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An elementary construction of complex patterns in nonlinear Schrödinger equations

Manuel del Pino[1], **Patricio Felmer**[1] **and Kazunaga Tanaka**[2]

[1] Departamento de Ingeniería Matemática and C.M.M., UMR2071 CNRS-UChile, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile

[2] Department of Mathematics, School of Science and Engineering, Waseda University, 3-4-1 Ohkubo, Shinjuku-ku, Tokyo 169-8555, Japan

E-mail: delpino@dim.uchile.cl, pfelmer@dim.uchile.cl and kazunaga@mn.waseda.ac.jp

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Abstract

We consider the problem of finding standing waves to a nonlinear Schrödinger equation. This leads to searching for solutions of the equation

$$-\varepsilon^2 u'' + V(x)u = |u|^{p-1}u \qquad \text{in } \boldsymbol{R},$$

$p > 1$, when ε is a small parameter. Given any finite set of points $x_1 < x_2 < \cdots < x_m$ constituted by isolated local maxima or minima of V, and corresponding arbitrary integers n_i, $i = 1, \ldots, m$, we prove that there is a finite energy solution exhibiting a cluster of n_i spikes concentrating around each x_i as $\varepsilon \to 0$. The clusters consist of spikes with alternating sign if the point is a minimum, and of constant sign if it is a maximum. This construction extends to infinitely many clusters of spikes under appropriate conditions. The proof follows an elementary variational matching approach, which resembles the so-called broken-geodesic method.

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1. Introduction

In this paper we are concerned with the following nonlinear Schrödinger equation:

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\psi_{xx} + W(x)\psi - \gamma|\psi|^{p-1}\psi. \tag{1.1}$$

One of the basic principles of quantum mechanics states that it contains classical mechanics as its limit when $\hbar \to 0$. The rigorous realization of this principle in particular cases is not an easy task because of the dispersive nature of the linear Schrödinger equation (see [13] for example). On the other hand, it is known that the presence of the nonlinear term in the equation compensates this dispersive character and makes possible the appearance of *solitary waves*, namely solutions whose energy travels as a localized packet, thus having 'particle-like' behaviour. Besides the many physical problems where this equation appears, this mathematical fact makes the analysis of solutions of (1.1) as $\hbar \to 0$ very interesting. Many works in the literature, in one and higher dimensions, have dealt with the analysis of the so-called standing waves, namely solutions of (1.1), of the form $\psi(x, t) = \exp(-\mathrm{i}Et/\hbar)u(x)$, with $u(x)$ real-valued, which is the simplest form of a solitary wave. After conveniently relabelling the parameters of the equation, $u(x)$ satisfies the scalar equation

$$-\varepsilon^2 u'' + V(x)u = |u|^{p-1}u \qquad \text{in } \boldsymbol{R}, \tag{1.2}$$

where $\varepsilon > 0$ is a small parameter. We assume henceforth $p > 1$ and that the potential $V(x)$ is of class $C^1(\boldsymbol{R})$ and satisfies $\inf_{x\in\boldsymbol{R}} V(x) > 0$.

Floer and Weinstein [11] studied (1.2) for the case $p = 3$. By means of a Lyapunov–Schmidt reduction they constructed single-peaked positive solutions of (1.2) which concentrate around each given non-degenerate maximum or minimum point of V as $\hbar \to 0$. This construction was extended by Oh to higher dimensions, and to *multi-peaked* solutions, with peaks associated to any given finite set of non-degenerate critical points of the potential (see [21–23]). It should be remarked that solutions with uniformly bounded mass $\mathrm{O}(\varepsilon)$ must necessarily concentrate around critical points of the potential (see [27]). A number of works have appeared in recent years dealing with the construction of multi-peaked solutions in a variety of situations, including relaxing non-degeneracy assumptions and more general nonlinearities (see, for instance, [2, 9, 10, 14 15, 16, 18, 25, 27,] and references therein). An interesting phenomenon was discovered by Kang and Wei [16]. They established the existence of positive solutions with any prescribed number of peaks clustering around each given local maximum point of the potential (in any space dimension). It was also proved in [16] that such a cluster solution does not exist near a non-degenerate local minimum (see also [14]).

In this paper we revisit the problem in the one-dimensional case, proposing a new variational construction of multi-peaked solutions of (1.2) associated to each given finite set of points which are local maxima or minima of $V(x)$, which recovers known constructions obtained with finite-dimensional reductions involving the implicit function theorem or via min–max methods. We consider not only positive solutions but also solutions that may change sign. The method presented allows the *gluing* of clusters of any prescribed numbers of spikes associated to each of these points. These clusters must be constituted of peaks of the same sign around a local maximum point, and with alternate signs at local minima. A main feature of our construction is its elementary character, which seems to make it feasible for considerable extension for the nonlinearities considered, while for simplicity we will only consider the case of problem (1.2).

Let us make our language more precise. The starting point is the following. A (positive) spike around a point ξ is a solution u_ε with a local maximum point $\xi_\varepsilon \to \xi$ and such that the scaled function $v_\varepsilon(y) = u_\varepsilon(\xi_\varepsilon + \varepsilon y)$, which satisfies

$$v_\varepsilon'' + V(\xi_\varepsilon + \varepsilon y)v_\varepsilon - |v_\varepsilon|^{p-1}v_\varepsilon = 0,$$

approaches locally over compact sets in y the unique solution $w = w(\xi; y)$ of

$$-w'' + V(\xi)w = w^p \qquad \text{in } \boldsymbol{R}, \tag{1.3}$$

$$w'(0) = 0, \quad w(y) > 0 \qquad \text{in } \boldsymbol{R}, \qquad w \in H^1(\boldsymbol{R}). \tag{1.4}$$

In other words, we have the resemblance $u_\varepsilon(x) \sim w(\xi, (x - \xi_\varepsilon)/\varepsilon)$ near ξ_ε.

The key definitions to state our main result are the following. We say that a solution $u_\varepsilon(x)$ of (1.2) has a *cluster of spikes of type* $(n,+)$ *with constant sign* at ξ if there are points $p_1^\varepsilon < p_2^\varepsilon < \cdots < p_n^\varepsilon$ with $p_i^\varepsilon \to \xi$ as $\varepsilon \to 0$ so that for certain $\delta > 0$,

$$\sup_{|x-\xi|<\delta} \left| u_\varepsilon(x) - \sum_{i=1}^{n} w\left(\xi; \frac{x-p_i^\varepsilon}{\varepsilon}\right) \right| \to 0, \qquad \text{as } \varepsilon \to 0.$$

Similarly, we say that $u_\varepsilon(x)$ has a *cluster of spikes of type* $(n,+)$ *with alternating sign at* ξ if there are points $p_1^\varepsilon < p_2^\varepsilon < \cdots < p_n^\varepsilon$ with $p_i^\varepsilon \to \xi$ as $\varepsilon \to 0$ so that for certain $\delta > 0$,

$$\sup_{|x-\xi|<\delta} \left| u_\varepsilon(x) - \sum_{i=1}^{n} (-1)^{i-1} w\left(\xi; \frac{x-p_i^\varepsilon}{\varepsilon}\right) \right| \to 0, \qquad \text{as } \varepsilon \to 0.$$

We say that $u_\varepsilon(x)$ has a cluster of spikes of type $(n,-)$, if $-u_\varepsilon(x)$ has a cluster of type $(n,+)$ (with constant or alternating sign) at ξ.

Theorem 1.1. *Let us consider m critical points of V,* $x_1 < x_2 < \cdots < x_m$, *such that for some* $h > 0$, $(x - x_i)V'(x) > 0$ *(or <0) whenever* $0 < |x - x_i| < h$. *Then given a collection of pairs* (n_i, r_i), $i = 1, \ldots, m$ *with* $r_i \in \{+,-\}$ *and* $n_i \in \boldsymbol{N}$, *there exists a solution* u_ε *of (1.2) which has a cluster of spikes of type* (n_i, r_i) *at* x_i, *with alternating sign if* x_i *is a local minimum and with constant sign if* x_i *is a local maximum.*

This result extends to the construction of solutions with infinitely many clusters of spikes. As a model, we have the validity of the following result showing the presence chaotic patterns of clusters of spikes.

Theorem 1.2. *Assume that V is periodic and consider a sequence of critical points of V* $\{x_i\}_{i\in\boldsymbol{N}}$ *such that for some* $h > 0$, $(x - x_i)V'(x) > 0$ *(or <0) whenever* $0 < |x - x_i| < h$. *Then given* $N \in \boldsymbol{N}$ *and a collection of pairs* (n_i, r_i), $i \in \boldsymbol{N}$ *with* $r_i \in \{+,-\}$ *and* $n_i \in \boldsymbol{N}$, $n_i \leqslant N$, *there exists a solution* u_ε *of (1.2) which has a cluster of spikes of type* (n_i, σ_i) *at* x_i, *with alternating sign if* x_i *is a local minimum and with constant sign if* x_i *is a local maximum.*

At this point we would like to look at our problem from a different point of view. If we consider the change of variables $x = \varepsilon t$, then we can see the problem as a slowly varying planar Hamiltonian system with potential $G(x,u) = G_\varepsilon(t,u) = V(\varepsilon t)u^2/2 - |u|^{p+1}/(p+1)$. This and more general Hamiltonian systems have been considered by several authors in physics and dynamical systems. In our situation, for fixed x the system has two separatrice loops (figure eight). When the area $A(x)$ inside the loops is extremal, it has been shown that the solution jumps from the inside to the outside of the loop. For earlier works in this direction, see Neistadt [20] and references therein. We note that $A'(x) = 0$ precisely when $V'(x) = 0$.

In this context, first works in studying multi-peaked solutions are due to Cherry [6] and later Palmer [24]. Through the study of a Poincaré–Melnikov function, the existence of single-peaked solutions, a transverse homoclinic orbit, is proved in [24]. Under this transversality condition existence of multi-peaked solutions as in theorems 1.1 and 1.2 can be proved by the Poincaré–Birkhoff–Smale theory of shadowing chains. In this paper we do not assume that the critical points are non-degenerate so that the theory in [24] is not applicable. Possibly the result can be still proved in this degenerate case using the notion of topological transversality from Burns and Weiss [5]. See also recent results by Batelli and Palmer [3].

