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Multi-Peak Solutions for Super-Critical Elliptic Problems in Domains with Small Holes

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This paper deals with the slightly super-critical elliptic problem

$$\begin{cases} -\Delta u = u^{\frac{(N+2)}{(N-2)}+\varepsilon} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\varepsilon > 0$ is a small parameter and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. Assuming that the domain exhibits k sufficiently small holes, multiple solutions are constructed by *gluing* double-spike patterns located near each of the holes. © 2002 Elsevier Science (USA)

Key Words: supercritical exponent; solution with multiple double-spikes.

1. INTRODUCTION

This paper deals with the construction of solutions of the problem

$$\begin{aligned} -\Delta u &= u^{\frac{(N+2)}{(N-2)}+\varepsilon} && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

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where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $N \geq 3$, and $\varepsilon > 0$ is a small parameter.

It is well known that the problem

$$\begin{aligned} -\Delta u = u^q \quad & \text{in } \Omega, \\ u > 0 \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega \end{aligned} \tag{2}$$

has at least one solution when $1 < q < \frac{N+2}{N-2}$. Instead, when $q \geq \frac{N+2}{N-2}$ the existence of solutions to problem (2) depends strongly on the topology or geometry of Ω. A well-known result by Pohozaev [13], asserts that (2) has no solutions if $q \geq \frac{N+2}{N-2}$ and Ω is star shaped. On the other hand, Kazdan and Warner [10] showed that (2) has a radially symmetric solution for any $q > 1$ when Ω is a symmetric annulus. Coron in [6] considered the case $q = \frac{N+2}{N-2}$, and showed that (2) is solvable when Ω is a (nonsymmetric) domain exhibiting a small hole, say $\Omega = \mathscr{D} \setminus \bar{B}(P_0, \mu)$, where $\mathscr{D}$ is a smooth bounded domain, $P_0 \in \mathscr{D}$ and μ is sufficiently small. In [2], Bahri and Coron considerably generalize this result proving that if $q = \frac{N+2}{N-2}$ and if some homology group of Ω with coefficients in $\mathbf{Z}_2$ is nontrivial, then problem (2) has a solution. While it may be expected that this solution *survives* a small super-critical perturbation of the exponent as in (1), the indirect variational arguments employed in [2, 6] do not seem to give in principle a clue on how to obtain this fact. Solvability when $q > \frac{N+2}{N-2}$ in domains "with topology" is not true, in general, as shown via counterexamples by Passaseo [11, 12], answering negatively the question stated by Brezis [4]. In our recent work [7], we have considered problem (1) in Coron's situation of a domain with a small perforation, and proved solvability whenever ε is sufficiently small. The proof is constructive and, rather puzzingly, the solutions found collapse as $\varepsilon \to 0$ in the form of a double spike: the solution tends to vanish everywhere except around two local maximum points which blow-up at the rate $O(\varepsilon^{\frac{-1}{2}})$. The perforation does not need to be symmetric or contained in a small ball; for instance, in $\mathbb{R}^3$ a domain with a torus with narrow section excised would suffice.

The purpose of this paper is to raise the issue of solvability of problem (1) in a domain exhibiting multiple holes. Our main result asserts that in such a situation, multi-peak solutions exist, consisting of the glueing of double-spikes associated with each of the holes. More precisely, our setting in problem (1) is as follows.

Let $\mathscr{D}$ be a bounded, smooth domain in $\mathbb{R}^N$, $N \geq 3$, and $P_1, P_2, \ldots, P_m$ points of $\mathscr{D}$. Let us consider the domain

$$\Omega = \mathscr{D} \Bigg\backslash \bigcup_{i=1}^{m} \bar{B}(P_i, \mu), \tag{3}$$

where $\mu > 0$ is a small number.

THEOREM 1.1. *There exists a* $\mu_0 > 0$, *which depends on* $\mathscr{D}$ *and the points* $P_1, \dots, P_m$ *such that if* $0 < \mu < \mu_0$ *is fixed and* Ω *is the domain given by* (3), *then the following holds*: *Given a number* $1 \leqslant k \leqslant m$, *there exists* $\varepsilon_0 > 0$ *and a family of solutions* u_ε, $0 < \varepsilon < \varepsilon_0$ *of* (1), *with the following property*: u_ε *has exactly k pairs of local maximum points* $(\xi^\varepsilon_{j1}, \xi^\varepsilon_{j2}) \in \Omega^2$ $j = 1, \dots, k$ *with* $c\mu < |\xi^\varepsilon_{ji} - P_j| < C\mu$ *for certain constants* c, C *independent of* μ, *and such that for each small* $\delta > 0$,

$$\sup_{\{|x-\xi^\varepsilon_{ij}|>\delta \ \ \forall i,j\}} u_\varepsilon(x) \to 0$$

and

$$\sup_{|x-\xi_{ij}|<\delta} u_\varepsilon(x) \to +\infty \qquad \forall\ i,j$$

as $\varepsilon \to 0$.

While it will be clear from the proofs that there is no need for the small excised domains to be balls of same radii, we will only consider this case for notational simplicity. Let us also observe that by relabeling the points $P_1, \dots, P_m$, the above result actually yields that for each $1 \leqslant k \leqslant m$ and any set of indices $i_1, \dots, i_k$ in $\{1, \dots, m\}$ a solution exhibiting double-spikes simultaneously near the points $P_{i_1}, \dots, P_{i_k}$ exists. This, in particular, yields the existence of at least $2^m - 1$ solutions of the problem whenever ε is sufficiently small.

The proof will provide much finer information on the asymptotic profile of the blow-up of these solutions as $\varepsilon \to 0$: after scaling and translation one sees around each ξ^ε_{ij} a solution in entire $\mathbb{R}^N$ of the equation at the critical exponent. More precisely, we will find

$$u_\varepsilon(x) = \sum_{i=1}^{k} \sum_{j=1}^{2} \left(\frac{\alpha_N \lambda_{ij} \varepsilon^{\frac{1}{N-2}}}{\varepsilon^{\frac{2}{N-2}} \lambda_{ij}^2 + |x - \xi^\varepsilon_{ij}|^2} \right)^{\frac{N-2}{2}} + \theta_\varepsilon(x), \tag{4}$$

where $\theta_\varepsilon(x) \to 0$ uniformly as $\varepsilon \to 0$. The numbers λ and the points ξ will be further identified as critical points of certain function built upon the Green's function of Ω.

The role of the Green's function in concentration phenomena associated with almost-critical problems on the subcritical side, i.e. $q = \frac{N+2}{N-2} - \varepsilon$, has already been considered in several works, [3, 6, 9, 14–16].

In what follows, we will denote by $G(x, y)$ the Green's function of Ω, namely G satisfies

$$\begin{aligned} \Delta_x G(x, y) &= \delta(x - y), \qquad && x \in \Omega, \\ G(x, y) &= 0, && x \in \partial\Omega, \end{aligned}$$

where $\delta(x)$ denotes the Dirac mass at the origin. We denote by $H(x, y)$ its regular part, namely

$$H(x, y) = \Gamma(x - y) - G(x, y),$$

where Γ denotes the fundamental solution of the Laplacian,

$$\Gamma(x) = b_N |x|^{2-N},$$

so that H satisfies

$$\begin{aligned} \Delta_x H(x, y) &= 0, && x \in \Omega, \\ H(x, y) &= \Gamma(x - y), && x \in \partial\Omega. \end{aligned}$$

Its *diagonal* $H(x, x)$ is usually called Robin's function of the domain.

The proof of Theorem 1.1 follows along the general lines of that we devised for the construction of a single two-spike: we work out a finite-dimensional reduction scheme in a suitable functional space, reducing the problem to that of finding critical points of a function which depends on points ξ and scaling parameters λ. The main part of the reduced function is explicitly given in terms of the Green's and Robin function. A critical point is finally found via a min–max characterization worked out with topological arguments. A technical point to be especially careful with is that of isolating the different pairs of spikes so that the min–max scheme does not see undesirable interactions between points associated with different holes.

Sections 2–4 will be devoted to discuss the finite-dimensional reduction scheme for the construction of a solution to (1) in the general case of h spikes. In Section 5 we will be back to our original setting, by considering the $2k$-spike case, with $1 \leqslant k \leqslant m$, and we will set up the min–max scheme to find a critical point of the reduced functional, which will let us to the proof of Theorem 1.1.

2. PRELIMINARIES AND BASIC ESTIMATES IN THE REDUCED ENERGY

Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$ and let us consider the enlarged domain

$$\Omega_\varepsilon = \varepsilon^{-\frac{1}{N-2}}\,\Omega, \qquad \varepsilon > 0.$$

If we make the change of variable

$$v(y) = \varepsilon^{\frac{1}{2+\varepsilon\frac{N-2}{2}}} u(\varepsilon^{\frac{1}{N-2}} y), \qquad y \in \Omega_\varepsilon,$$

we see that u solves (1) if and only if v satisfies

$$\begin{aligned} &\Delta v + v^{\frac{N+2}{N-2}+\varepsilon} = 0 && \text{in } \Omega_\varepsilon, \\ &v_\varepsilon > 0 && \text{in } \Omega_\varepsilon, \\ &v = 0 && \text{on } \partial\Omega_\varepsilon. \end{aligned} \tag{5}$$

