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"Bubble-tower" radial solutions in the slightly supercritical Brezis–Nirenberg problem

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Abstract

In this paper, we consider the Brezis–Nirenberg problem in dimension $N \geqslant 4$, in the supercritical case. We prove that if the exponent gets close to $\frac{N+2}{N-2}$ and if, simultaneously, the bifurcation parameter tends to zero at the appropriate rate, then there are radial solutions which behave like a superposition of *bubbles*, namely solutions of the form

$$\gamma \sum_{j=1}^{k} \left(\frac{1}{1 + M_j^{\frac{4}{N-2}} |y|^2} \right)^{(N-2)/2} M_j(1 + o(1)), \quad \gamma = (N(N-2))^{(N-2)/4},$$

where $M_j \to +\infty$ and $M_j = o(M_{j+1})$ for all j. These solutions lie close to turning points "to the right" of the associated bifurcation diagram.

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Keywords: Brezis–Nirenberg problem; Supercritical exponent; Bifurcation diagram; Lyapunov–Schmidt reduction

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1. Introduction

This paper deals with the analysis of solutions to the problem

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}+\varepsilon} + \lambda u & \text{in } B, \\ u>0 & \text{in } B, \\ u=0 & \text{on } \partial B, \end{cases} \tag{1.1}$$

where B denotes the unit ball in $\mathbb{R}^N$, $N\geqslant 4$, and $\varepsilon>0$ is a small parameter. In a celebrated paper, Brezis and Nirenberg [4] established that this problem for $\varepsilon=0$, in a general bounded smooth domain, is solvable for $0<\lambda<\lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ under Dirichlet boundary conditions. This result is optimal, since integrating the equation against a first eigenfunction of the Laplacian yields $\lambda<\lambda_1$. On the other hand, Pohozaev's identity [15], gives nonexistence for $\lambda\leqslant 0$, for any $\varepsilon\geqslant 0$, in star-shaped domains.

Let us consider a family of solutions u_ε of (1.1) for $\lambda=\lambda_\varepsilon\to 0$. It is well known that these solutions must be radially symmetric and radially decreasing [11], so that they maximize at the origin. Since the limiting problem $\lambda=0$, $\varepsilon=0$ does not possess any solution, it follows that

$$M_\varepsilon = \gamma^{-1}\max u_\varepsilon = \gamma^{-1}u_\varepsilon(0)\to +\infty$$

for some fixed constant $\gamma>0$, to be chosen later. Setting $p=\frac{N+2}{N-2}$, the scaled function $v_\varepsilon(z)=M_\varepsilon u_\varepsilon(M_\varepsilon^{(p+\varepsilon-1)/2}z)$, satisfies

$$\Delta v_\varepsilon + v_\varepsilon^{p+\varepsilon} + M_\varepsilon^{-(p+\varepsilon-1)}\lambda_\varepsilon v_\varepsilon = 0, \quad |z|<M_\varepsilon^{(p+\varepsilon-1)/2}.$$

Elliptic regularity implies that locally over compacts around the origin, $v_\varepsilon(z)$ converges to the unique positive radial solution of

$$\Delta w + w^p = 0$$

in entire space, with $w(0)=\gamma$. As it is well known, [2,17], for the convenient choice $\gamma=(N(N-2))^{\frac{N-2}{4}}$, this solution is explicitly given by

$$w(z)=\gamma\left(\frac{1}{1+|z|^2}\right)^{\frac{N-2}{2}}.$$

Coming back to the original variable, we expect then that "near the origin" the behavior of $u_\varepsilon(y)$ can be approximated as

$$u_\varepsilon(y)=\gamma\left(\frac{1}{1+M_\varepsilon^{\frac{4}{N-2}}|y|^2}\right)^{\frac{N-2}{2}} M_\varepsilon(1+o(1)). \tag{1.2}$$

A point to be made is that since the convergence in expanded variables is only local over compacts, it is not at all clear how far from the origin the approximation (1.2) holds true. Roughly speaking, we refer to a solution $u_\varepsilon(y)$ for which (1.2) holds with $o(1)\to 0$ uniformly in B, as a *single-bubble solution*.

One question we intend to respond in this work is in which range for $\lambda = o(1)$, depending on ε, one can actually see bubbling solutions. A new phenomenon, somewhat surprising, is that much more than single-bubble solutions is going on in this problem: we find the presence of *towers* constituted by superposition of bubbles of different blow-up orders, so that estimate (1.2) does not hold globally. In fact, given any number $k\geqslant 1$, there is an ε-dependent range for λ for which there exist solutions of the form

$$u_\varepsilon(y) = \gamma \sum_{j=1}^{k} \left(\frac{1}{1 + M_j^{\frac{4}{N-2}} |y|^2} \right)^{\frac{N-2}{2}} M_j (1 + o(1)) \quad \text{as } y \to 0,$$

where $M_j \to +\infty$ and $M_j = o(M_{j+1})$ for all j and $\gamma = (N(N-2))^{\frac{4}{N-2}}$ (see Fig. 8). This is in strong contrast with the case in which $\varepsilon = 0$ and one lets $\lambda \downarrow 0$ or $\lambda = 0$ and $\varepsilon \uparrow 0$ where only a single bubble is present, as established by Brezis and Peletier [5], also see [12,16].

For the precise statement of our results, we need to distinguish between the cases $N\geqslant 5$ and $N = 4$. For simplicity in the exposition, we restrict ourselves in this introduction to the case $N\geqslant 5$ and postpone to a last section the changes in statement and proof needed for $N = 4$.

Theorem 1. *Assume $N\geqslant 5$. Then, given an integer $k\geqslant 1$, there exists a number $\mu_k > 0$ such that if $\mu > \mu_k$ and*

$$\lambda = \mu \varepsilon^{\frac{N-4}{N-2}},$$

then there are constants $0 < \alpha_j^- < \alpha_j^+$, $j = 1, \dots, k$ which depend on k, N and μ and two solutions $u_\varepsilon^\pm$ of problem (1.1) *of the form*

$$u_\varepsilon^\pm(y) = \gamma \sum_{j=1}^{k} \left(\frac{1}{1 + [\alpha_j^\pm \varepsilon^{\frac{1}{2}-j}]^{\frac{4}{N-2}} |y|^2} \right)^{\frac{N-2}{2}} \alpha_j^\pm \varepsilon^{\frac{1}{2}-j} (1 + o(1)), \tag{1.3}$$

where $\gamma = (N(N-2))^{\frac{N-2}{4}}$ and $o(1)\to 0$ uniformly on B as $\varepsilon \to 0$.

The solutions predicted by the above result constitute a superposition of k bubbles, each of which has height of order $\varepsilon^{\frac{1}{2}-j}$ for $j = 1, \dots, k$. We mention that for $\lambda = 0$, nonradial solutions exhibiting multiple isolated bubbles blowing up like $\sqrt{\varepsilon}$ exist for some special domains, see [7,8].

The proof actually provides the explicit values of the constants $\alpha_j^{\pm}$ as follows. Given $k \geqslant 1$, let us consider the function

$$f_k(s) = kb_1 s^{\frac{4}{N-2}} + b_2 s^{-2\frac{N-4}{N-2}}, \tag{1.4}$$

where

$$b_1 = \left(\frac{N-2}{4}\right)^3 \frac{N-4}{N-1}, \quad b_2 = (N-2)\frac{\Gamma(N-1)}{\Gamma\left(\frac{N-4}{2}\right)\Gamma\left(\frac{N}{2}\right)}. \tag{1.5}$$

Let μ_k be the minimum value of the function $f_k(s)$, namely

$$\mu_k = (N-2)\left[\frac{b_1 k}{N-4}\right]^{\frac{N-4}{N-2}} \left[\frac{b_2}{2}\right]^{\frac{2}{N-2}}, \tag{1.6}$$

which is attained at $s = s_k$ given by

$$s_k = \left[\frac{(N-4)b_2}{2b_1 k}\right]^{\frac{1}{2}}.$$

Then, given $\mu > \mu_k$, the equation

$$\mu = f_k(s)$$

has exactly two solutions

$$0 < s_k^-(\mu) < s_k < s_k^+(\mu).$$

The numbers $\alpha_j^{\pm}$ can be expressed by the formulae

$$\alpha_j^{\pm} = b_3^{1-j} \frac{(k-j)!}{(k-1)!} s_k^{\pm}(\mu), \quad j = 1, \dots, k,$$

where

$$b_3 = \frac{(N-2)\sqrt{\pi}\,\Gamma\left(\frac{N}{2}\right)}{2^{N+2}\Gamma\left(\frac{N+1}{2}\right)}.$$

See Lemmas 1 and 2 in Section 2 for an explanation of why these numbers enter into the game. Let us notice that we also obtain the following multiplicity assertion: given $k \geqslant 1$ and $\lambda = \mu \varepsilon^{(N-4)/(N-2)}$ with $\mu > \mu_k$, then there is an $\varepsilon_0 > 0$ depending on k and μ, such that there are at least $2k$ solutions to problem (1.1).

