Spike Patterns in the Super-Critical Bahri–Coron Problem

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Dedicated to Antonio Marino

1 Introduction

This paper deals with the construction of solutions of the problem

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}+\varepsilon} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a smooth, bounded domain in $\mathbb{R}^N$, $N \geq 3$, and $\varepsilon > 0$ is a small parameter.

It is well known that the problem

$$\begin{cases} -\Delta u = u^q & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

has at least one solution when $1 < q < \frac{N+2}{N-2}$. However, when $q \geq \frac{N+2}{N-2}$, the existence of solutions to problem (1.2) depends strongly on the topology or geometry of Ω. A well-known result by Pohozaev [13], asserts that (1.2) has no solutions if $q \geq \frac{N+2}{N-2}$ and Ω is star-shaped. On the other hand Kazdan and Warner [10] showed that (1.2) has a radially symmetric solution for any $q > 1$ when Ω is a symmetric annulus. Coron in [5] considered the case $q = \frac{N+2}{N-2}$, and showed that (1.2) is solvable when Ω is a (non-symmetric) domain exhibiting a small hole, say $\Omega = \mathcal{D} \setminus \bar{B}(P_0, \mu)$, where $\mathcal{D}$ is a smooth bounded domain, $P_0 \in \mathcal{D}$ and μ is sufficiently small.

In [1], Bahri and Coron considerably generalize this result, proving that if $q = \frac{N+2}{N-2}$ and if some homology group of Ω with coefficients in $\mathbf{Z}_2$ is

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nontrivial, then problem (1.2) has a solution. While it may be expected that this solution *survives* a small supercritical perturbation of the exponent as in (1.1), the indirect variational arguments employed in [5] and [1] do not seem to give in principle a clue as to how to obtain this fact. Solvability when $q > \frac{N+2}{N-2}$ in domains "with topology" is not true in general as shown via counterexamples by Passaseo [11, 12], answering negatively the question posed by Brezis in [3]. In our recent work [6] we have considered problem (1.1) in Coron's situation of a domain with a small perforation, and proved solvability whenever ε is sufficiently small. The proof is constructive and, rather puzzlingly, the solutions found collapse as $\varepsilon \to 0$ in the form of a double spike: the solution tends to vanish everywhere except around two local maximum points which blow up at the rate $O(\varepsilon^{-\frac{1}{2}})$. This result generalizes to a domain exhibiting multiple holes, as we have recently established in [7]. In such a situation, multi-peak solutions exist, consisting of the gluing of double-spikes associated to each of the holes. More precisely, our setting in problem (1.1) is the following.

Let $\mathcal{D}$ be a bounded, smooth domain in $\mathbb{R}^N$, $N \geq 3$, and $P_1, P_2, \dots, P_m$ points of $\mathcal{D}$. Let us consider the domain

$$\Omega = \mathcal{D} \setminus \bigcup_{i=1}^{m} B(P_i, \mu) \tag{1.3}$$

where $\mu > 0$ is a small number.

Theorem 1.1. *There exists a $\mu_0 > 0$, which depends on $\mathcal{D}$ and the points $P_1, \dots, P_m$ such that if $0 < \mu < \mu_0$ is fixed and Ω is the domain given by (1.3), then the following holds: Given an integer $1 \leq k \leq m$, there exists $\varepsilon_0 > 0$ and a family of solutions u_ε, $0 < \varepsilon < \varepsilon_0$ of (1.1), with the following property: u_ε has exactly k pairs of local maximum points $(\xi_{j1}^\varepsilon, \xi_{j2}^\varepsilon) \in \Omega^2$ $j = 1, \dots, k$ with $c\mu < |\xi_{ji}^\varepsilon - P_j| < C\mu$, for certain constants c, C independent of μ and such that for each small $\delta > 0$,*

$$\sup_{\{|x-\xi_{ij}^\varepsilon|>\delta\,\forall i,j\}} u_\varepsilon(x) \to 0$$

and

$$\sup_{|x-\xi_{ij}|<\delta} u_\varepsilon(x) \to +\infty, \quad \forall\, i, j$$

as $\varepsilon \to 0$.

The proof provides much finer information on the asymptotic profile of the blowup of these solutions, as $\varepsilon \to 0$: after scaling and translation one sees around each ξ_{ij}^ε a solution in entire $\mathbb{R}^N$ of the equation at the critical exponent. More precisely, we will find,

$$u_\varepsilon(x) = \sum_{i=1}^{k} \sum_{j=1}^{2} \left(\frac{\alpha_N \lambda_{ij} \varepsilon^{\frac{1}{N-2}}}{\varepsilon^{\frac{2}{N-2}} \lambda_{ij}^2 + |x - \xi_{ij}^\varepsilon|^2} \right)^{\frac{N-2}{2}} + \theta_\varepsilon(x), \tag{1.4}$$

where $\theta_\varepsilon(x) \to 0$ uniformly as $\varepsilon \to 0$, for certain positive constants α_N. The numbers λ and the points ξ will be further identified as critical points of certain functionals built upon the Green function of Ω. The role of Green's function in concentration phenomena associated to almost-critical problems on the subcritical side, $q = \frac{N+2}{N-2} - \varepsilon$, has already been considered in several works; see Brezis and Peletier [4], Rey [14], [15], [16], Han [9] and Bahri, Li and Rey [2].