We mention the works by Kath [17] and by Gedeon *et al* [12], where slowly varying planar Hamiltonian systems are also studied. In particular, in [12] the existence of complex dynamical systems are constructed by means of the Conley index theory. Finally, we say that chaotic patterns with finitely or infinitely many spikes in related problems have also been detected via variational techniques by Séré [26], Coti-Zelati and Rabinowitz [7] and Alessio and Montecchiari [1].

A main feature of the technique presented is its elementary nature. It exhibits strong resemblance with the so-called broken-geodesic method in geometry. A variation of it has been used by Nakashima and Tanaka in [19] in studying clustering layers in some phase transition problems. Two main differences arise when we compare the situation we deal with here and the setting of the broken-geodesic method: first, the basic blocks, which are required to solve the equation under zero-Neumann boundary conditions, are not minimizers of the associated energy. Second, the *curve* obtained by joining the basic blocks is not continuous, but has matching derivatives. Broken-geodesic techniques were also used in a somewhat related construction by Buffoni and Séré [4].

2. Basic solutions and variational formulation of the problem

In this section we set up a finite-dimensional functional f_ε, whose critical points correspond to solutions of our problem. Then we see how a critical point of f_ε can be obtained by evaluating the Brouwer degree of some appropriate functions. Pending the evaluation of the degree, we then prove our theorems.

For notational convenience, we define

$$G(x,u) = \frac{1}{2}V(x)u^2 - \frac{1}{p+1}|u|^{p+1}.$$

Then, given positive numbers $a < b$, we consider the following boundary value problems:

$$-\varepsilon^2 u_{xx} + G'(x,u) = 0 \quad \text{in } (a,b), \qquad u_x(a) = u_x(b) = 0, \tag{2.1}$$

$$-\varepsilon^2 u_{xx} + G'(x,u) = 0 \quad \text{in } (-\infty,a), \qquad u(-\infty) = u_x(a) = 0, \tag{2.2}$$

$$-\varepsilon^2 u_{xx} + G'(x,u) = 0 \quad \text{in } (b,+\infty), \qquad u_x(b) = u(+\infty) = 0, \tag{2.3}$$

where $G'(x,u)$ denotes the derivative of G with respect to u. The following result concerns existence and uniqueness of solutions of the above problems exhibiting spikes at the endpoints of the corresponding intervals. We call them the *basic solutions*.

Lemma 2.1. *Let $M > 0$ and $\sigma = (s_1, s_2) \in \{+1,-1\}^2$ be given. Then there exist positive numbers δ_0, ε_0 and l_0 such that for any $0 < \varepsilon < \varepsilon_0$ and any a, b with $|a| + |b| \leqslant M$ and $(b-a)/\varepsilon \geqslant l_0$, problem (2.1) has a unique solution $u_\varepsilon(x) = u_{\varepsilon,\sigma}(a,b;x)$ which additionally satisfies*

$$\left\| u_\varepsilon(x) - s_1 w\left(a; \frac{x-a}{\varepsilon}\right) - s_2 w\left(b; \frac{x-b}{\varepsilon}\right)\right\|_{L^\infty[a,b]} < \delta_0. \tag{2.4}$$

Similarly, given $M > 0$, $\sigma \in \{+1,-1\}$, there is a $\delta_0 > 0$ such that for any a with $|a| \leqslant M$ and any ε sufficiently small, problem (2.2) has a unique solution $u_\varepsilon(x) = u_{\varepsilon,\sigma}(-\infty, a; x)$ with

$$\left\| u_\varepsilon(x) - \sigma w\left(a; \frac{x-a}{\varepsilon}\right)\right\|_{L^\infty(-\infty,a]} < \delta_0. \tag{2.5}$$

A similar statement holds for problem (2.3), for a solution $u_{\varepsilon,\sigma}(b;+\infty)$.

We postpone the proof of this result to section 3; instead we continue with the formulation of the finite-dimensional variational problem. We start recalling that the solutions of (1.2) are precisely the critical points in $H^1(\boldsymbol{R}^N)$ of the functional

$$I_\varepsilon(u) = \int_{\boldsymbol{R}^N} \frac{\varepsilon}{2}|u_x|^2 + \frac{1}{\varepsilon}G(x,u)\,\mathrm{d}x.$$

Let us consider m points $x_1 < x_2 < \cdots < x_m$, a number $h > 0$ and prescribed pairs (n_i, r_i) as in the statement of theorem 1.1. We assume that $M > 0$ is a constant such that $-M + 1 < x_1 < x_m < M - 1$. Let $n = \sum_{i=1}^{m} n_i$. We want to construct a solution having spikes (positive maxima or negative minima) precisely at n points $t_1 < \cdots < t_n$. We will find a critical point of I_ε by means of finding a critical point of a functional f_ε of the n-tuple $\boldsymbol{t} = (t_1, \ldots, t_n)$ constructed as the sum of energies of the basic solutions associated to each subinterval between the points.

Let us define $\nu_1 = 0$, $\nu_j = \sum_{i=1}^{j-1} n_i$, $j = 2, 3, \ldots, m$, so that $n = \nu_m + n_m$. Since we want to have a cluster of size n_j at x_j, we also impose

$$x_j - h < t_{\nu_j+1} < \cdots < t_{\nu_j+n_j} < x_j + h \qquad \text{for each } j. \tag{2.6}$$

Let us also consider signs $s_1, s_2, \ldots, s_n$ determined so that $s_{\nu_j+1} = r_j$ and

$$s_{\nu_j+1} = -s_{\nu_j+2} = s_{\nu_j+3} = \cdots = (-1)^{n_j-1} s_{\nu_j+n_j}$$

if x_j is a local minimum of $V(x)$ and

$$s_{\nu_j+1} = s_{\nu_j+2} = \cdots = s_{\nu_j+n_j}$$

if x_j is a local maximum. We may also assume that $|t_i| \leqslant M$, for all $1 \leqslant i \leqslant n$. We consider numbers δ_0, ε_0 and l_0 as in lemma 2.1. Thus assuming additionally that $\varepsilon < \varepsilon_0$ and

$$\frac{t_{i+1} - t_i}{\varepsilon} > l_0, \tag{2.7}$$

the following *building blocks* $u_\varepsilon^i(\boldsymbol{t})$ are uniquely defined: $u_\varepsilon^i(\boldsymbol{t}; x) = u_{\varepsilon,(s_i,s_{i+1})}(x)$ if $x \in (t_i, t_{i+1})$, $\nu_j \leqslant i \leqslant \nu_{j+1}$, $j = 1, \ldots, m-1$; $u_\varepsilon^0(\boldsymbol{t}; x) = u_{\varepsilon,s_1}(x)$ if $x \in (-\infty, t_1)$; and $u_\varepsilon^{n+1}(\boldsymbol{t}; x) = u_{\varepsilon,s_n}(x)$ if $(t_n, +\infty)$.

Let us consider the piecewise continuous function $u_\varepsilon(\boldsymbol{t}; x)$ defined as

$$u_\varepsilon(\boldsymbol{t}; x) = u_\varepsilon^i(\boldsymbol{t}; x) \qquad \text{if } t_i < x < t_{i+1}, \quad i = 0, \ldots, n+1,$$

where we understand $t_0 = -\infty$ and $t_{n+1} = +\infty$. We observe that this function is a solution of (1.2) if and only if it is continuous. Moreover, we can nicely characterize this requirement by means of the following 'broken energy':

$$f_\varepsilon(\boldsymbol{t}) = \sum_{j=0}^{n} I_{\varepsilon,(t_j,t_{j+1})}(u_\varepsilon^j(\boldsymbol{t})), \tag{2.8}$$

where

$$I_{\varepsilon,(a,b)}(u) = \int_a^b \frac{\varepsilon}{2}|u_x|^2 + \frac{1}{\varepsilon}G(x, u)\,\mathrm{d}x,$$

and $t_0 = -\infty$, $t_{n+1} = -\infty$. We have the following proposition.

Proposition 2.1. *Assume that $\varepsilon < \varepsilon_0$ and that the points $t_1 < \cdots < t_n$ satisfy conditions (2.6) and (2.7). Then f_ε is of class C^1 within this range and $u_\varepsilon(\boldsymbol{t})$ is a solution of (1.2) if and only if $\nabla f_\varepsilon(\boldsymbol{t}) = 0$.*

Proof. For numbers a, b and $\sigma \in \{+, -\}$, let us denote

$$m_{\varepsilon,\sigma}(a, b) = I_{\varepsilon,(a,b)}(u_{\varepsilon,\sigma}(a, b)), \tag{2.9}$$

allowing for $a = -\infty$ and $b = \infty$. A direct computation gives us the validity of the following facts: for $\varepsilon \in (0, \varepsilon_0]$ and $(b - a)/\varepsilon \geqslant l_0$, $m_{\varepsilon,\sigma}(a, b)$ is differentiable and

$$\begin{aligned} \frac{\partial}{\partial a} m_{\varepsilon,\sigma}(a, b) &= -\frac{1}{\varepsilon} G(a, u_{\varepsilon,\sigma}(a, b; a)), \\ \frac{\partial}{\partial b} m_{\varepsilon,\sigma}(a, b) &= \frac{1}{\varepsilon} G(b, u_{\varepsilon,\sigma}(a, b; b)). \end{aligned} \tag{2.10}$$

Similarly, $m_{\varepsilon,\sigma}(-\infty, b)$ and $m_{\varepsilon,\sigma}(a, \infty)$ are differentiable and

$$\frac{\partial}{\partial b} m_{\varepsilon,\sigma}(-\infty, b) = \frac{1}{\varepsilon} G(b, u_{\varepsilon,\sigma}(-\infty, b; b)),$$
$$\frac{\partial}{\partial a} m_{\varepsilon,\sigma}(a, \infty) = -\frac{1}{\varepsilon} G(a, u_{\varepsilon,\sigma}(a, \infty; a)). \tag{2.11}$$

