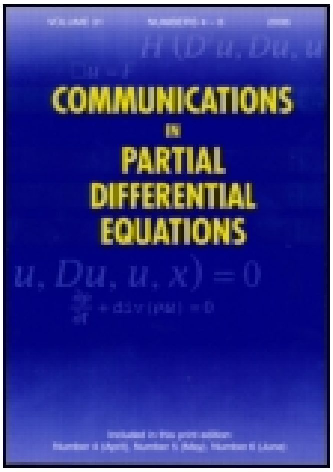

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ASYMPTOTIC BEHAVIOR OF BEST CONSTANTS AND EXTREMALS FOR TRACE EMBEDDINGS IN EXPANDING DOMAINS*

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1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$. For any $p > 1$, with $p \leq N/(N-2)$ if $N \geq 3$, we have the validity of the Sobolev trace embedding of $H^1(\Omega)$ into $L^{p+1}(\partial\Omega)$, namely there exists a positive constant S such that

$$S\|u\|^2_{L^{p+1}(\partial\Omega)} \leq \|u\|^2_{H^1(\Omega)}$$

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for all $u \in H^1(\Omega)$. The *best constant* for this embedding is the largest S for which the above relation holds, namely the number $S(\Omega)$ defined as

$$S(\Omega) = \inf_{u \in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 + u^2}{\left(\int_{\partial\Omega} |u|^{p+1}\right)^{\frac{2}{p+1}}}$$

Moreover, if $1 < p < N/(N-2)$ the embedding is compact which translates into existence of *extremals* for it, namely, functions u at which this infimum is achieved.

Let us fix $1 < p < N/(N-2)$ and a bounded smooth domain Ω. For a large positive number λ we consider the family of expanding domains

$$\Omega_\lambda = \lambda\Omega = \{\lambda x \mid x \in \Omega\}.$$

Our purpose in this paper is to describe the asymptotic behavior as $\lambda \to +\infty$ of the best constants $S(\Omega_\lambda)$ as well as that of the associated family of extremals u_λ. In what follows we shall denote by u_λ an extremal normalized so that the relation

$$\int_{\Omega_\lambda} |\nabla u_\lambda|^2 + u_\lambda^2 = \int_{\partial\Omega_\lambda} |u_\lambda|^{p+1}$$

holds. Of course, from the homogeneity of the Raleigh quotient, this relation is achieved by multiplying an arbitrary minimizer by an appropriate constant. This normalization is convenient, since the corresponding Euler-Lagrange equation satisfied by u_λ reduces simply to

$$\begin{aligned} &\Delta u_\lambda - u_\lambda = 0 \quad &&\text{in } \Omega_\lambda, \\ &\frac{\partial u_\lambda}{\partial \nu} = |u_\lambda|^{p-1} u_\lambda \quad &&\text{on } \partial\Omega_\lambda. \end{aligned} \tag{1.1}$$

Standard regularity theory and strong maximum principle then tell us that u_λ is smooth up to the boundary and strictly one signed. Thus, we assume from now on that $u_\lambda > 0$ in $\overline{\Omega}_\lambda$.

It is worth mentioning that problem 1.1 becomes, once conveniently scaled back to the variable in Ω,

$$\begin{aligned} &\varepsilon^2 \Delta u - u = 0 \quad &&\text{in } \Omega \\ &\frac{\partial u}{\partial \nu} = u^p &&\text{on } \partial\Omega \end{aligned}$$

where $\varepsilon = 1/\lambda$. This type of nonlinear boundary value condition arises in reaction-diffusion equations in heat transfer, chemical reactions and population dynamics, see [8] and references therein. For the associated heat flow of this problem, smallness of ε may be interpreted as slow diffusivity of the measured quantity.

Associated to problem (1.1) is the *energy functional*

$$J_\lambda(u) = \frac{1}{2}\int_{\Omega_\lambda} \left(|\nabla u|^2 + u^2\right) dy - \frac{1}{p+1}\int_{\partial\Omega_\lambda} u_+^{p+1}\, d\sigma. \tag{1.2}$$

Positive solutions of 1.1 are precisely the critical points of J_λ in $H^1(\Omega_\lambda)$. Now, it is straightforward to check that u_λ is a critical point of J_λ at the energy level

$$J_\lambda(u_\lambda) = c_\lambda \equiv \inf\{J_\lambda(u) \;/\; u \neq 0 \text{ is a critical point of } J_\lambda\},$$

namely, u_λ is a (positive) *least-energy solution* of 1.1 Moreover,

$$c_\lambda = \frac{p-1}{2p+2} S(\Omega_\lambda)^{\frac{p+1}{p-1}}.$$

Thus, our problem is reduced to studying the asymptotic behaviour of least-energy solutions of 1.1 and their associated energies.

If we stand at a point of $\partial\Omega_\lambda$ and we let $\lambda \to \infty$ then we will see the domain to become a half space which, after a convenient rotation and translation, may be assumed to be $\mathbb{R}^N_+ = \{(x', x_N) \mid x_N > 0\}$. Thus, it is natural to suspect that $S(\Omega_\lambda)$ converges to the corresponding quantity for the half-space, and u_λ to an associated extremal. Our first result states that $S(\Omega_\lambda)$ indeed approaches $S(\mathbb{R}^N_+)$, corrected by a negative factor of the maximum mean curvature of $\partial\Omega$.

Theorem 1.1. *There is a constant $\gamma = \gamma(p, N) > 0$ such that the following expansion holds*

$$S(\Omega_\lambda) = S(\mathbb{R}^N_+) - \lambda^{-1}\gamma \max_{x\in\partial\Omega} H(x) + o(\lambda^{-1}),$$

as $\lambda \to +\infty$. Here $H(x)$ denotes the mean curvature of the boundary at the point x.

Our second result says that, as expected, the extremals u_λ constitute a *single bump* at the boundary, whose shape is asymptotically that of an extremal for the half-space embedding. This bump is centered (in the Ω-coordinates) around a point of maximum mean curvature of $\partial\Omega$.

Theorem 1.2. *Let $y^\lambda \in \partial\Omega_\lambda$ be a maximum point of u_λ. Then $x^\lambda = \lambda^{-1}y^\lambda \in \partial\Omega$ satisfies*

$$H(x^\lambda) \to \max_{x\in\partial\Omega} H(x)$$

as $\lambda \to \infty$. Also, there are constants $\alpha, \beta > 0$ such that

$$u_\lambda(y) \le \alpha \exp\{-\beta\,|y - y^\lambda|\},$$

for all $y \in \Omega_\lambda$. Besides, given a sequence $\lambda_n \to \infty$ there is a subsequence, still denoted the same way, an extremal w of $S(\mathbb{R}^N_+)$ and a rotation Q such that

$$\sup_{y\in\Omega_{\lambda_n}} |u_{\lambda_n}(y) - w(Q(y - y_{\lambda_n}))| \to 0.$$

as $n \to \infty$.