In what follows we will denote by $G(x, y)$ the Green function of Ω, namely G satisfies

$$\Delta_x G(x,y) = \delta(x-y), \quad x \in \Omega,$$
$$G(x,y) = 0, \quad x \in \partial\Omega,$$

where $\delta(x)$ denotes the Dirac mass at the origin. We denote by $H(x, y)$ its regular part, namely

$$H(x,y) = \Gamma(x-y) - G(x,y)$$

where Γ denotes the fundamental solution of the Laplacian,

$$\Gamma(x) = b_N |x|^{2-N},$$

so that H satisfies

$$\Delta_x H(x,y) = 0, \quad x \in \Omega,$$
$$H(x,y) = \Gamma(x-y), \quad x \in \partial\Omega.$$

Its *diagonal* $H(x, x)$ is usually called the Robin function of the domain.

We shall concentrate next on the case of existence of a single two-spike solution, and state a general result derived in [6], which includes the case $k = 1$ in Theorem 1.1. In the two-spike concentration phenomenon, the following function will play a crucial role in our analysis:

$$\varphi(\xi_1, \xi_2) = H^{\frac{1}{2}}(\xi_1, \xi_1) H^{\frac{1}{2}}(\xi_2, \xi_2) - G(\xi_1, \xi_2). \tag{1.5}$$

We will construct solutions of (1.1) which as $\varepsilon \to 0$ develop a spike-shape, blowing up at exactly two distinct points ξ_1, ξ_2 while approaching zero elsewhere, provided that the set where $\varphi < 0$ is included in Ω^2 in a topologically nontrivial way. The pair (ξ_1, ξ_2) will be a critical point of φ with $\varphi(\xi_1, \xi_2) < 0$.

For a subspace B of Ω we will designate by $H^d(B)$ its d-th cohomology group with integral coefficients. We will consider the homomorphism $\iota^* : H^*(\Omega) \to H^*(B)$, induced by the inclusion $\iota : B \to \Omega$.

Theorem 1.2. *Assume $N \geq 3$ and let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$, with the following property: There exists a compact manifold $\mathcal{M} \subset \Omega$ and an integer $d \geq 1$ such that, $\varphi < 0$ on $\mathcal{M} \times \mathcal{M}$, $\iota^* : H^d(\Omega) \to H^d(\mathcal{M})$ is nontrivial and either d is odd or $H^{2d}(\Omega) = 0$.*

Then there exists $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$, problem (1.1) has at least one solution u_ε. Moreover, let $\mathcal{C}$ be the component of the set where $\varphi < 0$ which contains $\mathcal{M} \times \mathcal{M}$. Then, given any sequence $\varepsilon = \varepsilon_n \to 0$, there is a subsequence, which we denote in the same way, and a critical point $(\xi_1, \xi_2) \in \mathcal{C}$ of the function φ such that $u_\varepsilon(x) \to 0$ on compact subsets of $\Omega \setminus \{\xi_1, \xi_2\}$ and such that for any $\delta > 0$

$$\sup_{|x-\xi_i|<\delta} u_\varepsilon(x) \to +\infty, \quad i = 1, 2,$$

as $\varepsilon \to 0$.

The assumption of the above theorem does indeed hold true in the case of a small hole, as we explain next. Let us set

$$\Omega = \mathcal{D} \setminus \bar{B}(0, \mu). \tag{1.6}$$

Elementary properties of harmonic functions give the validity of the fact that

$$\lim_{\mu \to 0} H(x, y) = H_{\mathcal{D}}(x, y), \tag{1.7}$$

uniformly on x, y in compact subsets of $\bar{\mathcal{D}} \setminus \{0\}$, where $H_{\mathcal{D}}$ denotes the regular part of the Green function $G_{\mathcal{D}}$ on $\mathcal{D}$.

For any (fixed) sufficiently small number $\rho > 0$ there is a $\mu_0 > 0$ such that if $\mu < \mu_0$, and Ω is given by (1.6), then

$$\sup_{|\xi_1|=|\xi_2|=\rho} \varphi(\xi_1, \xi_2) < 0.$$

Hence, Theorem 1.2 applies to Ω given by (1.6), with

$$\mathcal{M} = \rho S^{N-1}.$$

This follows directly from (1.7) and the fact that $H_{\mathcal{D}}$ is smooth near $(0, 0)$ while $G_{\mathcal{D}}$ becomes unbounded as its arguments get close.

