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Semi-classical states of nonlinear Schrödinger equations: a variational reduction method

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Dedicated to Paul Rabinowitz on the occasion of his 60th birthday

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1. Introduction

This paper deals with the study of positive solutions of the equation

$$\varepsilon^2 \Delta u - V(x)u + f(u) = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

$$u(x) \to 0 \quad \text{as } |x| \to \infty.$$

Here and in what follows, V is a smooth function which is bounded and uniformly positive, let us say that for certain positive constants α, β

$$\alpha \le V(x) \le \beta \quad \text{for all } x \in \mathbb{R}^N.$$

The class of nonlinearities considered in this work includes, but it is not restricted to, the model $f(u) = u^p$ with $p > 1$, and $p < \frac{N+2}{N-2}$ if $N \ge 3$. Precise assumptions will be stated and discussed below.

A basic motivation for the study of this equation comes from the fact that it is satisfied by standing-wave solutions of the nonlinear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar^2}{2m} \Delta \psi - W(x)\psi + g(|\psi|)\psi, \tag{1.2}$$

namely solutions of the form $\psi(x,t) = u(x)e^{i\frac{Et}{\hbar}}$, whose amplitude $u(x)$ vanishes at infinity. Here then $u(x)$ satisfies (1.1) with $V = W + E$, $\varepsilon^2 = \frac{\hbar^2}{2m}$ and $f(u) = g(u)u$. In this context ε can be naturally regarded as a small parameter, see [13].

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An interesting class of solutions of (1.1) are the so called semi-classical states, which are families of solutions u_ε which develop a spike shape around one or more distinguished points of the space, while vanishing asymptotically elsewhere as $\varepsilon \to 0$. More precisely, scaling out ε around a point P, defining $v_\varepsilon(y) = u_\varepsilon(P + \varepsilon y)$, then v_ε satisfies

$$\Delta v - V(P + \varepsilon y)v + f(v) = 0 \quad \text{in} \quad \mathbb{R}^N,$$

$$v(x) \to 0 \quad \text{as } |x| \to \infty,$$

and one searches for solutions v_ε which approach as $\varepsilon \to 0$ a bell-shape given by a positive, ground-state solution of

$$\Delta w - V(P)w + f(w) = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{1.3}$$

For the power nonlinearity $f(s) = s^p$, $1 < p < \frac{N+2}{N-2}$, since $V(P) > 0$, it is well known that this problem has a positive solution which goes to zero at infinity. This solution is, besides, radially symmetric around some point and unique up to translations, see Gidas, Ni and Nirenberg [14] and Kwong [16]. Moreover, see [16] and Ni and Takagi [20], the linearized equation around w is nondegenerate in the sense that the problem

$$\Delta h - V(P)h + pw^{p-1}h = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{1.4}$$

has linear combinations of the functions $\frac{\partial w}{\partial x_i}$ as its only solutions which go to zero at infinity. These facts are crucial in the formulation of a Lyapunov-Schmidt type procedure, first introduced by Floer and Weinstein in [13] for the one-dimensional case, then extended by Oh to higher dimensions in [23], [24], which reduces the original problem to a finite dimensional one. This finite dimensional system of equations becomes, for the case of a single spike, one in P, which resembles $\nabla V(P) = 0$ as $\varepsilon \to 0$. The case of P a nondegenerate critical point of V was dealt with in [13], [23] and [24].

Rabinowitz was the first in dealing with the question from a global variational point of view. Roughly speaking, under only a basic set of assumptions in f existence is established for small ε whenever

$$\liminf_{|x| \to +\infty} V(x) > \inf_{x \in \mathbb{R}^N} V(x),$$

see [26]. Concentration takes place around a global minimizer of V, as established by Wang for $f(u) = u^p$, in [31]. In [8] the authors obtained local results for a rather general f for which no fine properties of the limiting equation were known. Assuming that in an open bounded set Λ one has

$$\inf_{x \in \partial\Lambda} V(x) > \inf_{x \in \Lambda} V(x),$$

a single-peaked solution around a minimizer of V in Λ is constructed. This result was extended in [9] to multi-peak solutions around any prescribed finite set of local minima. The variational approach was also found to work around any topologically non-trivial critical point of V in [10], without finite dimensional reduction, yet uniqueness of the asymptotic radial ground state was required.

On the other hand, the finite dimensional reduction method was also found suitable to find concentrating solutions around degenerate critical points of V when $f(u) = u^p$. Ambrosetti Badiale and Cingolani [1] do so at isolated local maxima with polynomial degeneracy and Y. Li [18] at general *stable* critical points of V. See also [17], [15] and [25] for recent results using this approach.

Throughout this paper the following hypotheses on $f : [0, \infty) \to \mathbb{R}$ will be assumed.

(f0) $f(s)$ is of class C^1 and $f'(s)s$ is locally Lipschitz on $[0, \infty)$.
(f1) There exists a number $p > 1$, with $p < \frac{N+2}{N-2}$ if $N \geq 3$, such that

$$\limsup_{s\to+\infty} \frac{f'(s)}{s^{p-1}} < +\infty.$$

(f2) There is a number $C > 0$ such that

$$f'(s)s \leq Cf(s) \quad \text{for } 0 < s < 1.$$

(f3) There exists a number $q > 1$ such that

$$0 < qf(s) \leq f'(s)s \quad \text{for all } s > 0.$$

Assumptions (f0)-(f3) guarantee, from standard variational arguments, existence of a least-energy ground state of (1.3), which is radially symmetric. Uniqueness is however not known. This is a delicate issue, for which affirmative answer is known from fine ODE analysis only for more restricted classes. Perhaps the most general result of this type is that recently obtained by Serrin and Tang in [28], which would guarantee radial uniqueness in (1.3) if additionally one assumes that the quotient $\frac{-s+f'(s)s}{-s+f(s)}$ is non-increasing. For instance, uniqueness does not seem to be known for the nonlinearity

$$f(s) = s^p + s^q, \quad 1 < q < p < \frac{N+2}{N-2},$$

situation in which (f0)-(f3) hold.

The purpose of the present work is to develop a variational method of construction of single and multiple-spike solutions, associated to general topologically nontrivial critical points of V, only under assumptions (f0)-(f3).

A difficulty faced with variational characterizations of critical values, is that they do not always allow easily to localize properties of associated critical points, especially if they do not enjoy a minimizing or least-energy character. On the other hand this is an advantage of the implicit-function Lyapunov-Schmidt type approach, which discovers the solutions around a small neighborhood of a well chosen first approximation. However, this approach relies heavily on nondegeneracy properties of the linearized problem around this first approximation, thus this reduction procedure is possible only with very fine information on the the limiting equation. In a number of interesting problems exhibiting point concentration this type of information is simply not available, and could be very hard to be obtained even for simplest possible nonlinearities, see for instance [12] and [2]. The need is then created of finding ways of *localizing without linearizing*.

The approach proposed here consists of dealing explicitly with a special negative gradient flow defined on Nehari's manifold for a properly penalized energy functional associated to the problem. Considered pointwise, this flow becomes a fairly explicit nonlocal evolution problem in $\mathbb{R}^N$ which turns out to have very nice properties (not shared by heat flow for instance). In particular, if we start from a well-chosen set of initial conditions obtained from a suitable test path associated to the linking situation assumed in V, then we we are able to follow the flow closely. Then we let time go to infinity and define a mimmax value along this deformed test path. Ekeland's variational principle then allows us to find almost-critical points which stay arbitrarily close to the deformed path. Close analysis of the flow finally leads us to capture the properties of these almost-critical points which in the limit in time will yield a solution of (1.1) with the desired characteristics, eliminating the penalization earlier introduced. This is done for the construction of both, single and multiple-spike solutions of (1.1).

At this point we would like to mention that in related work by Coti-Zelati and Rabinowitz in [6], [7] multi-bump solutions are constructed for equations including

$$\Delta u - u + k(x)u^p = 0,$$

with k periodic, and under certain nondegeneracy assumptions which cannot be dealt with via implicit-function finite dimensional reduction, see also the work by Spradlin [29], where infinite-bump solutions are obtained. These constructions are also based upon accurate considerations on the gradient flow.

We specify next what type of local linking we consider for the potential V. Let Λ be an open, bounded subset of $\mathbb{R}^N$ with C^1 boundary $\partial\Lambda$.

Definition 1.1 We say that there is a *local linking of V in Λ with critical value c^**, if the following conditions hold:

(a) There exist closed sets $B_0 \subset B \subset \Lambda$, B connected, such that if we consider the class of maps

$$\Gamma = \{\varphi \in C(B, \Lambda) \mid \varphi(x) = x \ \forall x \in B_0\} \tag{1.5}$$

then

$$\sup_{x\in B_0} V(x) < \inf_{\varphi\in\Gamma} \sup_{x\in\Lambda} V(\varphi(x)) \equiv c^*. \tag{1.6}$$

(b) For each $x \in \partial\Lambda$ with $V(x) = c^*$, there is a direction τ, tangent to $\partial\Lambda$, such that

$$\nabla V(x) \cdot \tau \neq 0. \tag{1.7}$$

Particular cases of local linking of V in Λ are local maxima, local minima or saddle points for V inside Λ, see below. On the other hand, (a) and (b) guarantee the existence of a critical point of V at level c^* inside Λ. Condition (b) is necessary in order to "seal" Λ at level c^*. so that standard deformation arguments indeed yield the presence inside Λ of such a critical point.

Theorem 1.1. *Assume f satisfies (f0)-(f3) and that Λ is a bounded, open subset of $\mathbb{R}^N$ with smooth boundary, in which there is linking for V with critical value c^*. Then there is $\varepsilon_0 > 0$, so that for every $0 < \varepsilon < \varepsilon_0$ a positive solution u_ε of (1.1) exists. If $u_\varepsilon(x_\varepsilon) = \max_{x\in\mathbb{R}^N} u_\varepsilon(x)$ then*

$$x_\varepsilon \in \Lambda, \;\; V(x_\varepsilon) \to c^* \;\; \textit{and} \;\; \nabla V(x_\varepsilon) \to 0 \;\; \textit{as} \;\; \varepsilon \to 0.$$

Moreover,

$$u_\varepsilon(x) \leq A e^{-b|x-x_\varepsilon|/\varepsilon}$$

for certain positive numbers A, b.

This result extends to the construction of multi-peak solutions. In order to avoid technicalities we will assume that the local linking corresponds to saddles. This allows to obtain in a simple way the Intersection Lemma in Sect. 7. More precisely consider

Definition 1.2 We say that there is a *local linking of V in Λ with critical value c^* of a saddle type,* if there is a local linking of V in Λ with critical value c^* and further:

There are complementary subspaces S, T such that $R^N = S \oplus T$, and $O \in \Lambda$ such that $B = B(O,r) \cap (S + \{O\}) \subset \Lambda$, $B_0 = (\partial B(O,r)) \cap (S + \{O\})$ and

$$c^* = \inf_{x\in(T+\{O\})\cap\Lambda} V(x). \tag{1.8}$$

Now we state our second result.