The facts described above have an interesting interpretation in terms of the bifurcation diagram for positive solutions of (1.1), for small ε. The positive solutions in the (λ, u) space can be identified with a C^1 curve in the (λ, m)-plane, where

$$m = u(0) = ||u||_\infty.$$

This curve stems from $(\lambda, m) = (\lambda_1, 0)$. For $\varepsilon = 0$ the positive solution of (1.1) is unique for each $0 < \lambda < \lambda_1$, see [14,18]. Hence, the curve goes left, without turning points, blowing up as $\lambda \to 0$, see Fig. 1.

Budd and Norbury [6] studied the supercritical case $\varepsilon > 0$ and derived qualitative properties of this bifurcation branch. In particular, formal asymptotics and numerical computations suggest that the following takes place: before reaching $\lambda = 0$, the curve turns right and then oscillates infinitely many times in the form of an exponentially damped oscillating curve along a line $\lambda = \lambda_*$, see Figs. 2 and 3. Merle and Peletier [13] established rigorously the existence of a unique value $\lambda = \lambda_* > 0$ for which necessarily $\lambda_n \to \lambda_*$ whenever u_n is an unbounded sequence of solutions of (1.1) with $\lambda = \lambda_n$. A radial, singular, positive solution exists for this value of λ (and only for this one).

Our result leads in particular to a rigorous description of the kth turning point P_k^ε "to the right" of the bifurcation curve in the (λ, m) quadrant (see Fig. 4): what we find then is that

$$P_k^\varepsilon \sim (\mu_k \varepsilon^{\frac{N-4}{N-2}}, c_k \varepsilon^{\frac{1}{2}-k}),$$

where c_k is given by

$$c_k = \frac{\gamma s_k b_3^{1-k}}{(k-1)!},$$

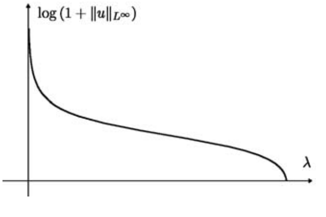

Fig. 1. $\varepsilon = 0$, the bifurcation diagram in the critical case.

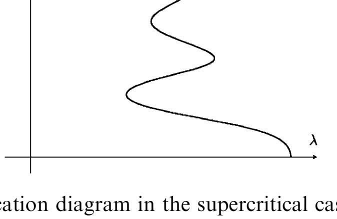

Fig. 2. $\varepsilon > 0$, the bifurcation diagram in the supercritical case (here $\varepsilon = \varepsilon_0 = 0.2$).

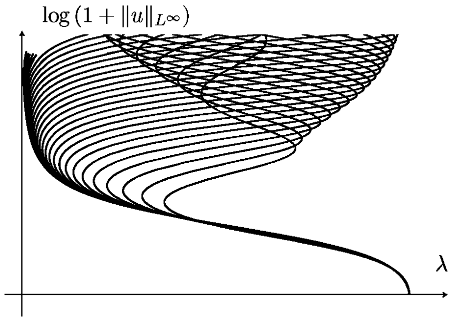

Fig. 3. Approximating the critical case: $\varepsilon = 2^{-q}\varepsilon_0$, $\mathbb{N} \ni q \to \infty$, $\varepsilon_0 = 0.2$.

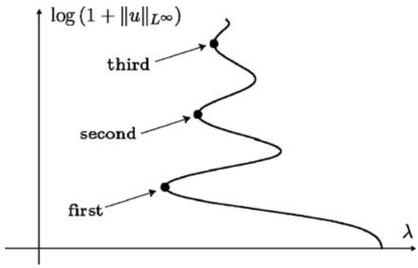

Fig. 4. The bifurcation diagram corresponding to $N = 5$, $p = 7/3$ and $\varepsilon = 0.2$, with the first three *turning* points *to the right* of the bifurcation diagram.

since for $\mu = \mu_k$, $s_k^{\pm}(\mu_k) = s_k$, so that $c_k = \gamma\alpha_k^{\pm}$. The curve itself is approximated in the (λ, m)-plane by the graph

$$\lambda = \varepsilon^{\frac{N-4}{N-2}} f_k(c_k^{-1}\varepsilon^{k-\frac{1}{2}}m) \quad \text{for } m \sim \varepsilon^{\frac{1}{2}-k}.$$

We may notice that consecutive turning points are spaced at distances that increase exponentially, so that for small ε the shape of the bifurcation curve is not quite a damped sinusoidal if one zooms down around the first given k right turns.

The method of proof of Theorem 1 consists of transforming the problem of finding a k-bubble solution into the problem of finding a k-bump solution of a second-order equation on the half-line obtained after the so-called Emden–Fowler transformation (see Figs. 5–8). After a procedure of finite-dimensional reduction, which has been used in the analysis of many singularly perturbed elliptic equations, introduced originally in the one-dimensional case by Floer and Weinstein [9], the problem becomes that of finding a critical point of a functional depending on k real parameters. In the predicted range for λ, this functional is a small C^1-perturbation of one having two nondegenerate critical points of Morse indices $k-1$ and k,

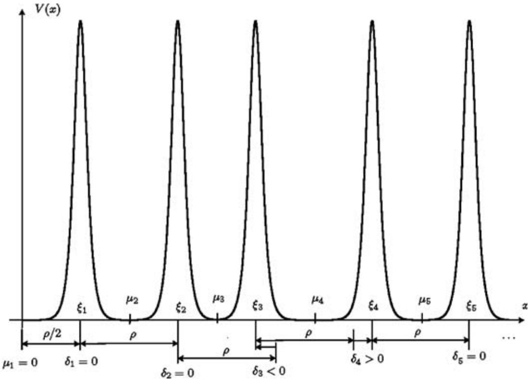

Fig. 5. The function V is defined by $V = \sum_{i=1}^{k} V_i$, where $V_i = U_i + \pi_i$, $U_i(x) = U(x - \xi_i)$ and $\pi_i(x) = -U(\xi_i)e^{-x}$. The function U is the unique solution of (2.5). The intervals are defined by $\xi_1 = \frac{\rho}{2} + \delta_1$ and $\xi_i = \xi_{i-1} + \rho + \delta_i$ for $i = 2, \ldots, k$, where $\rho = -\log \varepsilon$ tends to $+\infty$ and $|\delta_i| \leqslant K$ for some fixed constant K.

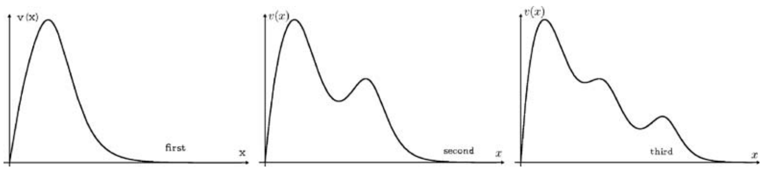

Fig. 6. Functions corresponding to the first three *turning* points *to the right* in the previous bifurcation diagram, with $\varepsilon = 0.2$, after the transformation:

$$v(x) = \left(\frac{2}{p-1}\right)^{\frac{2}{p-1+\varepsilon}} e^{-x} u(e^{-\frac{p-1}{2}x}).$$

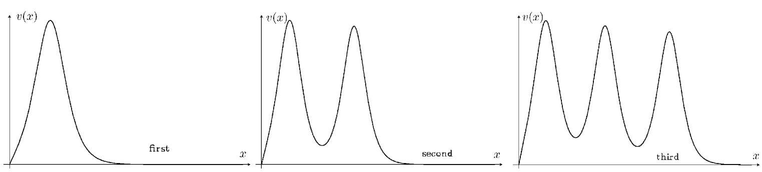

Fig. 7. Functions corresponding to the first three *turning* points *to the right* in the bifurcation diagram, corresponding now to $\varepsilon = 0.01$.

respectively. Although we will not elaborate around that point, it is an interesting by-product of the construction that the two corresponding k-tower solutions have inherited Morse indices, respectively, $2k - 1$ and $2k$ as critical points of the full energy functional of the problem.

We do not treat in this paper the case $N = 3$ nor do we attempt to describe the turning points "to the left" in the bifurcation curve, which are interesting questions

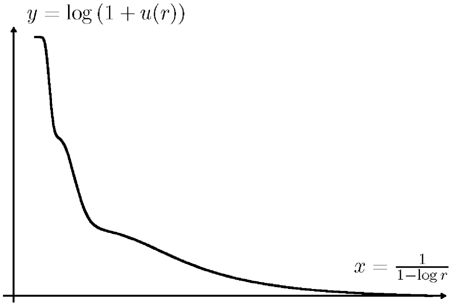

Fig. 8. A three-bubble solution u of (1.1) corresponding to the three bumps solution v of Fig. 7, with $\varepsilon = 0.01$. Appropriate scales have been chosen.

in their own right. We recall that in [4], existence is found for $\varepsilon = 0$ and $N = 3$ if and only if $\frac{\lambda_1}{4} < \lambda < \lambda_1$.