A second example is the following. Consider now a solid torus in $\mathbb{R}^3$ given by $T(l, r)$, where l is the radius of the axis circle, which we assume centered at 0, and r that of a cross-section. Assume now that there is an $r_0 > 0$ such that $T(l, r_0) \subset \mathcal{D}$. Consider now $\mathcal{D}_\delta$ defined as

$$\mathcal{D}_\delta = \mathcal{D} \setminus T(l, \delta).$$

Similarly, as in the previous example, the Green and Robin functions of $\mathcal{D}_\delta$ will approach that of $\mathcal{D}$. Then, fixing now a sufficiently small $\rho > 0$ and considering the boundary of a fixed section $S^1(\rho)$ of $T(l, \rho)$, we will have that if $\Omega = \mathcal{D}_\delta$ with δ sufficiently small, then

$$\sup_{\xi_1, \xi_2 \in S^1(\rho)} \varphi(\xi_1, \xi_2) < 0.$$

It follows that Theorem 1.2 applies now with

$$\mathcal{M} = S^1(\rho).$$

It is perhaps clear from the above argument that it suffices that for a torus not necessarily symmetric taken away, the same would be true, provided that it is "narrow" only in a certain region.

We explain next the main elements in the proofs of Theorems 1.1 and 1.2. One obvious difficulty to circumvent is the fact that Sobolev's embedding is no longer valid in our situation. We are able however to work out in "well-chosen" spaces a reduction to a finite dimensional problem, which we treat with a variational-topological approach. In the case of a single two-spike, the problem becomes basically reduced, as we will explain below, to that of finding a critical point of φ which persists under small C^1 perturbations. Such a critical point comes from a min-max quantity naturally defined from the assumptions of Theorem 1.2.

2 Recasting the problem: The finite-dimensional reduction

To find a multiple-spike solution, it is convenient to scale problem (1.1) into the expanding domain

$$\Omega_\varepsilon = \varepsilon^{-\frac{1}{N-2}}\Omega.$$

Let us consider the change of variables

$$v(y) = \varepsilon^{\frac{1}{2+\varepsilon\frac{N-2}{2}}} u(\varepsilon^{\frac{1}{N-2}} y), \quad y \in \Omega_\varepsilon.$$

Then u solves (1.1) if and only if v satisfies

$$\begin{cases} \Delta v + v^{\frac{N+2}{N-2}+\varepsilon} = 0 & \text{in } \Omega_\varepsilon \\ v_\varepsilon > 0 & \text{in } \Omega_\varepsilon \\ v = 0 & \text{on } \partial\Omega_\varepsilon\,. \end{cases} \tag{2.8}$$

Since Ω_ε expands to the whole $\mathbb{R}^N$, and all positive solutions of

$$\Delta v + v^{\frac{N+2}{N-2}} = 0 \quad \text{in } \mathbb{R}^N$$

are given by the functions

$$\bar{U}_{\lambda,\xi'}(y) = \lambda^{-\frac{N-2}{2}}\bar{U}\left(\frac{x-\xi'}{\lambda}\right)$$

with

$$\bar U(y) = \alpha_N \left(\frac{1}{1+|y|^2}\right)^{\frac{N-2}{2}}$$

and $\alpha_N = (N(N-2))^{\frac{N-2}{4}}$, $\xi' \in \mathbb{R}^N$, $\lambda > 0$, it is natural to seek solutions v of the form

$$v(y) \sim \sum_{i=1}^{h} \bar U_{\lambda_i,\xi_i'}(y) \tag{2.9}$$

for a certain set of h points $\xi_1,\dots,\xi_h$ in Ω and numbers $\lambda_1,\dots,\lambda_h > 0$, where now and in what follows we set, for $\xi \in \Omega$,

$$\xi' = \varepsilon^{-\frac{1}{N-2}}\xi \in \Omega_\varepsilon.$$

It turns out that this choice of scaling is precisely one at which we can find solutions satisfying (2.9), with the points ξ_i uniformly away from each other and from the boundary of Ω, and the positive scalars λ_i bounded above, and below away from zero. Such an approximation cannot be too good near the boundary, where v is supposed to vanish. A better approximation involves the orthogonal projections onto $H_0^1(\Omega_\varepsilon)$ of the functions $\bar U_{\lambda,\xi'}$. We denote by $V_{\lambda,\xi'}$ these projections, which are defined as the respective unique solutions of the equations

$$-\Delta V_{\lambda,\xi'} = \bar U_{\lambda,\xi'}^{\frac{N+2}{N-2}} \quad \text{in } \Omega_\varepsilon$$

$$V_{\lambda,\xi'} = 0 \quad \text{on } \partial\Omega_\varepsilon.$$

For a given set of points $\xi_1,\dots,\xi_h$ in Ω and numbers $\lambda_1,\dots,\lambda_h > 0$, we consider the functions

$$U_i = \bar U_{\lambda_i,\xi_i'}, \quad V_i = V_{\lambda_i,\xi_i'}, \quad i = 1,\dots,h. \tag{2.10}$$

