument, based upon standard elliptic estimates, allows us to end up with the situation that there exists a nonzero, bounded function ϕ which satisfies the equation $\Delta\phi + pw^{p-1}\phi = 0$ and which has no Fourier components in its first few Fourier components. Arguing mode by mode, we see that ϕ must be identically zero. This shows that the solution ϕ defined by (2.17) defines an operator in h with the desired property. $\square$

3. The proof of Theorem 1.1 when $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$

The proof presented in the previous section fails only in one step: in the construction of ϕ_k for $1 \leq k \leq N$. Formula (2.29) for ϕ_1 does not define a solution which decays like $r^{-\frac{2}{p-1}}$ *unless h_1 satisfies the orthogonality condition*

$$\int_0^\infty w'(\tau) h_1(\tau) \tau^{N-1}\, d\tau = 0. \tag{3.1}$$

This implies the following: Let us write

$$Z_i = \frac{\partial w}{\partial x_i}. \tag{3.2}$$

Then if $\frac{N+2}{N-3} < p < \frac{N+1}{N-3}$ and $0 < \sigma < N-2$, there is a linear operator $\phi = T(h)$ defined for h with $\|h\|_{**} < \infty$, with the property that for certain unique scalars $c_1, \ldots, c_N$,

$$\Delta\phi + pw^{p-1}\phi = h + \sum_{i=1}^N c_i Z_i \quad \text{in } \mathbb{R}^N, \tag{3.3}$$

and $\|\phi\|_* \leq C\|h\|_{**,}$. It turns out that this operator is also bounded in a variation of these norms which allows a singularity at a point different from the origin. We have that given $\Lambda > 0$ there is a $C > 0$ such that for all $\xi \in \mathbb{R}^N$ with $|\xi| \le \Lambda$ we have that $\|\phi\|_{*,\xi} \le C\|h\|_{**,\xi}$, where

$$\|\phi\|_{*,\xi} = \sup_{|x-\xi|\le 1} |x-\xi|^\sigma |\phi(x)| + \sup_{|x-\xi|\ge 1} |x-\xi|^{\frac{2}{p-1}} |\phi(x)|$$

$$\|h\|_{**,\xi} = \sup_{|x-\xi|\le 1} |x-\xi|^{2+\sigma} |h(x)| + \sup_{|x-\xi|\ge 1} |x-\xi|^{2+\frac{2}{p-1}} |h(x)|.$$

The existence of such an operator with similar bounds persists if one drills a small hole in $\mathbb{R}^N$ and imposes Dirichlet boundary conditions on its boundary. Let us consider, for given ξ the set

$$\mathcal{D}_{\lambda,\xi} = \{\xi + \lambda z | z \in \mathcal{D}\}.$$

Then, let us consider the linear problem

$$\begin{cases} \Delta\phi + pw^{p-1}\phi = h + \displaystyle\sum_{i=1}^{N} c_i Z_i & \text{in } \mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi} \\ \displaystyle\lim_{|x|\to+\infty} \phi(x) = 0, \quad \phi = 0 \text{ on } \partial\mathcal{D}_{\lambda,\xi}. \end{cases} \tag{3.4}$$

We have the following result, whose proof can be carried out with arguments similar to those in Lemma 2.1.

Lemma 3.1. *Assume that $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$. Given $\Lambda > 0$ there is a $C > 0$ such that for all $|\xi| \le \Lambda$, all small $\lambda > 0$, and any h with $\|h\|_{**,\xi} < \infty$, Problem* (3.4) *has a solution $\phi = \mathcal{T}(h)$ which depends linearly on h such that*

$$\|\phi\|_{*,\xi} + \max_{1\le i\le N} |c_i| \le C\|h\|_{**,\xi}.$$

In order to apply this result to solve Problem (1.3)–(1.4), we observe first that a translation and a dilation makes it equivalent to

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi} \\ \displaystyle\lim_{|x|\to+\infty} u(x) = 0, \quad u = 0 \text{ on } \partial\mathcal{D}_{\lambda,\xi} \end{cases} \tag{3.5}$$

Let $\varphi_\lambda(z)$ be the unique solution of

$$\Delta\varphi_\lambda = 0 \text{ in } \mathbb{R}^N \setminus \mathcal{D}, \quad \varphi_\lambda(z) = w(\xi + \lambda z) \text{ on } \partial\mathcal{D}, \quad \lim_{|x|\to+\infty} \varphi_\lambda(x) = 0. \tag{3.6}$$