Since Ω_ε is expanding to the whole $\mathbb{R}^N$, and all positive solutions of

$$\Delta v + v^{\frac{N+2}{N-2}} = 0 \qquad \text{in } \mathbb{R}^N$$

are given by the functions

$$\bar{U}(x) = \alpha_N \left(\frac{1}{1+|x|^2}\right)^{\frac{N-2}{2}} \qquad \text{and} \qquad \bar{U}_{\lambda,y}(x) = \lambda^{-\frac{N-2}{2}} \bar{U}\left(\frac{x-y}{\lambda}\right)$$

with $\alpha_N = (N(N-2))^{\frac{N-2}{4}}$, $y \in \mathbb{R}^N$ and $\lambda > 0$, it is natural to look for solutions v of the form

$$v(y) \sim \sum_{j=1}^{h} \bar{U}_{\lambda_j,\xi_j'}(y) \tag{6}$$

for certain set of h points $\xi_1, \dots, \xi_h$ in Ω and numbers $\lambda_1, \dots, \lambda_h > 0$, where from now on we use the letter ξ to denote a point in Ω and

$$\xi' = \varepsilon^{\frac{-1}{N-1}} \xi \in \Omega_\varepsilon.$$

A better approximation in (6) should be obtained by using the orthogonal projections onto $H_0^1(\Omega_\varepsilon)$ of the functions $\bar{U}_{\lambda,\xi'}$, denoted by $V_{\lambda,\xi'}$, namely the unique solution of the equation

$$\begin{aligned} -\Delta V_{\lambda,\xi'} &= \bar{U}_{\lambda,\xi'}^{\frac{N+2}{N-2}} && \text{in } \Omega_\varepsilon, \\ V_{\lambda,\xi'} &= 0 && \text{on } \partial\Omega_\varepsilon, \end{aligned}$$

so that the function $\phi_{\lambda,\xi'}$, defined as $\phi_{\lambda,\xi'} = \bar{U}_{\lambda,\xi'} - V_{\lambda,\xi'}$, will satisfy the equation

$$-\Delta \phi_{\lambda,\xi'} = 0 \qquad \text{in } \Omega_\varepsilon,$$

$$\phi_{\lambda,\xi'} = \bar{U}_{\lambda,\xi'} \qquad \text{on } \partial\Omega_\varepsilon.$$

Then, we have

$$\phi_{\lambda,\xi'}(x) = \varepsilon H(\varepsilon^{\frac{1}{N-2}}x, \xi)\lambda^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \bar{U}^p + o(\varepsilon) \tag{7}$$

and, away from $x = \xi'$,

$$V_{\lambda,\xi'}(x) = \varepsilon G(\varepsilon^{\frac{1}{N-2}}x, \xi)\lambda^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \bar{U}^p + o(\varepsilon), \tag{8}$$

uniformly for x on each compact subset of Ω_ε. Here G and H are, respectively, the Green's function of the Laplacian with the Dirichlet boundary condition on Ω and its regular part. For notational convenience from now on we denote $p = \frac{N+2}{N-2}$.

We consider the functions

$$\bar{U}_i = \bar{U}_{\lambda_i,\xi_i'}, \qquad V_i = V_{\lambda_i,\xi_i'}, \qquad i = 1, \ldots, h \tag{9}$$

and we write

$$\bar{V} = \sum_{j=1}^{h} \bar{U}_j, \qquad V = \sum_{j=1}^{h} V_j. \tag{10}$$

In what remains of this paper our goal is to find a solution v of problem (5) of the form

$$v = V + \phi, \tag{11}$$

which for suitable points ξ and scalars λ will have the remainder term ϕ of small order all over Ω_ε, in fact with magnitude not exceeding $O(\varepsilon)$ in any reasonable norm over Ω_ε. On the other hand, solutions of (5) correspond to stationary points of the functional $\mathscr{I}_\varepsilon$ defined as

$$\mathscr{I}_\varepsilon(u) = \frac{1}{2} \int_{\Omega_\varepsilon} |Du|^2 - \frac{1}{p+1+\varepsilon} \int_{\Omega_\varepsilon} u^{p+1+\varepsilon}. \tag{12}$$

If a solution of the form (11) exists, we should have $\mathscr{I}_\varepsilon(v) \sim \mathscr{I}_\varepsilon(V)$ and that the corresponding points (ξ, λ) in the definition of V are also "approximately stationary" for the finite-dimensional functional $(\xi, \lambda) \mapsto \mathscr{I}_\varepsilon(V)$. It is then a natural step toward the construction of the solution to understand the structure of this functional and to find critical points of it which survive small perturbations. Thus, our immediate goal is to estimate $\mathscr{I}_\varepsilon(V)$ where V is given by (10). If the points ξ_i are taken far apart from each other and also far away from the boundary, we have that as a first approximation

$$\mathscr{I}_\varepsilon(V) \sim \sum_{i=1}^{h} \mathscr{I}_\varepsilon(\bar{U}_i) \sim hC_N,$$

where

$$C_N = \frac{1}{2}\int_{\mathbb{R}^N} |D\bar{U}|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} |\bar{U}|^{p+1}.$$

To work out a more precise expansion, it will be convenient to recast the variables λ_i into the Λ_i's given by

$$\lambda_i = (a_N\Lambda_i)^{\frac{1}{N-2}} \tag{13}$$

with

$$a_N = \frac{1}{p+1}\frac{\int_{\mathbb{R}^N} \bar{U}^{p+1}}{(\int_{\mathbb{R}^N} \bar{U}^p)^2}.$$

Let us fix a small number $\delta > 0$. We will restrict ourselves to consider only points $\xi_i \in \Omega$ and positive numbers Λ_i, such that

$$|\xi_i - \xi_j| > \delta, \qquad \text{if } i \neq j, \qquad \text{dist}(\xi_i, \partial\Omega) > \delta, \qquad \delta < \Lambda_i < \delta^{-1} \tag{14}$$

for all $i = 1, \dots, h$.

Lemma 2.1. *The following expansion holds*:

$$\mathscr{I}_\varepsilon(V) = hC_N + \varepsilon[\gamma_N + \omega_N\Psi(\xi,\Lambda)] + o(\varepsilon) \tag{15}$$

uniformly with respect to (ξ,Λ) *satisfying* (13) *and* (14). *Here we have*

$$\Psi(\xi,\Lambda) = \frac{1}{2}\left\{\sum_{j=1}^{h} H(\xi_j,\xi_j)\Lambda_j^2 - 2\sum_{i<j} G(\xi_i,\xi_j)\Lambda_i\Lambda_j\right\} + \log(\Lambda_1\cdots\Lambda_h), \tag{16}$$

$$\gamma_N = \left\{\frac{h}{p+1}\omega_N + \frac{h}{2}\omega_N\log a_N - \frac{h}{p+1}\int_{\mathbb{R}^N} \bar{U}^{p+1}\log\bar{U}\right\} \tag{17}$$

and $\omega_N = \frac{1}{p+1}\int_{\mathbb{R}^N} \bar{U}^{p+1}$.

Proof. We first write

$$\mathscr{I}_\varepsilon(V) = \mathscr{I}_0(V) + \frac{1}{p+1}\int_{\Omega_\varepsilon} V^{p+1} - \frac{1}{p+1+\varepsilon}\int_{\Omega_\varepsilon} V^{p+1+\varepsilon}, \tag{18}$$

where

$$\mathscr{I}_0(V) = \frac{1}{2}\int_{\Omega_\varepsilon} |DV|^2 - \frac{1}{p+1}\int_{\Omega_\varepsilon} V^{p+1}.$$

Let us first estimate $\mathscr{I}_0(V)$; we have

$$\begin{aligned}\mathscr{I}_0(V) &= \mathscr{I}_0\left(\sum_{j=1}^{h} V_j\right)\\ &= \sum_{j=1}^{h}\left[\frac{1}{2}\int_{\Omega_\varepsilon} |DV_j|^2 - \frac{1}{p+1}\int_{\Omega_\varepsilon} |V_j|^{p+1}\right]\\ &\quad + \sum_{i\neq j}\int_{\Omega_\varepsilon} DV_i DV_j - \frac{1}{p+1}\int_{\Omega_\varepsilon}\left[\left(\sum_{j=1}^{h} V_j\right)^{p+1} - \sum_{j=1}^{h} V_j^{p+1}\right]. \end{aligned} \tag{19}$$

Arguing as in [1,3,7], and taking into account (7) and (8), one can prove that

$$\int_{\Omega_\varepsilon} |DV_i|^2 = \int_{\mathbb{R}^N} |D\bar{U}|^2 - \left(\int_{\mathbb{R}^N} \bar{U}^p\right)^2 H(\xi_i,\xi_i) a_N \Lambda_i^2 \varepsilon + o(\varepsilon), \tag{20}$$

$$\int_{\Omega_\varepsilon} DV_i DV_j = \left(\int_{\mathbb{R}^N} \bar{U}^p\right)^2 G(\xi_i,\xi_j) a_N \Lambda_i \Lambda_j \varepsilon + o(\varepsilon), \tag{21}$$

$$\int_{\Omega_\varepsilon} V_i^{p+1} = \int_{\mathbb{R}^N} \bar{U}^{p+1} - (p+1)\left(\int_{\mathbb{R}^N} \bar{U}^p\right)^2 H(\xi_i,\xi_i) a_N \Lambda_i^2 \varepsilon + o(\varepsilon) \tag{22}$$

and finally

$$\begin{aligned}&\frac{1}{p+1}\int_{\Omega_\varepsilon}\left[\left(\sum_{j=1}^{k} V_j\right)^{p+1} - \sum_{j=1}^{k} V_j^{p+1}\right]\\ &\qquad = 2\left(\int_{\mathbb{R}^N} \bar{U}^p\right)^2 G(\xi_i,\xi_j) a_N \Lambda_i \Lambda_j \varepsilon + (\varepsilon) \quad \forall i \neq j.\end{aligned} \tag{23}$$

From (19)–(23), we conclude that

$$\mathscr{I}_0(V) = hC_N + \frac{\omega_N}{2}\left\{\sum_{j=1}^{h} H(\xi_j,\xi_j)\Lambda_j^2 - 2\sum_{i<j} G(\xi_i,\xi_j)\Lambda_i\Lambda_j\right\} + o(\varepsilon).$$

Let us consider now the quantity

$$\mathscr{I}_\varepsilon(V) - \mathscr{I}_0(V) = \frac{\varepsilon}{(p+1)^2}\int_{\Omega_\varepsilon} V^{p+1} - \frac{\varepsilon}{p+1}\int_{\Omega_\varepsilon} V^{p+1}\log V + o(\varepsilon), \quad (24)$$

first we see that

$$\int_{\Omega_\varepsilon} V^{p+1} = h\int_{\mathbb{R}^N} \bar{U}^{p+1} + o(1).$$