The results here stated have a similar analogue for best constant and extremals of the usual embedding of $H^1(\Omega)$ into $L^{q+1}(\Omega)$. In fact, let us now fix $1 < q < (N+2)/(N-2)$ and define

$$C(\Omega_\lambda) = \inf_{u\in H^1(\Omega)\setminus\{0\}} \frac{\int_{\Omega_\lambda} |\nabla u|^2 + u^2}{\left(\int_{\Omega_\lambda} |u|^{q+1}\right)^{2/(q+1)}}.$$

Let us select a family of minimizers $u_\lambda \ge 0$ with

$$\int_{\Omega_\lambda} |\nabla u_\lambda|^2 + u_\lambda^2 = \int_{\Omega_\lambda} |u_\lambda|^{q+1},$$

then $u_\lambda > 0$ in Ω_λ and solves

$$\begin{aligned} &\Delta u_\lambda - u_\lambda + u_\lambda^q = 0 \quad &&\text{in } \Omega_\lambda,\\ &\frac{\partial u_\lambda}{\partial \nu} = 0 &&\text{on } \partial\Omega_\lambda. \end{aligned} \tag{1.3}$$

Moreover, u_λ is a least-energy solution of (1.3). These solutions and their asymptotic behavior have been analyzed by Lin, Ni and Takagi in [5], [6] and [7]. They find exactly the same: an expansion for $C(\Omega_\lambda)$ similar to Theorem 1.1 and the fact that u_λ is a single bump, whose maximum lies on the boundary, near a point of largest mean curvature, in the same sense as Theorem 1.2 states.

There is, however, an important difference between the situation treated in this paper and that just described which comes from the different nature of the *limiting equations*. In fact for Problem (1.1) it is important to

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understand least energy solutions of

$$\begin{aligned} &\Delta w - w = 0 && \text{in } \mathbb{R}^N_+, \\ &\frac{\partial w}{\partial \nu} = w^p && \text{on } \partial\mathbb{R}^N_+, \\ &w > 0 && \text{in } \mathbb{R}^N_+, \end{aligned} \tag{1.4}$$

while for problem (1.3) the corresponding equation is

$$\begin{aligned} &\Delta w - w + w^q = 0 && \text{in } \mathbb{R}^N_+, \\ &\frac{\partial w}{\partial \nu} = 0 && \text{on } \partial\mathbb{R}^N_+, \\ &w > 0 && \text{in } \mathbb{R}^N_+, \end{aligned} \tag{1.5}$$

In problem (1.5) it is well known that positive solutions are radially symmetric around some point, and that the radial solution is unique, a result due to Kwong, [4]. In the approach devised in the cited works this fact is of fundamental importance. Not only this, a by-product of Kwong's uniqueness proof is its *nondegeneracy up to translations*, namely the fact that only linear combinations of the partial derivatives of w lie in the kernel of the linearized operator. On the other hand, the proof of this fact relies on a delicate analysis for the ODE satisfied for the radial solution of (1.5). Problem (1.4) instead is of a different nature. As we shall see, a least energy solution is radial in the tangential variables but certainly not globally, so that an ODE approach to establish uniqueness or some form of nondegeneracy seems in principle hopeless. This condition in the framework of (1.5) for more general nonlinearities has been lifted in the work [3], and we borrow some of the ideas there employed in the last section. The scheme of the paper is as follows. In § 2 we work out a further variational characterization of extremals of the Sobolev embedding and establish existence of such extremals for the problem in the half space. In § 3 we work out uniform estimates for least energy solutions. Of special importance is their uniform exponential decay away from their maximum points. Finally, in § 4 we exploit the variational characterization of these solutions to conclude the proof of the main results.

2. EXISTENCE OF LEAST-ENERGY SOLUTIONS

Let $\Omega \leq \mathbb{R}^N$ be a bounded smooth domain or $\Omega = \mathbb{R}^N_+$. Let us consider the best Sobolev constant for the embedding of $H^1(\Omega)$ into $L^{p+1}(\Omega)$,

$$S(\Omega) = \inf_{u \in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 + u^2}{\left(\int_{\partial\Omega} |u|^{p+1}\right)^{2/(p+1)}} \tag{2.1}$$

with $1 < p < N/(N-2)$. As we have mentioned in the introduction, it is convenient for our purposes to obtain a further characterization of this value and its extremals in terms of the energy functional

$$J_\Omega(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + u^2)\,dx - \frac{1}{p+1}\int_{\partial\Omega} u_+^{p+1}\,d\sigma, \tag{2.2}$$

where $d\sigma$ is the surface measure on $\partial\Omega$. It is standard to check that J_Ω is of class C^1 in $H^1(\Omega)$, that it has a mountain-pass structure, and that its nontrivial critical points correspond precisely to the solutions of the problem

$$\begin{aligned} &\Delta u - u = 0 && \text{in } \Omega,\\ &\frac{\partial u}{\partial \nu} = u^p && \text{on } \partial\Omega,\\ &u > 0 && \text{in } \Omega. \end{aligned} \tag{2.3}$$

$C^\infty(\overline{\Omega})$-smoothness of the solutions of (2.3) follows for instance from general regularity results by Amann in [1]. Let us consider the number

$$c_\Omega \equiv \inf_{\substack{u\in H^1(\Omega)\\ u\neq 0}} \sup_{t>0} J_\Omega(tu). \tag{2.4}$$

It is easy to see that if $u_+ \neq 0$, the function $t \mapsto J_\Omega(tu)$ has a maximum $t = \bar{t} > 0$ which is its unique critical point. Then $\bar{t}u \in M_\Omega$, where

$$M_\Omega = \left\{ u \in H^1(\Omega)/u \neq 0,\ \int_\Omega [|\nabla u|^2 + u^2]\,dx = \int_{\partial\Omega} u_+^{p+1}\,d\sigma \right\}, \tag{2.5}$$

is the so-called *Nehari's manifold* of J_Ω. It follows from this fact that

$$c_\Omega = \inf_{u\in M_\Omega} J_\Omega(u).$$

Since all nontrivial solutions of (2.3) lie in M_Ω, the above number is called the *least energy* value for J_Ω and a solution u of (2.3) with $J_\Omega(u) = c_\Omega$, a *least energy solution*. These solutions and extremals of $S(\Omega)$ are related in the following way: if u is a least energy solution, then it is a extremal of $S(\Omega)$. Reciprocally, if $\bar{u} \geq 0$ minimizes the Raleigh quotient (2.1), then $u = t\bar{u}$ is a least energy solution of (2.3) where

$$t^{p-1} = \frac{\int_\Omega |\nabla \bar{u}|^2 + \bar{u}^2}{\int_\Omega \bar{u}^{p+1}}.$$