Now, it is clear that $u_\varepsilon(\boldsymbol{t})$ is a solution of (1.2) if and only if

$$u_\varepsilon^{j-1}(\boldsymbol{t}; t_j) = u_\varepsilon^j(\boldsymbol{t}; t_j), \qquad j = 1, \ldots, n. \tag{2.12}$$

On the other hand, $\nabla f_\varepsilon(\boldsymbol{t}) = 0$ is equivalent to

$$G(t_j, u_\varepsilon^{j-1}(\boldsymbol{t}; t_j)) = G(t_j, u_\varepsilon^j(\boldsymbol{t}; t_j)), \qquad j = 0, \ldots, n-1. \tag{2.13}$$

We observe that the values $u_\varepsilon^{j-1}(\boldsymbol{t}; t_j)$ and $u_\varepsilon^j(\boldsymbol{t}; t_j)$ are both very close to $s_j w(t_j; 0)$ by lemma 2.1. Also, $G(t_j, s_j w(t_j; 0)) = 0$ and $G_u(t_j, s_j w(t_j; 0)) \neq 0$. Thus, (2.13) and (2.12) are, indeed, equivalent. □

Our task in proving theorem 1.1 is therefore reduced to finding a critical point of the function $f_\varepsilon(\boldsymbol{t})$ on the set

$$\Delta^\varepsilon = \Delta_1^\varepsilon \times \cdots \times \Delta_m^\varepsilon,$$

where Δ_j^ε is the set of all n_j-tuples $\boldsymbol{t}^j = (t_{\nu_j+1}, \ldots, t_{\nu_j+n_j})$ such that

$$x_j - h < t_{\nu_j+1} < \cdots < t_{\nu_j+n_j} < x_j + h, \qquad \frac{t_{\nu_j+i+1} - t_{\nu_j+i}}{\varepsilon} > l_0,$$

for all $i = 1, \ldots, n_j$. To find such critical points we use Brouwer's degree theory, accordingly it will be enough to show that the degree of ∇f_ε over Δ^ε is not zero, i.e. $\deg(\nabla f_\varepsilon, \Delta^\varepsilon, 0) \neq 0$. See [8] for a discussion on Brouwer's degree theory.

The computation of this degree will be reduced to computing the degree of the gradient of partial functions, depending only on the group of variables associated to each critical point x_j of V. We fix j, and to simplify the notation, we write $\tau_i = t_{\nu_j+i}$, and the signs $s_i = s_{\nu_j+i}$, $i = 0, \ldots, n_j + 1$. We will assume in the arguments to follow that the points $\tau_0 = t_{\nu_j}$ and $\tau_{n+1} = t_{\nu_{j+1}+1}$ are kept fixed, and we define the partial energies $g_\varepsilon^j : \Delta_j^\varepsilon \to \boldsymbol{R}$ as

$$g_\varepsilon^j(\tau) = \sum_{i=0}^{n} m_{\varepsilon,\sigma_i}(\tau_i, \tau_{i+1}), \tag{2.14}$$

where $m_{\varepsilon,\sigma}$ was defined in (2.9), $\sigma_i = (s_i, s_{i+1})$, $\tau = (\tau_1, \ldots, \tau_n)$ and

$$\Delta_j^\varepsilon = \left\{ \tau / x_j - h \leqslant \tau_1 < \cdots < \tau_n \leqslant x_j + h,\ \frac{\tau_{i+1} - \tau_i}{\varepsilon} \geqslant l_0,\ i = 1, \ldots, n_j - 1 \right\}.$$

We observe that $\tau_0 = -\infty$ if $j = 1$ or $|x_{j-1} - \tau_0| < h$ otherwise. Similarly $\tau_{n_j+1} = +\infty$ if $j = m$ or lies in the h-neighbourhood of x_{j+1}. Thus these points are in relative terms very far away from the τ_i's if h was chosen small enough.

The following result is crucial and we postpone its proof to section 4.

Proposition 2.2. *There exist numbers $\kappa > 0$, $\varepsilon_0 > 0$ such that for each j, all $0 < \varepsilon < \varepsilon_0$ and any τ_0 and τ_{n_j+1} satisfying $|x_j - \tau_0| > 2h$, $|x_j - \tau_{n_j+1}| > 2h$, we have that $|\nabla g_\varepsilon^j(\tau)| \geqslant \kappa$ for all $\tau \in \partial \Delta_\varepsilon^j$ and*

$$\deg(\nabla g_\varepsilon^j, \Delta_j^\varepsilon, 0) = d_j \neq 0.$$

Before proceeding, let us use this result to prove our main results.

Proof of theorem 1.1. From the definition of the functions f_ε and g_j^ε, a direct computation shows that

$$\nabla_{t^j} f_\varepsilon(\boldsymbol{t}) = \nabla_{t^j} g_\varepsilon^j(\boldsymbol{t}^j) + \mathrm{o}(1), \tag{2.15}$$

where $\mathrm{o}(1) \to 0$ as $\varepsilon \to 0$, uniformly on $\boldsymbol{t} \in \Delta^\varepsilon$. Proposition 2.2 then implies that for all ε small enough $\nabla f_\varepsilon(\boldsymbol{t})$ does not vanish on $\partial\Delta^\varepsilon$ and that

$$\deg(\nabla f_\varepsilon, \Delta^\varepsilon, 0) = \prod_{j=1}^{m} \deg(\nabla g_\varepsilon^j, \Delta_j^\varepsilon, 0) \neq 0, \tag{2.16}$$

since the degree $\deg(\nabla g_\varepsilon^j(t_{\nu_j+1}, \dots, t_{\nu_j+n_j}), \Delta_j^\varepsilon, 0) = d_j$ does not depend on the points t_{ν_j} and $t_{\nu_{j+1}}$. Theorem 1.1 clearly follows from (2.16) and proposition 2.2. □

Proof of theorem 1.2. Fix $m \in \boldsymbol{N}$ and consider the points $x_1, x_2, \dots, x_m$. Proposition 2.2 then holds for the corresponding g_j^ε, $j = 1, 2, \dots, k$. Since V is periodic, and the size of the prescribed clusters is uniformly bounded, it follows that the numbers ε_0 and κ in proposition 2.2 may be chosen uniform in m. Similarly we see that the quantity $\mathrm{o}(1)$ in (2.15) also goes to zero uniformly in m. As a consequence we have the presence of a solution $u_\varepsilon^m(x)$ which develops the desired clusters at the x_i's as $\varepsilon \to 0$ uniformly in m. Finally, for fixed ε, sufficiently small independently of m we may pass using a standard compactness argument, to a subsequence which converges uniformly over compacts to a solution u_ε with the required properties. This concludes the proof. □

We devote the rest of the paper to proving lemma 2.1 and proposition 2.2.

3. Proof of lemma 2.1 and further properties of basic solutions

In this section we prove lemma 2.1, which allows us to reduce our problem to finding a critical point for the finite-dimensional function f_ε. We will also prove some results providing extra properties of the basic solutions that are needed in the computation of the degree in section 4.

We start by proving the uniqueness part of lemma 2.1. Let $\rho_0 > 0$ and $L_0 > 0$ be constants such that

$$G_{uu}(x, u) \geqslant \rho_0 \qquad \text{for all } |u| \leqslant 2\rho_0,\ x \in \boldsymbol{R}, \tag{3.1}$$

$$|w(a; L_0)| < \frac{\rho_0}{2} \qquad \text{for all } a \in [-M, M], \tag{3.2}$$

where $w(\xi, y)$ is the solution of (1.3) and (1.4).

Proof of lemma 2.1 (uniqueness). We only prove the first part, since the other is similar. We argue indirectly. Suppose that there exist sequences $(\varepsilon_n)_{n=1}^\infty$, $(\delta_n)_{n=1}^\infty$, $(a_n)_{n=1}^\infty$, $(b_n)_{n=1}^\infty$ satisfying

$$\varepsilon_n \to \infty, \qquad \delta_n \to 0, \qquad l_n \equiv \frac{b_n - a_n}{\varepsilon_n} \to \infty$$

and such that the equation

$$-\varepsilon_n^2 u_{xx} + G'(x, u) = 0 \quad \text{in } (a_n, b_n), \qquad u_x(a_n) = u_x(b_n) = 0, \tag{3.3}$$

has two solutions $u_n^{(1)}(x)$, $u_n^{(2)}(x)$ satisfying (2.4). We may assume that $a_n \to a_0$, $b_n \to b_0$ for suitable $a_0, b_0 \in [-M, M]$. We observe that then $w(a_n; y) \to w_1(y)$ and $w(b_n; y) \to w_2(y)$, where w_1 and w_2 are the unique positive solutions of

$$-w_{yy} + V_i w = w^p, \qquad w(y) > 0 \quad \text{in } \boldsymbol{R}, \qquad w_y(0) = 0,$$

where $V_1 = V(a_0)$ and $V_2 = V(b_0)$. By rescaling we introduce $v_n^{(i)}(y) = u_n^{(i)}(a_n + \varepsilon_n y) : [0, l_n] \to \boldsymbol{R}$. Since both $u_n^{(1)}(x)$, $u_n^{(2)}(x)$ satisfy (3.3), we have the function h_n defined as $h_n(y) = (v_n^{(2)}(y) - v_n^{(1)}(y))/\|v_n^{(2)} - v_n^{(1)}\|_{H^1}$ satisfies $\|h_n\|_{H^1} = 1$ and

$$\begin{aligned} &-h_{nyy} + V(a_n + \varepsilon_n y)h_n = p|\theta_n v_n^{(2)} + (1-\theta_n)v_n^{(1)}|^{p-1}h_n \quad \text{in } (0, l_n), \\ &h_{ny}(0) = h_{ny}(l_n) = 0, \end{aligned} \tag{3.4}$$

where $\theta_n = \theta_n(y) \in (0, 1)$. Multiplying by $h_n(y)$ and integrating over $[0, l_n]$, we find

$$\int_0^{l_n} |h_{ny}|^2 + G_{uu}(a_n + \varepsilon_n y, \theta_n v_n^{(2)} + (1-\theta_n)v_n^{(1)})h_n^2 \, dy = 0.$$

By the choice of L_0 and ρ_0, we have from here

$$\begin{aligned} &\int_0^{l_n} |h_{ny}|^2 \, dy + \rho_0 \int_{L_0}^{l_n - L_0} h_n^2 \, dy \\ &\qquad \leqslant -\int_{[0,L_0]\cup[l_n-L_0,l_n]} G_{uu}(a_n + \varepsilon_n y, \theta_n v_n^{(2)} + (1-\theta_n)v_n^{(1)})h_n^2 \, dy. \end{aligned} \tag{3.5}$$

Therefore, we have

$$\|h_n\|_{L^2(0,\,L_0)} \not\to 0 \qquad \text{or} \qquad \|h_n\|_{L^2(l_n - L_0,\, l_n)} \not\to 0,$$

since the contrary, together with (3.5), would imply that $\|h_n\|_{H^1} \to 0$.