On the other hand, for a number $\varrho > 0$ we can write

$$\int_{\Omega_\varepsilon} V^{p+1}\log V = \sum_{j=1}^{h}\int_{|x-\xi_j'|<\varrho} V^{p+1}\log V + o(\varepsilon).$$

For any index j, we have

$$\begin{aligned}
&\int_{|x-\xi_j'|<\varrho} V^{p+1}\log V \\
&\quad = -\frac{N-2}{2}\log\lambda_j\int_{|x-\xi_j'|<\varrho} V^{p+1} \\
&\qquad + \int_{|x-\xi_j'|<\varrho} V^{p+1}\log((\lambda_j)^{\frac{N-2}{2}}V_j + (\lambda_j)^{\frac{N-2}{2}}(V-V_j)) \\
&\quad = -\frac{N-2}{2}\log\lambda_j\left(\int_{\mathbb{R}^N}\bar{U}^{p+1} + o(\varepsilon)\right) + \int_{\mathbb{R}^N}\bar{U}^{p+1}\log\bar{U} + o(1).
\end{aligned}$$

Then we conclude

$$\begin{aligned}
&\int_{\Omega_\varepsilon} V^{p+1}\log V \\
&\quad = -\frac{N-2}{2}\log(\lambda_1\cdots\lambda_h)\left(\int_{\mathbb{R}^N}\bar{U}^{p+1}\right) + h\int_{\mathbb{R}^N}\bar{U}^{p+1}\log\bar{U} + o(1) \\
&\quad = -\frac{h}{2}(\log a_N)\int_{\mathbb{R}^N}\bar{U}^{p+1} - \left(\int_{\mathbb{R}^N}\bar{U}^{p+1}\right)\log(\Lambda_1\cdots\Lambda_h) \\
&\qquad + h\int_{\mathbb{R}^N}\bar{U}^{p+1}\log\bar{U} + o(1),
\end{aligned}$$

hence from (24) and the previous computation we get

$$\begin{aligned}
&\mathcal{I}_\varepsilon(V) - \mathcal{I}_0(V) \\
&= \varepsilon\Bigg[\frac{h}{(p+1)^2}\int_{\mathbb{R}^N} \bar{U}^{p+1} + \frac{h}{2(p+1)}\log a_N\left(\int_{\mathbb{R}^N} \bar{U}^{p+1}\right) \\
&\quad + \frac{\int_{\mathbb{R}^N} \bar{U}^{p+1}}{p+1}\log(\Lambda_1\cdots\Lambda_h) - \frac{h}{p+1}\int_{\mathbb{R}^N} \bar{U}^{p+1}\log\bar{U}\Bigg] + o(\varepsilon),
\end{aligned}$$

this concludes the proof. ∎

Remark 2.1. The quantity $o(\varepsilon)$ in the expansion of (15) is actually also of that size in the C^1-norm as a function of ξ and Λ in the considered region.

The next two sections will be devoted to reduce the problem of finding a solution of (5) of the form (11) to that of finding critical points (ξ, Λ) of a functional which is an $o(\varepsilon)$ perturbation of $\mathcal{I}_\varepsilon(V)$.

3. THE FINITE-DIMENSIONAL REDUCTION

Fix a small number $\delta > 0$ and consider points $\xi_i' \in \Omega_\varepsilon$, numbers $\Lambda_i > 0$, for $i = 1, \ldots, h$, such that

$$|\xi_i' - \xi_j'| > \delta\varepsilon^{\frac{-1}{N-1}}, \qquad \operatorname{dist}(\xi_i', \partial\Omega_\varepsilon) > \delta\varepsilon^{\frac{-1}{N-1}}, \qquad \delta < \Lambda_i < \delta^{-1}. \tag{25}$$

In this section, we deal with the following intermediate problem: Find a function ϕ such that for certain constants c_{ij} one has

$$\begin{cases}
\Delta(V+\phi) + (V+\phi)_+^{p+\varepsilon} = \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} & \text{in } \Omega_\varepsilon, \\
\phi = 0 & \text{on } \partial\Omega_\varepsilon, \\
\int_{\Omega_\varepsilon} \phi V_i^{p-1} Z_{ij} = 0 & \text{for all } i, j,
\end{cases} \tag{26}$$

where the functions V_i and V are defined in (9) and (10) and Z_{ij} will be defined below.

What we need to do is to solve (26) and then find points ξ and scalars Λ such that the associated c_{ij} are all zero, which yields a solution of (5).

Let us consider the functions

$$\bar{Z}_{ij} = \frac{\partial \bar{U}_i}{\partial \xi_{ij}'}, \quad j = 1, \ldots, N, \qquad \bar{Z}_{iN+1} = \frac{\partial \bar{U}_i}{\partial \lambda_i} = (x - \xi_i') \cdot \nabla \bar{U}_i + (N-2)\bar{U}_i$$

and then define the Z_{ij}'s in (26) to be their respective $H_0^1(\Omega_\varepsilon)$-projections, namely the unique solutions of

$$\begin{aligned} \Delta Z_{ij} &= \Delta \bar{Z}_{ij} && \text{in } \Omega_\varepsilon, \\ Z_{ij} &= 0 && \text{on } \partial\Omega_\varepsilon. \end{aligned}$$

The first equation in (26) can be rewritten in the following form:

$$\Delta\phi + (p+\varepsilon)V^{p+\varepsilon-1}\phi = -N_\varepsilon(\phi) - R^\varepsilon + \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} \qquad \text{in } \Omega_\varepsilon, \tag{27}$$

where

$$N_\varepsilon(\xi', \Lambda, \phi) = N_\varepsilon(\phi) = (V+\phi)_+^{p+\varepsilon} - V^{p+\varepsilon} - (p+\varepsilon)V^{p+\varepsilon-1}\phi \tag{28}$$

and

$$R^\varepsilon(\xi', \Lambda) = R^\varepsilon = V^{p+\varepsilon} - \sum_{j=1}^{h} \bar{U}_j^p. \tag{29}$$

Then we need to understand the following linear problem: given $h \in C^\alpha(\bar{\Omega}_\varepsilon)$, find a function ϕ such that

$$\begin{aligned} &\Delta\phi + (p+\varepsilon)V^{p+\varepsilon-1}\phi = h + \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} && \text{in } \Omega_\varepsilon, \\ &\phi = 0 && \text{on } \partial\Omega_\varepsilon, \\ &\int_{\Omega_\varepsilon} V_i^{p-1} Z_{ij}\phi = 0 && \text{for all } i, j, \end{aligned} \tag{30}$$

for certain constants c_{ij}, $i = 1, \ldots, h$, $j = 1, \ldots, N+1$. In order to get bounded solvability of (30), one needs to work in properly chosen functional spaces. Similarly as in [7], we introduce $L_*^\infty(\Omega_\varepsilon)$ and $L_{**}^\infty(\Omega_\varepsilon)$ to be, respectively, the spaces of functions defined on Ω_ε with finite $\|\cdot\|_*$-norm (respectively, $\|\cdot\|_{**}$-norm), where

$$\|\psi\|_* = \sup_{x\in\Omega_\varepsilon} \left| \left(\sum_{j=1}^{h} (1 + |x - \xi_j'|^2)^{\frac{N-2}{2}} \right)^{-1} \psi(x) \right|$$

and

$$\|\psi\|_{**} = \sup_{x\in\Omega_\varepsilon} \left| \left(\sum_{j=1}^{h} (1 + |x - \xi_j'|^2)^{\frac{N-2}{2}} \right)^{-\frac{N+1}{N-2}} \psi(x) \right|.$$

We then get the following result.

PROPOSITION 3.1. *There are numbers* $\varepsilon_0 > 0$, $C > 0$, *such that for each* $0 < \varepsilon < \varepsilon_0$, *any points* (ξ', Λ) *satisfying* (25), *and any* $h \in C^\alpha(\Omega_\varepsilon)$, *problem* (30) *has a unique solution*

$$\phi \equiv L_\varepsilon(h), \tag{31}$$

which besides satisfies

$$\|L_\varepsilon(h)\|_* \leqslant C\|h\|_{**}. \tag{32}$$

Moreover, *the operator* $S_\varepsilon(\xi', \Lambda, h) \equiv L_\varepsilon(h)$ *is of class* C^1 *in its arguments and*

$$\|\nabla_{\xi',\Lambda} S_\varepsilon(\xi', \Lambda, h)\|_* \leqslant C\|h\|_{**}. \tag{33}$$

The proof of this result is identical to that found in [7], except that there only the case $h = 2$ was considered. We therefore omit it. Now we return to the nonlinear problem (26).

PROPOSITION 3.2. *Assume the conditions of Proposition* 3.1 *are satisfied. Then there is a constant* $C > 0$ *such that*, *for all* $\varepsilon > 0$ *small enough*, *there exists a unique solution*

$$\phi = \phi(\xi', \Lambda) = \tilde{\phi} + \psi$$

to problem (26) *with* ψ *defined by* $\psi = -L_\varepsilon(R^\varepsilon)$ *and for points* ξ', Λ *satisfying* (25). *Besides*, *the map* $(\xi', \Lambda) \to \tilde{\phi}(\xi', \Lambda)$ *is of class* C^1 *for the* $\|\cdot\|_*$*-norm and*

$$\|\tilde{\phi}\|_* \leqslant C\varepsilon^{\min\{p,2\}}, \tag{34}$$

$$\|\nabla_{(\xi',\Lambda)}\tilde{\phi}\|_* \leqslant C\varepsilon^{\min\{p,2\}}. \tag{35}$$

Proof. Problem (26) is equivalent to solving a fixed point problem; indeed $\phi = \tilde{\phi} + \psi$ is a solution of (26) if

$$\tilde{\phi} = -L_\varepsilon(N_\varepsilon(\tilde{\phi} + \psi)) \equiv A_\varepsilon(\tilde{\phi}),$$

taking into account that $\psi = -L_\varepsilon(R^\varepsilon)$ and that L_ε is a linear operator.