In fact we always have the exact relation

$$c_\Omega = \frac{p-1}{2p+2} S(\Omega)^{\frac{p+1}{p-1}}.$$

As we have mentioned, for Ω bounded the compactness of the associated trace embedding yields the existence of extremals for $S(\Omega)$ and correspondingly of critical points of J_Ω at level c_Ω. The existence of extremals when $\Omega = \mathbb{R}^N_+$ is less obvious, but it is still true. To establish this, we shall make use of a concentration-compactness type argument borrowed from Coti-Zelati and Rabinowitz, Lemma 2.18 in [2], applied to a suitable P.S. sequence.

Proposition 2.1. *There exists a critical point of $J_{\mathbb{R}^N_+}$ at the level $c_{\mathbb{R}^N_+}$.*

For the proof we will make use of two preliminary results which we state and prove next.

Lemma 2.1. *There exists a sequence u_m in $M_{\mathbb{R}^N_+}$ such that*

$$J_{\mathbb{R}^N_+}(u_m) \searrow c_{\mathbb{R}^N_+}, \quad \text{and} \quad DJ_{\mathbb{R}^N_+}(u_m) \to 0. \tag{2.6}$$

Furthermore, u_m can be chosen satisfying

$$\int_{\mathbb{R}^{N-1}} (u_m)_+^{p+1} d\sigma > a \tag{2.7}$$

for all m, with $a > 0$ a positive constant.

Proof: The existence of u_m in $M_{\mathbb{R}^N_+}$ satisfying (2.6) is a consequence of Ekeland's variational principle applied to a minimizing sequence of $J_{\mathbb{R}^N_+}$ over $M_{\mathbb{R}^N_+}$. Now, for m sufficiently large,

$$\frac{c(\mathbb{R}^N_+)}{2} \le J_+(u_m) - DJ_+(u_m)u_m = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^{N-1}} (u_m)_+^{p+1}\, d\sigma,$$

which shows (2.7). $\square$

The following Lemma rules out the posibility that u_m asymptotically vanishes.

Lemma 2.2. *Let* u_m *be the sequence of Lemma 2.1. For* $y' \in \mathbb{R}^{N-1}$ *put* $\Gamma(y') = \{ x' \in \mathbb{R}^{N-1} \colon |x' - y'| \leq 1 \}$. *Then for each* $r \in]2, (2N-2)/(N-2)[$, *there exists* $a > 0$ *and* $\alpha \in]0, 1[$ *such that*

$$\|(u_m)_+\|_{L^r(\mathbb{R}^{N-1})} \leq a \left\{ \sup_{y' \in \mathbb{R}^{N-1}} \int_{\Gamma(y')} (u_m)_+^2 \, d\sigma \right\}^{\alpha/2}. \tag{2.8}$$

In particular

$$\sup_{y' \in \mathbb{R}^{N-1}} \int_{\Gamma(y')} (u_m)_+^2 \, d\sigma \geq b \tag{2.9}$$

for some $b > 0$.

Proof: Let us fix $r \in]2, \frac{2N-2}{N-2}[$. If $y' \in \mathbb{R}^{N-1}$ then by Hölder's inequality

$$\|u\|_{L^r(\Gamma(y'))} \leq \|u\|^{1-\alpha}_{L^2(\Gamma(y'))} \|u\|^{\alpha}_{L^{(2N-2)/(N-2)}(\Gamma(y'))}$$

for each $u \in H^1(\mathbb{R}^N_+)$, where $\alpha = (r - 2/r)(N-1)$. For $r \geq 2N/(N-1)$ we have $\alpha r \geq 2$ and by the limiting trace embedding $H^1(B(y')_+) \hookrightarrow L^{(2N-2)/(N-2)}(\Gamma(y'))$, where

$$B(y')_+ = \left\{ x \in \mathbb{R}^N_+ \ / \ |x - y'| \leq 1 \right\},$$

we obtain a constant C independent of y' with

$$\int_{\Gamma(y')} u^r \, d\sigma \leq C' \left(\sup_{y' \in \mathbb{R}^{N-1}} \int_{\Gamma(y')} u^2 \, d\sigma \right)^{(1-\alpha)r/2} \|u\|^{\alpha r - 2} \int_{B(y')_+} (u^2 + |\nabla u|^2) \, dx. \tag{2.10}$$

Now we choose a family $\{B(y_i')_+\}$ whose union covers $\mathbb{R}^{N-1}$ and such that each point of $\mathbb{R}^{N-1}$ is contained in at most k such balls. Summing up inequalities (2.10) over this family, we find that

$$\begin{aligned}
\int_{\mathbb{R}^{N-1}} u^r \, d\sigma &\leq kC \left(\sup_{y' \in \mathbb{R}^{N-1}} \int_{\Gamma(y')} u^2 \, d\sigma \right)^{(1-\alpha)r/2} \|u\|^{\alpha r - 2} \int_{\mathbb{R}^N_+} (u^2 + |\nabla u|^2) \, dx \\
&= kC \left(\sup_{y' \in \mathbb{R}^{N-1}} \int_{\Gamma(y')} u^2 \, d\sigma \right)^{(1-\alpha)r/2} \|u\|^{\alpha r}.
\end{aligned}$$

This yields (2.8) by setting $u = (u_m)_+$. If $2 < r < 2N/(N-1)$, the result follows from Hölder's inequality and the case just established. Finally, (2.9) is an inmediate consequence of Lemma 2.1 and (2.8) with $r = p+1$. $\square$

The invariance by tangencial translations of $\mathbb{R}^N_+$ and the above lemmas allow us to build up a relatively compact P.S. sequence for $J_{\mathbb{R}^N_+}$ at the level $c_{\mathbb{R}^N_+}$ which does not vanish uniformly. This implies the existence of a least-energy solution of (2.3) in $\Omega = \mathbb{R}^N_+$.