On the other hand, several works have dealt with changing-sign solutions, namely bifurcation from higher eigenvalues [1,3]. We believe that the method developed here may also apply to the construction of tower solutions with sign changes.

The next three sections will be devoted to the proof of Theorem 1: we first perform the asymptotic expansion which is the key of the method, then solve a nonlinear problem corresponding to a finite-dimensional reduction and finally solve the finite-dimensional problem. Section 5 is devoted to the statement and the proof of the result in the case $N = 4$.

Throughout this paper, we adopt the following *notations*. By y we denote the variable in the unit ball B in $\mathbb{R}^N$, $N \geqslant 4$, $z \in \mathbb{R}^N$ is given in terms of y after an appropriate scaling, $r = |y| \in (0,1)$ (resp. $r \in \mathbb{R}$) is transformed after a variant of the so-called Emden–Fowler transformation into a variable $x \in (0, +\infty)$ (resp. $x \in \mathbb{R}$). We take $p = \frac{N+2}{N-2}$ and in the rest of this paper, ε is a nonnegative small parameter.

2. The asymptotic expansion

The problem of finding radial solutions u to problem (1.1) corresponds to that of solving the boundary value problem

$$u'' + \frac{N-1}{r} u' + u^{p+\varepsilon} + \lambda u = 0, \quad u'(0) = 0, \; u(1) = 0. \tag{2.1}$$

Here and in what follows, $p = \frac{N+2}{N-2}$ and we write abusively $u = u(r)$ with $r = |y|$. We transform the problem by means of the following change of variable:

$$v(x) = \left(\frac{2}{p-1}\right)^{\frac{2}{p-1+\varepsilon}} r^{\frac{2}{p-1}} u(r) \quad \text{with } r = e^{-\frac{p-1}{2}x}, \; x \in (0, +\infty),$$

a variation of the so-called Emden–Fowler transformation, first introduced in [10]. Problem (2.1) then becomes

$$\begin{cases} v'' - v + e^{\varepsilon x} v^{p+\varepsilon} + \left(\dfrac{p-1}{2}\right)^2 \lambda e^{-(p-1)x} v = 0 \quad \text{on } (0, \infty), \\ v(0) = 0, \quad v > 0, \quad v(x) \to 0 \text{ as } x \to +\infty. \end{cases} \tag{2.2}$$

The functional associated to problem (2.2) is given by

$$E_\varepsilon(w) = I_\varepsilon(w) - \frac{1}{2}\left(\frac{p-1}{2}\right)^2 \lambda \int_0^\infty e^{-(p-1)x} |w|^2 \, dx \tag{2.3}$$

with

$$I_\varepsilon(w) = \frac{1}{2}\int_0^\infty |w'|^2 \, dx + \frac{1}{2}\int_0^\infty |w|^2 \, dx - \frac{1}{p+\varepsilon+1}\int_0^\infty e^{\varepsilon x} |w|^{p+\varepsilon+1} \, dx. \tag{2.4}$$

Let us consider the unique solution $U(x)$ to the problem

$$\begin{cases} U'' - U + U^p = 0 \quad \text{on } (-\infty, \infty), \\ U'(0) = 0, \\ U > 0, \quad U(x) \to 0 \quad \text{as } x \to \pm\infty. \end{cases} \tag{2.5}$$

This solution is nothing but the one given by the Emden–Fowler transformation (with $\varepsilon = 0$) of the radial solution of $\Delta w + w^p = 0$,

$$w(r) = \gamma \left(\frac{1}{1+r^2}\right)^{\frac{N-2}{2}} \quad \text{with } \gamma = (N(N-2))^{\frac{N-2}{4}},$$

namely

$$U(x) = \left(\frac{4N}{N-2}\right)^{\frac{N-2}{4}} e^{-x}\left(1 + e^{-\frac{4}{N-2}x}\right)^{-\frac{N-2}{2}}. \tag{2.6}$$

Let us consider points $0 < \xi_1 < \xi_2 < \cdots < \xi_k$. We look for a solution of (2.2) of the form

$$v(x) = \sum_{i=1}^{k} (U(x - \xi_i) + \pi_i) + \phi,$$

where ϕ is small and $\pi_i(x) = -U(\xi_i) e^{-x}$ (see Fig. 5). The correction π_i is meant to make the ansatz satisfy the Dirichlet boundary conditions. A main observation is that $v(x) \sim \sum_{i=1}^{k} U(x-\xi_i)$ solves (2.2) if and only if (going back in the change of variables)

$$u(r) \sim \gamma \sum_{i=1}^{k} \left(\frac{1}{1+e^{\frac{4\xi_i}{N-2}} r^2} \right)^{\frac{N-2}{2}} e^{\xi_i}$$

solves (2.2), see Fig. 8. Therefore, the ansatz given for v provides (for large values of the ξ_i's), a bubble-tower solution for (1.1) with $M_i = e^{\xi_i}$.

Let us write

$$U_i(x) = U(x-\xi_i), \quad V_i = U_i + \pi_i, \quad \pi_i(x) = -U(\xi_i)e^{-x}, \quad V = \sum_{i=1}^{k} V_i. \tag{2.7}$$

It is easily checked that V_i is nonnegative on $\mathbb{R}^+$. We shall work out asymptotics for the energy functional associated at the function V, assuming that the numbers ξ_i are large and also very far apart but at comparable distances from each other.

We make the following choices for the points ξ_i:

$$\begin{aligned} &\xi_1 = -\frac{1}{2}\log\varepsilon + \log \Lambda_1, \\ &\xi_{i+1} - \xi_i = -\log\varepsilon - \log \Lambda_{i+1}, \quad i = 1, \dots, k-1, \end{aligned} \tag{2.8}$$

where the Λ_i's are positive parameters. For notational convenience, we also set $\Lambda = (\Lambda_1, \Lambda_2, \dots, \Lambda_k)$. The advantage of the above choice is the validity of the expansion of the energy E_ε defined by (2.3) given as follows.

Lemma 1. *Let $N \geqslant 5$. Fix a small number $\delta > 0$ and assume that*

$$\delta < \Lambda_i < \delta^{-1} \quad \textit{for all } i = 1, \dots, k. \tag{2.9}$$

Assume also that $\lambda = \mu\varepsilon^{\frac{N-4}{N-2}}$ for some $\mu > 0$. Let V be given by (2.7). *Then, with the choice* (2.8) *of the points ξ_i, there are positive numbers a_i, $i = 0, \dots, 5$, depending only on N* (*which have the explicit expressions* (2.19)) *such that the following expansion holds*:

$$E_\varepsilon(V) = ka_0 + \varepsilon\Psi_k(\Lambda) + \frac{k^2}{2}a_3\varepsilon\log\varepsilon + a_5\varepsilon + \varepsilon\theta_\varepsilon(\Lambda), \tag{2.10}$$

where

$$\begin{aligned} \Psi_k(\Lambda) = &\, a_1\Lambda_1^{-2} - ka_3\log\Lambda_1 - a_4\mu\Lambda_1^{-(p-1)} \\ &+ \sum_{i=2}^{k} [(k-i+1)a_3\log\Lambda_i - a_2\Lambda_i], \end{aligned} \tag{2.11}$$

and as $\varepsilon \to 0$, the term $\theta_\varepsilon(\Lambda)$ converges to 0 uniformly and in the C^1-sense on the set of Λ_i's satisfying constraints (2.9).

Proof. We will estimate the different terms in the expansion of $E_\varepsilon(V)$ with V defined by (2.7), for the ξ_i's given by (2.8). Let I_ε be the functional in (2.4). We may write

$$I_\varepsilon(V) = I_0(V) - \frac{1}{p+1}\int_0^\infty (e^{\varepsilon x}-1)|V|^{p+\varepsilon+1}\,dx + A_\varepsilon,$$

$$A_\varepsilon = \left(\frac{1}{p+1} - \frac{1}{p+\varepsilon+1}\right)\int_0^\infty e^{\varepsilon x}|V|^{p+\varepsilon+1}\,dx$$

$$+ \frac{1}{p+1}\int_0^\infty e^{\varepsilon x}(|V|^{p+1} - |V|^{p+\varepsilon+1})\,dx.$$

Then, we find that

$$A_\varepsilon = k\varepsilon\left(\frac{1}{p+1}\int_{-\infty}^\infty U^{p+1}\log U\,dx + \frac{1}{(p+1)^2}\int_{-\infty}^\infty U^{p+1}\,dx\right) + o(\varepsilon). \tag{2.12}$$

On the other hand, for the same reason, we have

$$\begin{aligned}\int_0^\infty (e^{\varepsilon x}-1)V^{p+\varepsilon+1}\,dx &= \varepsilon\int_0^\infty xV^{p+\varepsilon+1}\,dx + o(\varepsilon)\\ &= \varepsilon\left(\sum_{i=1}^k \xi_i\right)\int_{-\infty}^\infty U^{p+1}\,dx + o(\varepsilon).\end{aligned} \tag{2.13}$$