Then we need to prove that the operator A_ε defined above is a contraction inside a properly chosen region. Arguing in [7], one can show that for all small $\varepsilon > 0$ and $\|\bar{\phi}\|_* \leqslant \frac{1}{4}$, we get

$$\|N_\varepsilon(\bar{\phi})\|_{**} \leqslant C\|\bar{\phi}\|_*^{\min\{p,2\}} \tag{36}$$

and

$$\|R^\varepsilon\|_{**} \leqslant C\varepsilon. \tag{37}$$

Hence, by definition of ψ and Proposition 3.1, we infer that

$$\|\psi\|_* \leqslant C\varepsilon$$

and

$$\|N_\varepsilon(\phi + \psi)\|_{**} \leqslant C(\|\phi\|_*^{\min\{p,2\}} + \varepsilon^{\min\{p,2\}}). \tag{38}$$

Let us now consider the set

$$\mathscr{F}_r = \{\tilde{\phi} \in H_0^1 \colon \|\tilde{\phi}\|_* \leqslant r\varepsilon^{\min\{p,2\}}\}$$

with r a positive number to be fixed later. From Proposition 3.1 and (38) we get

$$\|A_\varepsilon(\tilde{\phi})\|_* = \|L_\varepsilon(N_\varepsilon(\tilde{\phi} + \psi))\|_* \leqslant C\|N_\varepsilon(\tilde{\phi} + \psi)\|_{**}$$

$$\leqslant C[r^{\min\{p,2\}}\varepsilon^{\min\{p^2,4\}} + \varepsilon^{\min\{p,2\}}] < r\varepsilon^{\min\{p,2\}}$$

for small ε and any $\tilde{\phi} \in \mathscr{F}_r$, provided that r is chosen large enough, but independent of ε. A_ε turns out to be a contraction mapping in this region. This follows from the fact that N_ε defines a contraction in the $\|\cdot\|_{**}$-norm, which can be proved with a rather straightforward estimate, as done in detail in [7].

The proof of differentiability of the function $\tilde{\phi}(\xi', \Lambda)$ follows in approximately the same way as a similar result in [7], so we only sketch it. Let us write

$$B(\xi', \Lambda, \tilde{\phi}) \equiv \tilde{\phi} + L_\varepsilon(N_\varepsilon(\tilde{\phi} + \psi)),$$

we have $B(\xi', \Lambda, \tilde{\phi}) = 0$.

Now we write

$$D_{\tilde{\phi}}B(\xi', \Lambda, \tilde{\phi})[\theta] = \theta + L_\varepsilon(\theta D_{\tilde{\phi}}N_\varepsilon(\tilde{\phi} + \psi)) \equiv \theta + M(\theta).$$

It is not hard to check that the following estimate holds:

$$\|M(\theta)\|_* \leqslant C\varepsilon\|\theta\|_*.$$

It follows that for small ε, the linear operator $D_{\tilde{\phi}}B(\xi', \Lambda, \tilde{\phi})$ is invertible in L_*^∞, with uniformly bounded inverse. It also depends continuously on its parameters. Let us differentiate with respect to (ξ', Λ). We have

$$D_{\xi'}B(\xi', \Lambda, \tilde{\phi}) = (D_{\xi'}L_\varepsilon)(N_\varepsilon(\tilde{\phi} + \psi)) \circ L_\varepsilon[(D_{\xi'}N_\varepsilon)(\xi', \Lambda, \tilde{\phi} + \psi) \\ + L_\varepsilon(D_{\bar{\phi}}N_\varepsilon)(\xi', \Lambda, \tilde{\phi} + \psi)D_{\xi'}\psi],$$

where

$$D_{\xi'}\psi = -[(D_{\xi'}L_\varepsilon)(R^\varepsilon) \circ L_\varepsilon(D_{\xi'}R^\varepsilon)] \tag{39}$$

and

$$D_{\xi_i'}R^\varepsilon = (p + \varepsilon)V^{p+\varepsilon-1}D_{\xi_i'}V_i - p\bar{V}_i^{p-1}D_{\xi_i'}\bar{V}_i \quad \forall i = 1, \ldots, h. \tag{40}$$

These expressions depend continuously on their parameters; a similar computation holds for the derivative with respect to Λ. The implicit function theorem yields that $\tilde{\phi}(\xi', \Lambda)$ is a C^1 function into L_*^∞. Moreover, we have for instance

$$D_{\xi'}\tilde{\phi} = -(D_{\bar{\phi}}B(\xi', \Lambda, \tilde{\phi}))^{-1}[(D_{\xi'}L_\varepsilon)(N_\varepsilon(\tilde{\phi} + \psi))] \circ [L_\varepsilon(D_{\xi'}(N_\varepsilon(\xi', \Lambda, \tilde{\phi} + \psi))) \\ + L_\varepsilon((D_{\bar{\phi}}N_\varepsilon)(\xi', \Lambda, \tilde{\phi} + \psi)D_{\xi'}\psi)],$$

so that

$$\|D_{\xi'}\tilde{\phi}\|_* \leqslant C(\|N_\varepsilon(\tilde{\phi} + \psi)\|_{**} \\ + \|D_{\xi'}N_\varepsilon(\xi', \Lambda, \tilde{\phi} + \psi)\|_{**} + \|D_{\bar{\phi}}N_\varepsilon(\xi', \Lambda, \tilde{\phi} + \psi)D_{\xi'}\psi\|_{**}). \tag{41}$$

From (38) and (34) we get

$$\|N_\varepsilon(\tilde{\phi} + \psi)\|_{**} \leqslant C\varepsilon^{\min\{p,2\}}.$$

Straightforward computations allow us to estimate the other terms in (41), using, in particular, that by definition of ψ and Proposition 3.1,

$$\|D_{\xi'}\psi\|_* \leqslant C\varepsilon.$$

We finally obtain

$$\|D_{\xi'}\tilde{\phi}\|_* \leqslant C\varepsilon^{\min\{p,2\}}.$$

A similar estimate holds for differentiation with respect to Λ. This concludes the proof. ∎

4. THE REDUCED FUNCTIONAL

Let us consider points (ξ, Λ) which satisfy constraints (14) for some small fixed $\delta > 0$, and set $\xi' = \varepsilon^{\frac{-1}{N-1}}\xi$. Let $\phi(y) = \phi(\xi', \Lambda)(y)$ be the unique solution of problem

$$\begin{aligned} \Delta(V+\phi) + (V+\phi)_+^{p+\varepsilon} &= \textstyle\sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} && \text{in } \Omega_\varepsilon, \\ \phi &= 0 && \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \phi V_i^{p-1} Z_{ij} &= 0 && \text{for all } i,j, \end{aligned} \tag{42}$$

given by Proposition 3.2. Let us consider the functional

$$I(\xi, \Lambda) = \mathscr{I}_\varepsilon(V+\phi),$$

where $\mathscr{I}_\varepsilon$ was defined in (12). The definition of ϕ yields that

$$\mathscr{I}'_\varepsilon(V+\phi)[\eta] = 0$$

for all η which vanishes on $\partial\Omega_\varepsilon$ and such that

$$\int_{\Omega_\varepsilon} \eta V_i^{p-1} Z_{ij} = 0 \quad \text{for all } i,j.$$

The easily checked facts that

$$\frac{\partial V}{\partial \xi_{ij}} = Z_{ij} + o(1), \quad \frac{\partial V}{\partial \Lambda_i} = Z_{i(N+1)} + o(1)$$

with $o(1)$ small as $\varepsilon \to 0$, and the last part of Proposition 3.2 give the validity of the following.

LEMMA 4.1. *$v = V + \phi$ is a solution of problem (5), namely $c_{ij} = 0$ in (42) for all i, j, if and only if (ξ, Λ) is a critical point of I.*

Next step is then to give an asymptotic estimate for $I(\xi, \Lambda)$. Not too surprisingly, this functional and $\mathscr{I}_\varepsilon(V)$ coincide up to order $o(\varepsilon)$.

PROPOSITION 4.1. *We have the expansion*

$$I(\xi, \Lambda) = hC_N + \varepsilon[\gamma_N + w_N \Psi(\xi, \Lambda) + o(1)], \tag{43}$$

where $o(1) \to 0$ as $\varepsilon \to 0$ in the uniform C^1-sense with respect to (ξ, Λ) satisfying (25).

Here, we recall

$$\Psi(\xi,\Lambda)=\frac{1}{2}\left\{\sum_{j=1}^{h}H(\xi_j,\xi_j)\Lambda_j^2-2\sum_{i<j}G(\xi_i,\xi_j)\Lambda_i\Lambda_j\right\}+\log(\Lambda_1\cdots\Lambda_h)$$

and the constants in (43) *are those in Lemma* 2.1.

Proof. We start showing that

$$I(\xi,\Lambda)-\mathscr{I}_\varepsilon(V)=o(\varepsilon) \tag{44}$$

and

$$\nabla_{\xi,\Lambda}[I(\xi,\Lambda)-\mathscr{I}_\varepsilon(V)]=o(\varepsilon). \tag{45}$$

Taking into account that $0=D\mathscr{I}_\varepsilon(V+\psi+\tilde{\phi})[\tilde{\phi}]$, a Taylor expansion gives

$$\begin{aligned}\mathscr{I}_\varepsilon(V+\psi)-I(\xi,\Lambda)&=\int_0^1 t\,dt\,D^2\mathscr{I}_\varepsilon(V+\psi+t\tilde{\phi})[\tilde{\phi},\tilde{\phi}]\\&\quad\times\int_0^1 t\,dt\left[\int_{\Omega_\varepsilon}|\nabla\tilde{\phi}|^2-(p+\varepsilon)(V+\psi+t\tilde{\phi})^{p+\varepsilon-1}\tilde{\phi}^2\right]\\&=\int_0^1 t\,dt\left(\int_{\Omega_\varepsilon}N_\varepsilon(\tilde{\phi}+\psi)\tilde{\phi}\right.\\&\quad\left.+\int_{\Omega_\varepsilon}(p+\varepsilon)[V^{p+\varepsilon-1}-(V+\psi+t\tilde{\phi})^{p+\varepsilon-1}]\tilde{\phi}^2\right).\end{aligned} \tag{46}$$