Moreover, we write

$$U = \sum_{j=1}^{h} U_j, \qquad V = \sum_{j=1}^{h} V_j. \tag{2.11}$$

Consider further the functions

$$\bar Z_{ij} = \frac{\partial U_i}{\partial \xi_{ij}'},\ j = 1,\dots,N, \quad \bar Z_{iN+1} = \frac{\partial U_i}{\partial \lambda_i} = (x-\xi_i')\cdot\nabla U_i + (N-2)U_i,$$

and their respective $H_0^1(\Omega_\varepsilon)$-projections Z_{ij}, namely the unique solutions of

$$\Delta Z_{ij} = \Delta \bar Z_{ij} \quad \text{in } \Omega_\varepsilon$$

$$Z_{ij} = 0 \quad \text{on } \partial\Omega_\varepsilon.$$

We look for a solution v of problem (2.8) of the form

$$v = \sum_{i=1}^{h} V_i + \phi$$

where ϕ is some lower order term. In order to do so, we consider the following auxiliary problem: Find a (small) function ϕ such that for certain constants c_{ij}

$$\begin{cases} \Delta(V+\phi) + (V+\phi)_+^{p+\varepsilon} = \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} & \text{in } \Omega_\varepsilon \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon} \phi V_i^{p-1} Z_{ij} = 0 & \text{for all } i,j. \end{cases} \tag{2.12}$$

Here and in what follows we call $p = \frac{N+2}{N-2}$. Our task is then to solve (2.12) and find points ξ and scalars λ such that the associated c_{ij} are all zero, which determines a solution of (2.8).

The first equation in (2.12) can be rewritten in the following form:

$$\Delta\phi + (p+\varepsilon)V^{p+\varepsilon-1}\phi = -N_\varepsilon(\phi) - R^\varepsilon + \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} \quad \text{in } \Omega_\varepsilon \tag{2.13}$$

where

$$N_\varepsilon(\phi) = (V+\phi)_+^{p+\varepsilon} - V^{p+\varepsilon} - (p+\varepsilon)V^{p+\varepsilon-1}\phi, \tag{2.14}$$

and

$$R^\varepsilon = V^{p+\varepsilon} - \sum_{j=1}^{h} U_j^p. \tag{2.15}$$

It is then clear that we need to understand the following linear problem: given $h \in C^\alpha(\bar{\Omega}_\varepsilon)$, find a function ϕ such that

$$\begin{cases} \Delta\phi + (p+\varepsilon)V^{p+\varepsilon-1}\phi = h + \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} & \text{in } \Omega_\varepsilon \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon} V_i^{p-1} Z_{ij}\phi = 0 & \text{for all } i,j \end{cases} \tag{2.16}$$

for certain constants c_{ij}, $i = 1, \dots, h$, $j = 1, \dots, N+1$. In order to solve (boundedly) (2.16), it is convenient to work on functional spaces which depend on the chosen points ξ_i'. Let us consider the norms

$$\|\psi\|_* = \sup_{x\in\Omega_\varepsilon} \Big| \left(\sum_{j=1}^{h} (1+|x-\xi_j'|^2)^{-\frac{N-2}{2}} \right)^{-\beta} \psi(x) \Big|,$$

where $\beta = 1$ if $N = 3$, $\beta = \frac{2}{N-2}$ if $N > 3$, and

$$\|\psi\|_{**} = \sup_{x\in\Omega_\varepsilon} \Big| \left(\sum_{j=1}^{h} (1 + |x - \xi_j'|^2)^{-\frac{N-2}{2}} \right)^{-\frac{4}{N-2}} \psi(x) \Big|.$$

Let us fix a small number $\delta > 0$. From now on we will restrict ourselves to points $\xi_i' \in \Omega_\varepsilon$, and numbers $\lambda_i > 0$, $i = 1, \dots, h$, such that

$$|\xi_i' - \xi_j'| > \delta \varepsilon^{-\frac{1}{N-2}}, \quad \operatorname{dist}(\xi_i', \partial\Omega_\varepsilon) > \delta\varepsilon^{-\frac{1}{N-2}}, \quad \delta < \lambda_i < \delta^{-1}. \tag{2.17}$$

We have the validity of the following result.

Proposition 2.1. *There are numbers $\varepsilon_0 > 0$, $C > 0$, such that for all $0 < \varepsilon < \varepsilon_0$, points (ξ', Λ) satisfying condition (2.17) and $h \in C^\alpha(\Omega_\varepsilon)$, we have that (2.16) has a unique solution $\phi = L_\varepsilon(h)$. Besides,*

$$\|L_\varepsilon(h)\|_* \le C\|h\|_{**} \tag{2.18}$$

for any $h \in C^\alpha(\Omega_\varepsilon)$.