Now, we have the validity of the identity

$$I_0(V) = \sum_{i=1}^k I_0(V_i) + \frac{1}{p+1}B, \tag{2.14}$$

where

$$B = \int_0^\infty\left[\sum_{i=1}^k V_i^{p+1} - \left(\sum_{i=1}^k V_i\right)^{p+1} + (p+1)\sum_{i<j}\int_0^\infty U_i^pV_j\right]dx.$$

Indeed we have

$$\begin{aligned}&\frac{1}{p+1}B - \int_0^\infty\left[\sum_{i=1}^k V_i^{p+1} - \left(\sum_{i=1}^k V_i\right)^{p+1}\right]dx\\ &= \sum_{i<j}\int_0^\infty (V_i'V_j' + V_iV_j)\,dx\\ &= \sum_{i<j}\int_0^\infty(-V_i'' + V_i)V_j\,dx = \sum_{i<j}\int_0^\infty U_i^pV_j\,dx\end{aligned}$$

since $V_j(0) = 0$ and $\pi_i'' = \pi_i$. To estimate this latter quantity, we consider the numbers

$$\mu_1 = 0, \quad \mu_l = \frac{1}{2}(\xi_{l-1} + \xi_l), \;\; l = 2, \dots, k, \quad \mu_{k+1} = +\infty,$$

and decompose B as $B = -C_0 + C_1 + C_2$, where

$$C_0 = (p+1) \sum_{\substack{1 \leqslant l \leqslant k \\ j > l}} \int_{\mu_l}^{\mu_{l+1}} V_l^p V_j \, dx.$$

$$C_1 = \sum_{l=1}^{k} \int_{\mu_l}^{\mu_{l+1}} \left[V_l^{p+1} - \left(V_l + \sum_{i \neq l} V_i \right)^{p+1} + (p+1) \sum_{j \neq l} V_l^p V_j \right] dx.$$

and $C_2 \equiv B + C_0 - C_1$. Note that all these quantities depend on ε because of (2.8). First, let us estimate C_1. Using the mean value theorem, the fact that $V_i(x) \leqslant Ce^{-|x-\xi_i|}$ and setting $\rho = \log \frac{1}{\varepsilon}$, we get, using (2.8),

$$\begin{aligned}
|C_1| &\leqslant C \sum_{l=1}^{k} \int_{\mu_l}^{\mu_{l+1}} \left(V_l + \sum_{i \neq l} V_i \right)^{p-1} \left(\sum_{i \neq l} V_i \right)^2 dx \\
&\leqslant C \int_0^{\frac{\rho}{2}+K} e^{-(p-1)x} e^{-2|x-\rho|} \, dx \\
&\leqslant Ce^{-2\rho} \int_0^{\frac{\rho}{2}+K} e^{-(p-3)x} \, dx = O(e^{-\frac{p+1}{2}\rho}) = o(\varepsilon).
\end{aligned}$$

The above constant K depends only on δ. Similar considerations on the terms constituting C_2 yields $C_2 = o(\varepsilon)$. Let us now estimate C_0. First we observe that

$$C_0 = (p+1) \sum_{l=1}^{k} \int_{\mu_l}^{\mu_{l+1}} U_l^p U_{l+1} \, dx + o(\varepsilon).$$

Now, we have that

$$\int_{\mu_l}^{\mu_{l+1}} U_l^p U_{l+1} \, dx = \int_{\mu_l - \xi_l}^{\mu_{l+1} - \xi_l} U^p(x) U(x - (\xi_{l+1} - \xi_l)) \, dx.$$

On the other hand, according to (2.6), $U(x) = C_N [\mathrm{ch}(\frac{2x}{N-2})]^{-(N-2)/2}$, with $C_N = (\frac{N}{N-2})^{(N-2)/4}$. It is directly checked that

$$|U(x - \xi) - C_N e^{-|\xi - x|}| = O(e^{-p|\xi - x|})$$

as $\xi \to +\infty$. We conclude then that

$$C_0 = (p+1)\sum_{l=1}^{k-1} e^{-|\xi_{l+1}-\xi_l|} C_N \int_{-\infty}^{\infty} e^x U(x)^p\,dx + o(\varepsilon).$$

Collecting the above estimates, we find that

$$B = -a_2 \sum_{l=1}^{k-1} e^{-|\xi_{l+1}-\xi_l|} + o(\varepsilon) \tag{2.15}$$

with $a_2 = (p+1)C_N \int_{-\infty}^{\infty} e^x U(x)^p\,dx$.

Continuing our estimate of $I_\varepsilon(V)$, we have now to consider $I_0(V_i)$ for $i = 1, \ldots, k$. We begin with $i = 1$. We have

$$I_0(V_1) = I_0(U_1 + \pi_1) = I_0(U_1) + DI_0(U_1)[\pi_1] + \frac{1}{2} D^2 I_0(U_1 + s\pi_1)[\pi_1, \pi_1]$$

for some $s \in (0,1)$. We recall that $\pi_1(x) = -U_1(0)e^{-x}$. First we get

$$DI_0(U_1)[\pi_1] = \int_0^{\infty} (U_1{}'\pi_1{}' + U_1\pi_1 - U_1^p \pi_1)\,dx,$$

so that, integrating by parts and using the equation satisfied by U_1 we get

$$DI_0(U_1)[\pi_1] = U_1{}'(0)U_1(0) = U_1^2(0) + o(\varepsilon).$$

Now,

$$\frac{1}{2} D^2 I_0(U_1 + s\pi_1)[\pi_1, \pi_1] = \frac{1}{2}\int_0^{\infty} (|\pi_1{}'|^2 + \pi_1^2 - p(U_1 + s\pi_1)^{p-1}\pi_1^2)\,dx.$$

We observe that

$$\frac{1}{2}\int_0^{\infty} (|\pi_1{}'|^2 + \pi_1^2)\,dx = \frac{1}{2} U_1(0)^2,$$

and that $\int_0^{\infty} (U_1 + s\pi_1)^{p-1}\pi_1^2\,dx = o(\varepsilon)$. Now, let us set

$$a_0 = \frac{1}{2}\int_{-\infty}^{\infty} (|U'|^2 + U^2)\,dx - \frac{1}{p+1}\int_{-\infty}^{\infty} U^{p+1}\,dx.$$

Then

$$I_0(U_1) = a_0 - \left[\frac{1}{2}\int_{-\infty}^{0} (|U_1{}'|^2 + U_1^2)\,dx - \frac{1}{p+1}\int_{-\infty}^{0} U_1^{p+1}\,dx\right].$$

It turns out that $\int_{-\infty}^{0} U_1^{p+1}\,dx = o(\varepsilon)$ and $\frac{1}{2}\int_{-\infty}^{0}({U_1'}^2 + U_1^2)\,dx = \frac{1}{2}U_1^2(0) + o(\varepsilon)$. Combining the above estimates, we obtain

$$I_0(V_1) = a_0 + U_1^2(0) + o(\varepsilon). \tag{2.16}$$

Similar arguments give us that

$$I_0(V_i) = a_0 + o(\varepsilon) \quad \text{for all } i \geqslant 2. \tag{2.17}$$

Finally, as for the last term in the decomposition (2.3), we easily check that

$$\lambda \int_0^\infty e^{-(p-1)x}|V|^2\,dx = \lambda e^{-(p-1)\xi_1}\int_{-\infty}^{\infty} e^{-(p-1)x}|U(x)|^2\,dx + o(\varepsilon). \tag{2.18}$$

Summarizing, we obtain from estimates (2.12)–(2.18) the validity of the following expansion:

$$\begin{aligned} E_\varepsilon(V) = {} & ka_0 + a_1 e^{-2\xi_1} - a_2 \sum_{l=1}^{k} e^{-|\xi_{l+1}-\xi_l|} - a_3\varepsilon\left(\sum_{i=1}^{k}\xi_i\right) \\ & - \lambda a_4 e^{-(p-1)\xi_1} + ka_5\varepsilon + o(\varepsilon). \end{aligned}$$

Here the constants $a_i,\ i = 0, \dots, 5$ depend only on N and can be expressed as follows:

$$\begin{cases} a_0 = \frac{1}{2}\int_{-\infty}^{\infty}(|U'|^2 + U^2)\,dx - \frac{1}{p+1}\int_{-\infty}^{\infty} U^{p+1}\,dx, \\ a_1 = \left(\frac{4N}{N-2}\right)^{(N-2)/2}, \\ a_2 = \left(\frac{N}{N-2}\right)^{(N-2)/4}\int_{-\infty}^{\infty} e^x U^p\,dx, \\ a_3 = \frac{1}{p+1}\int_{-\infty}^{\infty} U^{p+1}\,dx, \\ a_4 = \frac{1}{2}\left(\frac{p-1}{2}\right)^2\int_{-\infty}^{\infty} e^{-(p-1)x}U^2\,dx, \\ a_5 = \frac{1}{p+1}\int_{-\infty}^{\infty} U^{p+1}\log U\,dx + \frac{1}{(p+1)^2}\int_{-\infty}^{\infty} U^{p+1}\,dx. \end{cases} \tag{2.19}$$