Since $\|\tilde{\phi}\|_*=O(\varepsilon^{\min\{p,2\}})$, we get

$$I(\xi,\Lambda)-\mathscr{I}_\varepsilon(V+\psi)=O(\varepsilon^{2\min\{p,2\}}). \tag{47}$$

Differentiating with respect to ξ variables we get from (46) that

$$\begin{aligned}&D_\xi[\mathscr{I}_\varepsilon(V+\psi)-I(\xi,\Lambda)]\\&\quad=\varepsilon^{-\frac{1}{N-1}}\int_0^1 t\,dt\left(\int_{\Omega_\varepsilon}D_{\xi'}[(N_\varepsilon(\tilde{\phi}+\psi))\tilde{\phi}]\right.\\&\qquad\left.+(p+\varepsilon)\int_{\Omega_\varepsilon}D_{\xi'}[((V+\psi+t\tilde{\phi})^{p+\varepsilon-1}-(V+\psi)^{p+\varepsilon-1})\tilde{\phi}^2]\right).\end{aligned} \tag{48}$$

Using the computations in the proof of Proposition 3.2 we get that the first integral in relation (48) can be estimated by $O(\varepsilon^{2\min\{p,2\}})$, so does the second; hence

$$D_\xi[I(\xi,\Lambda) - \mathscr{I}_\varepsilon(V+\psi)] = O(\varepsilon^{\min\{2p,4\}-\frac{1}{N-2}}).$$

Now, since $D\mathscr{I}_\varepsilon(V)[\psi] = \int_{\Omega_\varepsilon} R^\varepsilon\psi$,

$$\begin{aligned}&\mathscr{I}_\varepsilon(V+\psi) - \mathscr{I}_\varepsilon(V)\\&= \left\{\int_0^1 (1-t)\,dt\,[(p+\varepsilon)\int_{\Omega_\varepsilon} ((V+t\psi)^{p+\varepsilon-1} - V^{p+\varepsilon-1})\psi^2] - 2\int_{\Omega_\varepsilon} R^\varepsilon\psi\right\}.\end{aligned} \tag{49}$$

Since $\|\psi\|_* + \|R_\varepsilon\|_{**} = O(\varepsilon)$, the above term is $O(\varepsilon^2)$; then, (44) follows from (47) and (49). Using again (49), we see that

$$\begin{aligned}&D_\xi[\mathscr{I}_\varepsilon(V+\psi) - \mathscr{I}_\varepsilon(V)]\\&\quad = \varepsilon^{-\frac{1}{N-2}} D_{\xi'}\left\{\int_0^1 (1-t)\,dt\left[(p+\varepsilon)\int_{\Omega_\varepsilon} ((V+t\psi)^{p+\varepsilon-1} - V^{p+\varepsilon-1})\psi^2\right]\right.\\&\qquad \left. -2\int_{\Omega_\varepsilon} R^\varepsilon\psi\right\}.\end{aligned}$$

Since from Proposition 3.1 it follows that $\|D_{\xi'}\psi\|_* = O(\varepsilon)$, we get

$$D_\xi[\mathscr{I}_\varepsilon(V+\psi) - \mathscr{I}_\varepsilon(V)] = O(\varepsilon^2) - 2\varepsilon^{-\frac{1}{N-2}} D_{\xi'}\left(\int_{\Omega_\varepsilon} R^\varepsilon\psi\right).$$

Arguing as in [9], one gets that

$$D_{\xi'}\left(\int_{\Omega_\varepsilon} R^\varepsilon\psi\right) = o(\varepsilon^2),$$

which gives (45).

From Lemma 2.1, we can finally conclude that

$$I(\xi,\Lambda) = hC_N + \varepsilon[\gamma_N + w_N\Psi(\xi,\Lambda)] + o(\varepsilon). \tag{50}$$

On the other hand, as a consequence of (45) and the remark after Lemma 2.1 we also get

$$\nabla I(\xi,\Lambda) = \varepsilon\frac{1}{p+1}\left(\int_{\mathbb{R}^N} U^{p+1}\right)(\nabla\Psi(\xi,\Lambda) + o(1)). \quad ∎ \tag{51}$$

5. THE EXTERIOR DOMAIN

Let us consider the exterior domain

$$D_* = \mathbb{R}^N \setminus \bar{B}(0,1).$$

We denote by G_* and H_* the Green's function of D_* and its regular part. In this section, we will work out some estimates for these objects which will be useful for the resolution of the finite-dimensional variational problem derived in the previous section, in the situation of Theorem 1.1. Explicitly, we have

$$H_*(x,y) = \frac{b_N}{|\,|y|(x-\bar{y})|^{N-2}},$$

where $\bar{y} = \frac{y}{|y|^2}$, and

$$G_*(x,y) = \frac{b_N}{|x-y|^{N-2}} - H_*(x,y).$$

In particular,

$$H_*(x,x) = \frac{b_N}{(|x|^2-1)^{N-2}}.$$

More explicitly, let θ be the angle formed by the vectors x and y. Then,

$$H_*(x,y) = \frac{b_N}{(1+|x|^2|y|^2 - 2|x||y|\cos\theta)^{\frac{N-2}{2}}}.$$

We want to analyze the function

$$\varphi_*(x,y) = H_*(x,x)^{1/2} H_*(y,y)^{1/2} - G_*(x,y), \qquad x \neq y,$$

namely

$$\begin{aligned} b_N^{-1}\varphi_*(x,y) = & \frac{1}{(|x|^2-1)^{\frac{N-2}{2}}} \frac{1}{(|y|^2-1)^{\frac{N-2}{2}}} \\ & + \frac{1}{(1+|x|^2|y|^2-2|x||y|\cos\theta)^{\frac{N-2}{2}}} \\ & - \frac{1}{(|x|^2+|y|^2-2|x||y|\cos\theta)^{\frac{N-2}{2}}}. \end{aligned}$$

Now we make the following observation: let x and y vary letting their magnitudes remain constant. If we differentiate with respect to the angle θ, we obtain

$$a_N^{-1}\frac{\partial}{\partial\theta}\varphi_*(x,y) = \{(|x|^2+|y|^2-2|x||y|\cos\theta)^{-\frac{N}{2}} - (1+|x|^2|y|^2-2|x||y|\cos\theta)^{-\frac{N}{2}}\}\sin\theta > 0$$

for $0<\theta<\pi$. In particular, for given magnitudes $|x|$ and $|y|$ φ_* maximizes its value when $\theta=\pi$, in other words when x and y have opposite directions. Assume this is the situation, namely that for a unit vector $\mathbf{e}$, $x=s\mathbf{e}$, $y=-t\mathbf{e}$, with $s,t>1$. Then in this case $\varphi_*(x,y)$ reduces to

$$b_N^{-1}\varphi_*(x,y) = b_N^{-1}\tilde{\varphi}_*(s,t) = \frac{1}{(s^2-1)^{\frac{N-2}{2}}(t^2-1)^{\frac{N-2}{2}}} + \frac{1}{(st+1)^{N-2}} - \frac{1}{(s+t)^{N-2}}.$$

This function has a negative global minimum value, attained at a point of the form (ρ^*,ρ^*). Let

$$c^* = -\tilde{\varphi}_*(\rho^*,\rho^*) = -\min_{(x,y)\in D_*}\tilde{\varphi}_*(|x|,|y|). \tag{52}$$

Let us consider then a small value δ_* for which the level set $\{\tilde{\varphi}_*(s,t)=-\delta_*\}$ is a closed curve and that $\nabla\tilde{\varphi}_*(s,t)$ is nonzero on it. Set

$$\mathscr{A} = \{(x,y)\mid\tilde{\varphi}_*(|x|,|y|)< -\delta_*\}. \tag{53}$$

Thus, the above discussion shows that on this bounded region we have $\varphi_*(x,y)< -\delta_*$ and that if $(x,y)\in\partial\mathscr{A}$ one of the following two situations occurs: Either there is a tangential direction τ to $\partial\mathscr{A}$ such that $\nabla\varphi_*(x,y)\cdot\tau\neq 0$ or x,y lie in opposite directions, $\varphi_*(x,y)=-\delta_*$ and $\nabla\varphi_*(x,y)\neq 0$ points orthogonally outwards $\mathscr{A}$.

The following fact will be useful later. The matrix

$$M_*(x,y) = \begin{bmatrix} H_*(x,x) & -G_*(x,y) \\ -G_*(x,y) & H_*(y,y)\end{bmatrix}$$

is invertible in $\mathscr{A}$ and its inverse $M_*(x,y)^{-1}$ has a norm which is uniformly bounded. In fact, its eigenvalue with least absolute value is given by

$$\lambda = \frac{1}{2}(H_*(x,x)+H_*(y,y)-\sqrt{(H_*(x,x)+H_*(y,y))^2+4\Delta}),$$

where $\Delta = G_*^2(x,y) - H_*(x,x)H_*(y,y) > 0$ in $\mathscr{A}$. Thus,

$$|\lambda| \geqslant \frac{1}{4} \frac{4\Delta + \frac{1}{2}(H_*(x,x) + H_*(y,y))}{\sqrt{(H_*(x,x) + H_*(y,y))^2 + 4\Delta}}.$$

But Δ is uniformly bounded from below since $|\varphi_*(x,y)|$ is so over $\mathscr{A}$, and uniform bounds from above and below also hold true for H_* in this region.

Another observation is the following. Let $\mu > 0$ and consider now the exterior domain

$$D_\mu = \mathbb{R}^N \setminus \bar{B}(0,\mu).$$

Then we observe that if we denote by simply G_μ and H_μ its Green's function and regular parts, then $G_\mu(x,y) = \mu^{2-N} G_*(\mu^{-1}x, \mu^{-1}y)$, $H_\mu(x,y) = \mu^{2-N} H_*(\mu^{-1}x, \mu^{-1}y)$. In particular, the following holds. If we set $\mathscr{A}_\mu = \mu\mathscr{A}$ then $\mathscr{A}_\mu$ corresponds precisely to the set where $\varphi_\mu(|x|,|y|) < -\delta_*\mu^{2-N}$. Besides if

$$M_\mu(x,y) = \begin{bmatrix} H_\mu(x,x) & -G_\mu(x,y) \\ -G_\mu(x,y) & H_\mu(y,y) \end{bmatrix},$$

then

$$\|M_\mu(x,y)^{-1}\| \leqslant C\mu^{N-2}, \quad (x,y) \in \mathscr{A}_\mu. \tag{54}$$

We finish with a last observation. For the domain Ω given by

$$\Omega = \mathscr{D} \Big\backslash \bigcup_{j=1}^{m} \bar{B}(P_j,\mu) \tag{55}$$

with $P_1, P_2, \ldots, P_m$ points in the bounded, smooth domain $\mathscr{D}$, the Green's function G satisfies

$$G(x,y) = G_\mu(x - P_i, y - P_i) + O(1), \quad (x,y) \in (P_i, P_i) + \mathscr{A}_\mu,$$

where the quantity $O(1)$ is bounded independent of all small μ, in the C^1 sense. The same is true for the corresponding functions H and φ.