Proof of Proposition 2.1. First we note that by (2.9) there exists a sequence u_m in $M_{\mathbb{R}^N_+}$ satisfying (2.6) and such that

$$0 < \frac{b}{2} \leq \int_{\Gamma(y'_m)} u_m^2 \, d\sigma$$

for certain points $y'_m \in \mathbb{R}^{N-1}$. Thus, if we consider the translations $u_m(\cdot + y'_m)$, this new sequence of functions, which we denote the same way, satisfies (2.6) and

$$\frac{b}{2} \leq \int_{\Gamma(0)} u_m^2 \, d\sigma \tag{2.11}$$

for all $m \in \mathbb{N}$. This is the sequence with which we will work. Since u_m is bounded in $H^1(\mathbb{R}^N_+)$, we may assume that $u_m \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N_+)$. It is clear that u is a critical point of $J_{\mathbb{R}^N_+}$. It remains to show that $J_{\mathbb{R}^N_+}(u) = c(\mathbb{R}^N_+)$. The embeddings

$$H^1(\Omega) \hookrightarrow L^2(\Omega), \qquad H^1(\Omega) \hookrightarrow L^r(\partial\Omega \cap \mathbb{R}^{N-1})$$

are compact for all $r < (2N-2)/(N-2)$, provided Ω bounded. It follows that for each compactly supported smooth function η on $\mathbb{R}^N$ one has

$$\begin{aligned}
&\int_{\mathbb{R}^N_+} \{\eta|\nabla u_m|^2 + \eta u_m^2\} \, dx \\
&\quad = \int_{\mathbb{R}^N_+} \nabla u_m (u_m \nabla \eta) \, dx + \int_{\mathbb{R}^{N-1}} \eta u_m^{p+1} \, d\sigma - DJ_{\mathbb{R}^N_+}(u_m)\eta u_m \\
&\quad = \int_{\mathbb{R}^N_+} \nabla u (u \nabla \eta) \, dx + \int_{\mathbb{R}^{N-1}} \eta u^{p+1} \, d\sigma + o(1) \\
&\quad = -DJ_{\mathbb{R}^N_+}(u)\eta u + \int_{\mathbb{R}^N_+} \{\eta|\nabla u|^2 + \eta u^2\} \, dx + o(1).
\end{aligned}$$

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Thus we get that

$$\int_{\mathbb{R}^N_+} \{\eta|\nabla u_m|^2 + \eta u_m^2\}\, dx \to \int_{\mathbb{R}^N_+} \{\eta|\nabla u|^2 + \eta u^2\}\, dx.$$

From (2.11) we conclude that $u \neq 0$ and, furthermore, since $J_{\mathbb{R}^N_+}(u_m) = (1/2 - 1/(p+1)) \int_{\mathbb{R}^N_+} (|\nabla u_m|^2 + u_m^2)\, dx$, we obtain $J_{\mathbb{R}^N_+}(u) = c_{\mathbb{R}^N_+}$, as desired. □

3. ESTIMATES ON DECAY

First we use a Moser's iteration procedure to prove an important L^∞-estimate for the solutions of (1.1).

Lemma 3.1. *Let v be a solution of (1.1). Then there are constants $B = B(\Omega, p, N)$ and $\theta = \theta(\Omega, p, N)$, independent of $1 \leq \lambda < \infty$, such that*

$$\|v\|_\infty \leq BJ_\lambda(v)^\theta. \tag{3.1}$$

Proof: Let $\alpha > 1$. Multiplying (1.1) by $v^{2\alpha-1}$ and integrating by parts we obtain

$$\frac{2\alpha - 1}{\alpha^2} \int_{\Omega_\lambda} |\nabla v^\alpha|^2\, dy + \int_{\Omega_\lambda} v^{2\alpha}\, dy = \int_{\partial\Omega_\lambda} v^{p+2\alpha-1}\, d\sigma. \tag{3.2}$$

Hence,

$$\frac{1}{\alpha}\left[\int_{\Omega_\lambda} |\nabla v^\alpha|^2\, dy + \int_{\Omega_\lambda} v^{2\alpha}\, dy\right] \leq \int_{\partial\Omega_\lambda} v^{p+2\alpha-1}\, d\sigma. \tag{3.3}$$

Now, the trace embedding at the critical exponent $q^* = 2(N-1)/(N-2)$ tells us that

$$C \int_{\Omega_\lambda} [|\nabla z|^2 + z^2]\, dy \geq \left(\int_{\partial\Omega_\lambda} z^{q^*}\, d\sigma\right)^{2/q^*} \tag{3.4}$$

for all $z \in H^1(\Omega_\lambda)$. The constant C depends on N, Ω and λ, but it can be chosen invariant for $1 \leq \lambda < \infty$. Combining (3.3) and (3.4) we obtain the iterative inequality

$$\left(\int_{\partial\Omega_\lambda} v^{q^*}\, d\sigma\right)^{2/q^*} \leq C\alpha \int_{\partial\Omega_\lambda} v^{p+2\alpha-1}\, d\sigma. \tag{3.5}$$

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Next we consider the sequence of positive numbers α_j, $j = 0, 1, 2, \ldots,$ defined inductively as

$$p + 2\alpha_0 - 1 = q^*$$
$$p + 2\alpha_{j+1} - 1 = \alpha_j q^* , \quad \text{for } j = 0, 1, \ldots, \tag{3.6}$$

or, explicitly,

$$\alpha_j = \frac{(q^*/2)^{j+1}(q^* - p - 1) + p - 1}{q^* - 2}. \tag{3.7}$$

Note that $\alpha_j > 1$ and $\alpha_j \nearrow \infty$ as $j \to \infty$. By (3.4),

$$\begin{aligned} \left(\int_{\partial\Omega_\lambda} v^{p+2\alpha_0-1}\, d\sigma\right)^{2/q^*} &= \left(\int_{\partial\Omega_\lambda} v^{q^*}\, d\sigma\right)^{2/q^*} \\ &\leq C\int_{\Omega_\lambda} [|\nabla v|^2 + v^2]\, dx \\ &= C\left(\frac{1}{2} - \frac{1}{p+1}\right) J_\lambda(v). \end{aligned} \tag{3.8}$$