These constants can be explicitly computed using the explicit expression for U given by (2.6) and the identity

$$\int_0^\infty \left(\frac{r}{1+r^2}\right)^q \frac{dr}{r^{\alpha+1}} = \frac{\Gamma\left(\frac{q-\alpha}{2}\right)\Gamma\left(\frac{q+\alpha}{2}\right)}{2\Gamma(q)}.$$

The above decomposition of E_ε finally reads

$$E_\varepsilon(V) = ka_0 + \varepsilon\Psi_k(\Lambda) + \frac{k^2}{2}a_3\varepsilon\log\varepsilon + a_5\varepsilon + o(\varepsilon),$$

with Ψ_k given by (2.11). In fact, the term $o(\varepsilon)$ is uniform on the Λ_i's satisfying (2.9). A further computation along the same lines shows that differentiation with respect to the

Λ_i's leaves the term $o(\varepsilon)$ of the same order in the C^1-sense. This concludes the proof of Lemma 1. $\square$

If there is indeed a solution of (2.2) of the form $v = V + \phi$, with V as in the statement of the lemma, and ϕ small, it is natural to expect that this only occurs if the vector $\Lambda = (\Lambda_1, \dots, \Lambda_k)$ corresponds to a critical point of the function Ψ_k. This is in fact true, as we show in the following sections via a Lyapunov–Schmidt reduction procedure. Before, let us analyze the critical points of Ψ_k:

$$\Psi_k(\Lambda) = \varphi_k^{\mu}(\Lambda_1) + \sum_{i=2}^{k} \varphi_i(\Lambda_i),$$

$$\varphi_k^{\mu}(s) = a_1 s^{-2} - k a_3 \log s - a_4 \mu s^{-(p-1)} \quad \text{and} \quad \varphi_i(s) = (k-i+1) a_3 \log s - a_2 s.$$

Now the equation $\varphi_k^{\mu}(s)' = 0$ is exactly the equation $f_k(s) = \mu$ with f_k the function introduced in (1.4). In fact, we have

$$b_1 = \frac{1}{p-1}\frac{a_3}{a_4}, \quad b_2 = \frac{2}{p-1}\frac{a_1}{a_4},$$

where b_1 and b_2 are the numbers in (1.5), as can be checked using formulae (2.19). Then if $\mu > \mu_k$ with μ_k given by (1.6), φ_k^{μ} has exactly two critical points: a nondegenerate maximum, $s_k^+(\mu)$, and a nondegenerate minimum, $s_k^-(\mu)$. On the other hand, each of the functions φ_j has exactly one nondegenerate critical point, a maximum,

$$s = (k-j+1) b_3, \quad \text{for each } j = 2, \dots, k,$$

with $b_3 = \frac{a_3}{a_2} = \frac{(N-2)\sqrt{\pi}\,\Gamma(\frac{N}{2})}{2^{N+2}\Gamma(\frac{N+1}{2})}$.

Lemma 2. *Assume that* $\mu > \mu_k$ *with* μ_k *given by* (1.6). *Then, the function* $\Psi_k(\Lambda)$ *has exactly two critical points, given by*

$$\Lambda^{\pm} = (s_k^{\pm}(\mu), (k-1)b_3, (k-2)b_3, \dots, b_3).$$

These critical points are nondegenerate: Λ^+ *has Morse index* k *and* Λ^- *has Morse index* $k-1$.

3. The finite-dimensional reduction

In this section, we consider again points $0 < \xi_1 < \xi_2 < \dots < \xi_k$, which are for now arbitrary. We keep the notations U_i, V_i and V defined by (2.7) in the previous section. Additionally, we define

$$Z_i(x) = U_i'(x) - U_i'(0) e^{-x}, \quad i = 1, \dots, k$$

and consider the problem of finding a function ϕ for which there are constants c_i, $i = 1, \ldots, k$, such that, in $(0, \infty)$

$$\begin{cases} -(V+\phi)'' + (V+\phi) - e^{\varepsilon x}(V+\phi)_+^{p+\varepsilon} \\ \quad - \lambda(\frac{p-1}{2})^2 e^{-(p-1)x}(V+\phi) = \sum_{i=1}^{k} c_i Z_i, \\ \phi(0) = 0, \quad \lim_{x\to+\infty} \phi(x) = 0, \\ \int_0^\infty Z_i \phi \, dx = 0 \quad \text{for all } i = 1, \ldots, k. \end{cases} \tag{3.1}$$

The reason why we are interested in this intermediate problem will be made clear in the next section. This problem turns out to be solvable for points ξ_i chosen in a convenient range. After this, the original problem becomes reduced to adjust the points ξ_i so that $c_i = 0$ for all i. This section is devoted to solving problem (3.1). We will also establish differentiability properties which will be useful later. The choice of the points ξ_i will be carried out variationally in the next section.

Let us consider the linearized operator around V defined as

$$\mathscr{L}_\varepsilon \phi = -\phi'' + \phi - (p+\varepsilon)e^{\varepsilon x} V^{p+\varepsilon-1}\phi - \lambda\left(\frac{p-1}{2}\right)^2 e^{-(p-1)x}\phi.$$

Then problem (3.1) can be rewritten as

$$\begin{cases} \mathscr{L}_\varepsilon \phi = N_\varepsilon(\phi) + R_\varepsilon + \sum_{i=1}^{k} c_i Z_i \quad \text{in } (0, \infty), \\ \phi(0) = 0, \quad \lim_{x\to+\infty} \phi(x) = 0, \\ \int_0^\infty Z_i \phi \, dx = 0 \quad \text{for all } i = 1, \ldots, k, \end{cases} \tag{3.2}$$

where

$$N_\varepsilon(\phi) = e^{\varepsilon x}[(V+\phi)_+^{p+\varepsilon} - V^{p+\varepsilon} - (p+\varepsilon)V^{p+\varepsilon-1}\phi] \tag{3.3}$$

and $R_\varepsilon = e^{\varepsilon x}[V^{p+\varepsilon} - V^p] + V^p[e^{\varepsilon x} - 1] + [V^p - \sum_{i=1}^k V_i^p] + \lambda(\frac{p-1}{2})^2 e^{-(p-1)x} V$.

We will next analyze invertibility properties of the operator $\mathscr{L}_\varepsilon$ under the orthogonality conditions. To this end, it is convenient to introduce the following norm which depend on the points ξ_i. For a small positive number σ which will be fixed later and a function $\psi(x)$ defined on $(0, \infty)$, let us set

$$||\psi||_* = \sup_{x>0} \left(\sum_{i=1}^{k} e^{-\sigma|x-\xi_i|} \right)^{-1} |\psi(x)|. \tag{3.4}$$

Consider the linear problem of, given a function h, finding ϕ such that

$$\begin{cases} \mathscr{L}_\varepsilon \phi = h(x) + \sum_{i=1}^{k} c_i Z_i \quad \text{in } (0, \infty), \\ \phi(0) = 0, \quad \lim_{x \to +\infty} \phi(x) = 0, \\ \int_0^\infty Z_i \phi \, dx = 0 \quad \text{for all } i = 1, \dots, k, \end{cases} \tag{3.5}$$

for certain constants c_i. Then we have the validity of the following result.

Proposition 1. *There exist positive numbers ε_0, δ_0, δ_1, R_0, and a constant $C > 0$ such that if the scalar λ and the points $0 < \xi_1 < \xi_2 < \cdots < \xi_k$ satisfy*

$$R_0 < \xi_1, \quad R_0 < \min_{1 \leqslant i < k} (\xi_{i+1} - \xi_i), \qquad \xi_k < \frac{\delta_0}{\varepsilon}, \quad \lambda < \delta_1, \tag{3.6}$$

then for all $0 < \varepsilon < \varepsilon_0$ and all $h \in C[0, \infty)$ with $||h||_* < +\infty$, problem (3.5) admits a unique solution $\phi =: T_\varepsilon(h)$. Besides,

$$||T_\varepsilon(h)||_* \leqslant C ||h||_* \quad \text{and} \quad |c_i| \leqslant C ||h||_*.$$

For the proof we need the following result:

Lemma 3. *Assume the existence of sequences $\varepsilon_n \to 0$, $\lambda_n \to 0$, and points $0 < \xi_1^n < \xi_2^n < \cdots < \xi_k^n$ with*

$$\xi_1^n \to +\infty, \quad \min_{1 \leqslant i < k} (\xi_{i+1}^n - \xi_i^n) \to +\infty, \quad \xi_k^n = o(\varepsilon_n^{-1}),$$

such that for certain functions ϕ_n and h_n with $||h_n||_* \to 0$, and scalars c_i^n, one has on $(0, \infty)$

$$\begin{cases} -\phi_n'' + \phi_n - (p + \varepsilon_n) e^{\varepsilon_n x} V^{p+\varepsilon-1} \phi_n - \lambda_n (\frac{p-1}{2})^2 e^{-(p-1)x} \phi_n = h_n + \sum_{i=1}^{k} c_i^n Z_i^n, \\ \phi_n(0) = 0, \quad \lim_{x \to +\infty} \phi_n(x) = 0, \\ \int_0^\infty Z_i^n \phi_n \, dx = 0 \quad \forall i = 1, \dots, k. \end{cases} \tag{3.7}$$

Here Z_i^n is defined by $Z_i^n(x) = U'(x - \xi_i^n) + U'(\xi_i^n) e^{-x}$. Then $\lim_{n \to \infty} ||\phi_n||_* = 0$.