6. THE PROOF OF THE MAIN RESULT

Let us now fix $1 \leqslant k \leqslant m$; we are looking for solutions to problem (1) with k couples of spikes, each one of which is close to one of the points $P_1, \ldots, P_k$, when $\mu > 0$ is small.

The results obtained in Section 4 imply that our problem reduces to the study of critical points of the function Ψ, which in the case of $2k$ spikes takes the form

$$\Psi(\xi, \Lambda) = \sum_{i=1}^{k} \psi(\hat{\xi}_i, \hat{\Lambda}_i) - 2R(\xi, \Lambda),$$

where ξ is a k-tuple of pairs, say $\xi = (\hat{\xi}_1, \ldots, \hat{\xi}_k)$ with $\hat{\xi}_i = (\xi_{i1}, \xi_{i2}) \in \Omega^2$, and $\Lambda = (\hat{\Lambda}_1, \ldots, \hat{\Lambda}_k) = (\Lambda_{11}, \Lambda_{12}, \ldots, \Lambda_{k1}, \Lambda_{k2}) \in \mathbb{R}^{2k}_+$,

$$\psi(\hat{\xi}_i, \hat{\Lambda}_i) = \frac{1}{2}\{H(\xi_{i1}, \xi_{i1})\Lambda_{i1}^2 + H(\xi_{i2}, \xi_{i2})\Lambda_{i2}^2 - 2G(\xi_{i1}, \xi_{i2})\Lambda_{i1}\Lambda_{i2}\} + \log \Lambda_{i1}\Lambda_{i2} \tag{56}$$

and

$$R(\xi, \lambda) = \sum_{i<j} \sum_{1 \leqslant \ell_1, \ell_2 \leqslant 2} G(\xi_{i\ell_1}, \xi_{j\ell_2})\Lambda_{i\ell_1}\Lambda_{j\ell_2}.$$

Let us consider a small number $\mu > 0$ and the domain Ω given by (55). We define next a region $\Sigma \subset \Omega^{2k}$ where we will work out the variational problem introduced in Section 4. Let $\mathscr{A}$ be the region of $\mathbb{R}^{2N}$ defined in (53) and

$$\mathscr{A}_i = (P_i, P_i) + \mu\mathscr{A}.$$

In other words, $(x, y) \in \mathscr{A}_i$ if and only if

$$\tilde{\varphi}_*(\mu^{-1}|x - P_i|, \mu^{-1}|y - P_i|) \leqslant -\delta_*,$$

where $\tilde{\varphi}_*$ and δ_* were defined in the previous section. Let us set

$$\Sigma = \{\xi/(\xi_{i1}, \xi_{i2}) \in \mathscr{A}_i \ \forall i = 1, \ldots, k\}. \tag{57}$$

We shall consider the functional Ψ defined precisely over the class $\Sigma \times R^{2k}_+$; actually Ψ has some singularities that we avoid by replacing the term $G(\xi_{i1}, \xi_{i2})$ in (56) by

$$G_M(\xi_{i1}, \xi_{i2}) = \begin{cases} G(\xi_{i1}, \xi_{i2}) & \text{if } G(\xi_{i1}, \xi_{i2}) \leqslant M, \\ M & \text{if } G(\xi_{i1}, \xi_{i2}) > M, \end{cases} \tag{58}$$

where $M > 0$ is a very large number. For notational convenience, we still call Ψ the modified functional on $\Sigma \times R^{2k}_+$.

For every $\hat{\xi}_i \in \mathscr{A}_i$ we choose $d_i(\hat{\xi}_i) = (d_{i1}(\hat{\xi}_i), d_{i2}(\hat{\xi}_i)) \in \mathbb{R}^2_+$ to be a vector defining a negative direction of the quadratic form associate with ψ. Such a direction exists since the function φ, defined by

$$\varphi(x, y) = H(x,x)^{1/2}H(y, y)^{1/2} - G(x, y) \tag{59}$$

is negative over $\mathscr{A}_i$. Let us be more precise. For fixed $\hat{\xi}_i \in \mathscr{A}_i$, the function

$$\psi(\hat{\xi}_i, d) = \frac{1}{2}\{H(\xi_{i1}, \xi_{i1})d_1^2 H(\xi_{i2}, \xi_{i2})d_2^2 - 2G(\xi_{i1}, \xi_{i2})d_1 d_2\} + \log d_1 d_2 \tag{60}$$

regarded as a function of (d_1, d_2) only, with $d_1, d_2 > 0$, has a unique critical point $\bar{d}(\hat{\xi}_i)$ given by

$$\bar{d}_{i1}^2 = -\frac{H(\xi_{i2}, \xi_{i2})^{1/2}}{H(\xi_{i1}, \xi_{i1})^{1/2}\varphi(\xi_{i1}, \xi_{i2})}, \qquad \bar{d}_{i2}^2 = -\frac{H(\xi_{i1}, \xi_{i1})^{1/2}}{H(\xi_{i2}, \xi_{i2})^{1/2}\varphi(\xi_{i1}, \xi_{i2})}.$$

Note that, in particular,

$$H(\xi_{i1}, \xi_{i1})\bar{d}_{i1}^2 + H(\xi_{i2}, \xi_{i2})\bar{d}_{i2}^2 - 2G(\xi_{i1}, \xi_{i2})\bar{d}_{i1}\bar{d}_{i2} = -1$$

and

$$\psi(\hat{\xi}_i, \bar{d}(\hat{\xi}_i)) = -\frac{1}{2} + \log\frac{1}{|\varphi(\hat{\xi}_i)|}. \tag{61}$$

Then we simply choose $d_i(\bar{\xi}_i) = \bar{d}(\hat{\xi}_i)$.

Let ρ^* be the number given as in Eq. (52). Set

$$S_i = \{x/|x - P_i| = \mu\rho^*\}, \qquad S_i^2 = S_i \times S_i.$$

In what follows, we denote

$$\mathscr{S} = \prod_{i=1}^{k} S_i^2, \quad d(\xi) = (d_1(\hat{\xi}_1), \ldots, d_k(\hat{\xi}_k)) \in \mathbb{R}_+^{2k}.$$

Let Γ be the class of all continuous functions

$$\gamma : \mathscr{S} \times I_0^k \times [0, 1] \to \Sigma \times R_+^{2k},$$

such that

1. For all $\xi \in \mathscr{S}$, $t \in [0, 1]$ the following hold $\gamma(\xi, \sigma_0, t) = (\xi, \sigma_0 d(\xi))$, and $\gamma(\xi, \sigma_0^{-1}, t) = (\xi, \sigma_0^{-1} d(\xi))$.
2. $\gamma(\xi, \sigma, 0) = (\xi, \sigma d(\xi))$ for all $(\xi, \sigma) \in \mathscr{S} \times I_0^k$, where $I_0 = [\sigma_0, \sigma_0^{-1}]$ with σ_0 is a small number to be chosen later. Then we define the

min–max value as

$$c(\Omega) = \inf_{\gamma \in \Gamma} \sup_{(\xi,\sigma) \in S \times I_0^k} \Psi(\gamma(\xi, \sigma, 1)) \tag{62}$$

and we will prove in what follows that $c(\Omega)$ is a critical value of Ψ. We begin with an upper estimate for this value.

LEMMA 6.1. *For all sufficiently small* μ, *the following estimate holds*:

$$c(\Omega) \leqslant -\frac{k}{2} + k(N-2)\log\mu - k\log c^* + o(1),$$

where $o(1) \to 0$ *as* $\mu \to 0$.

Proof. We consider the test path defined for all $t \in [0,1]$ as $\gamma(\xi, \sigma, t) = (\xi, \sigma d(\xi))$. Maximizing $\Psi(\xi, \sigma d(\xi))$ in the variable σ, we observe that this maximum value is attained approximately at $\sigma = 1$, by our choice of the vector $d(\xi)$. Besides, by definition of S_i we see that

$$\varphi(\hat{\xi}_i) \leqslant -c^* \mu^{2-N}, \tag{63}$$

$d(\xi) = O(\mu^{\frac{N-2}{2}})$. This last fact gives that $R(\xi, d(\xi)) = O(\mu^{N-2})$. Hence

$$\max_{\sigma} \Psi(\xi, \sigma d(\xi)) = \sum_{i=1}^{k} \psi(\hat{\xi}_i, d_i(\hat{\xi}_i)) + o(1)$$

and the desired result follows from (61) and (63). ∎

A key step in the direction of proving that $c(\Omega)$ is indeed a critical value of Ψ is an intersection lemma. The idea behind this result is the topological continuation of the set of solution of an equation, and is based on the work of Fitzpatrick *et al.* [8]. For every $(\xi, \sigma, t) \in \mathcal{S} \times I_0^k \times [0,1]$ we denote $\gamma(\xi, \sigma, t) = (\tilde{\xi}(\xi, \sigma, t), \tilde{\Lambda}(\xi, \sigma, t)) \in \Sigma \times R_+^{2k}$, and we define the set

$$\mathcal{S}_1 = \{(\xi, \sigma) \in \mathcal{S} \times I_0^k / \tilde{\Lambda}_{i1}(\xi, \sigma, 1) \cdot \tilde{\Lambda}_{i2}(\xi, \sigma, 1) = 1\}.$$

LEMMA 6.2. *For every open neighborhood* V *of* $\mathcal{S}_1$ *in* $\mathcal{S} \times I_0^k$, *the projection* $g : V \to \mathcal{S}$ *induces a mono-morphism in cohomology*, *that is*

$$g^* : H^*(\mathcal{S}) \to H^*(V)$$

is injective.