Then $\varphi_\lambda(z) = (w(\xi) + O(\lambda))\varphi_0(z)$ where φ_0 is the unique solution of

$$\Delta\varphi_0 = 0 \text{ in } \mathbb{R}^N \setminus \mathcal{D}, \quad \varphi_0(x) = 1 \text{ on } \partial\mathcal{D}, \lim_{|x|\to+\infty} \varphi_0(x) = 0. \tag{3.7}$$

We also note that

$$\lim_{|x|\to+\infty} |x|^{N-2}\varphi_0(x) = f_0 := \frac{1}{(N-2)|S^{N-1}|}\int_{\mathbb{R}^N\setminus\mathcal{D}} |\nabla\varphi_0|^2 > 0. \tag{3.8}$$

The number $\int_{\mathbb{R}^N\setminus\mathcal{D}} |\nabla\varphi_0|^2$ corresponds precisely to the capacity of $\mathcal{D}$.

We look for a solution of the form $u = w - \varphi_\lambda(\frac{x-\xi}{\lambda}) + \phi$, which yields the following equation for ϕ

$$\Delta\phi + pw^{p-1}\phi = N(\phi) + E_\lambda$$

where

$$E_\lambda = pw^{p-1}\varphi_\lambda, \quad N(\phi) = -(w+\phi-\varphi_\lambda)^p + w^p + pw^{p-1}\phi - pw^{p-1}\varphi_\lambda. \tag{3.9}$$

We consider the intermediate linear problem

$$\begin{cases} \Delta\phi + pw^{p-1}\phi = N(\phi) + E_\lambda + \displaystyle\sum_{i=1}^{N} c_i Z_i \text{ in } \mathbb{R}^N \setminus (\mathcal{D}_{\lambda,\xi}) \\ \phi = 0 \text{ on } \partial\mathcal{D}_{\lambda,\xi}, \displaystyle\lim_{|x|\to+\infty} \phi(x) = 0. \end{cases} \tag{3.10}$$

This nonlinear problem can be solved via contraction mapping principle based on the operator $\mathcal{T}$ above introduced in similar way as in the previous section, to yield existence of a unique solution with

$$\|\phi_\lambda\|_{*,\xi} + \max_{1\le i\le N} |c_i(\lambda,\xi)| \to 0 \quad \text{as } \lambda \to 0,$$

uniformly on $|\xi| \le \Lambda$. Besides, the numbers $c_i(\lambda,\xi)$ define continuous functions of ξ. We also have the estimate

$$\|\phi_\lambda\|_{*,\xi} \le C_\sigma \lambda^\sigma.$$

We recall that in the definition of the norms we are using an arbitrary σ with $0 < \sigma < N-2$. The desired result will be concluded if we manage to choose the point ξ in such a way that

$$c_i(\lambda,\xi) = 0 \quad \text{for all } i = 1,\ldots,N.$$

Testing the equation against Z_i, and using the above stated estimate for ϕ we see that these numbers can be expanded as

$$c_i(\lambda,\xi) = \int_{\mathbb{R}^N\setminus\mathcal{D}_{\lambda,\xi}} E_\lambda Z_i + \lambda^{N-2}o(1),$$

where the quantity $o(1)$ is uniform on $|\xi| \le \Lambda$. Now, we have that

$$\int_{\mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi}} E_\lambda Z_i = \int_{\mathbb{R}^N \setminus (\mathcal{D}_{\lambda,\xi})} \varphi_\lambda(\tfrac{x}{\lambda}) w^{p-1}(x+\xi) \frac{\partial w}{\partial x_i}(x+\xi) + o(\lambda^{N-2})$$
$$= \lambda^{N-2} \Big(f_0 \int_{\mathbb{R}^N} |x|^{-(N-2)} w^{p-1}(x+\xi) \frac{\partial w}{\partial x_i}(x+\xi) + o(1) \Big).$$