We will construct a suitable sequence of positive numbers M_j such that

$$\int_{\partial\Omega_\lambda} v^{p+2\alpha_j-1}\, d\sigma \leq M_j \tag{3.9}$$

for all $j = 0, 1, \ldots$ Inequality (3.9) gives us M_0. Assuming that (3.9) holds for j we have by (3.5) that

$$\begin{aligned} \int_{\partial\Omega_\lambda} v^{p+2\alpha_{j+1}-1}\, d\sigma &= \int_{\partial\Omega_\lambda} v^{\alpha_j q^*}\, d\sigma \\ &\leq \left(C\alpha_j \int_{\partial\Omega_\lambda} v^{p+2\alpha_j-1}\, d\sigma\right)^{q^*/2} \\ &\leq \left(C\alpha_j M - j\right)^{q^*/2}. \end{aligned}$$

Hence, (3.9) holds for M_j defined by

$$M_0 = \left(C\left(\frac{1}{2} - \frac{1}{p+1}\right) J_\lambda(v)\right)^{q^*/2} \tag{3.10}$$
$$M_{j+1} = \left(C\alpha_j M_j\right)^{q^*/2} \quad \text{for } j = 0, 1, \ldots.$$

From these relations, we can obtain constants $C_1,\ C_2$ depending only on C, p, N such that

$$M_j \le \exp\left(-C \log M_0\right) \exp\left(C_2 \alpha_{j-1}(1 + \log M_0)\right) \tag{3.11}$$

and using (3.7), (3.9),

$$\|v\|_{L^{\alpha_j q*}(\partial\Omega_\lambda)} \le (\exp\left(C - 1 \log M_0\right))^{1/\alpha_j q^*} \exp\left(C_2/q^*(1 + \log M_0)\right). \tag{3.12}$$

Letting $j \to \infty$ we obtain the result applying the maximun principle. $\square$

Next we will establish three crucial properties of solutions of the limiting problem (1.4). The first one is a uniform estimate on least energy solutions.

Lemma 3.2. *There is a constant* $C = C(p, N)$ *such that*

$$\|\nabla^j w\|_\infty \le C$$

for each least energy solution of (*1.3*), *with* $0 \le j \le 3$.

Proof: The result follows from Proposition 3.1 and standard elliptic estimates. $\square$

The second property establishes uniform exponential decay of least energy solutions.

Lemma 3.3. *If* w *is a least energy solution of* (*1.4*) *then there exists positive constants* $a, b,$ *such that*

$$|w(x)| + |\nabla w(x)| \le a \exp\left(-b|x|\right)$$

for all $x \in \mathbb{R}^N_+$.

Proof: For $\delta > 0,$ let us set $z(x', x_N) = \exp\left(-\delta x_N\right)\bar{z}(x'),$ where $\bar{z}$ is a positive solution on $\mathbb{R}^{N-1}$ of

$$\Delta\bar{z} - (1 - \delta^2)\bar{z} = 0$$

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which decays exponentially at infinity. The function z solves the linear problem

$$\begin{aligned} &\Delta z - z = 0 \quad \text{in } \mathbb{R}^N_+, \\ &\frac{\partial z}{\partial \nu} = \delta z \qquad \text{on } \mathbb{R}^{N-1}, \end{aligned}$$

and decays exponentially at infinity. Let us consider the difference $\phi = Aw - z$, with $A > 0$. We have

$$\int_{\mathbb{R}^N_+} \left(\phi_+^2 + |\nabla \phi_+|^2\right) dx = \int_{\mathbb{R}^{N-1}} (Aw^p - \delta z)\phi_+ \, dx'.$$

Now, the Sobolev trace embedding of $H^1(\mathbb{R}^N_+)$ into $L^2(\mathbb{R}^{N-1})$, gives the existence of $\gamma > 0$ such that

$$\gamma \int_{\mathbb{R}^{N-1}} \phi_+^2 \, dx' \leq \int_{\mathbb{R}^{N-1}} (Aw^p - \delta z)\phi_+ \, dx'. \tag{3.13}$$

If we fix numbers $0 < \delta < \gamma$ and $R > 0$ such that $w^{p-1}(x) \leq \delta$ for $|x| \geq R$ and we choose A such that $Aw \leq \delta z$ for $|x| \leq R$, then we find that $\phi_+ = 0$ for $|x| \leq R$. Thus by (3.13),

$$\gamma \int_{|x| \geq R} \left(\phi_+^2\right) dx' \leq \delta \int_{|x| \geq R} \phi_+^2 \, dx'.$$

This implies that $\phi_+ \equiv 0$ and gives the desired exponential decay for w. The decay of $|\nabla w|$ is proved in a similar way, by comparing the functions $z_i = D_i w$ with z, using the equation satisfied by each of the functions z_i. □

Using the above established decay, a direct application of Theorem 0.1 in [9] gives the fact that least energy solutions of the limiting problem are radially symmetric with respect to the first $N - 1$ variables.

Lemma 3.4. *Let w be a least energy solution of (1.3) in $\mathbb{R}^N_+$ which maximizes at the origin. Then, $w = w(x', x_N)$ is radially symmetric with respect to the variable $x' \in \mathbb{R}^{N-1}$, and $w_r < 0$, $r = |x'|$.*

We end this section by establishing uniform exponential decay on the least energy solutions u_λ to (1.1).

Lemma 3.5. *There exists positive constants* α, β *such that*

$$u_\lambda(y) \leq \alpha \exp(-\beta|y - y^\lambda|)$$

for all $y \in \Omega_\lambda$ *and* λ *sufficiently large. Here* y^λ *denotes any maximum point of* u_λ.

Proof: First, we will see that the functions u_λ decay uniformly at infinty. By contradiction, we assume that there exist sequences $\overline{\lambda}_n \to \infty$ and $\overline{y}^n$ such that $|y^{\overline{\lambda}_n} - \overline{y}^n| \to \infty$ and $u_{\overline{\lambda}_n}(\overline{y}_n) \geq \epsilon$ for some $\epsilon > 0$ fixed. In this case, using an appropriate cut-off function and reasoning as in the proof of Proposition 2.1, we can see that $\lim_{n\to\infty} J_{\overline{\lambda}_n}(u_{\overline{\lambda}_n}) \geq 2c(\mathbb{R}^N_+)$, a contradiction. This proves that for each $\epsilon > 0$ there exists λ_0 and $R > 0$ such that $u_\lambda(y - y^\lambda) \leq \epsilon$ for $\lambda \geq \lambda_0$ and $|y - y^\lambda| \geq R$. The desired exponential decay will be a consequence of the following

Claim: There exists $R_0 > 0$ and $\nu_0 > 0$ such that for all $R > R_0$

$$\sup_{|y-y^\lambda|\geq R} u_\lambda(y) \geq 2 \sup_{|y-y^\lambda|\geq R+\nu_0} u_\lambda(y)$$

for all λ sufficiently large.