Proof. We shall establish first the weaker assertion that

$$\lim_{n \to \infty} ||\phi_n||_\infty = 0.$$

To do this, we assume the opposite, so that with no loss of generality we may take $||\phi_n||_\infty = 1$. Testing the above equation against Z_l^n, integrating by parts twice we get that

$$\sum_{i=1}^{k} c_i^n \int_0^\infty Z_i^n Z_l^n \, dx$$
$$= \int_0^\infty \left(-Z_l^n + \left[1 - (p+\varepsilon_n) e^{\varepsilon_n x} V^{p-1+\varepsilon_n} - \lambda_n \left(\frac{p-1}{2} \right)^2 e^{-(p-1)x} \right] Z_l^n \phi_n \right) dx$$
$$- \int_0^\infty h_n Z_l^n \, dx.$$

This defines a linear system in the c_i's which is "almost diagonal" as $n \to \infty$ approaches zero. Moreover, the assumptions made plus the fact that the function $x \mapsto Z_l^n(x) - U'(\xi_i^n) e^{-x} = U'(x - \xi_i^n)$ is a solution of

$$-Z'' + [1 - pU_l^{p-1}] Z = 0,$$

yield, after an application of dominated convergence, that $\lim_{n\to\infty} c_i^n = 0$. Assume that $x_n > 0$ is such that $\phi_n(x_n) = 1$, so that ϕ_n maximizes at this point. From (3.7) we may then assume that there is an l and a fixed $M > 0$ for which $|\xi_l^n - x_n| \leqslant M$. Set $\tilde{\phi}_n(x) = \phi_n(\xi_l^n + x)$. From (3.7) we see that passing to a suitable subsequence, $\tilde{\phi}_n$ converges uniformly over compacts to a nontrivial bounded solution $\tilde{\phi}$ of

$$-\tilde{\phi}'' + \tilde{\phi} - pU^p \tilde{\phi} = 0 \quad \text{in } (-\infty, +\infty).$$

Hence, for some $c \neq 0$, $\tilde{\phi} = cU'$. However, the orthogonality condition $\int_0^\infty Z_l^n \phi_n \, dx = 0$ passes to the limit exactly as

$$\int_{-\infty}^{\infty} U' \tilde{\phi} \, dx = 0.$$

We have thus reached a contradiction that shows that $||\phi_n||_\infty \to 0$. Now, we observe that Eq. (3.7) takes the form

$$-\phi_n'' + \phi_n = g_n, \quad \phi_n(0) = \phi_n(+\infty) = 0. \tag{3.8}$$

If $\sigma > 0$ is chosen a priori sufficiently small in the definition of the $*$-norm, then

$$|g_n(x)| \leqslant \theta_n \sum_{i=1}^{k} e^{-\sigma|x-\xi_i|} =: \psi_n(x),$$

with $\theta_n \to 0$. We see then that the function $C\psi_n$, for $C > 0$ sufficiently large, is a supersolution for (3.8), so that $\phi_n \leqslant C\psi_n$. Similarly, we may get $\phi_n \geqslant -C\psi_n$. This shows that $||\phi_n||_* \to 0$, and the proof of Lemma 3 is concluded. $\square$

Proof of Proposition 1. Let us consider the space

$$H = \left\{ \phi \in H_0^1(0,\infty) \colon \int_0^\infty Z_i \phi \, dx = 0 \ \forall i = 1, \dots, k \right\}$$

endowed with the usual inner product $[\phi, \psi] = \int_0^\infty (\phi' \psi' + \phi \psi) \, dx$. Problem (3.5) expressed in weak form is equivalent to that of finding a $\phi \in H$ such that

$$[\phi, \psi] = \int_0^\infty \left[(p+\varepsilon) e^{\varepsilon x} V^{p+\varepsilon-1} \phi + \lambda \left(\frac{p-1}{2} \right)^2 e^{-(p-1)x} \phi + h \right] \psi \, dx \quad \forall \psi \in H.$$

With the aid of Riesz's representation theorem, this equation gets rewritten in H in the operational form

$$\phi = K_\varepsilon(\phi) + \tilde{h} \tag{3.9}$$

with a certain $\tilde{h} \in H$ which depends linearly in h and where K_ε is a compact operator in H. Fredholm's alternative guarantees unique solvability of this problem for any h provided that the homogeneous equation $\phi = K_\varepsilon(\phi)$ has only the zero solution in H. Let us observe that this last equation is precisely equivalent to (3.5) with $h \equiv 0$. An indirect argument using the previous lemma shows that if the numbers R_0, ε_0, δ_0, δ_1 are suitably chosen then necessarily $\phi = K_\varepsilon(\phi)$ has the zero solution only in H. The fact that the unique solution $\phi =: T_\varepsilon(h)$ to (3.9) satisfies $||\phi||_* \leqslant C ||h||_*$ is again a consequence of Lemma 3. In fact, assuming the opposite, we can find functions (h_ε), with $||h_\varepsilon||_* \to 0$, and solutions (ϕ_ε) to problem (3.9) such that $||\phi_\varepsilon||_* = 1$, contradicting Lemma 3. This concludes the proof of Proposition 1. □

It is important for later purposes to understand the differentiability of the operator T_ε on the variables ξ_i. We shall use the notation $\xi = (\xi_1, \dots, \xi_k)$ and consider the Banach space $\mathscr{C}_*$ of all continuous functions ψ defined on $[0, \infty)$ for which $||\psi||_* < +\infty$, endowed with this norm. We also consider the space $\mathscr{L}(\mathscr{C}_*)$ of linear operators of $\mathscr{C}_*$.

Let us assume that conditions (3.6) hold. Fix $h \in \mathscr{C}_*$ and let $\phi = T_\varepsilon(h)$ for $\varepsilon < \varepsilon_0$. Consider differentiation with respect to the variable ξ_l. Let us recall that ϕ satisfies the equation

$$\mathscr{L}_\varepsilon \phi = h + \sum_{i=1}^k c_i Z_i,$$

plus the vanishing and orthogonality conditions, for some (uniquely determined) constants c_i. For some given $l \in \{1, \dots, k\}$, if we define the constant b_l defined by

$$b_l \int_0^\infty |Z_l|^2 \, dx = \int_0^\infty \phi \partial_{\xi_l} Z_l \, dx$$

and the function

$$f = -b_l \mathscr{L}_\varepsilon Z_l + c_l \partial_{\xi_l} Z_l + (p+\varepsilon) e^{\varepsilon x} \partial_{\xi_l} (V^{p-1+\varepsilon}) \phi,$$

one can then easily check that $\chi = \partial_{\xi_l} \phi$ satisfies

$$\chi = T_\varepsilon(f) + b_l Z_l.$$

Moreover, $||f||_* \leqslant C||h||_*$, $|b_l| \leqslant C||\phi||_*$, so that also $||\chi||_* \leqslant C||h||_*$. Besides, χ depends continuously on ξ_i, $i = 1, \ldots, k$, and h, for this norm. Thus, we have established the validity of the following result.

Proposition 2. *Under the assumptions of Proposition* 1, *consider the map* $\xi \mapsto T_\varepsilon$, *with values on* $\mathscr{L}(\mathscr{C}_*)$. *This map is of class* C^1. *Moreover, there is a constant* $C > 0$ *such that*

$$||D_\xi T_\varepsilon||_{\mathscr{L}(\mathscr{C}_*)} \leqslant C$$

uniformly on ξ *and* λ *satisfying conditions* (3.6).