Hence we obtain, setting

$$F(\xi) := \frac{f_0}{2} \int_{\mathbb{R}^N} |x|^{2-N} w(x+\xi)^p \, dx.$$

that

$$\mathbf{c}(\xi,\lambda) := (c_1, \dots, c_N) = \lambda^{N-2} \big(\nabla F(\xi) + o(1) \big)$$

where $o(1) \to 0$ uniformly on $|\xi| \le \Lambda$. Observe that F is radial and has a nondegenerate maximum at $\xi = 0$. It follows that the Brouwer degree of $\mathbf{c}(\xi,\lambda)$ in a small ball around the origin is non zero. Hence there exists a point $\xi = \xi_\lambda$, small with λ, that annihilates all c_i's simultaneously. This concludes the proof of the theorem. $\square$

4. Sketch of proof of Theorem 1.2

The proof of Theorem 1.2 is similar to that of Theorem 1.1 in the range $p < \frac{N+1}{N-3}$, except that we no longer have a continuum: the parameter λ *needs also adjustment*. The basic object is now the positive solution of

$$\Delta w + w^{\frac{N+2}{N-2}} = 0 \quad \text{in } \mathbb{R}^N, \quad w(0) = 1,$$

given by

$$w(x) = c_N \left(\frac{1}{1+|x|^2} \right)^{\frac{N-2}{2}},$$

which of course has fast decay. By abuse of notation, we use the same nomenclature as in the previous section. The main difference arises in the linearized problem: a solvability condition is required when working at mode 0 with fast decay solutions: The right hand sides must now be orthogonal to the generator of dilations,

$$Z_0(r) = r w'(r) + (N-2) w(r), \quad r = |x|.$$

We still denote $Z_i = w_{x_i}$. Appropriate norms are now

$$\|\phi\|_{*,\xi} = \sup_{|x-\xi|\le 1} |x-\xi|^{\sigma}|\phi(x)| + \sup_{|x-\xi|\ge 1} |x-\xi|^{N-2}|\phi(x)|$$
$$\|h\|_{**,\xi} = \sup_{|x-\xi|\le 1} |x-\xi|^{2+\sigma}|h(x)| + \sup_{|x-\xi|\ge 1} |x-\xi|^{N+2}|h(x)|.$$

Now the following holds:

Lemma 4.1. *Let $p = \frac{N+2}{N-2} + \epsilon$ and $\Lambda > 0$. Then there is $\varepsilon_0 >$ such that for $|\xi| < \Lambda$ and $\varepsilon, \lambda < \varepsilon_0$ there exists ϕ, solution of the problem*

$$\begin{cases} \Delta\phi + pw^{p-1}\phi = N(\phi) + E_\lambda + c_0 Z_0 + \displaystyle\sum_{i=1}^{N} c_i Z_i \ \text{in } \mathbb{R}^N \setminus (\mathcal{D}_{\lambda,\xi}) \\ \phi = 0 \ \text{on } \partial\mathcal{D}_{\lambda,\xi}, \ \displaystyle\lim_{|x|\to+\infty} \phi(x) = 0. \end{cases} \tag{4.1}$$

We have in addition

$$\|\phi_\lambda\|_{*,\xi} + \max_{1\le i\le N} |c_i(\lambda)| \to 0 \quad \text{as } \lambda \to 0,$$

and

$$\|\phi_\lambda\|_{*,\xi} \le C_\sigma \lambda^\sigma, \qquad \text{for all } 0 < \lambda, \varepsilon < \varepsilon_0$$

where

$$0 < \sigma < N-2. \tag{4.2}$$

After this is proven, we need to set the $N+1$ parameters $c_i(\xi, \lambda) = 0$, $i = 0, 1, \ldots, N$. The computation of c_i's for $i \ge 1$ is identical. On the other hand, c_0 is at main order the quantity

$$\int_{\mathbb{R}^N} (E_\lambda) Z_0 \sim a\lambda^{N-2} - b\varepsilon,$$

for certain positive constants a and b. Thus the system essentially decouples and one can find ξ as in the previous theorem, and now λ, a quantity of order $\varepsilon^{\frac{1}{N-2}}$. $\square$

5. Sketch of the proof of Theorem 1.3

The proof of this result is similar in spirit to that of the previous theorems. Now the basic point is to obtain a suitable invertibility theory for the linearized operator $\Delta + pw^{p-1}$ on $\mathbb{R}^N \setminus B_1(0)$ where, again with abuse of notation, we are calling w the unique solution w_* of Problem (1.12)–(1.13). Thus, we consider the problem

$$\Delta\phi + pw^{p-1}\phi = h \quad \text{in } \mathbb{R}^N \setminus \bar{B}_1(0)\,, \tag{5.1}$$

$$\phi = 0 \ \text{ on } \partial B_1(0)\,, \qquad \lim_{|x|\to+\infty} \phi(x) = 0\,. \tag{5.2}$$