By contradiction, let us assume that there exist sequences $R_n \to \infty$, $\overline{\lambda}_n \to \infty$, $\nu_n \to \infty$ and $\overline{y}^n \in \Omega_{\overline{\lambda}_n}$ with $|\overline{y}^n - y^{\overline{\lambda}_n}| \geq R_n + \nu_n$ such that $u_{\overline{\lambda}_n}(\overline{y}^n) > 1/2M_n$, where

$$M_n = \sup_{|y-y^{\overline{\lambda}_n}|>R_n} u_{\overline{\lambda}_n}(y).$$

From the uniform decay established above, we see that $M_n \to 0$. Let us set

$$v_n(y) = M_n^{-1} u_{\overline{\lambda}_n}(y + \overline{y}_n).$$

Then v_n is bounded, $v_n(0) > 1/2$ and satisfies $\Delta v_n - v_n - 0$ in $\Omega_{\overline{\lambda}_n} - \overline{y}_n$, with $\partial v_n/\partial \nu = M_n^{p-1} v_n^p$ on the boundary. Letting $n \to \infty$ we obtain a contradiction since v_n converges locally uniformly to a positive solution v of the limiting problem $\Delta v - v = 0$ in the entire space $\mathbb{R}^N$ or in the half-space $\mathbb{R}^N_+$, with $\partial v/\partial \nu = 0$ on $\mathbb{R}^{N-1}$. This object does not exist. Thus, the claim holds and we have the uniform exponential decay of the functions u_λ. By standard arguments involving local elliptic estimates, we obtain the validity of the same property for the derivatives of u_λ. $\square$

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4. PROOF OF THE MAIN RESULTS

In this section we will prove Theorems 1.1 and 1.2. In what follows u_λ will denote a least energy solution of (1.1), namely a critical point of J_λ defined in (1.2) such that $J_\lambda(u_\lambda) = c_\lambda = c_{\Omega_\lambda}$, see (2.4).

4.1. Upper Bound on c_λ

The first step is to obtain a good upper bound on c_λ. For this we use the characterization (2.4) that gives $c_\lambda \le \sup_{t>0} J_\lambda(t\,u)$ for all $u \in H^1(\Omega_\lambda)$. In order to find an upper bound we will construct a suitable test function. Given w a solution of the limiting equation (1.4) we consider the extension

$$\overline{w}(x', x_N) = \begin{cases} w(x', x_N) & \text{if } x_N > 0, \\ -x_N w^p(x', 0) + w(x', 0) & \text{if } x_N \le 0. \end{cases} \tag{4.1}$$

Fix $z \in \partial\Omega$. After a translation and rotation of the coordinate system we may assume that z is the origin and the inner normal to Ω at z is pointing in the direction of the positive x_N-axis. On the other hand, there exists a C^2 function $G\colon B' \to \mathbb{R}$ defined in a ball $B' = \{x' = (x_1, \ldots, x_{N-1})\colon |x'| < \alpha_0\}$, such that $G(0) = 0$, $\nabla G(0) = 0$ and

$$\begin{aligned} \partial\Omega \cap U &= \{(x', x_N)\colon x_N = G(x')\} \\ \Omega \cap U &= \{(x', x_N)\colon x_N > G(x')\}, \end{aligned} \tag{4.2}$$

where U is a neighborhood of z. Our test function (associated to the point z) will be $w_z(x) = \overline{w}(x - z)$ and if we denote by t_λ the only positive number such that

$$J_\lambda(t_\lambda \overline{w}) = \sup_{t>0} J_\lambda(t\overline{w}) \tag{4.3}$$

we have that $c_\lambda \le J_\lambda(t_\lambda \overline{w})$. By estimating the right-hand side of this inequality we will show the following fact.

Proposition 4.1. *There exists a positive constant γ, depending only on N and p, such that*

$$c_\lambda \le c(\mathbb{R}^N_+) - \frac{1}{\lambda}\gamma H(z) + o(1/\lambda) \tag{4.4}$$

as $\lambda \to \infty$, *for any* $z \in \partial\Omega$. *Here* $H(z)$ *denotes the mean curvature of* $\partial\Omega$ *at point* z.

With no loss of generality, we may assume that $H(z) \geq 0$. For the proof of this result we need the following lemma.

Lemma 4.1. *Let* w *be a least energy solution of* (*1.3*) *and*

$$g(x') = \langle D^2 G(0)x', x'\rangle.$$

Then, as $\lambda \to \infty$,

$$\begin{aligned} R_1(\lambda) := & \int_{\Omega_\lambda} (|\nabla \overline{w}|^2 + \overline{w}^2)\, dx - \int_{\mathbb{R}^N_+} (|\nabla w|^2 + w^2)\, dx \\ & = -\frac{1}{\lambda} \int_{\mathbb{R}^{N-1}} \big(|\nabla w(x',0)|^2 + w(x',0)^2\big) g(x')\, dx' + o\left(\frac{1}{\lambda}\right) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} R_2(\lambda) = & \frac{1}{p+1} \int_{\partial\Omega_\lambda} \overline{w}^{p+1}\, d\sigma - \frac{1}{p+1} \int_{\mathbb{R}^{N-1}} w^{p+1}\, dx' \\ & = -\frac{1}{\lambda} \int_{\mathbb{R}^{N-1}} w_{x_N}(x',0)^2 g(x')\, dx' + o\left(\frac{1}{\lambda}\right). \end{aligned} \tag{4.6}$$

Moreover, there exists a constant $\beta > 0$ *such that*

$$t_\lambda = 1 + \frac{1}{\lambda}\beta + o(1/\lambda). \tag{4.7}$$

Proof: Let us set $V = \Omega \cap U$ and $V_\lambda = \lambda V$. By the exponential decay of w we have

$$\begin{aligned} R_1 = & \int_{V_\lambda} (|\nabla \overline{w}|^2 + \overline{w}^2)\, dx - \int_{\mathbb{R}^N_+} (|\nabla w|^2 + w^2)\, dx + o(1/\lambda) \\ = & \int_{V_\lambda \setminus \mathbb{R}^N_+} (|\nabla \overline{w}|^2 + \overline{w}^2)\, dx - \int_{\mathbb{R}^N_+ \setminus V_\lambda} (|\nabla w|^2 + w^2)\, dx + o(1/\lambda). \end{aligned} \tag{4.8}$$