We assume $\|h_n\|_{L^2(0,\,L_0)} \not\to 0$. The case $\|h_n\|_{L^2(l_n - L_0, l_n)} \not\to 0$ can be treated in a similar way. Then we have $h_n \to h \neq 0$ weakly in H^1. By our assumptions we have

$$v_n^{(1)}(y) \to s_1 w_1(y) \qquad \text{and} \qquad v_n^{(2)}(y) \to s_1 w_1(y),$$

in L^∞_{loc}. Thus, from (3.4) we find that $h \in H^1(0, \infty)$ and it satisfies

$$-h_{yy} + V_1 h = p|w_1|^{p-1}h \quad \text{in } [0, \infty), \qquad h_y(0) = 0. \tag{3.6}$$

On the other hand, $w_{1y} \in H^1(0, \infty)$ and it satisfies

$$-(w_{1y})_{yy} + V_1 w_{1y} = p|w_1|^{p-1}w_{1y} \quad \text{in } [0, \infty), \qquad w_{1y}(0) = 0, \qquad w_{1yy}(0) \neq 0.$$

Thus $h(y)$ and $w_{1y}(y)$ are linearly independent solutions of

$$-v_{yy} + V_1 v = p|w_1|^{p-1}v.$$

Since $w_1(y) \to 0$ as $y \to \infty$, this equation has an unbounded solution $\zeta(y)$. However, $\zeta(y)$ must be a linear combination of $h(y)$ and $w_{1y}(y)$, which are both bounded in $[0, \infty)$. This is a contradiction that implies that the solution of (3.3) is unique, completing the proof. □

With a very similar proof, actually simpler, we can also consider the case when the spatial variable is frozen. For $a \in [-M, M]$ and $l > 0$ we consider the following problem:

$$-w_{yy} + G'(a, w(y)) = 0 \quad \text{in } (0, l), \qquad w_y(0) = w_y(l) = 0. \tag{3.7}$$

Then we have the following lemma.

Lemma 3.1. *There exist $l_0 > 0$ and $\delta_0 > 0$ such that if $l \geqslant l_0$ and $\sigma = (s_1, s_2) \in \{+, -\}^2$, then (3.7) has a unique solution, denoted by $w_\sigma(a, l; y)$, satisfying*

$$\|w_\sigma(a, l; y) - s_1 w(a; y) - s_2 w(a; l - y)\|_{L^\infty(0,l)} \leqslant \delta_0. \tag{3.8}$$

In the arguments to follow it will also be important to consider solutions of (2.1) but having only *one bump*. We can prove the following lemma using the arguments just given for lemma 2.1.

Lemma 3.2.

(a) For any $M > 0$ there exist δ_0, ε_0, $l_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and $-M \leqslant a < b \leqslant M$ satisfying $(b-a)/\varepsilon \geqslant l_0$, there is at most one solution u_ε of (2.1) satisfying

$$\left\| u_\varepsilon(x) - w\left(a; \frac{x-a}{\varepsilon}\right)\right\|_{L^\infty(a,b)} < \delta_0. \tag{3.9}$$

(b) A similar statement holds for the equation

$$-\varepsilon^2 u_{xx} + G'(x,u) = 0 \quad \text{in } (a,b), \qquad u_x(a) = u(b) = 0. \tag{3.10}$$

We continue the discussion of lemma 2.1, but now on the existence part. In our construction we will first obtain *one bump* solutions and then we will glue them conveniently to get the *two bump* solutions we are looking for. The construction of a *one bump* solution is done via a mountain pass argument on a penalized functional. We have the following existence result whose proof is given in the appendix.

Lemma 3.3. *Let δ_0, ε_0, $l_0 > 0$ so that the statement of lemma 3.2 holds. For every $\delta \in (0, \delta_0]$, there exist constants $\varepsilon_1 \in (0, \varepsilon_0]$, $l_1 \in [l_0, \infty)$ such that if $\varepsilon \in (0, \varepsilon_1]$ and $(b-a)/\varepsilon \geqslant l_1$, then*

(a) The Neumann boundary value problem (2.1) has a positive solution $u_\varepsilon(x) = w^l_{\varepsilon,\mathrm{N}}(a,b;x)$ satisfying

$$\|u_\varepsilon(a+\varepsilon y) - w(a;y)\|_{L^\infty(0,(b-a)/\varepsilon)} < \delta. \tag{3.11}$$

(b) The Dirichlet–Neumann boundary value problem (3.10) has a positive solution $u_\varepsilon = w^l_{\varepsilon,\mathrm{D}}(a,b;x)$ satisfying (3.11).

Remark 3.1. We can also construct a solution satisfying (3.11) in $(0,\infty)$ in a similar way to the proof of lemma 3.3.

On the other hand, as in lemma 3.3, we can also find positive solutions $u_\varepsilon = w^r_{\varepsilon,\mathrm{N}}(a,b;x)$ to the Neumann problem (2.1) satisfying

$$\left\| u_\varepsilon(x) - w\left(b; \frac{x-b}{\varepsilon}\right)\right\|_{L^\infty(a,b)} < \delta, \tag{3.12}$$

and solutions $u_\varepsilon = w^r_{\varepsilon,\mathrm{D}}(a,b;x)$ to the Dirichlet–Neumann boundary value problem

$$-\varepsilon^2 u_{xx} + G'(x,u) = 0 \quad \text{in } (a,b), \qquad u(a) = u_x(b) = 0, \tag{3.13}$$

and so that also satisfies (3.12).

Proof of lemma 2.1 (existence). We first discuss the case of a solution of type $\sigma = (+,+)$. We will obtain such a solution by joining $w^l_{\varepsilon,\mathrm{N}}(a,\tau;x)$ with $w^r_{\varepsilon,\mathrm{N}}(\tau,b;x)$ for a suitable $\tau \in (a,b)$. That is we find $u_{\varepsilon,(+,+)}(a,b;x)$ of the form

$$u_{\varepsilon,(+,+)}(a,b;x) = \begin{cases} w^l_{\varepsilon,\mathrm{N}}(a,\tau;x), & x \in [a,\tau], \\ w^r_{\varepsilon,\mathrm{N}}(\tau,b;x), & x \in (\tau,b], \end{cases} \tag{3.14}$$

where $\tau \in (a,b)$ is such that

$$w^l_{\varepsilon,\mathrm{N}}(a,\tau;\tau) = w^r_{\varepsilon,\mathrm{N}}(\tau,b;\tau). \tag{3.15}$$

In what follows we show that such a $\tau \in (a,b)$ exists. First we observe that the functions $(a,y) \mapsto w^l_{\varepsilon,\mathrm{N}}(a, a+\varepsilon y; a+\varepsilon y)$ and $(b,y) \mapsto w^r_{\varepsilon,\mathrm{N}}(b-\varepsilon y, b; b-\varepsilon y)$ are continuous $\boldsymbol{R} \times [l_1,\infty)$ as a consequence of their uniqueness.

Next we obtain some asymptotic estimates of $w^l_{\varepsilon,\mathrm{N}}$ and $w^r_{\varepsilon,\mathrm{N}}$. We claim that there are numbers $\varepsilon_2 \in (0, \varepsilon_1]$, $l_2 > l_1$ and $\nu_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_2]$, we have

$$w^l_{\varepsilon,\mathrm{N}}(a, a + \varepsilon l_1; a + \varepsilon l_1) \geqslant \nu_0, \qquad 0 < w^r_{\varepsilon,\mathrm{N}}(b - \varepsilon y, b; b - \varepsilon y) \leqslant \frac{\nu_0}{2}, \tag{3.16}$$

for $y \geqslant l_2$. To prove the claim we first see that

$$w^l_{\varepsilon,\mathrm{N}}(a, a + \varepsilon l_1; a + \varepsilon y) \to w_{(+,+)}(a, 2l_1; y),$$

in $C^2(0, l_1)$ as $\varepsilon \to 0$, where $w_{(+,+)}(a, 2l_1; y)$ is a solution of (3.7) satisfying (3.8). Setting

$$\nu_0 = \tfrac{1}{2} \inf_{a\in[-M,M]} w_{(+,+)}(a, 2l_1; l_1),$$

we can show the first inequality in (3.16), if $\varepsilon_2 \leqslant \varepsilon_1$ is sufficiently small. Next we observe that for $\varepsilon \in (0, \varepsilon_1]$, $w(z) = w^r_{\varepsilon,\mathrm{N}}(b - \varepsilon y, b; b - \varepsilon z)$ satisfies

$$-w_{zz} + V(b - \varepsilon z)w = w^p \quad \text{in } (0, y), \qquad |w(z)| \leqslant \rho_0 \quad \text{in } (L_0, y).$$