5.1. Condition for non-resonance

We want to investigate under what conditions the homogeneous problem with $h = 0$ in (5.1)–(5.2) admits only the trivial solution. To this end, we consider the first eigenvalue of the problem

$$\psi'' + \frac{N-1}{r}\psi' + pw^{p-1}\psi + \nu\frac{\psi}{r^2} = 0 \tag{5.3}$$

$$\psi(1) = 0, \quad \psi(+\infty) = 0\ . \tag{5.4}$$

This eigenvalue is variationally characterized as

$$\nu(p) = \inf_{\psi\in\mathcal{E}} \frac{\int_1^\infty |\psi'|^2 r^{N-1}dr - p\int_1^\infty w^{p-1}|\psi|^2 r^{N-1}dr}{\int_1^\infty \psi^2 r^{N-3}dr}\,, \tag{5.5}$$

with

$$\mathcal{E} \ = \{\psi \in C^1[1,\infty) \ / \ \psi(1) = 0,\ \int_1^\infty |\psi^{'}(r)|^2 r^{N-1}dr < +\infty\,\}.$$

This quantity is well defined thanks to Hardy's inequality,

$$\frac{(N-2)^2}{4}\int_1^\infty \psi^2 r^{N-3}dr \le \int_1^\infty |\psi'|^2 r^{N-1}dr.$$

The number $\nu(p)$ is negative, since this Rayleigh quotient gets negative when evaluated at $\psi = w$. An extremal is easily found, using the fast decay of $w^{p-1} = O(r^{-4})$. This extremal represents a positive solution to problem (5.3)–(5.4) for $\nu = \nu(p)$. Let us consider now Problem (5.1)–(5.2) for $h = 0$, and assume that we have a solution ϕ. The symmetry of the domain $\mathbb{R}^N \setminus B_1(0)$ allows us to expand ϕ into spherical harmonics. We write again ϕ as

$$\phi(x) = \sum_{k=0}^{\infty} \phi_k(r)\Theta_k(\theta), \quad r > 0,\ \theta \in S^{N-1}.$$

The components ϕ_k then satisfy the differential equations

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \left(pw^{p-1} - \frac{\lambda_k}{r^2}\right)\phi_k = 0, \quad r \in (1,\infty), \tag{5.6}$$

$$\phi_k(1) = 0, \quad \phi_k(+\infty) = 0.$$

Let us consider first the radial mode $k = 0$, namely $\lambda_k = 0$. We observe that the function

$$Z_1(r) = rw'(r) + \frac{2}{p-1}w$$

satisfies

$$Z_1'' + \frac{N-1}{r}Z_1' + pw^{p-1}Z_1 = 0, \qquad \text{for all } r > 1,$$

but $Z_1(1) \neq 0$. We notice that Z_1 is one-signed for all large r. It follows then that a second generator of the solutions of this ODE is given, for large r, by the reduction of order formula,

$$Z_2 = Z_1(r) \int_R^r \frac{dr}{r^{N-1}Z^2}$$

but since at main order $Z_1(r) \sim cr^{2-N}$ we see that $Z_2(+\infty) \neq 0$. Since ϕ_0 is a linear combination of Z_1 and Z_2 it follows that the only possibility is $\phi_0 = 0$. Let us consider now mode 1, namely $k = 1, \dots, N-1$, for which $\lambda_k = (N-1)$. In this case we also have an explicit solution which does not vanish at $r = 1$ but it does at $r = +\infty$. Simply $Z_1(r) = w'(r)$. But the same argument as above gives us a second generator $Z_2(r) \sim r$ as $r \to +\infty$, hence again, the only possibility is that $\phi_k \equiv 0$ for all $k = 1, \dots, N$.