Once this result is established, we see that Problem (2.12) is equivalent to the fixed point problem

$$\phi = -L_\varepsilon(N_\varepsilon(\phi) + R^\varepsilon).$$

We set even further

$$\psi_\varepsilon = -L_\varepsilon(R^\varepsilon), \quad \tilde\phi = \phi - \psi_\varepsilon \tag{2.19}$$

and rewrite the problem as

$$\tilde\phi = -L_\varepsilon(N_\varepsilon(\tilde\phi + \psi_\varepsilon)) \equiv T_\varepsilon(\tilde\phi).$$

It is not hard to check that $\|R_\varepsilon\|_{**} = O(\varepsilon)$, so that $\|\psi_\varepsilon\|_* = O(\varepsilon)$. From the fact that N_ε has a power behavior greater than one for small values of its argument, it can be shown that the operator T_ε defines a contraction mapping of a certain small ball in the $\|\ \|_*$ norm into itself. More precisely, we have that

$$\|N_\varepsilon(\phi)\|_{**} \le C\|\phi\|_*^{\min\{p\beta+1,2\}};$$

hence T_ε applies a ball with radius $O(\varepsilon^{\min\{p,2\}})$ into itself. Then the result follows from the Banach fixed point theorem applied in such a ball:

Proposition 2.2. *Assume the conditions of Proposition 2.1 are satisfied. Then there is a constant $C > 0$ such that, for all $\varepsilon > 0$ small enough, and all points ξ', λ satisfying (2.17) there exists a unique solution*

$$\phi = \phi(\xi', \lambda) = \tilde\phi + \psi_\varepsilon$$

to problem (2.12) with $\psi_\varepsilon = -L_\varepsilon(R_\varepsilon)$ such that

$$\|\tilde\phi\|_* \le C\varepsilon$$

It can be shown that the map $(\xi', \lambda) \to \tilde{\phi}(\xi', \lambda)$ is of class C^1 for the $\|\cdot\|_*$-norm and

$$\|\nabla_{(\xi',\lambda)}\tilde{\phi}\|_* \leq C\varepsilon. \tag{2.20}$$

We also have that $\|\nabla_{(\xi',\lambda)}\psi_\varepsilon\|_* \leq C\varepsilon$.

3 The energy approach

The functional associated to Problem (2.8) is given by

$$\mathcal{I}_\varepsilon(v) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u|^2 - \frac{1}{p+1+\varepsilon}\int_{\Omega_\varepsilon} v^{p+1+\varepsilon}. \tag{3.21}$$

Regular critical points of it correspond exactly to the solutions of (2.8). Let us also observe that given points ξ and scalars λ, ϕ satisfies (2.12) if and only if

$$\mathcal{I}_\varepsilon'(V+\phi)[\eta] = 0 \tag{3.22}$$

for all η, which satisfies the orthogonality relations $\int_{\Omega_\varepsilon} V_i^{p-1} Z_{ij}\eta = 0$. On the other hand, it is readily checked that the scalars c_{ij} in (2.8) are all zero if and only if $\mathcal{I}_\varepsilon'(V+\phi)[Z_{ij}] = 0$ for all i, j. This last relation and (3.22) combined, plus the relationship up to lower order terms between the derivatives of V with respect to ξ and λ and the Z_{ij}'s, plus the smallness of these derivatives in ϕ, yield that the c_{ij}'s are zero in (2.8) if and only if

$$D_{\xi',\lambda}\mathcal{I}_\varepsilon(V+\phi) = 0.$$

Now we recall that we want to consider points

$$\xi_i' = \varepsilon^{-\frac{1}{N-2}}\xi_i, \tag{3.23}$$

with $\xi_i \in \Omega$. It will also be convenient, rather than working with the numbers λ_i, to do so with the Λ_i's given by

$$\lambda_i = (a_N \Lambda_i)^{\frac{1}{N-2}} \tag{3.24}$$

with

$$a_N = \frac{1}{p+1}\frac{\int_{\mathbb{R}^N} \bar{U}^{p+1}}{(\int_{\mathbb{R}^N} \bar{U}^p)^2}.$$

The role of this constant is to provide a simpler form for the expansion of the functional. Thus we search for critical points (ξ, Λ) of

$$I(\xi, \Lambda) \equiv \mathcal{I}_\varepsilon(V+\phi). \tag{3.25}$$

A crucial step is to give an asymptotic estimate for $I(\xi,\Lambda)$. Let us set

$$C_N = \frac{1}{2}\int_{\mathbb{R}^N} |D\bar{U}|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} |\bar{U}|^{p+1}.$$

Since the points ξ_i' are very far away from each other and from the boundary of the expanding domain Ω_ε, and the perturbation ϕ is a lower order term, then at first order

$$I(\xi,\Lambda) \sim \sum_{i=1}^{h} \mathcal{I}_\varepsilon(V_i) \sim hC_N.$$

A precise account of lower order terms in this expansion is given in the result below.