Recalling (3.1) and using a suitable comparison argument in (L_0, y), we can find $l_2 \geqslant l_1$ independent of $b \in [-M, M]$ such that

$$0 < w^r_{\varepsilon,\mathrm{N}}(b - \varepsilon y, b; b - \varepsilon y) \leqslant \frac{\nu_0}{2} \qquad \text{for all } b \in [-M, M] \text{ and } y \geqslant l_2.$$

This proves the second inequality in (3.16). With similar arguments we can also prove that

$$w^r_{\varepsilon,\mathrm{N}}(b - \varepsilon l_1, b, b - \varepsilon l_1) \geqslant \nu_0, \qquad 0 < w^l_{\varepsilon,\mathrm{N}}(a, a + \varepsilon y, a + \varepsilon y) \leqslant \frac{\nu_0}{2}, \tag{3.17}$$

for $y \geqslant l_2$. Thus, we obtain from (3.16) and (3.17) that

$$w^l_{\varepsilon,\mathrm{N}}(a, a + \varepsilon l_1; a + \varepsilon l_1) - w^r_{\varepsilon,\mathrm{N}}(a + \varepsilon l_1, b; a + \varepsilon l_1) \geqslant \tfrac{1}{2}\nu_0 > 0,$$

$$w^l_{\varepsilon,\mathrm{N}}(a, b - \varepsilon l_1; b - \varepsilon l_1) - w^r_{\varepsilon,\mathrm{N}}(b - \varepsilon l_1, b; b - \varepsilon l_1) \leqslant -\tfrac{1}{2}\nu_0 < 0,$$

if $(b - a)/\varepsilon \geqslant 2l_1 + l_2$. Then by continuity it follows the existence of $\tau \in (a + \varepsilon l_1, b - \varepsilon l_1)$ which satisfies (3.15). Then we get (2.4) from (3.11).

For the construction of a solution of type $\sigma = (+, -)$, we proceed in an analogous way. That is, we find $u_{\varepsilon,(+,-)}(a, b; x)$ of the form

$$u_{\varepsilon,(+,-)}(a, b; x) = \begin{cases} w^l_{\varepsilon,\mathrm{D}}(a, \tau; x), & x \in [a, \tau], \\ -w^r_{\varepsilon,\mathrm{D}}(\tau, b; x), & x \in (\tau, b]. \end{cases} \tag{3.18}$$

In order for such a function to be the solution of (2.1), we need to have

$$(w^l_{\varepsilon,\mathrm{D}})_x(a, \tau; \tau) = (w^r_{\varepsilon,\mathrm{D}})_x(\tau, b; \tau). \tag{3.19}$$

The existence of $\tau \in (a, b)$ is based in continuity properties of the functions $(w^l_{\varepsilon,\mathrm{D}})_x(a, a + \varepsilon y; a + \varepsilon y)$ and $(w^r_{\varepsilon,\mathrm{D}})_x(b - \varepsilon y, b; b - \varepsilon y)$ in a, b, y, and the inequalities

$$(w^l_{\varepsilon,\mathrm{D}})_x(a, a + \varepsilon l_1; a + \varepsilon l_1) \leqslant -\nu_0 \qquad \text{and} \qquad |(w^r_{\varepsilon,\mathrm{D}})_x(b - \varepsilon y, b; b - \varepsilon y)| \leqslant \frac{\nu_0}{2}$$

for $y \geqslant l_2$,

$$(w^r_{\varepsilon,\mathrm{D}})_x(b - \varepsilon l_1, b; b - \varepsilon l_1) \geqslant \nu_0 \qquad \text{and} \qquad |(w^l_{\varepsilon,\mathrm{D}})_x(a, a + \varepsilon y; a + \varepsilon y)| \leqslant \frac{\nu_0}{2}$$

for $y \geqslant l_2$. The proof of these inequalities can be done as that of (3.16) and (3.17) and we omit it. Condition (2.4) is obtained as before.

The construction of solutions of type $(-, +)$ and $(-, -)$ is similar. The second part of lemma 2.1 is essentially lemma 3.3. □

Without difficulty lemma 2.1 can be refined to obtain the following.

Lemma 3.4.

(a) *For any* $M > 0$ *and* $\delta > 0$ *there exists* $\varepsilon_1 = \varepsilon_1(\delta, M) > 0$ *and* $l_1 = l_1(\delta, M)$ *such that if* $\varepsilon \in (0, \varepsilon_1]$, $-M \leqslant a < b \leqslant M$, $(b-a)/\varepsilon \geqslant l_1$ *and* $\sigma = (s_1, s_2) \in \{+1, -1\}^2$, *then there exists a solution* $u_\varepsilon(x) = u_{\varepsilon,\sigma}(a, b; x)$ *of (2.1) satisfying*

$$\left\| u_\varepsilon(a + \varepsilon y) - s_1 w(a; y) - s_2 w\left(b; \frac{b-a}{\varepsilon} - y\right)\right\|_{C^2(0,(b-a)/\varepsilon)} < \delta,$$

(b) *For any* $M > 0$ *and* $\delta > 0$ *there exists* $\varepsilon_1 = \varepsilon_1(\delta, M) > 0$ *such that if* $\varepsilon \in (0, \varepsilon_1]$, $b \in [-M, M]$ *and* $\sigma \in \{+, -\}$, *then there exists a solution* $u_\varepsilon(x) = u_{\varepsilon,\sigma}(-\infty, b; x)$ *of (2.2) satisfying*

$$\|u_\varepsilon(b + \varepsilon y) - \sigma w(b; y)\|_{C^2(-\infty,0)} < \delta.$$

A similar statement holds for the equation on (a, ∞).

A combination of lemmas 3.1 and 3.4 leads to the following.

Lemma 3.5. *For any* $l > l_0$ *and* $\delta > 0$, *there exists* $\varepsilon_1 = \varepsilon_1(l, \delta) > 0$ *independent of* a, $b \in [-M, M]$ *such that for* $\varepsilon \in (0, \varepsilon_1]$, $(b-a)/\varepsilon \in [l_0, l]$ *and* $\sigma \in \{+, -\}^2$

$$\left\| u_{\varepsilon,\sigma}(a, b; a + \varepsilon y) - w_\sigma\left(a, \frac{b-a}{\varepsilon}; y\right)\right\|_{C^2(0,(b-a)/\varepsilon)} \leqslant \delta.$$

Finally we establish the exponential decay of the basic solutions. We have the following lemma.

Lemma 3.6. *There are constants* $\alpha > 0$ *and* $\beta > 0$, *independent of* a, b, ε, σ, *such that for* $\varepsilon \in (0, \varepsilon_0]$, $(b-a)/\varepsilon \geqslant l_0$ *it holds*

$$|u_{\varepsilon,\sigma}(a, b; x)| + \varepsilon|(u_{\varepsilon,\sigma})_x(a, b; x)| \leqslant \beta\left\{\exp\left(-\frac{\alpha(x-a)}{\varepsilon}\right) + \exp\left(-\frac{\alpha(b-x)}{\varepsilon}\right)\right\}$$

for all $x \in [a, b]$, *and*

$$|u_{\varepsilon,\sigma}(-\infty, b; x)| + \varepsilon|(u_{\varepsilon,\sigma})_x(-\infty, b; x)| \leqslant \beta \exp\left(-\frac{\alpha(b-x)}{\varepsilon}\right)$$

for all $x \in (-\infty, b]$. *A similar statement holds for the interval* (a, ∞).

Proof. We recall that the solution $w(\cdot; \cdot)$ to the limiting equations (1.3) and (1.4) is exponentially decaying. Then, noting that $u_{\varepsilon,\sigma}$ satisfies

$$-\varepsilon^2 u_{xx} + G_u(x, u(x)) = 0,$$

the use of (3.1) and a standard comparison argument allows us to prove the lemma. □

4. Computing the degree. Proof of proposition 2.2

In this section we compute the degree of the function g_ε^j on Δ_j^ε. This leads us to prove proposition 2.2. For notational convenience we shall drop the index j from g_ε^j, Δ_j^ε and n_j.

The first step is the study of the behaviour of ∇g_ε over $\partial\Delta^\varepsilon$. With this information, we later compute the $\deg(\nabla g_\varepsilon, \Delta^\varepsilon, 0)$ by introducing a suitable homotopy with an explicit function.

In what follows, we fix $\tau_0 \in [-\infty, x_j - 2h]$ and $\tau_{n+1} \in [x_j + 2h, \infty]$. By definition of Δ^ε we see that

$$\partial\Delta^\varepsilon = \{\tau;\ \tau_1 = x_j - h\} \cup \{\tau;\ \tau_n = x_j + h\} \cup \left\{\tau;\ \frac{\tau_{i+1} - \tau_i}{\varepsilon} = l_0 \text{ for some } i\right\},$$

We study next the behaviour of ∇g_ε on each of the above three sets. We have two cases.

Case 1: ∇g_ε *when* $\tau_1 = x_j - h$ *or* $\tau_n = x_j + h$. We deal first with the case $\tau_1 = x_j - h$. We start with some preliminaries. Recalling that $w(a; y)$ is the solution of (1.3) and (1.4) with $\xi = a$, we define

$$H(a) = \int_0^\infty \frac{1}{2}|w_y(a;y)|^2 + G(a, w(a;y))\,\mathrm{d}y.$$

Making a change of variables, we easily see that

$$w(a;y) = V(a)^{1/(p-1)} w(V(a)^{1/2} y),$$

where $w(y)$ is the unique positive solution in $H^1(\boldsymbol{R})$ of

$$-w_{yy} + w = w^p \quad \text{in } \boldsymbol{R}, \qquad w'(0) = 0.$$

Then we can see that

$$H(a) = C_0 V(a)^{(p+3)/2(p-1)}, \tag{4.1}$$

where

$$C_0 = \int_0^\infty \frac{1}{2}|w_y|^2 + \frac{1}{2}|w|^2 - \frac{1}{p+1}|w|^{p+1}\,\mathrm{d}y.$$

Now we can state the following lemma.