Proof. Let us define the set

$$Z([0,1]) = \{(\xi,\sigma) \in \mathscr{S} \times I_0^k / f(\xi,\sigma,t) \neq \mathbb{1} \quad \text{for all } t \in [0,1]\},$$

where $\mathbb{1} = (1,\ldots,1)$, $f = (f_1,\ldots,f_k)$ and $f_i(\xi,\sigma,t) = \tilde{\Lambda}_{i1}(\xi,\sigma,t) \cdot \tilde{\Lambda}_{i2}(\xi,\sigma,t)$. The function h defined by $h(\xi,\sigma,t) = (g(\xi,\sigma), f(\xi,\sigma,t))$ is a homotopy of pairs

$$h : (\mathscr{S} \times I_0^k, Z([0,1])) \times [0,1] \to (\mathscr{S} \times \mathbb{R}_+^k, \mathscr{S} \times (\mathbb{R}_+^k \setminus \{\mathbb{1}\})).$$

By choosing σ_0 small enough we have that the following inclusion is well defined:

$$j : (\mathscr{S} \times I_0^k, \mathscr{S} \times \partial I_0^k) \to (\mathscr{S} \times I_0^k, Z([0,1])).$$

If i is also an inclusion map and $h_0(\cdot) = h(\cdot,0)$, then we have the following commutative diagram in cohomology:

$$\begin{array}{ccc} H^*(\mathscr{S} \times I_0^k, Z([0,1])) & \xleftarrow{h_0^*} & H^*(\mathscr{S} \times \mathbb{R}_+^k, \mathscr{S} \times (\mathbb{R}_+^k \setminus \{\mathbb{1}\})) \\ \searrow^{j^*} & & {}^{i^*}\swarrow \\ & H^*(\mathscr{S} \times I_0^k, \mathscr{S} \times \partial I_0^k). & \end{array}$$

Since i^* is an isomorphism, we conclude that h_0^* is a mono-morphism and then from the homotopy axiom, we find that

$$h_1 = (g, f_1) : (\mathscr{S} \times I_0^k, Z([0,1])) \to (\mathscr{S} \times \mathbb{R}_+^k, \mathscr{S} \times (\mathbb{R}_+^k \setminus \{\mathbb{1}\}))$$

induces a mono-morphism in cohomology, where $h_1(\cdot) = h(\cdot,1)$. Next, defining

$$Z(1) = \{(\xi,\sigma) \in \mathscr{S} \times I_0^k / f(\xi,\sigma,1) \neq \mathbb{1}\}$$

and noting that $Z([0,1]) \subset Z(1)$, we also find that

$$h_1 : (\mathscr{S} \times I_0^k, Z(1)) \to (\mathscr{S} \times \mathbb{R}_+^k, \mathscr{S} \times (\mathbb{R}_+^k \setminus \{\mathbb{1}\}))$$

induces a mono-morphism in cohomology. Since V and $Z(1)$ are open, and $V^c \subset Z(1)$, defining $Z = Z(1) \cap V$ and using the excision axiom, we conclude that

$$h_1^* : H^*(\mathscr{S} \times \mathbb{R}_+^k, \mathscr{S} \times (\mathbb{R}_+^k \setminus \{\mathbb{1}\})) \to H^*(V, Z)$$

is a mono-morphism. Let e be a generator of $H^k(\mathbb{R}_+^k, \mathbb{R}_+^k \setminus \{\mathbb{1}\})$ and $u \in H^i(\mathscr{S})$, with $i \geqslant 0$, then following from the basic relation between cross product and cup product in cohomology, we have

$$h_1^*(u \times e) = d^*(g^*(u) \times f_1^*(e)) = g^*(u) \smile f_1^*(e).$$

Since h_1^* is a mono-morphism, it follows that g^* is also a mono-morphism. ∎

PROPOSITION 6.1. *There is a constant K so that*

$$\sup_{(\xi,\sigma)\in\mathcal{S}\times I_0^k} \Psi(\gamma(\xi,\sigma,1)) \geqslant -K \quad \textit{for all } \gamma \in \Gamma.$$

Proof. We observe that $\hat{\xi}_i \in \mathcal{A}_i$ implies that $\xi_{il} \in B(P_i, \rho_2\mu)\backslash B(P_i, \rho_1\mu)$, for $l = 1, 2$, with $1 < \rho_1 < \rho^* < \rho_2$ independent of μ. Hence we can find $\delta_0 > 0$ such that $(\xi_{i1} - P_i) \cdot (\xi_{i2} - P_i) > 0$ whenever $|\xi_{i1} - \xi_{i2}| < \delta_0$. Next let $K_0 > 0$ so that $G(x, y) \geqslant K_0$ implies $|x - y| < \delta_0$.

Assume, by contradiction, that for certain $\gamma \in \Gamma$

$$\Psi(\gamma(\xi,\sigma,1)) \leqslant -kK_0 \quad \text{for all } (\xi,\sigma) \in \mathcal{S} \times I_0^k.$$

This implies that for all $(\xi,\sigma) \in \mathcal{S}_1$, $(\tilde{\xi}, \tilde{\Lambda}) = (\tilde{\xi}(\xi,\sigma,1), \tilde{\Lambda}(\xi,\sigma,1))$, we have

$$2\sum_{i=1}^{k} G(\tilde{\xi}_i) - \sum_{i=1}^{k}\{H(\tilde{\xi}_{i1}, \tilde{\xi}_{i1})\tilde{\Lambda}_{i1}^2 + H(\tilde{\xi}_{i2}, \tilde{\xi}_{i2})\tilde{\Lambda}_{i2}^2\} + 2R(\tilde{\xi}, \tilde{\Lambda}) \geqslant kK_0$$

and then taking a small neighborhood V of $\mathcal{S}_1$ in $\mathcal{S} \times I_0^k$,

$$\sum_{i=1}^{k} G(\tilde{\xi}_i(\xi,\sigma,1)) \geqslant kK_0 \qquad \text{for all } (\xi,\sigma) \in V.$$

Note that $R(\tilde{\xi}, \tilde{\Lambda})$ is small compared to $G(\tilde{\xi}_i)$. From here we conclude that for every $(\xi,\sigma) \in V$ there exists $i \in \{1, \dots, k\}$ such that

$$G(\tilde{\xi}_i(\xi,\sigma,1)) \geqslant K_0$$

and consequently $|\tilde{\xi}_{i1} - \tilde{\xi}_{i2}| < \delta_0$. Let us fix a point $\bar{x}$ such that $|\bar{x}| = \rho^*\mu$, then $\bar{\xi}_i = (P_i + \bar{x}, P_i - \bar{x}) \in S_i^2$ and $\bar{\xi} = (\bar{\xi}_1, \dots, \bar{\xi}_k) \in \mathcal{S}$. We note that because of the above conclusion $\gamma_1(V) \subset (\Sigma\backslash T(\bar{\xi})) \times R_+^{2k}$, where $\gamma_1 = \gamma(\cdot, 1)$ and $T(\bar{\xi}) = \{t\bar{\xi}/\rho_1 < t < \rho_2\}$.

Let us consider the map $r\colon \Sigma \times R_+^{2k} \to \mathcal{S}$ defined componentwise as $r_i(\xi, \Lambda) = \rho^*\mu(\xi_{i1}/|\xi_{i1}|, \xi_{i2}/|\xi_{i2}|)$. Then $\gamma_0^* \circ r^* : H^*(\mathcal{S}) \to H^*(\mathcal{S} \times I_0^k)$, where $\gamma_0 = \gamma(\cdot, 0)$ is an isomorphism. Denoting $\gamma_1 = \gamma(\cdot, 1)$, by homotopy axiom we see then that $\gamma_1^* \circ r^*$ is also an isomorphism. Consider the following commutative diagram:

$$\begin{array}{ccccc} H^*(\mathcal{S}\times I_0^k) & \stackrel{\gamma_1^*}{\longleftarrow} & H^*(\Sigma\times R_+^{2k}) & \stackrel{\gamma^*}{\longleftarrow} & H^*(\mathcal{S}) \\ i_1^*\downarrow & & i_2^*\downarrow & & i_3^*\downarrow \\ H^*(V) & \stackrel{\tilde\gamma_1^*}{\longleftarrow} & H^*(\gamma_1(V)) & \stackrel{\tilde r^*}{\longleftarrow} & H^*(\mathcal{S}\setminus\{\bar\xi\}), \end{array}$$

where i_1, i_2 and i_3 are inclusion maps, $\tilde\gamma_1=\gamma_1|_V$ and $\tilde r=r|_{\gamma_1(V)}$. Since i_1^* is a mono-morphism by Lemma 6.2, we obtain a contradiction with the fact that $H^{2Nk}(\mathcal{S}\setminus\{\bar\xi\})=0$. ∎

In view of Proposition 6.1, in order to prove that the min–max number (62) is a critical value, we need to care about the fact that the domain in which Ψ is defined is not necessarily closed for the gradient flow of Ψ. The following lemma is a step in this direction.

LEMMA 6.3. *Let $(\xi^n,\Lambda^n)\in\Sigma\times R_+^{2k}$ be a sequence such that*

$$\nabla_\Lambda\Psi(\xi^n,\Lambda^n)\to 0. \tag{64}$$

Then each component of Λ^n is bounded above and below by positive constants.