Theorem 1.2. *Assume that f satisfies (f0)-(f3). Let Λ_i, $i = 1, \ldots, \ell$, be smooth bounded domains in R^N with $\bar\Lambda_i \cap \bar\Lambda_j = \emptyset$, such that there is linking of V inside each Λ_i with critical values c_i^* of saddle type. Then there exists $\varepsilon_0 >$ such that,* *for all* $0 < \varepsilon < \varepsilon_0$ *a positive solution* u_ε *of (1.1) exists and, if* $x_{\varepsilon i} \in \Lambda_i$ *is such that* $u_\varepsilon(x_{\varepsilon i}) = \max_{x \in \Lambda_i} u_\varepsilon(x)$ *then*

$$V(x_{\varepsilon i}) \to c_i^* \quad \textit{and} \quad \nabla V(x_{\varepsilon i}) \to 0, \quad \textit{as } \varepsilon \to 0, \quad \textit{for all .}$$

Moreover

$$u_\varepsilon(x) \le A e^{-b \min_i |x - x_{\varepsilon i}|/\varepsilon} \quad \textit{for all } x \in \mathbb{R}^N,$$

for certain positive numbers A, b.

We should also mention that equations of the form (1.1), in bounded domains under Dirichlet or Neumann boundary condition with a constant potential have also drawn considerable attention. Many results on existence of spike-layered patterns been established in recent years for those problems, starting with the works of Ni and Takagi [20] and [21] and Ni and Wei [22]. These results, as well as most of the subsequent progress found in the literature, make essential use the nondegeneracy condition on the limiting equation. In [11] we have obtained concentration results for least energy solutions under conditions not ensuring uniqueness or nondegeneracy, in the spirit of (f0)-(f3), which also considerably simplifies the original proofs. We believe that the techniques developed in the present paper may be adapted to attack this type of problems. They may also be of use in the study of related point-concentration phenomena, like nearly critical elliptic equations, or Ginzburg-Landau vortices, see [27], [4].

The rest of this paper will be devoted to the proof of the above results. In Sect. 2 we introduce the variational framework. Rather than the usual energy functional for problem (1.1), we consider a penalized modification J_ε, defined as in [8] and [10]. The Nehari's manifold is defined here as the set of those $u \neq 0$ for which $J_\varepsilon'(u_\varepsilon)u_\varepsilon = 0$. A min-max quantity eventually leading to our seeked single-peak solution is defined, making use of a suitable gradient flow. In Sect. 3, estimates for this flow are found, in particular leading to its global definiteness in time. In Sect. 4 local control of the flow is obtained which leads to the proof of Theorem 1.1. In Sect. 5 and Sect. 6 we generalize this analysis to the case of multiple concentration, leading to the proofs of Theorems 1.2 and Theorem 1.3.

2. The min-max

In this section we will define a min-max quantity which will later be established to yield a single-spike solution as that predicted in Theorem 1.1. Thus we assume in what follows that V has nontrivial linking with critical value c^* in a bounded open set Λ. We begin with a useful observation: with no loss of generality we may assume that the least value of V on $\bar{\Lambda}$ is very close to c^*. In fact, let us consider sets B and B_0 as in the definition of linking in Λ: let $\delta > 0$ be an arbitrary small number and consider the set

$$\Lambda_\delta = \{x \in \Lambda \ / \ V(x) > c^* - \delta\}.$$

Then we can replace Λ by Λ_δ without affecting condition (b) in the definition of linking. In fact, let $c^* - \delta < c_1 < c^*$ and define

$$B_\delta = B \cap \{x \in \Lambda \mid V(x) \geq c_1\}, \quad B_{0\delta} = B \cap \{x \in \Lambda \mid V(x) = c_1\}.$$

Let us notice that $B_{0\delta}$ is non-empty thanks to the connectedness of B. Then, given a continuous function $\varphi : B_\delta \to \Lambda_\delta$ satisfying $\varphi(x) = x$ on $B_{0\delta}$, we define $\tilde{\varphi}$ on B by extending φ as the identity on $B \setminus B_\delta$. Then $\tilde{\varphi} : B \to \Lambda$, and $\sup_{x \in B} V(\tilde{\varphi}(x)) = \sup_{x \in B_\delta} V(\tilde{\varphi}(x))$. This proves our claim.

Let us consider the usual energy functional E_ε associated to equation (1.1),

$$E_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} \varepsilon^2 |\nabla u|^2 + V(x) u^2 - \int_{\mathbb{R}^N} F(u), \quad u \in H^1. \tag{2.1}$$

We have set $f(s) = 0$ for all $s < 0$ and $F(s) = \int_0^s f(t)dt$. Standard arguments show that nonzero critical points of E_ε correspond precisely to the positive solutions of (1.1) in $H^1(\mathbb{R}^N)$. In what follows, for functions u and v in $H^1(\mathbb{R}^N)$ we will denote

$$\|u\|^2_{H^1,\varepsilon} = \int \varepsilon^2 |\nabla u|^2 + u^2, \quad \|v\|^2_{H^1} = \int |\nabla v|^2 + v^2.$$

Similarly, we denote

$$\|v\|^2_{H^2} = \|v\|^2_{H^1} + \int |D^2 v|^2, \quad \|u\|^2_{H^2,\varepsilon} = \|u\|^2_{H^1,\varepsilon} + \varepsilon^4 \int |D^2 u|^2.$$

As in [8], [9] and [10], we will work with a a modified version J_ε of E_ε, which penalizes with high values concentration outside Λ. Let q be as in assumption (f3), and fix a number $r > (q+1)/(q-1)$. Let $a > 0$ be so that $f(a) = \alpha/r$, where α is a positive lower bound of V. Then we define the functions

$$\tilde{f}(s) = \begin{cases} f(s) & s \leq a \\ f(a) + f'(a)(s-a) & s > a \end{cases}$$

and

$$g(x,s) = \chi_\Lambda(x) f(s) + (1 - \chi_\Lambda(x)) \tilde{f}(s), \quad (x,s) \in \mathbb{R}^N \times \mathbb{R},$$

where χ_Λ denotes the characteristic function of the set Λ. Thanks to assumption (f1), we have that if a is chosen small enough then

$$\tilde{F}(s) - \frac{1}{2} \tilde{f}(s) s \leq 0 \quad \text{and} \quad \frac{\tilde{f}(s)}{s} \leq \frac{V_0}{2}, \quad \text{for all} \quad s \geq 0. \tag{2.2}$$

Let $G(x,s)=\int_0^s g(x,t)dt$ and consider the modified functional J_ε given by

$$J_\varepsilon(u)=\frac{1}{2}\int_{\mathbb{R}^N}\varepsilon^2|\nabla u|^2+V(x)u^2-\int_{\mathbb{R}^N}G(x,u),\quad u\in H^1. \tag{2.3}$$

Let us observe that f satisfies the so-called Ambrosetti-Rabinowitz condition

$$0<(q+1)F(s)\le f(s)s\quad\text{for all}\quad s>0, \tag{2.4}$$

as can be easily seen from assumption (f3). It follows that J_ε satisfies the Palais-Smale condition, as can be shown by slightly modifying the argument given in Lemma 1.1 in [8]. This is an important advantage of J_ε with respect to E_ε, for which the P.S. condition could typically fail. Our strategy consists of finding a critical point of J_ε. These critical points are weak solutions of the equation

$$\varepsilon^2\Delta u-V(x)u+g(x,u)=0\quad\text{in}\quad\mathbb{R}^N. \tag{2.5}$$

Thus, if they additionally satisfy

$$0\le u(x)\le a\quad x\in\mathbb{R}^N\setminus\Lambda, \tag{2.6}$$

then they are solutions of (1.1). A consequence of assumption (f3) is that the function $f(s)/s$ is strictly increasing on $s>0$. This fact and a standard argument reduces the search of nontrivial critical points of J_ε to that of critical points of J_ε on its *Nehari manifold* $\mathcal{M}_\varepsilon$ defined as

$$\mathcal{M}_\varepsilon=\{u\in H^1\setminus\{0\}/\mathcal{F}_\varepsilon(u)=0\}, \tag{2.7}$$

where

$$\mathcal{F}_\varepsilon(u)=\int_{\mathbb{R}^N}\varepsilon^2|\nabla u|^2+V(x)u^2-\int_{\mathbb{R}^N}g(x,u)u, \tag{2.8}$$

see [10]. We observe that $u|_\Lambda\not\equiv 0$ if $u\in\mathcal{M}_\varepsilon$. This and the fact that $f(s)/s$ is strictly increasing imply that $u\in\mathcal{M}_\varepsilon$ if and only if

$$J_\varepsilon(u)=\max_{t\ge 0}J_\varepsilon(tu).$$

Next we see that for $u\in\mathcal{M}_\varepsilon$ we have

$$\mathcal{F}_\varepsilon'(u)u=\mathcal{F}_\varepsilon'(u)u-\mathcal{F}_\varepsilon(u)=\int_{\mathbb{R}^N}g'(x,u)u^2-g(x,u)u,$$

and then, from hypothesis (f1), we see that

$$\mathcal{F}_\varepsilon'(u)u\ge(q-1)\int_\Lambda f(u)u>0. \tag{2.9}$$

This fact and assumption (f0) imply that $\mathcal{M}_\varepsilon$ is locally a $C^{1,1}$ manifold. We actually have more. Given $K > 0$ let us consider consider the set

$$\mathcal{M}_\varepsilon^K = \{u \in \mathcal{M}_\varepsilon \,/\, J_\varepsilon(u) \le \varepsilon^N K\}.$$

Then the following fact holds.

Lemma 2.1. *There exist positive numbers k_1 and k_2 such that for all sufficiently small ε and all $u \in \mathcal{M}_\varepsilon^K$ one has*

$$\mathcal{F}_\varepsilon'(u)u \ge \varepsilon^N k_2 \tag{2.10}$$

and

$$\varepsilon^N k_1 \ge \|u\|_{H^1,\varepsilon} \ge \varepsilon^N k_2 \tag{2.11}$$

Proof. Let $u \in \mathcal{M}_\varepsilon^K$. For the proof of this fact it is convenient to rescale the function u defining $v_\varepsilon(y) = u(\varepsilon y)$, and $\Lambda_\varepsilon = \varepsilon^{-1}\Lambda$. Since $u \in \mathcal{M}_\varepsilon$ and $J_\varepsilon(u) \le \varepsilon^N K$, from relations (2.4) and (2.2) we find that

$$(\frac{1}{2} - \frac{1}{q+1}) \int_\Lambda f(u)u \le K\varepsilon^N$$

Using again that $u \in \mathcal{M}_\varepsilon$ and (2.2) we find that

$$N_\varepsilon \equiv \int_{\mathbb{R}^N} \varepsilon^2|\nabla u|^2 + V_0 u^2 \le k$$

Now, let us observe that assumptions (f1) and (f3) imply the existence of a constant A so that

$$0 < f(s) \le A(|s|^q + |s|^p), \quad s > 0. \tag{2.12}$$

Using that $u \in \mathcal{M}_\varepsilon$ and Sobolev's embedding we find that

$$N_\varepsilon \le 2A \int_{\Lambda_\varepsilon} (v_\varepsilon^{p+1} + v_\varepsilon^{q+1}) \le K(N_\varepsilon^{(p+1)/2} + N_\varepsilon^{(q+1)/2}).$$

It follows that $\|u\|_{H^1,\varepsilon}$ is bounded below by a constant independent of ε, times ε^N. This proves the estimates (2.11). On the other hand, we have that

$$\int \varepsilon^2|\nabla u|^2 + V(x)u^2 = \int g(u,x)u \le \int f(u)u,$$

hence inequality (2.10) follows from relation (2.9). □

Important role in our analysis will be played by the limiting functionals I_ν, $\nu > 0$, given by

$$I_\nu(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \nu u^2 - \int_{\mathbb{R}^N} F(u), \quad u \in H^1.$$

Let us consider the numbers

$$b^\nu = \inf_{v\neq 0,\, v\in H^1} \max_{t>0} I_\nu(tv). \tag{2.13}$$

Then $b^\nu > 0$ is the least value of I_ν, at which there are nontrivial critical points. These are the *least energy solutions* of the equation

$$\Delta u - \nu u + f(u) = 0 \qquad \text{in } \mathbb{R}^N,$$

$$0 < u(x) \to 0 \quad \text{as } |x| \to \infty,$$

see for instance [3]. By comparison with suitable barriers, we find that for each of these solutions there are positive numbers m_1, m_2 such that

$$0 \le u(x) \le m_1 e^{-m_2|x|}, \quad \text{for} \quad x \in \mathbb{R}^N,$$

besides they are radially symmetric up to translations thanks to a classical result in [14].