But

$$V_\lambda = \{(x', x_N) \;/\; x_N/\lambda > G(x'/\lambda)\},$$

with $x' \in B'_\lambda = \lambda B'$ (see (4.2)), and using the exponential decay of w and that $G(0) = 0,\ \nabla G(0) = 0$ we can show that

$$\int_{V_\lambda \setminus \mathbb{R}^N_+} (|\nabla \overline{w}|^2 + \overline{w}^2)\, dx \tag{4.9}$$
$$= -\frac{1}{\lambda} \int_{B'_\lambda \cap \{x/G(x'/\lambda)<0\}} (|\nabla_{x'} w(x',0)|^2 + w(x',0)^{2p} + w(x',0)^2) g(x')\, dx' + o\left(\frac{1}{\lambda}\right),$$

and

$$\int_{\mathbb{R}^N_+ \setminus V_\lambda} (|\nabla w|^2 + w^2)\, dx$$
$$= \frac{1}{\lambda} \int_{B'_\lambda \cap \{x'/G(x'/\lambda)>0\}} \left(|\nabla w(x',0)|^2 + w(x',0)^2\right) g(x')\, dx' + o\left(\frac{1}{\lambda}\right). \tag{4.10}$$

Combining (4.9), (4.9) and (4.10) we obtain (4.6), after noticing that $-w_{x_N} = w^p$ at $x_N = 0$. Estimate (4.7) is found in a similar way. Finally, to see (4.7), let us observe that, explicitly,

$$t_\lambda^{p-1} = \frac{\int_{\Omega_\lambda} (|\nabla \overline{w}|^2 + \overline{w}^2)\, dx}{\int_{\partial\Omega_\lambda} \overline{w}^{p+1}\, dx}.$$

Since w satisfies

$$\int_{\mathbb{R}^N_+} (|\nabla w|^2 + w^2)\, dx = \int_{\mathbb{R}^{N-1}} |w(x',0)|^{p+1}\, dx',$$

the desired result readily follows from estimates (4.6) and (4.7). $\square$

Proof of Proposition 4.1. Let us consider w, a least energy solution of (1.4). From Lemma 4.1 we obtain that

$$c_\lambda \le \frac{1}{2}\left(\int_{\mathbb{R}^N_+} \left(|\nabla w|^2 + w^2\right) dx + 2\beta/\lambda \int_{\mathbb{R}^N_+} \left(|\nabla w|^2 + w^2\right) dx\right.$$
$$\left. -1/\lambda \int_{\mathbb{R}^{N-1}} (|\nabla w|^2 + w^2)\, g(x')\, dx'\right) - \frac{1}{p+1}\left(\int_{\mathbb{R}^{N-1}} w^{p+1}\, dx'\right.$$
$$\left. +(p+1)\beta/\lambda \int_{\mathbb{R}^{N-1}} w^{p+1}\, dx'\right) + 1/\lambda \int_{\mathbb{R}^{N-1}} w_{x_N}^2 g(x')\, dx' + o(1/\lambda)$$

$$= c(\mathbb{R}^N_+) - 1/\lambda \int_{\mathbb{R}^{N-1}} \left(\frac{1}{2} (|\nabla w|^2 + w^2) - w_{x_N}^2 \right) g(x')\, dx' + o(1/\lambda).$$

Since w is radially symmetric with respect to x', we can compute

$$\int_{\mathbb{R}^{N-1}} \left(\frac{1}{2} (|\nabla w|^2 + w^2) - w_{x_N}^2 \right) g(x')\, dx' = (N-1)H(z) \int_0^\infty r^N E_w(r) dr,$$

where $E_w(r) = 1/2(|\nabla w(r,0)|^2 + w(r,0)^2) - w_{x_N}(r,0)^2$, $r = |x'|$. Now, we put

$$\gamma = (N-1) \max_{w \in K} \int_0^\infty r^N E_w(r)\, dr. \tag{4.11}$$

Here K designates the set of all least energy solutions of (1.4). It is not hard to check that the number γ is indeed finite and it is achieved, arguing similarly as in the proof of Proposition 2.1. Furthermore, multiplying (1.4) by $|x'|^2 w_{x_N}$ and integrating by parts, we find

$$\int_{\mathbb{R}^{N-1}} |x'|^2 \left[\frac{1}{2} (|\nabla w|^2 + w^2) - w_{x_N}^2 \right] dx' = 2 \int_{\mathbb{R}^N_+} w_{x_N} x' \cdot \nabla_{x'} w\, dx,$$

which implies that $\gamma > 0$. This proves the result. □

4.2. Lower Bound on c_λ

This is the crucial part of the proof. We consider points $y^\lambda \in \partial \Omega_\lambda$ at which u_λ maximizes and a subsequence $y^n = y^{\lambda_n}$ such that $x^n = \lambda_n y^n$ converges to $\bar{x} \in \partial \Omega$. Let us set $u_n(y) = u_{\lambda_n}(y + y^n)$. After a rotation and a translation n-dependent, we may assume that $x^n = 0$ and that Ω can be described in a fixed neighborhood U of $\bar{x}$ as the set $\{(x', x_N) \,/\, x_N > G_n(x')\}$, with G_n smooth, $G_n(0) = 0$ and $\nabla G_n(0) = 0$. We can take G_n such that G_n converges in C^2_{loc} to a corresponding parametrization at $\bar{x}$, G. We put $\Omega_n = \Omega_{\lambda_n}$, $U_n = \lambda_n U$, $U_n^+ = U_n \cap \mathbb{R}^N_+$, $U_n^0 = \lambda_n (U \cap \mathbb{R}^N_+)$, $\Gamma_n = \lambda_n (U \cap \partial \Omega)$ and $V_n = \lambda_n (U \cap \Omega)$. Ω_n can be described in the neighborhood U_n of the origin as the set $\{(y', y_N) \colon y_N > \lambda_n G_n(\lambda_n^{-1} y')\}$. Now we extend u_n

from V_n to U_n by defining $\overline{u}_n(y', y_N)$ in the following way:

$$\begin{array}{ll} u_n(y', y_N) & \text{if } y_N \geq \lambda_n G_n\left(\dfrac{y'}{\lambda_n}\right) \\ u_n\left(y', G_n\left(\dfrac{y'}{\lambda_n}\right)\right) + \left(G_n\left(\dfrac{y'}{\lambda_n}\right) - y_N\right) u_n^p\left(y', G_n\left(\dfrac{y'}{\lambda_n}\right)\right) & \text{if } y_N < \lambda_n G_n\left(\dfrac{y'}{\lambda_n}\right). \end{array} \tag{4.12}$$

Proposition 4.2. *With the notations above, we have that*

$$c_n \geq c_{\mathbb{R}^N_+} - \frac{1}{\lambda_n}\gamma H(x_n) + o(1/\lambda_n), \tag{4.13}$$

where γ is the constant defined in (4.11).