Now we are ready to solve problem (3.1). We shall do this after restricting conveniently the range of the parameters ξ_i and λ. Let us consider for a number M large but fixed, the following conditions:

$$\begin{cases} \xi_1 > \frac{1}{2}\log(M\varepsilon)^{-1}, \quad \log(M\varepsilon)^{-1} < \min\limits_{1 \leqslant i < k} (\xi_{i+1} - \xi_i), \\ \qquad \xi_k < k \log(M\varepsilon)^{-1}, \quad \lambda < M\varepsilon^{\frac{3-p}{2}}. \end{cases} \tag{3.10}$$

Useful facts that we easily check is that under relations (3.10), N_ε and R_ε defined by (3.3) satisfy for all small $\varepsilon > 0$ and $||\phi||_* \leqslant \frac{1}{4}$ the estimates

$$||N_\varepsilon(\phi)||_* \leqslant C||\phi||_*^{\min\{p,2\}} \quad \text{and} \quad ||R^\varepsilon||_* \leqslant C\varepsilon^{\frac{3-p}{2}}, \tag{3.11}$$

provided that σ is chosen small enough.

Proposition 3. *Assume that relations* (3.10) *hold. Then there is a constant* $C > 0$ *such that, for all* $\varepsilon > 0$ *small enough, there exists a unique solution* $\phi = \phi(\xi)$ *to problem* (3.1) *which, besides, satisfies*

$$||\phi||_* \leqslant C\varepsilon.$$

Moreover, the map $\xi \mapsto \phi(\xi)$ *is of class* C^1 *for the* $||\cdot||_*$*-norm and*

$$||D_\xi \phi||_* \leqslant C\varepsilon.$$

Proof. Problem (3.1) is equivalent to solving a fixed point problem. Indeed ϕ is a solution of (3.1) if and only if

$$\phi = T_\varepsilon(N_\varepsilon(\phi) + R_\varepsilon) =: A_\varepsilon(\phi).$$

Thus, we need to prove that the operator A_ε defined above is a contraction in a proper region. Let us consider the set

$$\mathscr{F}_r = \{\phi \in C[0, \infty) \colon \ ||\phi||_* \leqslant r\varepsilon^a\}$$

with r a positive number to be fixed later and a given $a \in (0, \frac{3-p}{2})$. From Proposition 1 and (3.11), we get

$$||A_\varepsilon(\phi)||_* \leqslant C||N_\varepsilon(\phi) + R_\varepsilon||_* \leqslant C[(r\varepsilon)^{\min\{p,2\}} + \varepsilon^{\frac{3-p}{2}}] < r\varepsilon^a$$

for all small ε, provided that r is chosen large enough, but independent of ε. Thus, A_ε maps $\mathscr{F}_r$ into itself for this choice of r. Moreover, A_ε turns out to be a contraction mapping in this region. This follows from the fact that N_ε defines a contraction in the $|| \cdot ||_*$-norm, which can be proved in a straightforward way.

Concerning now the differentiability of the function $\phi(\xi)$, let us write

$$B(\xi, \phi) \coloneqq \phi - T_\varepsilon(N_\varepsilon(\phi) + R_\varepsilon).$$

Of course we have $B(\xi, \phi) = 0$. Now we write

$$D_\phi B(\xi, \phi)[\theta] = \theta - T_\varepsilon(\theta D_\phi(N_\varepsilon(\phi)) =: \theta + M(\theta).$$

It is not hard to check that the following estimate holds:

$$||M(\theta)||_* \leqslant C\varepsilon||\theta||_*.$$

It follows that for small ε, the linear operator $D_\phi B(\xi, \phi)$ is invertible in $\mathscr{C}_*$, with uniformly bounded inverse. It also depends continuously on its parameters. Let us differentiate with respect to ξ. We have

$$D_\xi B(\xi, \phi) = -(D_\xi T_\varepsilon)(N_\varepsilon(\phi) + R_\varepsilon) - T_\varepsilon((D_\xi N_\varepsilon)(\xi, \phi) + D_\xi R_\varepsilon),$$

where all the previous expressions depend continuously on their parameters. Hence, the implicit function theorem yields that $\phi(\xi)$ is a C^1 function into $\mathscr{C}_*$. Moreover, we have

$$D_\xi \phi = -(D_\phi B(\xi, \phi))^{-1}[D_\xi B(\xi, \phi)],$$

so that

$$||D_\xi \phi||_* \leqslant C(||N_\varepsilon(\phi) + R_\varepsilon||_* + ||D_\xi N_\varepsilon(\xi, \phi)||_*) \leqslant C\varepsilon.$$

This concludes the proof of Proposition 3. $\square$

4. The finite-dimensional variational problem

In this section, we fix a large number M and assume that conditions (3.10) hold true for $\xi = (\xi_1, \dots, \xi_k)$ and λ. According to the results of the previous section, our problem has been reduced to that of finding points ξ_i so that the constants c_i which appear in (3.2), for the solution ϕ given by Proposition 3, are all zero. Thus, we need to solve the system of equations

$$c_i(\xi) = 0 \quad \text{for all } i = 1, \dots, k. \tag{4.1}$$

If (4.1) holds, then $v = V + \phi$ will be a solution to (3.1) with the desired form. This system turns out to be equivalent to a variational problem, which we introduce next.

Let us consider the functional

$$\mathscr{I}_\varepsilon(\boldsymbol{\xi}) = E_\varepsilon(V + \phi),$$

where $\phi = \phi(\xi)$ is given by Proposition 2 and E_ε is defined by (2.3). We claim that solving system (4.1) is equivalent to finding a critical point of this functional. In fact, integrating (3.1) against Z_i and using the definition of E_ε and ϕ, we obtain

$$DE_\varepsilon(V + \phi)[Z_i] = 0 \quad \text{for all } i = 1, \dots, k. \tag{4.2}$$

Now, it is easily checked that

$$\frac{\partial}{\partial \xi_i}(V + \phi) = Z_i + o(1),$$

with $o(1) \to 0$ in the *-norm as $\varepsilon \to 0$. We can decompose each of the $o(1)$ terms above as the sum of a small term which lies in the vector space spanned by the Z_i's, and a function η with $\int_0^{+\infty} Z_i \eta \, dx = 0$ for all i. Again, from Eq. (3.1), we get $DJ_\varepsilon(V + \phi)[\eta] = 0$. What we have shown is that system (4.2) is equivalent to

$$\nabla \mathscr{I}_\varepsilon(\boldsymbol{\xi}) = 0.$$

The following fact is crucial to find critical points of $\mathscr{I}_\varepsilon$.

Lemma 4. *The following expansion holds*:

$$\mathscr{I}_\varepsilon(\boldsymbol{\xi}) = E_\varepsilon(V) + o(\varepsilon),$$

where the term $o(\varepsilon)$ is uniform in the C^1-sense over all points satisfying constraint (3.10), *for given $M > 0$.*

Proof. Taking into account that $0 = DE_\varepsilon(V+\phi)[\phi]$, a Taylor expansion gives

$$
\begin{aligned}
&E_\varepsilon(V+\phi) - E_\varepsilon(V)\\
&\quad = \int_0^1 D^2E_\varepsilon(V+t\phi)[\phi^2]t\,dt\\
&\quad = \int_0^1 \left(\int_0^\infty [N_\varepsilon(\phi)+R_\varepsilon]\phi + \int_0^\infty (p+\varepsilon)[V^{p+\varepsilon-1} - (V+t\phi)^{p+\varepsilon-1}]\phi^2 \right) t\,dt. \qquad (4.3)
\end{aligned}
$$

Since $||\phi||_* = O(\varepsilon)$, we get

$$
\mathscr{I}_\varepsilon(\boldsymbol{\xi}) - E_\varepsilon(V) = O(\varepsilon^2),
$$

uniformly on points satisfying (3.10). Differentiating now with respect to the ξ variables, we get from (4.3) that

$$
\begin{aligned}
D_\xi[\mathscr{I}_\varepsilon(\boldsymbol{\xi}) - E_\varepsilon(V)] = \int_0^1 &\left(\int_{\Omega_\varepsilon} D_\xi[(N_\varepsilon(\phi)+R_\varepsilon)\phi]t\,dt \right.\\
&\left. + (p+\varepsilon)\int_0^\infty D_\xi[((V+t\phi)^{p+\varepsilon-1} - V^{p+\varepsilon-1})\phi^2] \right).
\end{aligned}
$$

Using the computations in the proof of Proposition 2, we get that the first integral can be estimated by $O(\varepsilon^2)$, so does the second. Hence, the proof of Lemma 4 is complete. $\square$

Proof of Theorem 1. Let us assume $\mu > \mu_k$ with μ_k given by (1.6). We need to find a critical point of $\mathscr{I}_\varepsilon(\xi)$. We consider the change of variable $\xi = \xi(\Lambda)$:

$$
\xi_1 = -\frac{1}{2}\log\varepsilon - \log\Lambda_1, \quad \xi_{i+1} - \xi_i = -\log\varepsilon - \log\Lambda_i, \;\; i \geqslant 2.
$$

where the Λ_i's are positive parameters, and we denote $\Lambda = (\Lambda_1, \dots, \Lambda_k)$. Thus, it suffices to find a critical point of

$$
\Phi_\varepsilon(\Lambda) \equiv \varepsilon^{-1}\nabla\mathscr{I}_\varepsilon(\xi(\Lambda)).
$$

From the above lemma and the decomposition (2.10) given in Lemma 1, which actually holds with the $o(\varepsilon)$ term in the C^1 sense uniformly on points satisfying constraints (3.10), we obtain

$$
\nabla\Phi_\varepsilon(\Lambda) = \nabla\Psi_k(\Lambda) + o(1),
$$

where $o(1)\to 0$ uniformly on points Λ satisfying $M^{-1} < \Lambda_i < M$, for any fixed large M. We assume that for our fixed $\mu > \mu_k$, the critical points $\Lambda^\pm$ of Ψ_k in Lemma 4 satisfy this constraint. Since the critical points $\Lambda^\pm$ are nondegenerate, it follows that the local degrees $\deg(\nabla\Psi_k, \mathscr{V}_\pm, 0)$ are well defined and they are nonzero.