Now we will build up a min-max quantity for J_ε. To begin with, we consider a path φ_ε in the class Γ given by (1.5) with the property that

$$V(\varphi_\varepsilon(x)) \le c^* + \varepsilon^2, \quad \text{for all } x \in B. \tag{2.14}$$

Using a deformation argument we may also assume, with no loss of generality that φ_ε is so that whenever $x \in B$ and

$$V(\varphi_\varepsilon(x)) \in [c^*, c^* + \varepsilon^2] \quad \text{implies} \quad |\nabla V(\varphi_\varepsilon(x))| \le \varepsilon^2.$$

Next we consider a fixed critical point of the functional I_{c^*} at level b^{c^*} which we choose radially symmetric around the origin and denote w_{c^*}. Associated to φ_ε, we consider the path $p_\varepsilon : B \to \mathcal{M}_\varepsilon$ defined as

$$p_\varepsilon(x)(y) = w_{c^*}\left(\frac{y - \varphi_\varepsilon(x)}{\varepsilon}\right) t_{x,\varepsilon}, \quad y \in \mathbb{R}^N, \tag{2.15}$$

where $t_{x,\varepsilon}$ is the unique $t > 0$ such that

$$t_{x,\varepsilon} w_{c^*}\left(\frac{\cdot - \varphi_\varepsilon(x)}{\varepsilon}\right) \in \mathcal{M}_\varepsilon.$$

Our method consists of deforming the path p_ε along a suitable negative gradient flow for J_ε on $\mathcal{M}_\varepsilon$. For this purpose, it is convenient to endow $H^1(\mathbb{R}^N)$ with the inner product

$$(u, v)_* = \int_{\mathbb{R}^N} \varepsilon^2 \nabla u \nabla v + V(x)uv. \tag{2.16}$$

In what follows ∇J_ε and $\nabla \mathcal{F}_\varepsilon$ will denote the gradients of J_ε and $\mathcal{F}_\varepsilon$ with respect to the inner product (2.16). We consider the following initial value problem in H^1.

$$\dot{\eta}_\varepsilon(x, t) = \mathcal{G}_\varepsilon(\eta_\varepsilon(x, t)), \qquad \eta(x, 0) = p_\varepsilon(x), \quad x \in B, \tag{2.17}$$

where

$$\mathcal{G}_\varepsilon(u) = \nabla J_\varepsilon(u) - d(u)\nabla \mathcal{F}_\varepsilon(u), \tag{2.18}$$

with

$$d(u) = \frac{(\nabla \mathcal{F}_\varepsilon(u), \nabla J_\varepsilon(u))_*}{\|\nabla \mathcal{F}_\varepsilon(u)\|_*^2}. \tag{2.19}$$

The vector field $\mathcal{G}_\varepsilon$ corresponds to the orthogonal projection of ∇J_ε onto the tangent space to the Nehari manifold $\mathcal{M}_\varepsilon$. It is clearly locally Lipschitz continuous and satisfies $\mathcal{G}_\varepsilon(u) = 0$ if and only if $\nabla J_\varepsilon(u) = 0$. Thus (2.17) has a unique solution $\eta_\varepsilon(x, t)$ defined on some time interval. It is easily checked that $\eta_\varepsilon(x, t)$ belongs to $\mathcal{M}_\varepsilon$ at all times and that $J_\varepsilon(\eta(x, t))$ is decreasing in t. In Proposition 3.1 (i) we will prove that $\eta(x, t)$ is defined for all $t \geq 0$. Accepting this fact for the moment we define the min-max value

$$C_\varepsilon = \inf_{t \geq 0} \sup_{x \in B} J_\varepsilon(\eta_\varepsilon(x, t)). \tag{2.20}$$

Now we will establish that C_ε is a critical value of J_ε. is a direct consequence of the following result.

Lemma 2.2. (i) *For all sufficiently small ε we have the validity of the estimates,*

$$\varepsilon^N(b^{c^*} + o(1)) \leq C_\varepsilon \leq \varepsilon^N(b^{c^*} + o(\varepsilon)). \tag{2.21}$$

(ii) *There exists a number $\sigma > 0$ such that for all sufficiently small ε,*

$$\sup_{x \in B_0} J_\varepsilon(p_\varepsilon(x)) \leq \varepsilon^N(b^{c^*} - \sigma).$$

Lemma 2.2, the fact that J_ε satisfies the Palais-Smale condition and a standard deformation argument readily yield that C_ε is a critical value of J_ε. In the next sections we will find an associated critical point which actually solves equation (1.1) and satisfies the conditions of Theorem 1.1.

A technical point we would like to emphasize, which constitutes a crucial difference with the min-max quantity defined in [10], is the fact that the elements of the basic path $p_\varepsilon(x)$ *do not* resemble, after the proper scaling, a least energy solution of I_ν for $\nu = V(\varphi_\varepsilon(x))$, except when this value equals c^*. Making the choice of a "path of least energy spikes" would be in our situation hopeless since we do not have an uniqueness assumption that would allow to make such a selection in a continuous way. Lemma 2.1 shows that the original linking situation of V remains respected through deformations of this "energetically rough" path at the level of the functional J_ε. We may call this approach a *variational finite dimensional reduction*. The moral is perhaps that in the study of this type of point-concentration phenomena in the presence of variational structure, linking in the finite dimensional guiding energy may be seen transmitted to the functional counting only with very rough information, in opposition to the fine facts needed for the Lyapunov-Schmidt reduction procedure.

Proof of Lemma 2.2. Let us prove part (i). To establish the upper estimate in (2.21), let us consider the test path $p_\varepsilon(x)$ defined in (2.15). Clearly we have

$$C_\varepsilon \le \sup_{x\in B} J_\varepsilon(p_\varepsilon(x)).$$

For $x \in B$ we have

$$J_\varepsilon(p_\varepsilon(x)) = \varepsilon^N \left\{ \int\limits_{B(0,R_\varepsilon)} |\nabla w_{c^*}|^2 + V(\varphi_\varepsilon(x) + \varepsilon y) w_{c^*}^2 - \int\limits_{B(0,R_\varepsilon)} F(w_{c^*}) + o(\varepsilon) \right\},$$

where $R_\varepsilon = -k \log \varepsilon$, with k large enough. Here we use the exponential decay of w_{c^*} and we notice that, since B is closed and $B \subset \Lambda$, for ε small enough the test path p_ε does not touch the penalization. We have that

$$V(\varphi_\varepsilon(x) + \varepsilon y) = V(\varphi_\varepsilon(x)) + \varepsilon \nabla V(\varphi_\varepsilon(x)) \cdot y + O(\varepsilon^2 |y|^2).$$

Then, using the radial symmetry of w_{c^*}, its exponential decay, and (2.14) we find that

$$\int\limits_{B(0,R_\varepsilon)} V(\varphi_\varepsilon(x) + \varepsilon y) w_{c^*}^2 \le c^* \int\limits_{B(0,R_\varepsilon)} w_{c^*}^2 + o(\varepsilon). \tag{2.22}$$

From here we obtain

$$J_\varepsilon(p_\varepsilon(x)) \le \varepsilon^N \{I_{c^*}(w_{c^*}) + o(\varepsilon)\},$$

and then we finally get the desired upper bound

$$J_\varepsilon(p_\varepsilon(x)) \le \varepsilon^N (b^{c^*} + o(\varepsilon)).$$

A similar argument yields the validity of the estimate in part (ii). Next we prove the lower bound in (2.21). For this purpose we define the *center of mass* of a function in $L^2(\mathbb{R}^N)$ which is not identically zero as the quantity

$$\beta(u) = \frac{\int\limits_{\Lambda^+} x u^2(x)dx}{\int\limits_{\mathbb{R}^N} u^2(x)dx},$$

where Λ^+ is a small neighborhood of $\bar{\Lambda}$. In order to obtain the lower bound, we will first study the auxiliary minimization problem

$$m_\varepsilon = \mathrm{Inf}\{J_\varepsilon(u)/u \in \mathcal{M}_\varepsilon, \beta(u) = z_\varepsilon\}, \tag{2.23}$$

where $z_\varepsilon \in \Lambda$. Standard arguments yield that problem (2.23) has indeed a minimizer u_ε. We scale this function and define

$$w_\varepsilon(y) = u_\varepsilon(z_\varepsilon + \varepsilon y).$$

The functions w_ε have a uniform bound in H^1. Since $u_\varepsilon \in \mathcal{M}_\varepsilon$, an argument of concentration-compactness type (see for instance Lemma 3.1 below) gives that

$$\lim_{R\to\infty} \int_{\mathbb{R}^N \setminus B(0,R)} |\nabla w_\varepsilon|^2 + V(z_\varepsilon + \varepsilon y) w_\varepsilon^2 = 0, \tag{2.24}$$

uniformly in ε, and that every sequence of $\{w_\varepsilon\}$ has a weakly convergent subsequence, whose limit w is nonzero. Dichotomy is excluded thanks to the minimizing character of u_ε and the fact that the variation of V on Λ below the level c^* can be assumed small enough. Given $R > 0$ we have

$$\varepsilon^{-N} \int_{\mathbb{R}^N} \varepsilon^2 |\nabla u_\varepsilon|^2 + V(x) u_\varepsilon^2 \ge \int_{B(z_\varepsilon, R)} |\nabla w_\varepsilon|^2 + V(z_\varepsilon + \varepsilon y) w_\varepsilon^2 \tag{2.25}$$

and

$$-\varepsilon^{-N} \int_{\mathbb{R}^N} G(x, u_\varepsilon) \ge -\int_{B(z_\varepsilon, R)} F(w_\varepsilon) - \int_{\mathbb{R}^N \setminus B(z_\varepsilon, R)} F(w_\varepsilon). \tag{2.26}$$

By (2.24) we have that

$$\int_{\mathbb{R}^N \setminus B(z_\varepsilon, R)} F(w_\varepsilon) = o_R(1),$$

uniformly in ε. Let us take a sequence $\varepsilon_n \to 0$ so that, dropping the dependence of n in the notation, $z_\varepsilon \to z$ and $V(z_\varepsilon) \to V(z)$. Using (2.25) y (2.26) we obtain

$$\liminf_{\varepsilon\to 0} \varepsilon^{-N} J_\varepsilon(u_\varepsilon) \geq I_{V(z)}(w) + o_R(1).$$

Consequently

$$\liminf_{\varepsilon\to 0} \varepsilon^{-N} m_\varepsilon = \liminf_{\varepsilon\to 0} \varepsilon^{-N} J_\varepsilon(u_\varepsilon) \geq I_{V(z)}(w) \geq b^{V(z)}. \tag{2.27}$$

Now we complete the argument. We define the function $\varphi_{\varepsilon,t} : B \to \Lambda$ as $\varphi_{\varepsilon,t}(x) = \beta(\eta_\varepsilon(x,t))$. We see that $\varphi_{\varepsilon,t} : B \to \Lambda^+$, so that it does not quite belong to Γ. However, we see from the linking assumption, that if the class Γ^+ is defined in the same way as Γ, but allowing the test paths to take values in Λ^+, then the min-max value c^* remains unchanged if we replace Γ by Γ^+. Thus, for every $\varepsilon > 0$ we can find a point $x_\varepsilon \in B$ such that

$$V(\varphi_{\varepsilon,t}(x_\varepsilon)) \geq c^*.$$

Taking $z_\varepsilon = \varphi_{\varepsilon,t}(x_\varepsilon)$ we have that for every $t \geq 0$

$$\sup_{x\in B} J_\varepsilon(\eta_\varepsilon(x,t)) \geq J_\varepsilon(\eta_\varepsilon(x_\varepsilon,t)) \geq m_\varepsilon,$$

and then the result follows from (2.27). $\square$

3. Estimates on the flow

The purpose of this section is to prove the following basic facts of the flow in (2.17).