Proposition 3.1. *Let us fix $\delta > 0$. Then there exist positive constants γ_N and w_N such that the following expansion holds.*

$$I(\xi,\Lambda) = hC_N + \varepsilon\big[\gamma_N + w_N\Psi(\xi,\Lambda) + o(1)\big], \tag{3.26}$$

where the quantity $o(1)$ tends to zero as $\varepsilon \to 0$ uniformly in the C^1-sense in the variables (ξ,Λ) for which ξ' given by (3.23) and λ given by (3.24). satisfy constraints (2.17). Here

$$\Psi(\xi,\Lambda) = \frac{1}{2}\{\sum_{j=1}^{h} H(\xi_j,\xi_j)\Lambda_j^2 - 2\sum_{i<j} G(\xi_i,\xi_j)\Lambda_i\Lambda_j\} + \log(\Lambda_1\cdots\Lambda_h), \tag{3.27}$$

The estimate given by the last proposition tells us that it is sufficient to find a critical point for Ψ which is *stable* under small C^1-perturbations. We construct such a critical point through a min-max characterization in the following section. We will sketch how to do so only for the case of a two-spike, under the assumption of Theorem 1.2. In that case the function Ψ becomes

$$\Psi(\xi,\Lambda) = \frac{1}{2}\{\sum_{j=1}^{2} H(\xi_j,\xi_j)\Lambda_j^2 - 2G(\xi_1,\xi_2)\Lambda_1\Lambda_2\} + \log(\Lambda_1\Lambda_2). \tag{3.28}$$

4 The min-max

In this section we set up a min-max scheme to find a critical point of the function Ψ given by (3.28). This scheme is then used to find a critical point for the reduced functional I (see (3.25), (3.26)). We recall that the function Ψ is well defined in $(\Omega\times\Omega\setminus\Delta)\times\mathbb{R}^2_+$, where Δ is the diagonal

$\Delta = \{(\xi_1, \xi_2) \in \Omega \times \Omega \,/\, \xi_1 = \xi_2\}$. In order to avoid the singularity of Ψ over Δ, we let $M > 0$ be a very large number, and we define

$$G_M(\xi) = \begin{cases} G(\xi) \text{ if } \; G(\xi) \le M \\ M \quad \text{if } \; G(\xi) > M, \end{cases} \tag{4.29}$$

and we consider $\Psi_{M,\rho} : \Omega_\rho \times \Omega_\rho \times \mathbb{R}^2_+ \to \mathbb{R}$ defined by

$$\Psi_{M,\rho}(\xi, \Lambda) = \Psi(\xi, \Lambda) - G_M(\xi)\Lambda_1\Lambda_2 + G(\xi)\Lambda_1\Lambda_2, \tag{4.30}$$

where $\rho > 0$ and $\Omega_\rho = \{\xi \in \Omega \,/\, \text{dist}(\xi, \Omega) > \rho\}$. We will specify ρ later, and for notational convenience we will simply write $\Psi_{M,\rho} = \Psi$ and $D = \Omega_\rho \times \Omega_\rho \times \mathbb{R}^2_+$. We consider a further restriction $D_\varphi = \{(\xi, \Lambda) \in D \,/\, \varphi(\xi) < -\rho_0\}$, where $\rho_0 = \min\{\frac{1}{2}\exp(-2C_0 - 1), -\frac{1}{2}\max\{\varphi \,/\, \text{in } \mathcal{M}^2\}\}$, with

$$C_0 = \sup_{(\xi,\sigma) \in \mathcal{M}^2 \times I_0} \Psi(\xi, \sigma).$$

With this choice certainly $\mathcal{M}^2 \times \mathbb{R}^2_+ \subset D_\varphi$.

Aiming to define the min-max class, for every $\xi \in \mathcal{M}^2$ we let $d(\xi) = (d_1(\xi), d_2(\xi)) \in S^1 \subset \mathbb{R}^2$ be the negative direction of the quadratic form defining Ψ. Such a direction exists since, by hypothesis of Theorem 1.2, the function φ is negative over $\mathcal{M}^2$. We easily see that there is a constant $c > 0$ such that $c < d_1(\xi) d_2(\xi) < c^{-1}$ for all $\xi \in \mathcal{M}^2$.

Next we let Γ be the class of continuous functions $\gamma : \mathcal{M}^2 \times I_0 \times [0,1] \to D_\varphi$, such that

1. $\gamma(\xi, \sigma_0, t) = (\xi, \sigma_0 d(\xi))$, and $\gamma(\xi, \sigma_0^{-1}, t) = (\xi, \sigma_0^{-1} d(\xi))$ for all $\xi \in \mathcal{M}^2$, $t \in [0,1]$, and
2. $\gamma(\xi, \sigma, 0) = (\xi, \sigma d(\xi))$ for all $(\xi, \sigma) \in \mathcal{M}^2 \times I_0$,

where $I_0 = [\sigma_0, \sigma_0^{-1}]$ with σ_0 is a small number to be chosen later. Then we define the min-max value

$$c(\Omega) = \inf_{\gamma \in \Gamma} \sup_{(\xi,\sigma) \in \mathcal{M}^2 \times I_0} \Psi(\gamma(\xi, \sigma, 1)) \tag{4.31}$$