Let us consider now modes $N+1$ or higher. This case is harder. Not only we do not have an explicit solution to the ODE to rely on, but it could be the case that a non-trivial solution exists. Let us assume this is the case for an arbitrary mode $k \geq N$. We claim that ϕ_k cannot change sign in $(1, \infty)$. In fact if it did, we begin by observing that it can only do it a finite number of times, since its behavior at infinity must be eventually like that of a decaying solution of the Euler's ODE

$$Z'' + \frac{N-1}{r}Z' - \frac{\lambda_k}{r^2}Z = 0$$

namely, at main order we must have

$$Z(r) = cr^{-\mu}(1 + o(1)), \quad \mu = -\frac{N-2}{2} - \frac{1}{2}\sqrt{(N-2)^2 + 4\lambda_k}\,.$$

Let $r_0 > 1$ be the last zero of ϕ_k, and let us assume that $\phi > 0$ on (r_0, ∞) We observe now that since $\Delta w < 0$, $w'(r)$ has exactly one zero in $(1, \infty)$. Thanks to Sturm's theorem this zero must be less than r_0. Hence $w' < 0$ in (r_0, ∞). Let us observe now that

$$W(r) = r^{N-1}(w'\phi_k' - w''\phi_k)$$

satisfies in (r, ∞)

$$W'(r) = r^{N-3}(\lambda_k - \lambda_1)w'\phi_k < 0 \quad \text{in } (r_0, \infty),$$

while $W(r_0) < 0$ and $W(+\infty) = 0$, which is impossible. This shows that ϕ_k must be one-signed. Thus the only possibility for equation (5.6) to have a nontrivial solution for a given $k \geq N$ is that $\lambda_k = -\nu(p)$. Thus we have proven the following result

Lemma 5.1. *Assume that p is such that*

$$\nu(p) \neq -j(N-2+j) \quad \text{for all } j = 2, 3, \dots \tag{5.7}$$

where $\nu(p)$ is the principal eigenvalue defined by (5.5)*. Then Problem* (5.3)–(5.4) *with $h = 0$ admits only the solution $\phi = 0$.*

This non-resonance condition produces a good solvability theory for equation (5.1)–(5.2). We can describe qualitatively the set of exponents p for which condition (5.7) fails. We have:

Lemma 5.2. *For each $j \geq 2$ the set of numbers p for which $\nu(p) = -j(N-2+j)$ is non-empty and finite. In particular, there exists a sequence of the form*

$$\frac{N+2}{N-2} < p_1 < p_2 < p_3 < \cdots ; \quad p_j \to +\infty \quad \text{as } j \to +\infty\,, \tag{5.8}$$

such that condition (5.7) *holds if and only if $p \neq p_j$ for all $j = 1, 2, \dots$.*

The proof of this result is contained in [11]. It consists of showing that the eigenvalue $\nu(p)$ is a real analytic function of the parameter p. A basic ingredient is the proof of analytic dependence of w as a function of p, in appropriate spaces, which follows basically form an analysis due to Dancer [5].

5.2. Solvability of (5.1)–(5.2)

We consider now the full problem (5.1)–(5.2), namely

$$\Delta\phi + pw^{p-1}\phi = h \quad \text{in } \mathbb{R}^N \setminus \bar{B}_1(0)\,,$$
$$\phi = 0 \ \text{ on } \partial B_1(0)\,, \quad \lim_{|x|\to+\infty} \phi(x) = 0\,.$$

Let us fix a small number $\sigma > 0$ and consider the norms

$$\|\phi\|_* = \sup_{|x|>1} |x|^{N-2-\sigma}|\phi(x)| + \sup_{|x|>1} |x|^{N-1-\sigma}|\nabla\phi(x)| \tag{5.9}$$

and

$$\|h\|_{**} = \sup_{|x|>1} |x|^{N-\sigma}|h(x)|. \tag{5.10}$$

Lemma 5.3. *Assume that p satisfies condition* (5.7)*. Then for any h with $\|h\|_{**} < +\infty$, Problem* (5.1)*–*(5.2) *has a unique solution $\phi = T(h)$ with $\|\phi\|_* < +\infty$. Besides, there exists a constant $C(p) > 0$ such that*

$$\|T(h)\|_* \leq C\|h\|_{**}.$$

5.3. The operator $\Delta + pw^{p-1}$ in $\delta^{-1}\mathcal{D} \setminus B_1(0)$

We assume that $Q = 0$, and consider the large expanded domain $\mathcal{D}_\delta = \delta^{-1}\mathcal{D}$. We shall carry out a gluing procedure that will permit to establish the same conclusion of Proposition 5.3 in this domain, provided that δ is taken sufficiently small. Thus we consider now the linear problem

$$\Delta\phi + pw^{p-1}\phi = h \quad \text{in } \mathcal{D}_\delta \setminus \bar{B}_1(0)\,, \tag{5.11}$$

$$\phi = 0 \ \text{ on } \partial B_1(0) \cup \partial\mathcal{D}_\delta. \tag{5.12}$$

We consider the same norms as in (5.9), (5.10) restricted to this domain.