Proposition 3.1. (i) *The flow $\eta_\varepsilon(x,t)$ given by (2.17) is well defined for all $t \geq 0$.*
(ii) *There is a constant K independent of ε so that*

$$\|\eta_\varepsilon(x,t)\|_{H^2,\varepsilon} \leq \varepsilon^N K \quad \textit{and} \quad \|\eta_\varepsilon(x,t)\|_{L^\infty} \leq K,$$

for all $t \geq 0$ and all $x \in B$.

Before proving the proposition lets us consider the following estimate of the gradient of J_ε.

Lemma 3.2. *Given any $\alpha_0 > 0$ there are $\varepsilon_0 > 0$, $\sigma > 0$ and $\delta > 0$ such that*

$$\|\nabla J_\varepsilon(u)\|_* \leq \alpha_0$$

for all $0 < \varepsilon < \varepsilon_0$ and $u \in \mathcal{M}_\varepsilon^{b^{c^*}+\sigma}$.

Proof. It is not hard to check that $\nabla J_\varepsilon(u)$ is Lipschitz continuous over $\mathcal{M}_\varepsilon^{b^{c^*}+\sigma}$. Let us assume that the result of the lemma is not true and let u_ε be a minimizer of J_ε over $\mathcal{M}_\varepsilon$. A concentration compactness argument allows then to prove that $\varepsilon^{-N} J_\varepsilon(u_\varepsilon) \to c^* - \delta$, as $\varepsilon \to 0$. Thus, choosing $\delta + \sigma$ small enough and using that ∇J_ε is Lipschitz continuous over $\mathcal{M}_\varepsilon^{b^{c^*}+\sigma}$ we find a contradiction with $\|\nabla J_\varepsilon(u)\|_* > \alpha_0$. □

Proof of Proposition 3.1. Let us recall that the vector field $\mathcal{G}_\varepsilon$ defines a negative gradient on $\mathcal{M}_\varepsilon$. Thus, while $\eta(x,t)$ is well defined, $\eta(x,t) \in \mathcal{M}_\varepsilon^K$, for certain $K > 0$ independent of x. From Lemma 2.1 we see then that in this time interval $\eta(x,t)$ is bounded in H^1 norm, uniformly in x. On the other hand, also from that lemma we have that the number $d(\eta(t))$, given in (2.19) is well defined and bounded. Consequently $\mathcal{G}_\varepsilon$ is bounded along the flow, and thus global existence of it follows from standard ODE theory. This completes the proof of part (i).

In the proof of (ii) it will be more convenient to work with stretched variables. We consider the change of variables $x = \varepsilon y$. Let $V_\varepsilon(y) = V(\varepsilon y)$. We make this change of variables in η, p_ε, $g(x,u)$, J_ε, $\mathcal{F}_\varepsilon$, $\mathcal{G}_\varepsilon$ and $(\cdot,\cdot)_*$. We shall avoid relabelling these objects after the change of variables is made in order to keep notational simplicity. For instance now de denote

$$(v_1, v_2)_* = \int (\nabla v_1 \nabla v_2 + V(\varepsilon y) v_1 v_2) dy \tag{3.1}$$

$$J_\varepsilon(v) = \frac{1}{2} \int (|\nabla v|^2 + V(\varepsilon y) v^2) dy - \int G(\varepsilon y, v) dy.$$

First we make the vector field $\mathcal{G}_\varepsilon$ more explicit. Using the definition of the inner product (3.1) we find

$$\nabla J_\varepsilon(u) = u - A_\varepsilon g(y,u)$$

and

$$\nabla \mathcal{F}_\varepsilon(u) = 2u - A_\varepsilon(g'(y,u)u - g(y,u)),$$

where $A_\varepsilon = (\Delta - V_\varepsilon)^{-1}$. Using local elliptic estimates, and taking into account that V is bounded, we find that for $r > 1$ $A_\varepsilon : L^r \to W^{2,r}$ defines a bounded operator whose norm is bounded independently of ε. If we define

$$h(y,u) = -g(y,u) - d(u)(g'(y,u)u - g(y,u)),$$

then equation (2.17) can be written as

$$\frac{d\eta}{dt} = -(1 - 2d(\eta))\eta + A_\varepsilon h(y,\eta), \qquad \eta(0) = p_\varepsilon(x), \tag{3.2}$$

for each $x \in B$. We denote its solution below simply as $\eta(t)$, dropping the dependence on x. Since J_ε decreases along the flow, which stays on the Nehari manifold, η remains bounded in H^1, for all $t \geq 0$. This estimate is independent of x, t and ε, that is, there is a constant K such that

$$\|\eta(t)\|_{H^1} \leq K \quad \text{for all} \quad t \geq 0. \tag{3.3}$$

This global estimate implies by means of the Sobolev embeddings that for $p_0 = 2N/(N-2)$ we have

$$\|\eta(t)\|_{L^{p_0}} \leq K \quad \text{for all} \quad t \geq 0. \tag{3.4}$$

A better estimate satisfies the initial condition. In fact, $p_\varepsilon(x) \in W^{2,r}(\mathbb{R}^N)$ for all $r \geq 1$. Moreover the $W^{2,r}$ norm is independent of ε and of r in a closed, bounded interval contained in $(0, \infty)$.

Next we regard $d(\eta(t))$ as a function of t, and we define $b(t) = t - 2\int_0^t d(\tau)d\tau$. Thanks to Lemma 2.1, there is a constant K such that

$$|d(t)| = \left|\frac{(\nabla J_\varepsilon(\eta), \nabla \mathcal{F}_\varepsilon(\eta))_*}{\|\nabla \mathcal{F}_\varepsilon(\eta)\|_*^2}\right| \leq K\|\nabla J_\varepsilon(\eta)\|_*,$$

and then, assuming that an appropriate choice of δ in Lemma 3.2 has been made, we see that d satisfies $d(t) \leq 1/4$ for all $t \geq 0$. Here we note that the constant k_2 appearing in (2.10) does not depend on δ. As a consequence we have that $b(t) \geq t/2$ and that

$$e^{-b(t)} \int_0^t e^{b(\tau)}(1 + d(\tau))d\tau \leq \frac{5}{2}e^{-b(t)}(e^{b(t)} - 1) \leq \frac{5}{2}. \tag{3.5}$$

Continuing with the analysis of the flow, we use $e^{b(t)}$ as multiplier in (3.2) and we obtain that η satisfies

$$e^{b(t)}\eta(t) = \eta(0) + \int_0^t e^{b(\tau)} A_\varepsilon h(y, \eta)d\tau. \tag{3.6}$$

On the other hand, from assumptions (f1) and (f3) we get that

$$|f'(s)| \leq A|s|^{q-1} \quad \text{if } s \leq 1 \qquad \text{and} \qquad |f'(s)| \leq A|s|^{p-1} \quad \text{if } s > 1,$$

thus

$$|h(y, \eta(t))| \leq A(1 + d(t))|\eta(t)|^q \text{ if } \eta(t) \leq 1 \quad \text{and}$$
$$|h(y, \eta(t))| \leq A(1 + d(t))|\eta(t)|^p \text{ if } \eta(t) > 1.$$

Next we decompose $h(t) = h(\cdot, \eta(t))$ setting for each $t > 0$,

$$\Omega_1(t) = \{y \,/\, \eta(t)(y) \le 1\} \quad \text{and} \quad \Omega_2(t) = \{y \,/\, \eta(t)(y) > 1\},$$

and then $h_i(t) = \chi_{\Omega_i(t)} h(t)$ for $i = 1, 2$. Certainly we have $h = h_1 + h_2$. We note that if $r_1 = p_0/q$ and $r_2 = p_0/p$ then $h_i : \mathbb{R}_+ \to L^{r_i}(\mathbb{R}^N)$ is continuous and bounded for $i = 1, 2$. Here we use the global H^1 bound of the flow given in (3.3).

From (3.6) we can define the following decomposition of η as $\eta(t) = \eta_1(t) + \eta_2(t)$ with

$$e^{b(t)} \eta_i(t) = \eta_i(0) + \int_0^t e^{b(\tau)} A_\varepsilon h_i(\tau) d\tau, \tag{3.7}$$

$i = 1, 2$, where $\eta_1(0) = \eta(0)$ and $\eta_2(0) = 0$. Next we perform a bootstrap iteration. From the discussion above and the properties of the operator A_ε we have

$$\|A_\varepsilon h_i\|_{W^{2,r_i}} \le K(1 + d(t)).$$

Thus, from (3.7) we have

$$\|\eta_i(t)\|_{W^{2,r_i}} \le e^{-b(t)} \|\eta_i(0)\|_{W^{2,r_i}} + K e^{-b(t)} \int_0^t e^{b(\tau)} (1 + d(\tau)) d\tau.$$

But then, from (3.5), we conclude that $\|\eta_i(t)\|_{W^{2,r_i}} \le K$ for $i = 1, 2$. Next we use the Sobolev embedding to find $p_1 > p_0$ such that

$$\|\eta(t)\|_{L^{p_1}} \le K \quad \text{for all} \quad t \ge 0. \tag{3.8}$$

We can repeat this bootstrap procedure until obtaining

$$\|\eta(t)\|_{H^2(\mathbb{R}^N)} \le K \quad \forall t \ge 0$$

for a certain constant K. We do not do the details. With some more iterations we also get the L^∞ estimate. $\square$

4. The proof of Theorem 1.1

As we have already shown, the min-max C_ε is a critical value for the functional J_ε. Hence to complete the proof of Theorem 1.1 we only need to prove that there is a critical point of J_ε associated to this level, which is also a critical point of E_ε. Let us define

$$\mathcal{K}^{c^*} = \{x \in \Lambda \;/\; V(x) = c^*, \nabla V(x) = 0\}$$

and let Λ_0 be a small neighborhood of $\mathcal{K}^{c^*}$ such that $\operatorname{dist}(\Lambda_0, \partial\Lambda) > 0$.