Here $\mathscr{V}_{\pm}$ are arbitrarily small neighborhoods of the points $\Lambda^{\pm}$ in $\mathbb{R}^k$. We also conclude that $\deg(\nabla \mathscr{I}_\varepsilon, \mathscr{V}_{\pm}, 0) \neq 0$ for all sufficiently small ε. Hence, we may find critical points $\Lambda_\varepsilon^{\pm}$ of Φ_ε with

$$\Lambda_\varepsilon^{\pm} = \Lambda^{\pm} + o(1), \quad \lim_{\varepsilon \to 0} o(1) = 0.$$

For $\xi_\varepsilon^{\pm} = \xi(\Lambda_\varepsilon^{\pm})$, the functions $v^{\pm} = V + \phi(\xi_\varepsilon^{\pm})$ are solutions of problem (2.2). From the equation satisfied by ϕ, (3.1), and its smallness in the $*$-norm, we derive that $v = V(1+o(1))$, where $o(1) \to 0$ uniformly on $(0, \infty)$. Further, if we set simply $\xi^{\pm} \equiv \xi(\Lambda^{\pm})$, then it is also true that

$$v^{\pm}(x) = \sum_{1=1}^{k} U(x - \xi_i^{\pm})(1 + o(1)),$$

again with $o(1) \to 0$ uniformly on $(0, \infty)$. Finally, if we go back via the change of variables

$$u^{\pm}(r) = \left(\frac{p-1}{2r}\right)^{2/(p-1+\varepsilon)} v^{\pm}\left(-\frac{2}{p-1}\log r\right),$$

to a solution of (1.1), the explicit form of the parameters $\Lambda^{\pm}$ found in Lemma 2 provides expression (1.3) for the solutions. This concludes the proof of Theorem 1. $\square$

5. The case $N = 4$

In this section, we show the modifications needed in Theorem 1 and its proof for the case $N = 4$. In that case, our main result reads as follows.

Theorem 2. *Let $N = 4$. Given a number $k \geqslant 1$, if $\mu > \mu_k$, where*

$$\mu_k = k \frac{\pi}{2^5} e^2 \tag{5.1}$$

and

$$\lambda e^{-2/\lambda} = \mu \varepsilon, \tag{5.2}$$

then there are constants $0 < \alpha_j^- < \alpha_j^+$, $j = 1, \ldots, k$, which depend on k and μ, and two solutions $u_\varepsilon^{\pm}$ of problem (1.1) of the form

$$u_\varepsilon^{\pm}(y) = \gamma \sum_{j=1}^{k} \left(\frac{1}{1 + M_j^2 |y|^2}\right) M_j (1 + o(1)),$$

where $o(1) \to 0$ holds uniformly on B as $\varepsilon \to 0$ and $M_j^{\pm} = \alpha_j^{\pm} \varepsilon^{\frac{1}{2} - j} |\log \varepsilon|^{-\frac{1}{2}}$.

As in the case $N \geqslant 5$, these solutions are superposition of k bubbles. However, if $N = 4$, the order of the height of each bubble is corrected with a logarithmic term, namely $\varepsilon^{\frac{1}{2}-j}|\log \varepsilon|^{-\frac{1}{2}}$. The constants $\alpha_j^{\pm}$ are also found explicitly as explained next.

Given $k \geqslant 1$, the number μ_k in (5.1) is the minimum value of the function

$$f_k(s) = k\frac{\pi}{2^5}\frac{s^2}{2\log s - 1}$$

in the range $s \in (\sqrt{e}, \infty)$, and this minimum value is attained at $s = e$. Then, given $\mu > \mu_k$, the equation

$$\mu = f_k(s)$$

has exactly two solutions

$$e^{\frac{1}{2}} < s_k^-(\mu) < s_k < s_k^+(\mu).$$

The numbers $\alpha_j^{\pm}$ can be expressed by the formulae

$$\alpha_j^{\pm} = \left(\frac{2^3}{\pi}\right)^{1-j}\frac{(k-j)!}{(k-1)!}s_k^{\pm}(\mu), \quad j = 1, \dots, k.$$

For the proof of Theorem 2, we proceed exactly as for $N \geqslant 5$, except that now the choice of the points ξ_i has to be made differently to that in (2.8). More precisely, Lemma 1 has to be replaced by the following:

Lemma 5. *Let $N = 4$ and $\delta > 0$. Take λ as in* (5.2) *with μ a fixed positive number, and the points ξ_i to be*

$$\xi_1 = \lambda^{-1} + \log \Lambda_1, \quad \xi_{i+1} - \xi_i = 2\lambda^{-1} - \log \lambda - \log(\mu^{-1}\Lambda_{i+1}). \tag{5.3}$$

Then, with V be given by (2.7), *the following expansion holds*

$$E_\varepsilon(V) = ka + \varepsilon\Psi_k(\Lambda) + b\varepsilon\log\varepsilon + c\varepsilon\lambda + d\varepsilon + o(\varepsilon)$$

uniformly with respect to $\delta < \Lambda_i < \delta^{-1}$ and for certain absolute positive constants a, b, c, d. Here the function $\Psi_k(\Lambda)$ is given by

$$\Psi_k(\Lambda) = -\mu a_1\Lambda_1^{-2}\log\Lambda_1 + ka_3\log\Lambda_1 + \sum_{i=2}^{k}\left[(k-i+1)a_3\log\Lambda_i - a_2\Lambda_i\right]$$

and the constants a_1, a_2 and a_3 are given by formulae (2.19).

The above expansion differs from the case $N \geqslant 5$ only in the estimate of the term $\lambda \int_0^\infty e^{-(p-1)x}|V|^2\,dx$. In fact, for $N = 4$, estimate (2.18) becomes

$$\lambda \int_0^\infty e^{-(p-1)x}|V|^2\,dx = \lambda\left(\frac{4N}{N-2}\right)^{\frac{N-2}{2}} \xi_1 e^{-2\xi_1} + o(\varepsilon).$$

Direct examination of the results in Section 3, show that they still hold true for the choice of points ξ_i as in (5.3) and parameter λ as in (5.2). The results of Section 4 follow exactly in the same way, now yielding Theorem 2.

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Appendix

Let B be the unit ball in $\mathbb{R}^N$, $N \geqslant 4$, and consider the positive solutions of

$$\begin{cases} \Delta u + u^{p+\varepsilon} + \lambda u = 0 & \text{in } B, \\ u > 0 \text{ in } B, \quad u = 0 \text{ on } \partial B \end{cases} \tag{A.1}$$

with $p = \frac{N+2}{N-2}$ and $\varepsilon \geqslant 0$. According to the theorem of Gidas et al. [11], all solutions are radial and decreasing along any radius, so there exists a unique branch of solutions as shown by the following parametrization method (see for instance [3] for more details). Consider the solutions of

$$\begin{cases} v'' + \frac{N-1}{r} v' + v^{p+\varepsilon} + v = 0 & \text{in } [0, +\infty), \\ v(0) = a > 0, \quad v'(0) = 0 \end{cases} \tag{A.2}$$

and denote by $\rho = \rho(a) > 0$ the first zero of v, which is well defined for any $a > 0$ (see for instance [3]). Then to any solution u of (A.1) corresponds a function v defined on $[0, \sqrt{\lambda})$ such that $v(|x|) = \lambda^{-1/(p+\varepsilon-1)} u(x/\sqrt{\lambda})$ for any $x \in B$, which can be extended to $[0, +\infty)$ as a solution of (A.2). Reciprocally, if v is a solution of (A.2), then $u(x) = \rho^{2/(p+\varepsilon-1)} v(\rho|x|)$ for any $x \in B$ is a solution of (A.1) with $\lambda = \rho^2$. The bifurcation diagram $(\lambda, ||u||_{L^\infty})$ is therefore fully parametrized by $a \mapsto (\rho^2, a\rho^{2/(p+\varepsilon-1)})$ with $\rho = \rho^2(a)$. For convenience, we use a logarithmic scale for the plots and take $N = 5$ (qualitative aspect of the bifurcation diagrams does not depend much on the dimension).

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