and we will prove in what follows that $c(\Omega)$ is a critical value of Ψ. For this purpose we will first prove an intersection lemma based on a topological continuation result of Fitzpatrick, Massabó and Pejsachowicz [8]. For every $(\xi, \sigma, t) \in \mathcal{M}^2 \times I_0 \times [0,1]$, we denote $\gamma(\xi, \sigma, t) = (\tilde{\xi}(\xi, \sigma, t), \tilde{\Lambda}(\xi, \sigma, t)) \in D_\varphi$, and we define $\mathcal{S} = \{(\xi, \sigma) \in \mathcal{M}^2 \times I_0 \,/\, \tilde{\Lambda}_1(\xi, \sigma, 1) \cdot \tilde{\Lambda}_2(\xi, \sigma, 1) = 1\}$,; then we have

Lemma 4.1. *For every open neighborhood V of $\mathcal{S}$ in $\mathcal{M}^2 \times I_0$, the map $g^* : H^*(\mathcal{M}^2) \longrightarrow H^*(V)$, induced by the projection $g : V \to \mathcal{M}^2$, is a monomorphism.*

As a consequence, we have

Proposition 4.2. *There is a constant K, independent of σ_0, so that*

$$\sup_{(\xi,\sigma)\in\mathcal{M}^2\times I_0} \Psi(\gamma(\xi,\sigma,1)) \geq -K \quad \textit{for all } \gamma\in\Gamma.$$

Proof. Since Ω is smooth, there is a $\delta_0>0$ such that if $\xi_1,\xi_2\in\Omega_\rho$ and $|\xi_1-\xi_2|<\delta_0$, then the line segment $[\xi_1,\xi_2]\subset\Omega$. Then we let $K>0$ so that $G(\xi_1,\xi_2)\geq K$ implies $|\xi_1-\xi_2|<\delta_0$.

Assume, for contradiction, that for certain $\gamma\in\Gamma$

$$\Psi(\gamma(\xi,\sigma,1))\leq -K \qquad \text{for all} \quad (\xi,\sigma)\in\mathcal{M}^2\times I_0.$$

This implies that, for a small neighborhood V of $\mathcal{S}$ in $\mathcal{M}^2\times I_0$, we have

$$G(\tilde{\xi}(\xi,\sigma,1))\geq K \quad \text{for all} \quad (\xi,\sigma)\in V. \tag{4.32}$$

Let $D_0=\Omega\times\Omega\times\mathbb{R}^2_+$ and $\gamma_1=\gamma(\cdot,1)$. Consider the inclusion $i_2:\gamma_1(V)\to D_0$ and the maps $p:\gamma_1(V)\to\Omega\times\mathbb{R}^2_+$ and $\delta:\Omega\times\mathbb{R}^2_+\to D_0$ defined as $p(\xi_1,\xi_2,\Lambda)=(\xi_1,\Lambda)$ and $\delta(\xi_1,\Lambda)=(\xi_1,\xi_1,\Lambda)$. From (4.32) we find that the function $h:\gamma_1(V)\times[0,1]\to D_0$ defined as $h(\xi_1,\xi_2,\Lambda,t)=(\xi_1,\xi_2+t(\xi_1-\xi_2),\Lambda)$ is a homotopy between i_2 and $\delta\circ p$. Let d be the integer given in Theorem 1.2 and consider the following commutative diagram:

$$\begin{array}{ccc}
H^{2d}(\mathcal{M}^2\times I_0) & \xleftarrow{\gamma_1{}^*} & H^{2d}(D_0)\\
i_1^*\downarrow & & i_2^*\downarrow\\
H^{2d}(V) & \xleftarrow{\gamma_2{}^*} & H^{2d}(\gamma_1(V)),
\end{array}$$

where i_1 is an inclusion map and $\gamma_2=\gamma_1|_V$. From the hypothesis of Theorem 1.2 we find $u\in H^d(\mathcal{M})$ and $v\in H^d(\Omega)$ are nontrivial elements such that $\iota^*(v)=u$. If $\hat{v}\times\hat{v}\in H^{2d}(D_0)$ is the corresponding element, then by the homotopy axiom and Lemma 4.1 we have $i_1^*\circ\gamma_1^*(\hat{v}\times\hat{v})\neq 0$. On the other hand we see that $\delta^*(\hat{v}\times\hat{v})=\hat{v}\smile\hat{v}\in H^{2d}(\Omega\times\mathbb{R}^2_+)$ is zero, either because d is odd or because $H^{2d}(\Omega)=0$. In both cases we have then $\gamma_2^*\circ i_2^*(\hat{v}\times\hat{v})=0$, providing a contradiction. □

In proving that $c(\Omega)$ is a critical value for Ψ, the next key step is to show that Ψ satisfies the Palais–Smale (P.S.) condition in D_φ. We do this now.