The following is the main step in the proof of the theorem.

Proposition 4.1. *Let Λ_0 be as above and Λ_1 a domain such that*

$$\overline{\Lambda}_0 \subset \Lambda_1 \subset \bar{\Lambda}_1 \subset \Lambda.$$

Then there exists a positive number k such that whenever $\beta(\eta_\varepsilon(x, \bar{t})) \in \partial\Lambda_1$ for $x \in B$ and $\bar{t} \geq 0$ we have

$$J_\varepsilon(\eta_\varepsilon(x, \bar{t})) \leq \varepsilon^N (b^{c^*} - k)$$

Let us assume for the moment the validity of this result and conclude the proof of our first main result.

Proof of Theorem 1.1. From Ekeland's variational principle we can find sequences u_ε^n, $t_n \geq 0$ and $x_n \in B$ such that

$$u_\varepsilon^n \to u_\varepsilon, \quad \nabla J_\varepsilon(u_\varepsilon^n) \to 0, \quad J_\varepsilon(u_\varepsilon^n) \to C_\varepsilon \geq (b^{c^*} + o(1))\varepsilon^N$$

and

$$\text{dist}\,(u_\varepsilon^n, \eta_\varepsilon(x_n, t_n)) \to 0.$$

Thus $J_\varepsilon(\eta_\varepsilon(x_n, t_n)) \to C_\varepsilon \geq (b^{c^*} + o(1))\varepsilon^N$. Then, by Proposition 4.1, we conclude that $\beta(u_\varepsilon^n)$ is away from the boundary. This together with the decay of u_ε, being a solution of (2.5), implies that u_ε is a solution of (1.1). Choosing Λ_1 closer to Λ_0 if necessary, we can prove that $\beta(u_\varepsilon)$ belongs to a small neighborhood of Λ_0. Finally, shrinking successively the set Λ_0 towards $\mathcal{K}^{c^*}$, we get the rest of the statement of Theorem 1.1. □

It remains to prove Proposition 4.1. To this end, some lemmas are in order. The next result shows the presence of a *collar* around the critical points of V at level c^*. On this collar the gradient of V is nonzero.

Lemma 4.2. *There exist numbers $k > 0$, $\sigma > 0$ and a closed set $D \subset \Lambda_1$ such that*

(i) $\nabla V(x) \neq 0$, *for all $x \in D$,*
(ii) *If γ is a curve in Λ so that $\gamma(0) \in \Lambda_0$ and $\gamma(\bar{t}) \in \partial\Lambda_1$ with*

$$V(\gamma(\bar{t})) \leq c^* + \sigma,$$

then there exists $t_1, t_2 \in [0, \bar{t}]$ such that $\gamma(t) \in D$ for all $t \in [t_1, t_2]$ and $|\gamma(t_1) - \gamma(t_2)| \geq k$.

Proof. Let us denote

$$\mathcal{R} = \{x \in \bar{\Lambda}_1 / \nabla V(x) \neq 0\}.$$

Using hypothesis on V b) we find $\sigma > 0$ so that $L = \partial\Lambda_1 \cap \{x / |V(x) - c| \leq \sigma\} \subset \mathcal{R}$. Then we find $k > 0$ so small that $L_k = \{x \in \bar{\Lambda}_1 / k \leq \text{dist}(x, L) \leq 2k\} \subset \mathcal{R}$.

On the other hand we define $c_1 = \max_{x\in\partial\Lambda_1} V(x)$ and $c_2 = \inf_{x\in\Lambda_0} V(x)$. Then we use Sard's Lemma to find $c_1 < c_3 < c_2$ so that

$$V^{c_3} = \{x \in \bar{\Lambda}_1 / V(x) = c_3\} \subset \mathcal{R}.$$

Modifying Λ_0 if necessary, we may assume that $c^* - \sigma < c_3$. Next we decrease $k > 0$, if necessary, so that

$$L'_k = \{x \in \bar{\Lambda} / k \le \operatorname{dist}(x, V^{c_3}) \le 2k\} \subset \mathcal{R}.$$

Then, defining $D = L_k \cup L'_k$ we see that (i) is satisfied automatically. To prove (ii) we just observe that if γ is as in (ii), then it has to cross either L_k or L'_k, from where the conclusion readily follows. □

Lemma 4.3. *Given* K *there exist* $\varepsilon_0 > 0, \sigma > 0$ *and* $k > 0$ *such that the following holds. If* $0 < \varepsilon < \varepsilon_0$ *and* $w_\varepsilon \in H^2(\mathbb{R}^N)$ *satisfy*

(i) $\|w_\varepsilon\|_{H^2} \le K$,
(ii) *For* $u_\varepsilon(x) = w_\varepsilon(\frac{x}{\varepsilon})$ *we have* $u_\varepsilon \in \mathcal{M}_\varepsilon^{b^{c^*}+\sigma}$,
(iii) $\beta(u_\varepsilon) \in D$,

then

$$\|\mathcal{G}_\varepsilon(u_\varepsilon)\|_* \ge k\varepsilon.$$

Proof. Suppose the lemma is not true, then there are sequences $k_n \to 0$ and $\varepsilon_n \to 0$ such that $\beta(u_{\varepsilon_n}) \in D$ and $\|\mathcal{G}_{\varepsilon_n}(u_{\varepsilon_n})\|_* \le k_n\varepsilon_n$. This implies that $\|\nabla J_{\varepsilon_n}(u_{\varepsilon_n})\|_* \le \bar{k}_m\varepsilon_m$, with $\bar{k}_n \to 0$, as can be seen from the fact that $u_{\varepsilon_n} \in \mathcal{M}_{\varepsilon_n}^{b^{c^*}+\sigma}$ and Lemma 2.1.

Next we assume, without loss of generality, that $\beta(u_{\varepsilon_n}) \to x_0$ and we show that $\nabla V(x_0) = 0$, reaching a contradiction. Using (ii) and a concentration compactness argument we can prove that $w_{\varepsilon_n}(y) = u_{\varepsilon_n}(x_0 + \varepsilon_n y)$ converges in H^1 to a solution w of the equation

$$\Delta w - V(x_0)w + f(w) = 0.$$

Now we use $\partial w_{\varepsilon_n}/\partial x_i$ as a test function on the gradient of J_ε to find

$$\left| \int_{\mathbb{R}^N} \nabla w_{\varepsilon_n} \nabla \frac{\partial w_{\varepsilon_n}}{\partial x_i} + V(x_0 + \varepsilon y) w_{\varepsilon_n} \frac{\partial w_{\varepsilon_n}}{\partial x_i} - f(w_{\varepsilon_n}) \frac{\partial w_{\varepsilon_n}}{\partial x_i} \right| \le \bar{k}_n \varepsilon_n \|w_{\varepsilon_n}\|_{H^2}.$$

Using that $w_{\varepsilon_n} \in H^2$, we integrate by parts to obtain

$$\left| \int_{\mathbb{R}^N} \frac{\partial}{\partial x_i} V(x_0 + \varepsilon_n y) w_{\varepsilon_n}^2(y) \right| \le \bar{k}_n \|w_{\varepsilon_n}\|_{H^2},$$

and then taking limits we get $\frac{\partial}{\partial x_i} V(x_0) = 0$. Note that we have also used that V is bounded. □

Lemma 4.4. *Given $K > 0$ there are numbers $\alpha > 0$, $R > 0$ such that, given any sufficiently small ε and $u \in \mathcal{M}_\varepsilon^K$, there exists $\bar{x} \in \Lambda$ satisfying*

$$\int_{B(\bar{x},\varepsilon R)} u^2 > \varepsilon^N \alpha. \tag{4.1}$$

Proof. For the proof of this fact, it is convenient to rescale the function u defining $v_\varepsilon(y) = u(\varepsilon y)$, and $\Lambda_\varepsilon = \varepsilon^{-1}\Lambda$.

Since $u \in \mathcal{M}_\varepsilon$ and $J_\varepsilon(u) \le \varepsilon^N K$, we find From Lemma (2.1) that $\|v_\varepsilon\|_{H^1}$ is bounded below by a constant independent of ε.

On the other hand, we have the validity of the following fact: *Let $\{v_n\}$ be a bounded sequence in H^1 and $\varepsilon = \varepsilon_n \to 0$ be such that for some $R > 0$ one has*

$$\lim_{n\to\infty} \sup_{y\in\Lambda_\varepsilon} \int_{B(y,R)} |v_n|^2 = 0, \tag{4.2}$$

then $\int_{\Lambda_\varepsilon} |v_n|^{r+1} \to 0$ for each $r \in (1, \frac{N+2}{N-2})$. This is actually a slight variation of Lemma 2.18 in [6] for which the same proof applies, so we omit it.

To complete the proof, we assume by contradiction that for some $R > 0$ there are sequences $\varepsilon = \varepsilon_n \to 0$ and $\{u_n\} \in \mathcal{M}_\varepsilon$ with $J_\varepsilon(u_n) \le K\varepsilon^N$ such that for $v_n(y) = u_n(\varepsilon_n y)$ one has

$$\lim_{n\to\infty} \sup_{y\in\Lambda_\varepsilon} \int_{B(y,R)} |v_{\varepsilon,n}|^2 = 0. \tag{4.3}$$

Then by the above result, we find that $v_n \to 0$ in $L^{r+1}(\Lambda_\varepsilon)$. But this and the fact that u_n lies on Nehari's manifold imply that $v_{\varepsilon,n} \to 0$ in H^1. We have thus reached a contradiction, which proves the validity of the lemma. □

Lemma 4.5. *Given positive numbers α_1, α_2 there exist positive numbers R_0, ε_0 and k such that for all $R > R_0$, the following holds. If $0 < \varepsilon < \varepsilon_0$, $u \in \mathcal{M}_\varepsilon^{bc^*+\sigma}$ are so that*

$$\int_{B(x_i,\varepsilon R)} u^2 \ge \varepsilon^N \alpha_i, \quad i = 1, 2, \tag{4.4}$$

with $B(x_1, 2\varepsilon R) \cap B(x_2, 2\varepsilon R) = \emptyset$, then $\|\nabla J_\varepsilon(u)\|_* \ge k$.