Proof: We have $c_n \geq J_{\Omega_n}(t\, u_n)$ for all $t > 0$. Thus, by the exponential decay of u_n,

$$c_n \geq J_{V_n}(t\, u_n) + o(\lambda_n^{-1}).$$

We can make the following decomposition:

$$\begin{aligned} c_n \geq\ & J_{U_n^+}(t\overline{u}_n) + \frac{1}{2}\int_{V_n \setminus \mathbb{R}^N_+} (|t\nabla u_n|^2 + |tu_n|^2)\, dx \\ & - \frac{1}{2}\int_{U_n^+ \setminus V_n} (|t\nabla \overline{u}_n|^2 + |t\overline{u}_n|^2)\, dx - \frac{1}{p+1}\int_{\Gamma} n(tu_n)^{p+1}\, d\sigma \\ & + \frac{1}{p+1}\int_{U_n^0} (t\,\overline{u}_n)^{p+1}\, d\sigma + o(\lambda_n^{-1}). \end{aligned} \tag{4.14}$$

Now, let us choose $t = t_n$ in a such way that

$$J_{U_n^+}(t_n\overline{u}_n) = \sup_{t>0} J_{U_n^+}(t\,\overline{u}_n).$$

Passing to a subsequence, we may assume that $u_n \to w$ in the H^1-sense, where w is a least-energy solution of (1.4). Since

$$c_{\mathbb{R}^N_+} = \inf_{v \in H^1(\mathbb{R}^N_+)} \sup_{t>0} J_+(tv),$$

it is not hard to check that $t_n \to 1$ and $J_{U_n^+}(t_n\overline{u}_n) \geq c(\mathbb{R}_+^N) + o(\lambda_n^{-1})$. From these facts, it follows that

$$C_n \geq c(\mathbb{R}_+^N) + R_1(n) + R_2(n) + o(\lambda_n^{-1}), \tag{4.15}$$

where

$$2R_1(n) = \int_{V_n\setminus\mathbb{R}_+^N} (|t_n\nabla u_n|^2 + |t_n u_n|^2)\,dx - \int_{U_n^+\setminus V_n} (|t_n\nabla\overline{u}_n|^2 + |t_n\overline{u}_n|^2)\,dx \tag{4.16}$$

and

$$(p+1)R_2(n) = -\int_{\Gamma_n} (t_n u_n)^{p+1}\,d\sigma + \int_{U_n^0} (t_n\overline{u}_n)^{p+1}\,d\sigma. \tag{4.17}$$

Now, since $\overline{u}_n \to w$ C^1-locally, with uniform exponential decay, $G_n \to G$, and $t_n \to 1$, we can do the following computation:

$$\begin{aligned}
&\int_{U_n^+\setminus V_n} (|t_n\nabla\overline{u}_n|^2 + |t_n\overline{u}_n|^2)\,dy \\
&\quad = \int_{B_n'\cap\{y'/G_n(y'/\lambda_n)>0\}} \int_0^{\lambda_n G_n(y'/\lambda_n)} (|\nabla_{y'}u_n|^2(1 - py_N u_n^{p-1})^2 \\
&\qquad + u_n^{2p} + (u_n - y_N u_n^p)^2)\,dy_n\,dy' + o(\lambda_n^{-1}) \\
&\quad = \int_{B_n'\cap\{y'/G(y'/\lambda_n)>0\}} (\lambda_n G(y'/\lambda_n)\big[\,|\nabla_{y'}w(y',0)|^2 + w(y',0)^{2p} + w(y',0)^2\,\big] \\
&\qquad - \lambda_n^2 G(y'/\lambda_n)^2\big[\,pw(y',0)^{p-1}|\nabla_{y'}w(y',0)|^2 + w(y',0)^p\,\big] \\
&\qquad + \lambda_n^3/3G(y'/\lambda_n)^3[p^2w(y',0)^{2p-2}|\nabla_{y'}w(y',0)|^2 \\
&\qquad + w(y',0)^{2p}\,])\,dy' + o(\lambda_n^{-1}).
\end{aligned}$$

But $G(0) = 0$ and $\nabla G(0) = 0$, thus

$$\begin{aligned}
&\int_{U_n^+\setminus V_n} (|t_n\nabla\overline{u}_n|^2 + |t_n\overline{u}_n|^2)\,dy \\
&\quad = \frac{1}{\lambda_n} \int_{B_n'\cap\{y'/\,G(y'/\lambda_n)>0\}} \Big(|\nabla_{y'}w(y',0)|^2 + w(y',0)^{2p} + w(y',0)^2\Big) g(y')\,dy' \\
&\qquad + o\left(\frac{1}{\lambda_n}\right).
\end{aligned}$$

where $g(y') = \langle D^2 G(0)y', y'\rangle$. Similarly, we can prove that

$$\int_{V_n h \mathbb{R}^N_+} (|t_n \nabla u_n|^2 + |t_n u_n|^2)\, dy$$
$$= -\frac{1}{\lambda_n} \int_{B'_n \cap \{y'/G(y'/\lambda_n)<0\}} (|\nabla w|^2 + w^2) g(y')\, dy' + o\left(\frac{1}{\lambda_n}\right) \tag{4.18}$$

and we conclude from here that

$$R_1(n) = -1/\lambda_n \int_{\mathbb{R}^{N-1}} \big(|\nabla w|^2 + w^2\big) g(y')\, dy' + o\left(\frac{1}{\lambda_n}\right). \tag{4.19}$$

We can analyze the quantity R_2 in the same fashion and then conclude the result, similarly as in the proof of Proposition 4.1. □

We are now ready to finish the proofs of Theorems 1.1 and 1.2. Combining Propositions 4.1 and 4.2, we have

$$J_\lambda(u_\lambda) \geq c_{\mathbb{R}^N_+} - \frac{1}{\lambda} \gamma H(y^\lambda) + o(\lambda^{-1}),$$

$$J_\lambda(u_\lambda) \leq c_{\mathbb{R}^N_+} - \frac{1}{\lambda} \gamma \max_{z \in \partial\Omega} H(z) + o(\lambda^{-1}).$$

This implies, $H(x_\lambda) \to \max_{z\in\partial\Omega} H(z)$. From these estimates, the proof of Proposition 4.2 and the exponential decay given by Lemma 3.5, we obtain the validity of all the assertions of the theorems. The proof is thus concluded.

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