Lemma 4.1. *For any* $\delta > 0$, *there exists* $\varepsilon_1 = \varepsilon_1(\delta) \in (0, \varepsilon_0]$ *and* $L = L(\delta) > l_0$ *such that*

(a) if $\varepsilon \in (0, \varepsilon_1]$ *and* $(b-a)/\varepsilon \geqslant 3L|\log\varepsilon|$, *then*

$$\left|\frac{\partial}{\partial a} m_{\varepsilon,\sigma}(a,b) - H'(a)\right| \leqslant \delta \qquad \textit{and} \qquad \left|\frac{\partial}{\partial b} m_{\varepsilon,\sigma}(a,b) - H'(b)\right| \leqslant \delta;$$

(b) if $\varepsilon \in (0, \varepsilon_1]$ *and* $(b-a)/\varepsilon \geqslant l_0$, *then*

$$\left|\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b}\right) m_{\varepsilon,\sigma}(a,b) - (H'(a) + H'(b))\right| \leqslant \delta.$$

Proof.

(a) By (2.10), we have

$$\frac{\partial}{\partial a} m_{\varepsilon,\sigma}(a,b) = -\frac{1}{\varepsilon} G(a, u_{\varepsilon,\sigma}(a,b;a)).$$

Setting $v(y) = u_{\varepsilon,\sigma}(a,b;a+\varepsilon y)$ and choosing a cut-off function $\varphi(y) : [0,\infty) \to [0,1]$ such that $\varphi(\tau) = 1$ for $\tau \in [0,1]$, $\varphi(\tau) = 0$ for $\tau \in [2,\infty)$ and $\varphi'(\tau) \leqslant 0$ for $\tau \in [0,\infty)$), we have

$$\begin{aligned}
\frac{\partial}{\partial a} m_{\varepsilon,\sigma}(a,b) &= \frac{1}{\varepsilon}\left(\frac{1}{2}|v_y(0)|^2 - G(a, v(0))\right) \\
&= \int_0^{2L|\log\varepsilon|} \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}y}\left\{\varphi\left(\frac{y}{L|\log\varepsilon|}\right)\left(-\frac{1}{2}|v_y|^2 + G(a+\varepsilon y, v(y))\right)\right\}\mathrm{d}y \\
&= \int_0^{2L|\log\varepsilon|} \frac{1}{L\varepsilon|\log\varepsilon|}\varphi'\left(\frac{y}{L|\log\varepsilon|}\right)\left(-\frac{1}{2}|v_y|^2 + G(a+\varepsilon y, v(y))\right)\mathrm{d}y \\
&\quad + \int_0^{2L|\log\varepsilon|} \varphi\left(\frac{y}{L|\log\varepsilon|}\right) G_x(a+\varepsilon y, v(y))\,\mathrm{d}y = (I) + (II).
\end{aligned}$$

By lemma 3.6, there exists $L = L(\delta) > 0$ such that if $(b-a)/\varepsilon \geqslant 3L|\log\varepsilon|$, then

$$|-\tfrac{1}{2}|v_y|^2 + G(a+\varepsilon y, v(y))| \leqslant \tfrac{1}{2}\delta\varepsilon \qquad \text{for } y \in [L|\log\varepsilon|, 2L|\log\varepsilon|].$$

Thus we have

$$|(I)| \leqslant \frac{1}{2}\int_{L|\log\varepsilon|}^{2L|\log\varepsilon|} \frac{1}{L|\log\varepsilon|}|\varphi'\left(\frac{y}{L|\log\varepsilon|}\right)|\delta\,\mathrm{d}y \leqslant \frac{\delta}{2}.$$

By lemmas 3.4 and 3.6, we can also see

$$\left|(II) - \int_0^\infty G_x(a, w(a;y))\,\mathrm{d}y\right| \leqslant \frac{\delta}{2},$$

provided $(b-a)/\varepsilon \gg 1$ and ε is sufficiently small. By the definition of $H(a)$, we have

$$H'(a) = \int_0^\infty G_x(a, w(a;y))\,\mathrm{d}y.$$

Thus we get the first inequality in (a). The second inequality can be obtained in a similar way.

(b) Now we prove (b). By (2.10), we have

$$\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b}\right) m_{\varepsilon,\sigma}(a,b) = -\frac{1}{\varepsilon}G(a, u_{\varepsilon,\sigma}(a,b;a)) + \frac{1}{\varepsilon}G(b, u_{\varepsilon,\sigma}(a,b;b))$$
$$= \int_0^{(b-a)/\varepsilon} \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}y}\left\{-\frac{1}{2}|v_y|^2 + G(a+\varepsilon y, v(y))\right\}\mathrm{d}y,$$

where $v(y) = u_{\varepsilon,\sigma}(a,b;a+\varepsilon y)$. Arguing as in part (a), we obtain the desired result. □

Now we recall (4.1) and choose $\delta > 0$ such that

$$|H'(x)| > 2\delta \qquad \text{for } x \in [x_j - h, x_j - \tfrac{1}{2}h].$$

Then we choose $\varepsilon_1 = \varepsilon_1(\delta)$ and $L = L(\delta)$ as given in lemma 4.1. Then, by making ε_1 smaller if necessary, we see that for $\tau_1 = x_j - h$ and $\varepsilon \in (0, \varepsilon_1]$, there exists $k \in \{1, 2, \ldots, n\}$ satisfying

$$x_j - h = \tau_1 < \tau_2 < \cdots < \tau_k \leqslant x_j - \tfrac{1}{2}h, \qquad \text{and} \qquad \frac{\tau_{k+1} - \tau_k}{\varepsilon} \geqslant 3L|\log\varepsilon|. \tag{4.2}$$

Now we state first our estimate.

Proposition 4.1. *Assume that (4.2) holds then, for $\varepsilon \in (0, \varepsilon_1]$ we have*

$$\sum_{i=1}^k \frac{\partial}{\partial\tau_i} g_\varepsilon(\tau_1, \ldots, \tau_n) \begin{cases} < 0, & \text{if } x_j \text{ is a local minimum,} \\ > 0, & \text{if } x_j \text{ is a local maximum.} \end{cases}$$

We remark that in our analysis we consider $\tau_0 > -\infty$ and $\tau_{n+1} < \infty$, but a similar result holds if we take $\tau_0 = -\infty$ or $\tau_{n+1} = \infty$.

Proof. We just deal with the case where x_j is a local minimum of V. In view of (4.2), we can apply lemma 4.1 to obtain

$$\sum_{i=1}^k \frac{\partial}{\partial\tau_i} g_\varepsilon(\tau_1, \ldots, \tau_n) = \frac{\partial}{\partial\tau_1} m_{\varepsilon,\sigma_0}(\tau_0, \tau_1) + \sum_{i=1}^{k-1}\left(\frac{\partial}{\partial\tau_i} + \frac{\partial}{\partial\tau_{i+1}}\right) m_{\varepsilon,\sigma_i}(\tau_i, \tau_{i+1})$$
$$+ \frac{\partial}{\partial\tau_k} m_{\varepsilon,\sigma_k}(\tau_k, \tau_{k+1}) \leqslant 2\sum_{i=1}^k (H'(\tau_i) + \delta) < 0$$

completing the proof. □

In the case $\tau_n = x_j + h$, we proceed in a similar way. First we obtain $k \in \{1, \ldots, n\}$ such that

$$\begin{aligned} &x_j + \tfrac{1}{2}h \leqslant \tau_k < \tau_{k+1} < \cdots < \tau_n = x_j + \tfrac{1}{2}h, \\ &\frac{\tau_k - \tau_{k-1}}{\varepsilon} \geqslant 3L|\log\varepsilon|. \end{aligned} \tag{4.3}$$

Then we have the following proposition.

Proposition 4.2. *Assume that (4.3) holds then, for $\varepsilon \in (0, \varepsilon_1]$ we have*

$$\sum_{i=k}^{n} \frac{\partial}{\partial \tau_i} g_\varepsilon(\tau_1, \ldots, \tau_n) \begin{cases} > 0 & if\ x_j \text{ is a local minimum}, \\ < 0 & if\ x_j \text{ is a local maximum}. \end{cases}$$

Case 2: ∇g_ε *when* $(\tau_{i+1} - \tau_i)/\varepsilon = l_0$. To estimate ∇g_ε when τ_{i+1} and τ_i are relatively close, we need some preliminaries. By elementary phase plane analysis on the function $w_\sigma(a, l; y)$, introduced in lemma 3.1, we have the following lemma.

Lemma 4.2. *For any $a \in \boldsymbol{R}$, $l \geqslant l_0$ and $\sigma \in \{+, -\}^2$,*

$$E_\sigma(a, l) \equiv \tfrac{1}{2}|w'_\sigma(a, l; y)|^2 - G(a, w_\sigma(a, l; y))$$

is independent of y. Moreover

(a) if $\sigma = (+, +)$ or $(-, -)$, then $E_\sigma(a, l) < 0$,
(b) if $\sigma = (+, -)$ or $(-, +)$, then $E_\sigma(a, l) > 0$.

Now we can prove the following important lemma.

Lemma 4.3.

(a) For any $l \geqslant l_0$ there exists $\rho(l) > 0$ and $\varepsilon_2(l) > 0$ such that for $(b - a)/\varepsilon \in [l_0, l]$ and $\varepsilon \in (0, \varepsilon_2]$,

$$\varepsilon \frac{\partial}{\partial a} m_{\varepsilon,\sigma}(a, b) \begin{cases} \leqslant -\rho(l), & \text{if } \sigma = (+, +) \text{ or } (-, -), \\ \geqslant \rho(l), & \text{if } \sigma = (+, -) \text{ or } (-, +), \end{cases} \tag{4.4}$$

$$\varepsilon \frac{\partial}{\partial b} m_{\varepsilon,\sigma}(a, b) \begin{cases} \geqslant \rho(l), & \text{if } \sigma = (+, +) \text{ or } (-, -), \\ \leqslant -\rho(l), & \text{if } \sigma = (+, -) \text{ or } (-, +). \end{cases} \tag{4.5}$$

(b) For any $\delta > 0$ there exist $l(\delta) \geqslant l_0$ and $\varepsilon_2 > 0$ such that for $(b - a)/\varepsilon \geqslant l(\delta)$ and $\varepsilon \in (0, \varepsilon_2]$,

$$\varepsilon \left| \frac{\partial}{\partial a} m_{\varepsilon,\sigma}(a, b) \right| \leqslant \delta \qquad \text{and} \qquad \varepsilon \left| \frac{\partial}{\partial b} m_{\varepsilon,\sigma}(a, b) \right| \leqslant \delta.$$

Proof.