Proof. Suppose the lemma is false. Then there are $\alpha_1 > 0$ and $\alpha_2 > 0$ such that for a given $R > 0$ we can find sequences $\varepsilon_n \to 0$, $k_n \to 0$ and $u_n \in \mathcal{M}_{\varepsilon_n}$ such that (4.4) holds for certain x_{in}, $i = 1, 2$, and $\|\nabla J_{\varepsilon_n}(u)\|_* \to 0$. We rescale

u_n defining $v_n(y) = u_n(x_{1n} + \varepsilon_n y)$. By standard arguments we prove that v_n converges in H^1, after taking a subsequence, to a non-trivial solution of the limiting equation. Here we use that the values of V do not vary much over Λ to avoid dichotomy. Now we choose $R > 0$ large enough so that for every solution z of the limiting equation centered at the origin we have

$$\int_{\mathbb{R}^N \setminus B(0,R)} z^2 \leq \frac{\alpha_2}{2}.$$

Then we get a contradiction, since on one hand $\varepsilon_n^{-1}|x_{1n} - x_{2n}| \to \infty$ an on other hand $\varepsilon_n^{-N} J_{\varepsilon_n}(u_n) \leq c^* + \sigma$. □

Proof of Proposition 4.1. A deformation argument allows us to assume that the basic path φ_ε satisfies

$$V(\varphi_\varepsilon(x)) \leq c^* - \sigma \quad \text{for all} \quad \varphi_\varepsilon(x) \notin \Lambda_0.$$

for certain small constant $\sigma > 0$. With this in mind we observe that if $\varphi_\varepsilon(x) \notin \Lambda_0$ then

$$J_\varepsilon(\eta_\varepsilon(x,t)) \leq \varepsilon^N(b^c - \sigma) \quad \text{for all} \quad t \geq 0.$$

Thus the only interesting case appears when $\varphi_\varepsilon(x) \in \Lambda_0$, and we assume this now. For simplicity we will write $\eta_\varepsilon(x,t) = \eta_\varepsilon(t)$

An application of Lemmas 4.4 and 4.5 yields the existence of $\alpha > 0$ and $R > 0$ such that if we set $\alpha_N = \alpha/2(20)^N$, then for each $t \geq 0$ there is a point $\bar{x}_t \in \Lambda$ with

$$\int_{B(\bar{x}_t,\varepsilon R)} \eta_\varepsilon(t)^2 \geq \varepsilon^N \alpha, \quad \text{and} \quad \int_{B(\bar{x},\varepsilon R)} \eta_\varepsilon(t)^2 \leq \varepsilon^N \alpha_N, \tag{4.5}$$

for all $\bar{x} \in \mathbb{R}^N$ with $|\bar{x} - \bar{x}_t| \geq 4R$. At this point we consider a slightly different notion of center of mass, which is more robust than β in front of small variations of the function far away from the center of mass. Let us consider a partition $\{\mathcal{R}_i \ / \ i \in \mathbb{N}\}$ of $\mathbb{R}^N$ consisting of cubes with side $R/10$. Let us fix points $x_i^\varepsilon \in \varepsilon\mathcal{R}_i$ for each i, and a number $\alpha > 0$. For $u \in \mathcal{M}_\varepsilon$ we let its modified center of mass be

$$\beta_R(u) = \frac{\sum_{i\in\mathbb{N}} x_i^\varepsilon (\int_{\varepsilon\mathcal{R}_i} u^2 - \varepsilon^N \alpha_N)_+}{\sum_{i\in\mathbb{N}} (\int_{\varepsilon\mathcal{R}_i} u^2 - \varepsilon^N \alpha_N)_+},$$

well defined whenever the denominator of the above quotient is non-zero. In particular this object is well defined on the functions $\eta_\varepsilon(t)$, provided that α was chosen sufficiently small. Moreover, if we set $\gamma(t) = \beta_R(\eta_\varepsilon(t))$, we see that there is a constant $M > 0$ such that

$$|\gamma_\varepsilon(t_1) - \gamma_\varepsilon(t_2)| \geq 2R\varepsilon \quad \text{implies} \quad \|\eta_\varepsilon(t_1) - \eta_\varepsilon(t_2)\| \geq \varepsilon^N M. \tag{4.6}$$

On the other hand, the result of Lemma 4.3 holds with β replaced by β_R, since $\beta - \beta_R = o(1)$ as $\varepsilon \to 0$. Thus we may assume in the remaining of the proof, that $\beta_R(\eta_\varepsilon(\bar{t})) \in \partial \Lambda_1$.

Since $\gamma_\varepsilon(0) \in \Lambda_0$, $\gamma_\varepsilon(\bar{t}) \in \partial\Lambda_1$ and $V(\gamma_\varepsilon(\bar{t})) \le c^* + \sigma$, by Lemma 4.2 there exist t_1, t_2 such that $\gamma_\varepsilon(t) \in D$ for $t \in [t_1, t_2]$ and $|\gamma_\varepsilon(t_1) - \gamma_\varepsilon(t_2)| \ge k_1$. Then we use Lemma 4.3 to find k such that

$$\|\mathcal{G}_\varepsilon(\eta_\varepsilon(t))\| \ge k\varepsilon \quad \forall t \in [t_1, t_2].$$

Thus

$$J_\varepsilon(\eta_\varepsilon(t_2)) - J_\varepsilon(\eta_\varepsilon(t_1)) = \int_{t_1}^{t_2} (\nabla J_\varepsilon(\eta_\varepsilon(t)), \dot{\eta}_\varepsilon) \le -\varepsilon k \int_{t_1}^{t_2} \|\dot{\eta}_\varepsilon(t)\| dt. \tag{4.7}$$

On the other hand, there exists a partition of the interval $[t_1, t_2]$ such that $s_0 = t_1 < s_1 < \ldots < s_{n_\varepsilon} = t_2$, $n_\varepsilon + 1 \ge k_1/2R\varepsilon$ and $|\gamma(s_{i+1}) - \gamma(s_i)| \ge 2R\varepsilon$. Then, from (4.6) we have, for all $0 \le i \le n_\varepsilon - 1$, that

$$\varepsilon^N M \le \|\eta_\varepsilon(s_{i+1}) - \eta_\varepsilon(s_i)\| \le \int_{s_i}^{s_{i+1}} \|\dot{\eta}_\varepsilon(t)\| dt. \tag{4.8}$$

Combining equation (4.8) and (4.7) we obtain a constant k such that

$$J_\varepsilon(\eta_\varepsilon(t_2)) - J_\varepsilon(\eta_\varepsilon(t_1)) \le k\varepsilon^N$$

and consequently

$$J_\varepsilon(\eta_\varepsilon(\bar{t})) - J_\varepsilon(\eta_\varepsilon(0) \le -k\varepsilon^N,$$

from where the result follows. □

5. Some preliminaries for the study of multipeak solutions

In this section we introduce some definitions and we do a preliminary analysis leading to the proof of Theorem 1.2 on the existence of multipeak solutions. Thus, we consider sets Λ_i, $i = 1, \cdots, \ell$ as in the statement of the theorem, and corresponding sets B_i, $B_{0,i}$ as in the definition of local linking, with corresponding classes of paths Γ_i and critical values c_i^*. See Definition 1.1.

Let us choose small neighborhoods Λ_i^+ of $\bar{\Lambda}_i$, in such a way that $\bar{\Lambda}_i^+ \cap \bar{\Lambda}_j^+ = \emptyset$, for all $i \ne j$. We define $\Sigma = \mathbb{R}^N \setminus \bigcup_{i=1}^{\ell} \Lambda_i^+$ and we let $\rho_i : \mathbb{R}^N \to [0, 1]$ be C^∞ functions such that

$$\rho_i(x) = \begin{cases} 1 & x \in \Lambda_i^+ \\ 0 & x \in \bigcup\limits_{i \ne j} \Lambda_j^+, \end{cases}$$

$i = 1, \ldots, \ell$, and $\sum_{i=1}^{\ell} \rho_i(x) \equiv 1$ for all $x \in \mathbb{R}^N$. We consider $\Lambda = \bigcup_{i=1}^{\ell} \Lambda_i$, and define a penalized functional J_ε as in (2.3) and the Nehari manifold as in (2.7) and (2.8). We will also consider *local Nehari manifolds*

$$\mathcal{M}_{i\varepsilon} = \{u \neq 0 \ / \ \mathcal{F}_{i\varepsilon}(u) = 0\},$$

where

$$\mathcal{F}_{i\varepsilon}(u) = J_\varepsilon'(u)(\rho_i u), \quad i = 1, ..., \ell.$$

We observe that and that all critical points of J_ε belong to all $\mathcal{M}_{i\varepsilon}$'s. These sets will be actual manifolds only at some regions. On the other hand, we clearly have that $\bigcap_{i=1}^{\ell} \mathcal{M}_{i\varepsilon} \subset \mathcal{M}_\varepsilon$.

Now construct next a suitable test path p_ε. We first choose for each $i \in \{1, \cdots, \ell\}$ a path $\varphi_{i\varepsilon} \in \Gamma_i$ such that

$$V(\varphi_{i\varepsilon}(x)) \leq c_i^* + \varepsilon^2 \quad \text{for all} \quad x \in B_i.$$

As in Sect. 2, we assume that $\varphi_{i\varepsilon}$ has been chosen so that $x \in B_i$ and $V(\varphi_{i\varepsilon}(x)) \in [c_i^*, c_i^* + \varepsilon^2]$ implies $|\nabla V(\varphi_{i\varepsilon}(x))| \leq \varepsilon^2$. Then we define

$$p_\varepsilon : \prod_{i=1}^{\ell} B_i \to \bigcap_{i=1}^{\ell} \mathcal{M}_{i\varepsilon}$$

as

$$p_\varepsilon(\mathbf{x}) = \sum_{i=1}^{\ell} w_{c_i^*}\left(\frac{\cdot - \varphi_{i\varepsilon}(x_i)}{\varepsilon}\right) \bar{\rho}_i(\cdot) t_\varepsilon(x_i),$$

where $\mathbf{x} = (x_1, \ldots, x_\ell)$, $\bar{\rho}_i$ is a cut-off function such that $\bar{\rho}_i \equiv 1$ over a small neighborhood of $\bar{\Lambda}_i$, $\operatorname{supp}(\bar{\rho}_i) \subset \Lambda_i^+$, so that in particular $\rho_i \bar{\rho}_i = \bar{\rho}_i$. $w_{c_i^*}$ is a least energy critical point of $I_{c_i^*}$ and $t_\varepsilon(x_i)$ is chosen so that $p_\varepsilon(\mathbf{x})\rho_i \in \mathcal{M}_\varepsilon$ for all $i = 1, \ell$. We observe that $p_\varepsilon(\mathbf{x}) \in \mathcal{M}_{\varepsilon i}$ for all $i = 1, ..., \ell$, by the choice of $\bar{\rho}_i$. We also note that $p_\varepsilon(\mathbf{x})(y) = 0$ for all $y \in \Sigma$.

Next we define the gradient flow for J_ε projected onto the manifolds $\mathcal{M}_{i\varepsilon}$. Unlike the single peak case, here we shall introduce an inner product depending on the point. The evolution problem defining the gradient flow will become fairly simple, allowing the extra properties we need for multipeaks.