Lemma 5.4. *Assume that p satisfies condition* (5.7)*. Then there is a number δ_0 such that for all $\delta < \delta_0$ and any h with $\|h\|_{**} < +\infty$, Problem* (5.11)*–*(5.12) *has a unique solution $\phi = T_\delta(h)$ with $\|\phi\|_* < +\infty$. Besides, there exists a constant $C(p, \mathcal{D}) > 0$ such that*

$$\|T_\delta(h)\|_* \leq C\|h\|_{**}.$$

The proof of this result follows a similar scheme to that of Lemma 2.1. The point now is that the fact that the linear theory involves faster decays makes the contribution of the far-away part of $\mathcal{D}_\delta$ to enter at a substantially small order. An analysis of this type *is not possible* if the basic cell w was taken as a slow-decaying solution.

5.4. Conclusion of the proof of Theorem 1.3

Let us assume the validity of condition (5.7) or, equivalently, that $p \neq p_j$ for all j, with p_j the sequence in (5.8). Problem (1.3)–(1.4) is, after setting $v(x) = \delta^{\frac{2}{p-1}}u(\delta x)$, equivalent to

$$\Delta v + v^p = 0 \quad \text{in } \mathcal{D}_\delta \setminus \bar{B}_1(0)\,, \tag{5.13}$$

$$v = 0 \ \text{ on } \partial B_1(0) \cup \partial\mathcal{D}_\delta. \tag{5.14}$$

Let us consider the smooth cut-off function η_δ, introduced in the previous section, which equals 1 in $B(0, 2\delta^{-1})$ and 0 outside $B(0, 3\delta^{-1})$. We search for a solution v to problem (5.13)–(5.14) of the form

$$v = \eta_\delta w + \phi,$$

which is equivalent to the following problem for ϕ:

$$\Delta\phi + pw^{p-1}\phi = N(\phi) + E \quad \text{in } \mathcal{D}_\delta \setminus \bar{B}_1(0)\,, \tag{5.15}$$

$$\phi = 0 \quad \text{on } \partial B_1(0) \cup \partial\mathcal{D}_\delta. \tag{5.16}$$

where

$$N(\phi) = N_1(\phi) + N_2(\phi)\,,$$

$$N_1(\phi) = -(\eta_\delta w + \phi)^p + (\eta_\delta w)^p + p(\eta_\delta w)^{p-1}\phi,$$

$$N_2(\phi) = p(1 - \eta_\delta^{p-1})w^{p-1}\phi,$$

and

$$E = -\Delta(\eta_\delta w) - (\eta_\delta w)^p.$$

According to Proposition 5.4 we thus have a solution to (5.13)–(5.14) if ϕ solves the fixed point problem

$$\phi = T_\delta(N(\phi) + E)\,. \tag{5.17}$$

We get

$$\|E\|_{**} \le C\delta^\sigma\,. \tag{5.18}$$

On the other hand, we also find

$$\|N_2(\phi)\|_{**} \le C\delta^2\|\phi\|_*$$

and so that

$$\|N_1(\phi)\|_{**} \le C\,(\,\|\phi\|_*^p + \|\phi\|_*^2\,)\,. \tag{5.19}$$

Let us consider now the operator

$$\mathcal{T}(\phi) = T_\delta(N(\phi) + E)$$

defined in the region

$$\mathcal{B} = \{\phi \in C^1(\bar{\mathcal{D}}_\delta \setminus B_1(0)) \;/\; \|\phi\|_* \le \delta^{\frac{\sigma}{2}}\,\}\,.$$

We immediately get that $\mathcal{T}(\mathcal{B}) \subset \mathcal{B}$, provided that δ is sufficiently small. The existence of a fixed point thus follows from Schauder's theorem. $\square$

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