(a) We prove just for (4.4) and $\sigma = (+, -)$. We argue indirectly. If the proposition does not hold, there exist sequences a_j, b_j, ε_j such that $(b_j - a_j)/\varepsilon_j \in [l_0, l]$, $\varepsilon_j \to 0$ and

$$\limsup_{j \to \infty} \varepsilon_j \frac{\partial}{\partial a} m_{\varepsilon_j,\sigma}(a_j, b_j) \leqslant 0.$$

We may assume $a_j \to \tilde{a}$, $b_j \to \tilde{a}$ and $(b_j - a_j)/\varepsilon_j \to \tilde{l} \in [l_0, l]$. By lemma 3.5, we have $u_{\varepsilon_j,\sigma}(a_j, b_j; a_j + \varepsilon_j y) \to w_\sigma(\tilde{a}, \tilde{l}; y)$. Thus

$$\varepsilon_j \frac{\partial}{\partial a} m_{\varepsilon_j,\sigma}(a_j, b_j) = -G(a_j, u_{\varepsilon_j,\sigma}(a_j, b_j; a_j)) \to -G(\tilde{a}, w_\sigma(\tilde{a}, \tilde{l}; 0)) = E_\sigma(\tilde{a}, \tilde{l}) > 0,$$

which is a contradiction proving (a). Here we used lemma 4.2.

(b) Observing that $w(a; y)$ satisfies

$$\tfrac{1}{2}|w_y(a; y)|^2 - G(a; w(a; y)) \equiv 0,$$

We can deduce (b) from lemmas 2.1 and 3.4.

□

Using lemma 4.3 we choose numbers

$$\rho_0 > \rho_1 > \cdots > \rho_n \qquad \text{and} \qquad l_0 < l_1 < \cdots < l_n$$

as follows: first we apply (a) of lemma 4.3 and set $\rho_0 = \rho(l_0)$. Next we apply (b) and set $l_1 = l(\rho_0/2)$. We continue this process obtaining

$$\rho_1 = \rho(l_1), \qquad l_2 = l\left(\frac{\rho_1}{2}\right), \qquad \rho_2 = \rho(l_2), \ldots, l_n = l\left(\frac{\rho_{n-1}}{2}\right), \qquad \rho_n = \rho(l_n).$$

Then we have the following proposition.

Proposition 4.3. *Suppose that* $(\tau_1, \ldots, \tau_n) \in \Delta^\varepsilon$ *satisfies* $(\tau_{i+1} - \tau_i)/\varepsilon = l_0$ *for some* $i \in \{1, 2, \ldots, n-1\}$. *Then there exists* $j \in \{2, 3, \ldots, n\}$ *and* $k \in \{0, 1, \ldots, n\}$ *such that*

$$\frac{\tau_j - \tau_{j-1}}{\varepsilon} \in [l_0, l_k] \qquad \text{and} \qquad \frac{\tau_{j+1} - \tau_j}{\varepsilon} \in [l_{k+1}, \infty). \tag{4.6}$$

For such a j *we have*

$$\frac{\partial}{\partial \tau_j} g_\varepsilon(\tau_1, \ldots, \tau_n) > 0, \qquad \text{if } (s_1, \ldots, s_n) = (+, +, \ldots) \text{ or } (-, -, \ldots), \tag{4.7}$$

$$\frac{\partial}{\partial \tau_j} g_\varepsilon(\tau_1, \ldots, \tau_n) < 0, \qquad \text{if } (s_1, \ldots, s_n) = (+, -, \ldots) \text{ or } (-, +, \ldots). \tag{4.8}$$

Proof. First we look at (4.6). If it does not hold for all j and k, then we have

$$\frac{\tau_{i+1} - \tau_i}{\varepsilon} = l_0, \qquad \frac{\tau_{i+2} - \tau_{i+1}}{\varepsilon} \leqslant l_1, \ldots, \frac{\tau_{i+2} - \tau_{i+1}}{\varepsilon} \leqslant l_{n-i-1}, \qquad \frac{\tau_{n+1} - \tau_n}{\varepsilon} \leqslant l_{n-i},$$

and then we obtain

$$\frac{\tau_{n+1} - \tau_n}{\varepsilon} \leqslant l_n. \tag{4.9}$$

But this is impossible for sufficiently small ε since we have $\tau_{n+1} - \tau_n \geqslant h$.

Thus, let j and k so that (4.6) holds. Then by lemma 4.3, we have

$$\varepsilon \frac{\partial}{\partial \tau_j} m_{\varepsilon, \sigma_{j-1}}(\tau_{j-1}, \tau_j) \begin{cases} \geqslant \rho_k, & \text{if } \sigma_{j-1} = (+,+) \text{ or } (-,-), \\ \leqslant -\rho_k, & \text{if } \sigma_{j-1} = (+,-) \text{ or } (-,+), \end{cases}$$

and also

$$\varepsilon \left| \frac{\partial}{\partial \tau_j} m_{\varepsilon, \sigma_j}(\tau_j, \tau_{j+1}) \right| \leqslant \frac{\rho_k}{2}.$$

Since

$$\frac{\partial}{\partial \tau_j} g_\varepsilon(\tau_1, \ldots, \tau_n) = \frac{\partial}{\partial \tau_j} m_{\varepsilon, \sigma_{j-1}}(\tau_{j-1}, \tau_j) + \frac{\partial}{\partial \tau_j} m_{\varepsilon, \sigma_j}(\tau_j, \tau_{j+1}),$$

we obtain (4.7) and (4.8). □

This completes the study of the behaviour of ∇g_ε over $\partial \Delta^\varepsilon$. We can now see from propositions 4.1–4.3 that, as desired, there is a number $\kappa > 0$ such that for all small ε,

$$|\nabla g_\varepsilon(\tau)| \geqslant \kappa \qquad \text{for all } \tau \in \partial \Delta^\varepsilon,$$

and then $\deg(\nabla g_\varepsilon, \Delta^\varepsilon, 0)$ is well defined. As in [19], we define $\Phi_\varepsilon : \Delta^\varepsilon \to \boldsymbol{R}$ as

$$\Phi_\varepsilon(\tau) = \frac{1}{2}(\tau_1 - x_j)^2 + \frac{1}{2}(\tau_n - x_j)^2 + \sum_{j=1}^{n-1} \exp\left(-\frac{\tau_{j+1} - \tau_j}{\varepsilon}\right). \tag{4.10}$$

Then we have the following crucial proposition.

Proposition 4.4. . *For sufficiently small $\varepsilon > 0$, we have:*

(a) if x_j is a local minimum of $V(x)$ and $(s_1, s_2, \ldots, s_n) = (+, +, \ldots)$ or $(-, -, \ldots)$, then ∇g_ε and $-\nabla\Phi_\varepsilon$ are homotopic in Δ^ε, i.e. for all $\theta \in [0, 1]$ and $(\tau_1, \ldots, \tau_n) \in \partial\Delta^\varepsilon$,

$$\theta\nabla g_\varepsilon(\tau_1, \ldots, \tau_n) - (1-\theta)\nabla\Phi_\varepsilon(\tau_1, \ldots, \tau_n) \neq 0; \tag{4.11}$$

(b) if x_j is a local maximum of $V(x)$ and $(s_1, s_2, \ldots, s_n) = (+, -, \ldots)$ or $(-, +, \ldots)$, then ∇g_ε and $\nabla\Phi_\varepsilon$ are homotopic in Δ^ε, i.e. for all $\theta \in [0, 1]$ and $(\tau_1, \ldots, \tau_n) \in \partial\Delta^\varepsilon$

$$\theta\nabla g_\varepsilon(\tau_1, \ldots, \tau_n) + (1-\theta)\nabla\Phi_\varepsilon(\tau_1, \ldots, \tau_n) \neq 0. \tag{4.12}$$

Proof. It is not hard to prove that

$$\text{(4.2) implies } \left(\frac{\partial}{\partial\tau_1} + \cdots + \frac{\partial}{\partial\tau_k}\right)\Phi_\varepsilon(\tau_1, \ldots, \tau_n) < 0, \tag{4.13}$$

$$\text{(4.3) implies } \left(\frac{\partial}{\partial\tau_k} + \cdots + \frac{\partial}{\partial\tau_n}\right)\Phi_\varepsilon(\tau_1, \ldots, \tau_n) < 0, \tag{4.14}$$

$$\text{(4.6) implies } \frac{\partial}{\partial\tau_j}\Phi_\varepsilon(\tau_1, \ldots, \tau_n) < 0. \tag{4.15}$$

From here (4.11) and (4.12) follow easily. □

Thus we can see from the homotopy invariance of the Brouwer degree that

$$\deg(\nabla g_\varepsilon, \Delta^\varepsilon) = \deg(\pm\nabla\Phi_\varepsilon, \Delta_\varepsilon).$$

But, since $\Phi_\varepsilon : \Delta^\varepsilon \to \boldsymbol{R}$ has a unique non-degenerate critical point in Δ_ε, we get $\deg(\nabla g_\varepsilon, \Delta^\varepsilon) = \pm 1$ and the proof of proposition 2.2 is thus concluded.