Given a vector $\mathbf{d} = (d_1, \ldots, d_\ell) \in \mathbb{R}^\ell$ with small norm, we define the following inner product in H^1, for $\phi, \varphi \in H^1$

$$(\phi, \varphi)_{\mathbf{d}} = \int_{\mathbb{R}^N} (1 - d)\varepsilon^2 \nabla\phi \nabla\varphi + \{(1 - d)V(x) - \varepsilon^2\omega\}\phi\varphi, \tag{5.1}$$

where $d = 2\sum_{i=1}^{\ell} d_i \rho_i$, and $\omega = \sum_{i=1}^{\ell} d_i \Delta\rho_i$. We denote by $\|\cdot\|_{\mathbf{d}}$ the norm associated to $(\cdot,\cdot)_{\mathbf{d}}$. For a given function $\psi \in \cap_{i=1}^{\ell} \mathcal{M}_{i\varepsilon}$ we let $\nabla J_\varepsilon(\psi)$ and $\nabla \mathcal{F}_{i\varepsilon}(\psi)$ be the gradients of J_ε and $\mathcal{F}_{i\varepsilon}$ with respect to the inner (5.1). Then we define the vector field $\mathcal{G}_\varepsilon$ as the projection of the gradient of J_ε onto $\cap_{i=1}^{\ell} \mathcal{M}_{i\varepsilon}$. Explicitely, we let

$$\mathcal{G}_\varepsilon(\psi) = \nabla J_\varepsilon(\psi) - \sum_{i=1}^{\ell} d_i \nabla \mathcal{F}_{i\varepsilon}(\psi), \tag{5.2}$$

where $\mathbf{d} = (d_1, \dots, d_\ell)$ is chosen so that it satisfies the nonlinear system

$$\sum_{j=1}^{\ell} d_j a_{ij}(\mathbf{d}) = b_j(\mathbf{d}), \qquad j = 1, ..., \ell, \tag{5.3}$$

where

$$a_{ij}(\mathbf{d}) = (\nabla \mathcal{F}_{j\varepsilon}(\psi), \nabla \mathcal{F}_{i\varepsilon}(\psi))_{\mathbf{d}} \quad \text{and} \quad b_j(\mathbf{d}) = (\nabla J_\varepsilon(\psi), \nabla F_{j\varepsilon}(\psi))_{\mathbf{d}},$$

whenever system (5.3) is uniquely solvable. Assuming for the moment that this is the case for all ψ in certain class of functions S which contains $p_\varepsilon(\mathbf{x})$, and assuming additionally that $\mathbf{d}$ depends on ψ in a Lipschitz manner, we may consider for each given $\mathbf{x}$ the differential equation

$$\dot{\eta} = -\mathcal{G}_\varepsilon(\eta), \qquad \eta(0) = p_\varepsilon(\mathbf{x}). \tag{5.4}$$

Equivalently, while the flow is defined we have

$$(\dot{\eta}, \varphi)_{\mathbf{d}} = -(\eta, \varphi)_{\mathbf{d}} + \int_{\mathbb{R}^N} h(x, \eta)\varphi, \quad \text{for } \varphi \in H^1,$$

where

$$h(x, s) = g(x, s) - \sum_{i=1}^{\ell} d_i \rho_i (s g'(x, s) + g(s)).$$

Let us consider the divergence form operator

$$L_\varepsilon \phi = -\varepsilon^2 \mathrm{div}((1-d)\nabla\phi) + \{(1-d)V(x) - \varepsilon^2\omega\}\phi, \quad \text{for } \phi \in H^1.$$

It can be proved that the operator L_ε is invertible and that its inverse A_ε is bounded as an operator from L^r into $W^{2,r}$ for all $r > 1$. Thus (5.4) is equivalent to

$$\frac{d\eta}{dt} = -\eta + A_\varepsilon h(\eta), \qquad \eta(0) = p_\varepsilon(\mathbf{x}). \tag{5.5}$$

We point out here that this equation makes sense only if the inner product $(\cdot,\cdot)_{\mathbf{d}}$ and the operator A_ε are well defined. This is the case when $\eta(t) \in S$, where S is to be precisely defined later.

In the next lemma we study system (5.3), giving a crucial information on the projections and the flow defined by (5.5). For a set $\Omega \subset \mathbb{R}^N$ we denote by $\|\cdot\|_{\mathbf{d},\Omega}$ the norm in $H^1(\Omega)$ obtained as $\|\cdot\|_{\mathbf{d}}$ but integrating only over Ω.

Given positive numbers k_0, K and δ_1 we define now the set

$$\begin{aligned} S = \{\psi \in \cap_{i=1}^{\ell} \mathcal{M}_{i\varepsilon} \,/\, \|\psi\|_{\mathbf{0},\Sigma} \le \varepsilon^N \delta_1, \quad \|\psi\|_{\mathbf{0}} \le \varepsilon^N K, \quad \|\nabla J_\varepsilon(\psi)\|_{\mathbf{0}} \le \delta_1 \\ \text{and } \|\psi\|_{\mathbf{0},\Lambda_i^+} \ge \varepsilon^N k_0 \quad \text{for all } i = 1, \ldots, \ell\}. \end{aligned}$$

We observe that when $\mathbf{d}$ is small, the norms $\|\cdot\|_{\mathbf{d}}$ and $\|\cdot\|_{\mathbf{0}}$ are equivalent.

Lemma 5.1. *There is $\gamma_0 > 0$, $\varepsilon_0 > 0$, and $K > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, $0 < \gamma < \gamma_0$ there exists $k_0 > 0$ and $\delta_1 > 0$ so that whenever $\psi \in S$ we have that system (5.3) has a unique solution $\mathbf{d} \in B(0, \gamma)$.*

Proof. We fix k, δ_1 so that

$$\int_{\Lambda_i} f(\psi)\psi \ge k\varepsilon^N \tag{5.6}$$

for all $\psi \in S$. We notice that for any given $\sigma > 0$ there exists $\delta_1 > 0$ such that for all $\psi \in S$ we have $\operatorname{dist}(\rho_i\psi, \mathcal{M}_\varepsilon) \le \varepsilon^N\sigma$ for all i. Then (5.6) also implies that $\|\nabla\mathcal{F}_\varepsilon(\rho_i\psi)\|_{\mathbf{d}} \ge \bar{k}$, for certain $\bar{k} > 0$. Then, making δ_1 smaller if necessary, we have $\|\nabla\mathcal{F}_{i\varepsilon}(\psi)\|_d \ge \bar{k}/2$. Thus we have that for some $k_1 > 0$

$$a_{ii}(\mathbf{d}) \ge k_1 > 0 \quad i = 1, \ldots, \ell. \tag{5.7}$$

Next we divide each equation in (5.3) and we define $T : \mathbb{R}^N \to \mathbb{R}^N$ as

$$T_i(\mathbf{d}) = -\sum_{\substack{j=i \\ j\ne i}}^{\ell} \left(\frac{a_{ij}(\mathbf{d})}{a_{ii}(\mathbf{d})} d_j - \frac{b_j(\mathbf{d})}{a_{ii}(\mathbf{d})} \right) \quad i = 1, \ldots, \ell.$$

Then (5.3) is equivalent to the fixed point problem

$$\mathbf{d} = T(\mathbf{d}),$$

We will prove next that $T : \bar{B}(0,\gamma) \to \bar{B}(0,\gamma)$. We define $\phi_i = \nabla\mathcal{F}_{i\varepsilon}(\psi)$ and we see that

$$\operatorname{supp}(\phi_i) \subset \Lambda_i^+ \cup \Sigma, \quad \text{for all } i = 1, \ldots, \ell.$$

Then, for $i \ne j$, we find that

$$|a_{ij}(\mathbf{d})| \le |(\phi_i, \phi_j)_{\mathbf{d}}| \le \|\phi_i\|_{\mathbf{d},\Sigma} \|\phi_j\|_{\mathbf{d},\Sigma},$$

but

$$\|\phi_i\|_{\mathbf{d},\Sigma} = \sup_{\varphi \in H^1(\Sigma)} \frac{(\phi_i, \varphi)_{\mathbf{d}}}{\|\varphi\|_{\mathbf{d}}} \le m\|\psi\|_{\mathbf{d},\Sigma}. \tag{5.8}$$

On the other hand, for some $\bar{m} > 0$

$$|b_i(\mathbf{d})| \leq \|\nabla J_\varepsilon(\psi)\|_{\mathbf{d}} \|\nabla \mathcal{F}_{i\varepsilon}(\psi)\|_{\mathbf{d}} \leq \bar{m}\delta_1 \|\psi\|_{\mathbf{d}}. \tag{5.9}$$

Thus, we have that $T(\mathbf{d}) \in B(0, \gamma)$ for all $\mathbf{d} \in B(0, \gamma)$, if δ_1 is small enough.

To prove that T is a contraction we differentiate T and use the Mean Value Theorem. We take into account the definition of a_{ij} and b_j and we use an argument as above. We omit the details. □

Remark 5.2. Lemma 5.1 above defines a function

$$\mathbf{d} : S \subset H^1 \to \mathbb{R}^\ell,$$

which is locally Lipschitz.

6. Analysis of the gradient flow for multipeak solutions

In this section we make the analysis of the flow defined by (5.5). In particular we will prove that the flow exists for all $t \geq 0$, by proving that $\eta(t) \in S$ for all $t \geq 0$. In proving this we will obtain a close control of $\eta(t)$ away from the set Λ. It will be more convenient to work with stretched variables. Thus we change variables as $x = \varepsilon y$. We denote $V_\varepsilon(y) = V(\varepsilon y)$ and we make corresponding change of variables in η, p_ε, $g(x, u)$, J_ε, $\mathcal{M}_\varepsilon$, $\mathcal{M}_{i\varepsilon}$, $\mathcal{F}_\varepsilon$, $\mathcal{F}_{i\varepsilon}$, A_ε, L_ε, S and $(\cdot, \cdot)_{\mathbf{d}}$. For simplicity we shall keep the same notation for these objects in the new variables. In particular now we denote

$$J_\varepsilon(v) = \frac{1}{2}\int (|\nabla v|^2 + V(\varepsilon y)v^2)dy - \int G(\varepsilon y, v)dy,$$

$$(\phi, \varphi)_{\mathbf{d}} = \int_{\mathbb{R}^N} (1-d)\nabla\phi\nabla\varphi + \{(1-d)V(\varepsilon y) - \varepsilon^2\omega(\varepsilon y)\}\phi\varphi,$$

$$L_\varepsilon\phi = -\mathrm{div}((1 - d(\varepsilon y))\nabla\phi) + \{(1 - d(\varepsilon y))V(\varepsilon y) - \varepsilon^2\omega\}\phi.$$

Let us fix a point $\mathbf{x}$ and let $T > 0$ be so that the solution $\eta(t)$ of (5.5) exists and $\eta(t) \in S$ for all $t \in [0, T]$. Such a $T > 0$ indeed exists. In the next lemma we establish the positivity of η.

Lemma 6.1. *If the number γ_0 in Lemma 5.1 is fixed sufficiently small, then $\eta(t) \geq 0$ for all $t \in [0, T]$.*

Proof. If γ_0 is chosen small enough, then $|d|$ is small, thus $h(y, s) \geq 0$ for all $s \in \mathbb{R}$. Here we note that $h(y, s) = 0$ if $s \leq 0$ and we use hypothesis (f2). Multiplying the equation

$$L_\varepsilon\psi = h(\eta)$$

by $\psi_- = \min\{\psi, 0\}$ and integrating, we easily conclude that $\psi = A_\varepsilon h(\eta) \geq 0$. Now we consider in H^1 the problem

$$(e^t\eta)' = e^t A_\varepsilon h(\eta), \qquad \eta(0) = p_\varepsilon(\mathbf{x}),$$

whose unique solution satisfies $(e^t\eta)' \geq 0$. Noting that $p_\varepsilon(\mathbf{x}) \geq 0$ in $\mathbb{R}^N$ we find then that $\eta(t) \geq 0$ in $\mathbb{R}^N$, for all $t \in [0, T]$. □

Next we obtain an estimate from above for the flow. As we have already mentioned, for each $r > 1$ L_ε has a bounded inverse $A_\varepsilon : L^r \to W^{2,r}$ whose norm is bounded independently of ε as follows from local elliptic estimates. Reproducing the proof of Proposition 3.1 with only minor changes, we can find a constant K such that $\|\eta(t)\|_{L^\infty} \leq K$ for all $t \in [0, T]$. Let us consider the function $\psi_\varepsilon = KA_\varepsilon\chi_{\Lambda_\varepsilon}$, where $\Lambda_\varepsilon = (1/\varepsilon)\Lambda$.