Proof. For notational simplicity in the proof, we shall drop from the sequences the dependence on n. Let us denote here that

$$\xi=(\xi_{11},\xi_{12},\dots,\xi_{k1},\xi_{k2}),\quad \Lambda=(\Lambda_{11},\Lambda_{12},\dots,\Lambda_{k1},\Lambda_{k2}),$$

let us also denote

$$H_{il}=H(\xi_{il},\xi_{il}),\quad G_{il,jm}=G(\xi_{il},\xi_{jm}).$$

Then (64) corresponds to the system

$$H_{il}\Lambda_{il}+\frac{1}{\Lambda_{il}}-\sum_{jm\neq il}G_{il,jm}\Lambda_{jm}=o(1).$$

Assume that the sequence Λ^n is not bounded above or below component-wise. Since the numbers H and G remain uniformly controlled (we are working with fixed μ), we easily see that either $\Lambda_{il}\to 0$ or $\Lambda_{il}\to+\infty$, and that at least for one index il $\Lambda_{il}\to+\infty$. Set $\tilde\Lambda_{il}=\Lambda_{il}/|\Lambda|$. Passing to a subsequence we may assume that this sequence of vectors approaches a nonzero vector $\hat\Lambda$. Relabeling if necessary, after dropping those equations corresponding to zero coordinate in Λ, we obtain that the resulting system has the following form:

$$M\hat{\Lambda} + R\hat{\Lambda} = 0,$$

where M is a block matrix of the form

$$M = \begin{bmatrix} M_1 & & & & & & & \\ & M_2 & & & & & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & M_s & & & \\ & & & & & H_1 & & \\ & & & & & & \cdot & \\ & & & & & & & \cdot \\ & & & & & & & & H_t \end{bmatrix}$$

and the M_i's are two by two blocks of the form

$$M_i = \begin{bmatrix} H_{i1} & -G_{i1,i2} \\ -G_{i1,i2} & H_{i2} \end{bmatrix},$$

associated with a pairs of coordinates ξ_{i1}, ξ_{i2} for which both coordinates in $\hat{\Lambda}$ are nonzero. The H_i's instead correspond to numbers of the form H_{il} for $l = 1$ or 2, corresponding to those coordinates in which one and only one of the components l in the vector $\hat{\Lambda}$ became nonzero. The matrix R has entries bounded independent of μ, while the entries in the blocks of M are comparatively very large. From the analysis in the previous section, the matrix M turns out invertible, and M^{-1} has a matrix norm which is uniformly small if μ was chosen small enough, see (54). It follows that the above system has only $\hat{\Lambda} = 0$ as a solution, a contradiction that proves the lemma. ∎

We finally can prove

PROPOSITION 6.2. *The functional Ψ satisfies the Palais–Smale condition in the region $\Sigma \times R_+^{2k}$ at the level $c(\Omega)$ given in* (62), *provided that μ was chosen sufficiently small.*

Proof. Let us consider a sequence $(\xi^n, \Lambda^n) \in \Sigma \times \mathbb{R}_+^{2k}$ such that

$$\nabla_\Lambda \Psi(\xi^n, \Lambda^n) \to 0$$

and

$$\nabla_\xi^\tau \Psi(\xi^n, \Lambda^n) \to 0,$$

where $\nabla_\xi^\tau \Psi$ corresponds to the tangential gradient of Ψ to $(\partial\Sigma) \times R_+^{2k}$ in case that ξ^n is approaching a boundary point of Σ or the full gradient otherwise. From the previous lemma, the components of Λ^n are bounded above and below by positive constants, so that we may assume, passing to a subsequence, $(\xi^n, \Lambda^n) \to (\bar{\xi}, \bar{\Lambda}) \in \bar{\Sigma} \times \mathbb{R}_+^{2k}$ and $\Psi(\xi^n, \Lambda^n) \to c(\Omega)$. Then

$$\nabla_\Lambda \Psi(\bar{\xi}, \bar{\Lambda}) = 0.$$

If $\bar{\xi}$ lies in the interior of Σ we would have converged to a critical point of Ψ. Assume that $\bar{\xi} \in \partial\Sigma$. It means that

$$\tilde{\varphi}_*(\mu^{-1}|\bar{\xi}_{i_0 1} - P_{i_0}|, \mu^{-1}|\bar{\xi}_{i_0 2} - P_{i_0}|) = -\delta_*$$

for some index i_0.

We first observe that since $\nabla_\Lambda \Psi(\bar{\xi}, \bar{\Lambda}) = 0$, $\bar{\Lambda}$ satisfies

$$\bar{\Lambda}_{i1}^2 = -\frac{H(\bar{\xi}_{i2}, \bar{\xi}_{i2})^{1/2}}{H(\bar{\xi}_{i1}, \bar{\xi}_{i1})^{1/2} \varphi(\bar{\xi}_{i1}, \bar{\xi}_{i2})} + \theta_{i1}, \qquad \bar{\Lambda}_2^2 = -\frac{H(\bar{\xi}_{i1}, \bar{\xi}_{i1})^{1/2}}{H(\bar{\xi}_{i2}, \bar{\xi}_{i2})^{1/2} \varphi(\bar{\xi}_{i1}, \bar{\xi}_{i2})} + \theta_{i2},$$

where, with μ chosen sufficiently small, the quantity θ_{il} is of small order. Substituting back in Ψ, we get

$$c(\Omega) = \Psi(\bar{\xi}, \bar{\Lambda}) = -\frac{k}{2} + \sum_{i=1}^{k} \log \frac{1}{|\varphi(\bar{\xi}_{i1}, \bar{\xi}_{i2})|} + \theta(\bar{\xi}), \tag{65}$$

where $\theta(\xi)$ is small in the C^1-sense, as μ becomes smaller. Hence for each i either $\nabla\varphi(\bar{\xi}_{i1}, \bar{\xi}_{i2}) \sim 0$ if $(\bar{\xi}_{i1}, \bar{\xi}_{i2})$ lies in the interior of $\mathscr{A}_i$ or $\nabla\varphi(\bar{\xi}_{i1}, \bar{\xi}_{i2}) \cdot T \sim 0$ for any direction tangential to $\partial\mathscr{A}_i$ otherwise. Thus, the angle formed by the vectors $\bar{\xi}_{i_0 1} - P_{i_0}$ and $\bar{\xi}_{i_0 2} - P_{i_0}$ must be close to π since otherwise, the analysis in the previous section would yield that some tangential derivative of φ would be away from 0. This implies that

$$\varphi(\bar{\xi}_{i1}, \bar{\xi}_{i2}) \sim \mu^{2-N} \tilde{\varphi}_*(\mu^{-1}|\bar{\xi}_{i_0 1} - P_{i_0}|, \mu^{-1}|\xi_{i_0 2} - P_{i_0}|) = -\delta_* \mu^{2-N}.$$

But combining this last relation with the upper estimate for $c(\Omega)$ in Lemma 6.1, we see that for some index i_1 we have that $|\varphi(\bar{\xi}_{i_1 1}, \bar{\xi}_{i_1 2})|$ must be very large, say greater than $2c^* \mu^{2-N}$ if δ^* was originally chosen sufficiently small. Finally, the definition of c^* would then tell us that the angle formed by the vectors $\bar{\xi}_{i_1 1} - P_{i_1}$ and $\bar{\xi}_{i_1 2} - P_{i_1}$ must be away from π. Again, this would imply that some inner or tangential derivative of φ would be away from zero. This is a contradiction. Hence the point $\bar{\xi}$ lies in the interior of Σ. Hence Palais Smale (PS) holds, and the proposition has been proven. ∎

Proof of Theorem 1.1 is now completed. We consider the domain $\Sigma_{r,R} = \Sigma \times [r, R]^{2k}$ with r, R to be chosen later. The functional I is well defined on $\Sigma_{r,R}$ except on the set

$$\Delta_{\tilde{\rho}} = \{(\xi, \Lambda) \in \Sigma_{r,R} / |\xi_{i1} - \xi_{i2}| < \tilde{\rho} \text{ for some } 1 \leqslant i \leqslant k\}.$$

Modifying I in (50), by extending Ψ to all $\Sigma_{r,R}$, as in (58), we extend I and keep relations (50) and (51) over $\Sigma_{r,R}$.

By the Palais Smale condition for Ψ proved in Proposition 6.2 there are numbers $R > 0$, $c > 0$ and $\alpha_0 > 0$ such that for all $0 < \alpha < \alpha_0$, and $(\xi, \Lambda) \in \Sigma$ satisfying $|\Lambda| \geqslant R$ and $c(\Omega) - 2\alpha \leqslant \Psi(\xi, \Lambda) \leqslant c(\Omega) + 2\alpha$, we have $|\nabla \Psi(\xi, \Lambda)| \geqslant c$.

Next we use the min–max characterization of $c(\Omega)$ to choose $\gamma \in \Gamma$ so that

$$c(\Omega) \leqslant \sup_{(\xi,\sigma) \in \mathscr{S} \times I_0^k} \Psi(\gamma(\xi, \sigma, 1)) \leqslant c(\Omega) + \alpha.$$

By making r small and R larger if necessary, we can assume that $\gamma(\xi, \sigma, 1) \in \Sigma_{2r,R/2} \subset \Sigma_{r,R}$ for all $(\xi, \sigma) \in \mathscr{S} \times I_0^k$.

We define a min–max value for the functional I using γ and the negative gradient flow for I. More precisely we consider $\eta : \Sigma_{r,R} \times [0, \infty] \to \Sigma_{r,R}$ being the solution of the equation $\dot{\eta} = -h(\eta) \nabla I(\eta)$ with initial condition $\eta(\xi, \Lambda, 0) = (\xi, \Lambda)$. Here the function h is defined in Σ so that $h(\xi, \Lambda) = 0$ for all (ξ, Λ) with $\Psi(\xi, \Lambda) \leq c(\Omega) - 2\alpha$ and $h(\xi, \Lambda) = 1$ if $\Psi(\xi, \Lambda) \geqslant c(\Omega) - \alpha$, satisfying $0 \leqslant h \leqslant 1$.

By the choice of r and R and taking into account (50) and (51), we have $\eta(\xi, \Lambda, t) \in \Sigma_{r,R}$ for all $t \geqslant 0$. Then the following min–max value

$$C(\Omega) = \inf_{t \geqslant 0} \sup_{(\xi,\sigma) \in \mathscr{S} \times I_0^k} I(\eta(\gamma(\xi, \sigma, 1), t))$$

is a critical value for I. In all this reasoning, we are assuming that ε is small enough to make the errors in (50) and (51) sufficiently small.

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