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Appendix

In this appendix we will give a proof of the existence lemma 3.3. Our proof consists of finding a critical point for the functional

$$J(\varepsilon, a, l; v) = \int_0^l \frac{1}{2}|v_y|^2 + V(a+\varepsilon y)v^2 - \frac{1}{p+1}v_+^{p+1}\,\mathrm{d}y,$$

where $u_+ = \max\{u, 0\}$ and $l = (b-a)/\varepsilon$, so that $u(x) = v((x-a)/\varepsilon)$ satisfies (3.11). We consider the functional over the Sobolev space H to be the space $H_{\mathrm{N}} = H^1(0, l)$ in the case of the Neumann boundary value problem (2.1) and the space $H_{\mathrm{D}} = \{v \in H^1(0, l);\ v(l) = 0\}$ in the case of Dirichlet–Neumann boundary problem (3.10).

We proceed to modify the functional J according to an idea from del Pino and Felmer [9]. We consider $\rho_0 > 0$ and $L_0 > 0$ such that (3.1) and (3.2) hold. We set

$$f(u) = \begin{cases} |u|^{p-1}u, & \text{if } |u| \leqslant \rho_0, \\ \rho_0^{p-1}u, & \text{if } |u| > \rho_0 \end{cases}$$

and

$$f(y,u) = \chi_{[0,L_0]}(y)u_+^p + (1-\chi_{[0,L_0]}(y))f(u_+),$$

where $\chi_{[0,L_0]}(y) = 1$ if $y \in [0, L_0]$ and $\chi_{[0,L_0]}(y) = 0$ if $y \notin [0, L_0]$. Finally we set the penalized functional

$$\tilde{J}(\varepsilon, a, l; v) = \int_0^l \frac{1}{2}|v_y|^2 + V(a+\varepsilon y)v^2 - F(y,v)\,\mathrm{d}y,$$

where $F(y,\xi) = \int_0^\xi f(y,\tau)\,\mathrm{d}\tau$. We remark that critical points of $\tilde{J}(\varepsilon, a, l; v)$ satisfy the equation

$$-v_{yy} + V(a+\varepsilon y)v = f(y,v) \qquad \text{in } (0,l),$$

with boundary conditions depending on $H = H_\mathrm{D}$ or $H = H_\mathrm{N}$. Thus a critical point v of $\tilde{J}(\varepsilon, a, l; \cdot)$ is a critical point of $J(\varepsilon, a, l; \cdot)$ if

$$\|v(y)\|_{L^\infty(L_0,l)} \leqslant \rho_0 \tag{A.1}$$

is satisfied.

It is easy to see that $\tilde{J}(\varepsilon, a, l; v)$ has the mountain pass geometry and satisfies the Palais–Smale compactness condition. Then, by the mountain pass theorem, there is a non-trivial critical point $v^* = v(\varepsilon, a, l; y)$ of $\tilde{J}$ characterized as follows:

$$\tilde{J}(\varepsilon, a, l; v^*) = b(\varepsilon, a, l) \equiv \inf_{\gamma\in\Gamma}\max_{\tau\in[0,1]} \tilde{J}(\varepsilon, a, l; \gamma(\tau)), \tag{A.2}$$

where

$$\Gamma = \{\gamma \in C([0,1], H);\ \gamma(0) = 0,\ \tilde{J}(\varepsilon, a, l; \gamma(1)) < 0\}. \tag{A.3}$$

We will show that if l is sufficiently large and ε is sufficiently small then $v(\varepsilon, a, l; y)$ satisfies

$$\|v(\varepsilon, a, l; y) - w(a; y)\|_{L^\infty(0,l)} \leqslant \delta.$$

This implies, on the one hand, that $v(\varepsilon, a, l; y)$ satisfies (A.1) and so the penalization does not act, and, on the other hand, that the rescaled function $u(x) = v((x-a)/\varepsilon)$ satisfies (3.11), as required.

For our purposes, we introduce the following limiting functionals:

$$J_\infty(a; v) = \int_0^\infty \frac{1}{2}|v_y|^2 + V(a)v^2 - \frac{1}{p+1}v_+^{p+1}\,\mathrm{d}y,$$

$$\tilde{J}_\infty(a; v) = \int_0^\infty \frac{1}{2}|v_y|^2 + V(a)v^2 - F(y, v_+)\,\mathrm{d}y.$$

Both functionals are defined on $H^1(0,\infty)$ and have a mountain pass geometry. We also see that

$$J_\infty(a; v) \leqslant \tilde{J}_\infty(a; v) \qquad \text{for all } v \in H^1(0,\infty). \tag{A.4}$$

We define

$$c_\infty(a) = \inf_{\gamma\in\Gamma_{a,\infty}}\max_{\tau\in[0,1]} J_\infty(a; \gamma(\tau)) \qquad \text{and} \qquad b_\infty(a) = \inf_{\gamma\in\tilde{\Gamma}_{a,\infty}}\max_{\tau\in[0,1]} \tilde{J}_\infty(a; \gamma(\tau)),$$

where $\Gamma_{a,\infty}$ and $\tilde{\Gamma}_{a,\infty}$ are defined in a similar way to (A.3).

Next we claim that $w(a; y)$ is the unique critical point of $\tilde{J}_\infty(a; \cdot)$ satisfying $\tilde{J}_\infty(a; v) = b_\infty(a)$. To prove this claim we first recall a well-known fact, i.e. $c_\infty(a)$ is attained by $w(a; y)$, which is the unique critical point of $J_\infty(a; v)$. Next we see that

$$b_\infty(a) = c_\infty(a). \tag{A.5}$$

In fact, by (A.4), it is clear that $c_\infty(a) \leqslant b_\infty(a)$. On the other hand, setting

$$\gamma(\tau) = Rw(a; y)\tau \qquad \text{for large constant } R > 1,$$

we see that $\gamma \in \tilde{\Gamma}_{a,\infty}$ and

$$b_\infty(a) \leqslant \max_{\tau\in[0,1]} \tilde{J}_\infty(a; \gamma(\tau)) = c_\infty(a).$$

Thus we get (A.5). Now we complete the proof of the claim: let $v(y)$ be a critical point of $\tilde{J}_\infty(a; v)$ with a critical value $b_\infty(a) = c_\infty(a)$. Setting $\gamma(\tau) = Rv(y)\tau$, we see that

$$\max_{\tau\in[0,1]} J_\infty(a; \gamma(\tau)) \leqslant \max_{\tau\in[0,1]} \tilde{J}_\infty(a; \gamma(\tau)) = c_\infty(a).$$

Thus we have $\max_{\tau\in[0,1]} J_\infty(a; \gamma(\tau)) = c_\infty(a)$, from where $v(y) = rw(a; y)$ for some $r > 0$. But then, since $v \mapsto f(y, v)/v$ is strictly increasing, we have $v(y) = w(a; y)$, proving the claim.

Now we complete the proof of the lemma. We argue indirectly and suppose that there exist $\delta > 0$ and sequences $(\varepsilon_n)_{n=1}^\infty$, $(a_n)_{n=1}^\infty \subset [-M, M]$, $(l_n)_{n=1}^\infty$ and $(v_n(y))_{n=1}^\infty$ such that (a) $\varepsilon_n \to 0$, $l_n \to \infty$ and $a_n \to a_0$ for some $a_0 \in [-M, M]$. (b) $v_n(x)$ is a critical point of $\tilde{J}(\varepsilon_n, a_n, l_n; v)$ with $\tilde{J}(\varepsilon_n, a_n, l_n; v_n) = b(\varepsilon_n, a_n, l_n)$. (c) $\|v_n - w(a_n; y)\|_{L^\infty(0,l_n)} \geqslant \delta$.

By using proper test functions we can easily see that

$$b(\varepsilon_n, a_n, l_n) \to b_\infty(a_0). \tag{A.6}$$

Let us assume that, extracting a subsequence if necessary, we have

$$v_n(y) \to v_0(y) \qquad \text{in } C^2_{\text{loc}}.$$

Since $v_n(y)$ is a critical point of $\tilde{J}(\varepsilon_n, a_n, l_n; v)$, by analysing the corresponding differential equations, and the decay properties of $v_n(y)$ implied by the form of $f(y, v)$, we see that $v_n(y)$ converges to $v_0(y)$ in $H^1(0, \infty)$ and then $v_0(y)$ is a critical point of $\tilde{J}_\infty(a_0; v)$. Moreover,

$$\tilde{J}_\infty(a_0; v_0) = \lim_{n\to\infty} \tilde{J}(\varepsilon_n, a, l_n; v_n) = b_\infty(a_0).$$

Thus by our earlier claim $v_0(y) = w(a_0; y)$ contradicting (c) and (A.6). □

Remark A.1. We can show that $v(\varepsilon, a, l; y)$ has only one local maximum which is located at boundary 0 for sufficiently small ε. In fact, if not, there exist sequences $\varepsilon_j \to 0$, $a_j \to a_0 \in [-M, M]$ and $l_j \to \infty$ such that $v_j(y) = v(\varepsilon_j, a_j, l_j; y)$ attains a local maximum at $y_j \in (0, l_j]$. Of course, $v_{jy}(y_j) = 0$ and $v_{jyy}(y_j) \leqslant 0$. Since v_j satisfies the rescaled equation, we have $v_j(y_j) \geqslant V(a_j + \varepsilon_j y_j)^{1/(p-1)}$. On the other hand, v_j satisfies (3.11) and we can see that y_j is bounded as $j \to \infty$. We assume $y_j \to y_0 \geqslant 0$. Since $v_j(y)$ converges to $w(a_0; y)$ in C^2_{loc}, we have $w_y(a_0; y_0) = \lim_{j\to\infty} v_{jy}(y_j) = 0$. Thus we have $y_0 = 0$. Since $v_{jy}(0) = v_{jy}(y_j) = 0$, we can find $s_j \in (0, y_j)$ such that $v_{jyy}(s_j) = 0$. We have $s_j \to 0$ and $0 = v_{jyy}(s_j) \to w_{yy}(a; 0) < 0$. This is a contradiction.

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