Lemma 6.2. *For each sufficiently small σ there is a function $w_\varepsilon(y)$ and positive constants b and C such that*

$$w_\varepsilon - \sigma A_\varepsilon w_\varepsilon \geq \psi_\varepsilon \tag{6.1}$$

and

$$p_\varepsilon(\mathbf{x})(y) \leq w_\varepsilon(y) \leq Ce^{-b\,\mathrm{dist}\,(y,\Lambda_\varepsilon)} \quad \textit{for all } y \in \mathbb{R}^N, \tag{6.2}$$

for all $\mathbf{x} \in \prod \mathbf{B_i}$.

Proof. For a fixed small $\sigma > 0$, we consider the fundamental solution ϱ of

$$-\Delta\varrho + \sigma\varrho = \delta_0$$

in $\mathbb{R}^N$. Then set

$$w_\varepsilon(y) = M\int_{\Lambda_\varepsilon} \varrho(y-z)dz,$$

for $M > 0$ also fixed. It is easily checked that for M chosen sufficiently large, estimates (6.2) hold for w_ε. On the other hand,

$$L_\varepsilon w_\varepsilon \geq -(1 - d(\varepsilon y))\Delta w_\varepsilon + \nabla_y d(\varepsilon y)\nabla w_\varepsilon + \alpha w_\varepsilon,$$

for certain $\alpha > 0$. Now, the asymptotic behavior of $\nabla\varrho$ is dominated by that of ϱ, and $\nabla_y d(\varepsilon y)$ gets small as ε does. It follows that

$$(1-d)^{-1}(L_\varepsilon w_\varepsilon - \sigma w_\varepsilon) \geq -\Delta w_\varepsilon + \frac{\alpha}{2}w_\varepsilon \geq M\chi_{\Lambda_\varepsilon}.$$

Thus choosing $\sigma \leq \frac{\alpha}{2}$ the conclusion follows from the monotonicity of A_ε. □

Lemma 6.3. *$\eta(t)(y) \le w_\varepsilon(y)$
and*

$$|\nabla_y \eta(t)(y)| \le m_1 \exp(-m_2 dist(y, \Lambda_\varepsilon)), \tag{6.3}$$

for all $y \in \mathbb{R}^N$, $t \in [0, T]$.

Proof. We first choose $\sigma \ge f'(a)$ so that

$$h(y, \eta(t)) \le \sigma\eta(t) + K\chi_{\Lambda_\varepsilon}(y), \text{ for all } y \in \mathbb{R}^N,$$

where K has been enlarged if necessary and we have used the L^∞ estimate of $\eta(t)$. Next we define

$$\varphi = h(y, \eta) - \sigma\eta - K\chi_{\Lambda_\varepsilon} \quad \text{and} \quad \vartheta = w_\varepsilon - \sigma A_\varepsilon w_\varepsilon - \psi_\varepsilon.$$

We note that $A_\varepsilon\varphi - \vartheta \in H^1$ and $A_\varepsilon\varphi - \vartheta \le 0$ for all $(y, t) \in \mathbb{R}^N \times [0, T]$. We consider next the ordinary differential equation in H^1

$$\dot\mu = -\mu + \sigma A_\varepsilon\mu + A_\varepsilon\varphi - \vartheta, \qquad \mu(0) = \eta(0) - w_\varepsilon. \tag{6.4}$$

Certainly the function $\eta(t) - w_\varepsilon$ is the solution (6.4). On the other hand the equation

$$\dot\mu = -\mu + \sigma A_\varepsilon\mu_- + A_\varepsilon\varphi - \vartheta, \qquad \mu(0) = \eta(0) - w_\varepsilon, \tag{6.5}$$

has a unique solution, say $\bar\mu$. Since $\bar\mu$ also satisfies

$$\frac{d}{dt}e^t\bar\mu = e^t(\sigma A_\varepsilon\bar\mu_- + A_\varepsilon\varphi - \vartheta) \le 0 \quad \text{and} \quad \bar\mu(0) \le 0,$$

we find that $\bar\mu(t) \le 0$. But then $\bar\mu$ satisfies (6.4), which also has a unique solution, and consequently $\eta(t) \le w_\varepsilon$ for all $t \in [0, T]$. To prove (6.3) we just use elliptic estimates in the equation

$$L_\varepsilon \frac{d}{dt}(e^t\eta) = e^t h(y, \eta)$$

then we integrate. □

The following lemma states that the flow η exists for all time.

Lemma 6.4. $\sup\{\tau \ge 0 \,/\, \eta(t) \in S$ *for* $t \in [0, \tau]\} = +\infty$.

Proof. Assume the supremum above is $T < +\infty$. Then, using Lemma 6.3 we find that

$$\operatorname{dist}(\rho_i\eta(t), \mathcal{M}_\varepsilon) \le m_1 e^{-\frac{m_2}{\varepsilon}}, \tag{6.6}$$

for certain m_1 and m_2. Then, assuming the constant k_0 in Lemma 5.1 has been chosen small enough, the use of (5.6) yields

$$\|\eta(t)\|_{\mathbf{d}, \Lambda_i^+} \ge k_0 \qquad \forall t \in [0, T].$$

Since the other inequalities defining S are also kept true we obtain a contradiction with the definition of T. □

7. Existence of multi-peak solutions: proof of Theorems 1.2

In this section we complete the proof of Theorem 1.2. We use the same idea as in Sect. 4, after we know that along the flow we keep ℓ bumps, thanks to the exponential control of the flow obtained in Lemma 6.3.

Proof of Theorem 1.2 Let η be the solution of (5.4) that exists for all $t \geq 0$. We consider the min-max value

$$C_\varepsilon = \inf_{t\geq 0} \sup_{\mathbf{x}\in\prod B_i} J_\varepsilon(\eta(p_\varepsilon(\mathbf{x}), t)). \tag{7.1}$$

We first show that C_ε is a critical value for J_ε. In this direction we see that estimating the energy of test path p_ε with computations as in Sect. 2 we have

$$J_\varepsilon(\eta(t)) \leq \varepsilon^N \sum_{i=1}^{\ell} (b^{c_i^*} + o(\varepsilon)), \tag{7.2}$$

and also that there exists $\sigma > 0$ such that

$$\sup_{\mathbf{x}\in\partial\Pi B_i} J_\varepsilon(p_\varepsilon(\mathbf{x})) \leq \varepsilon^N \sum_{i=1}^{\ell} (b^{c_i^*} - \sigma). \tag{7.3}$$

Next we need an Intersection Lemma which reads as follows:

Lemma 7.1.

$$\sup_{\mathbf{x}\in\prod B_i} J_\varepsilon(\eta(p_\varepsilon(\mathbf{x}), t)) \geq \varepsilon^N \sum_{i=1}^{\ell} (b^{c_i^*} + o(1)), \qquad \textit{for all } t \geq 0. \tag{7.4}$$

Proof. Using Lemma 6.3 and the fact that $\eta(t) \in S$, we find k such that

$$\|\eta(t)\|_{H^1(\Lambda_i^+)} \geq \varepsilon^N k \qquad \forall t \in [0, T],$$

for all $i = 1, \ldots, \ell$. We define a *local* center of mass, for $u \in L^2(R^N)$ and $1 \leq i \leq \ell$

$$\beta_i(u) = \frac{\int_{\Lambda_i^+} xu^2(x)dx}{\int_{\mathbb{R}^N} u^2(x)\rho_i(x)dx}.$$

We observe that the local center of mass are well defined for $\eta(t)$, for all t, thus we may consider the function $\varphi : \prod B_i \times [0, \infty) \to \prod \Lambda_i^+$ defined as

$$\varphi_i(\mathbf{x}, t) = \beta_i \eta(p_\varepsilon(\mathbf{x}), t)) \in \Lambda_i^+.$$

If $t = 0$ then $\eta(p_\varepsilon(\mathbf{x}), 0)) = p_\varepsilon(\mathbf{x})$ and consequently φ is homotopic to the identity.

We are interested in finding, for each $t \geq 0$ and $0 < \varepsilon \leq \varepsilon_0$ a point $\mathbf{x} \in \prod B_i$ such that

$$V(\varphi_i(\mathbf{x}, t) \geq c_i^*. \tag{7.5}$$

This can be achieved solving the systems of equations

$$f_i(\mathbf{x}) \equiv q_i(\varphi_i(\mathbf{x}, t) - O_i) = 0, \tag{7.6}$$

where $q_i : R^N \to S_i$ is the orthogonal projection onto S_i. Here we use the notation S_i, T_i for the subspaces of R^N and O_i for the point in Λ_i in Definition 1.2. The function $f : \prod B_i \times [0, \infty) \to \prod S_i$ is homotopic to the identity and it satisfies

$$f(\mathbf{x}, t) \neq 0 \qquad \text{for all } \mathbf{x} \in \partial \prod B_i \qquad t \geq 0.$$

This last fact is a consequence of (7.3) and an argument as in Lemma 2.2, estimate (2.27). Then, using degree theory we obtain a solution $\bar{\mathbf{x}}$ of system (7.6).

Next, considering $z_\varepsilon = \varphi_i(\bar{\mathbf{x}}, t)$ and use the argument in Lemma 2.2 again we obtain

$$\sup_{\mathbf{x} \in \prod B_i} J_\varepsilon(\eta(t)\rho_i) \geq \varepsilon^N (b^{c_i^*} + o(1))$$

for all $1 \leq i \leq \ell$, and then

$$J_\varepsilon(\eta(t)) \geq \varepsilon^N \sum_{i=1}^{\ell} (b^{c_i^*} + o(1)),$$

finishing the proof of the lemma. □

Estimates (7.3) and (7.4) imply that C_ε is a critical value. All we have to prove is that associated to C_ε there is a critical point of J_ε which is also a critical point of the original functional. But this can be achieved by an argument as in the case of a single peak. Without presenting all details we just point out that, as a consequence of Lemma 6.3, η can be decomposed as

$$\eta(t) = \sum_{i=1}^{\ell} \eta_i(t) + r(\varepsilon)$$

where $\eta_i(t)$ satisfies $\mathrm{supp}(\eta_i) \subset \Lambda_i^+$ and $\mathrm{dist}(\eta_i(t), \mathcal{M}_\varepsilon) \leq \varepsilon^N m_1 e^{-\frac{m_2}{\varepsilon}}$. The error term $r(\varepsilon)$ is supported in Σ and satisfies

$$|r(\varepsilon)(y)| \leq m_1 e^{-\frac{m_2}{\varepsilon} dist(y, \Lambda)} \quad \text{for all } y \in \mathbb{R}^N.$$

We conclude from here that, except for an exponentially small error, each η_i can be treated is in the case of a single peak. Then the ideas used in the proof of Theorem 1.1 can be used to prove that there is a critical point associated to C_ε that does not touch the penalization. □

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