Proposition 4.3. *The function Ψ satisfies the P.S. condition in D_φ at level $c(\Omega)$.*

Proof. The following preliminary fact fixes the value of the parameter $\rho>0$: Given $c\in\mathbb{R}$ there exists $\rho>0$ sufficiently small so that if $(\bar{\xi}_1,\bar{\xi}_2)\in\partial(\Omega_\rho\times\Omega_\rho)$ is such that $\varphi(\bar{\xi}_1,\bar{\xi}_2)=c$, then there is a vector τ, tangent to $\partial(\Omega_\rho\times\Omega_\rho)$ at the point $(\bar{\xi}_1,\bar{\xi}_2)$, so that

$$\nabla\varphi(\bar{\xi}_1,\bar{\xi}_2)\cdot\tau\neq 0. \tag{4.33}$$

This choice of ρ allows us to prove a related property for Ψ. That is, given a sequence $\{(\xi_n, \Lambda_n)\} \subset D_\varphi$ such that $(\xi_n, \Lambda_n) \to (\bar{\xi}, \bar{\Lambda}) \in \partial D_\varphi$ and $\Psi(\xi_n, \Lambda_n) \to c(\Omega)$, there is a vector T, tangent to ∂D_φ at $(\bar{\xi}, \bar{\Lambda})$, such that

$$\nabla\Psi(\bar{\xi}, \bar{\Lambda}) \cdot T \neq 0. \tag{4.34}$$

In order to prove (4.34) we first observe that if $\Lambda_n \to \bar{\Lambda} \in \partial\mathbb{R}^2_+$ then $\Psi(\xi_n, \Lambda_n) \to -\infty$. Thus we can assume that $\bar{\Lambda} \in \mathbb{R}^2_+$, $\bar{\xi} \in \bar{\Omega}_\rho \times \bar{\Omega}_\rho$ and $\varphi(\bar{\xi}) \leq -\rho_0$. Two cases arise: if $\nabla_\Lambda \Psi(\bar{\xi}, \bar{\Lambda}) \neq 0$, then T can be chosen parallel to $\nabla_\Lambda \Psi(\bar{\xi}, \bar{\Lambda})$. Otherwise, when $\nabla_\Lambda \Psi(\bar{\xi}, \bar{\Lambda}) = 0$ we have that $\bar{\Lambda}$ satisfies

$$\bar{\Lambda}_1^2 = -\frac{H(\bar{\xi}_2, \bar{\xi}_2)^{1/2}}{H(\bar{\xi}_1, \bar{\xi}_1)^{1/2}\varphi(\bar{\xi}_1, \bar{\xi}_2)}, \quad \bar{\Lambda}_2^2 = -\frac{H(\bar{\xi}_1, \bar{\xi}_1)^{1/2}}{H(\bar{\xi}_2, \bar{\xi}_2)^{1/2}\varphi(\bar{\xi}_1, \bar{\xi}_2)},$$

and $\bar{\xi}$ satisfies $\varphi(\bar{\xi}) < 0$. Substituting back in Ψ, we get

$$\Psi(\bar{\xi}_1, \bar{\xi}_2, \bar{\Lambda}_1, \bar{\Lambda}_2) = -\frac{1}{2} + \frac{1}{2}\log\frac{1}{|\varphi(\bar{\xi}_1, \bar{\xi}_2)|}$$

and then $\varphi(\bar{\xi}) = -\exp(-2c(\Omega) - 1) \leq -2\rho_0 < -\rho_0$. Thus $\bar{\xi} \in \partial(\Omega_\rho \times \Omega_\rho)$ and the application of (4.33) completes the proof of (4.34). Now we can define an appropriate negative gradient flow that will remain in D_φ at level $c(\Omega)$.

To finish, we mention that the Palais–Smale condition indeed holds: if $\{(\xi_n, \Lambda_n)\} \subset D_\varphi$ satisfies $\Psi(\xi_n, \Lambda_n) \to c(\Omega)$ and $\nabla\Psi(\xi_n, \Lambda_n) \to 0$, then $\{(\xi_n, \Lambda_n)\}$ has a subsequence converging to some $(\bar{\xi}, \bar{\Lambda}) \in D$. In fact, it can be shown that the sequence Λ_n remains bounded. Finally we conclude using (4.34). □

In view of Proposition 4.1 and 4.2 we have that the number $c(\Omega)$ given in (4.31) is a critical value for Ψ in D. This min-max setting does survive a small C^1-perturbation of Ψ in the considered region, yielding a critical point of the functional I (see (3.25), (3.26)) as well, as required.

We finish this note by mentioning that extra care needs to be taken in the construction of multiple pairs of spikes as in Theorem 1.1. We need to work on a region for the reduced functional which indeed *isolates* pairs of spikes associated to distinct holes. This is possible provided that μ is chosen sufficiently small: in such a case, interactions of far away spikes become negligible, and the functional Ψ basically decouples into the sum of several functionals of the form φ.

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