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Boundary Concentration for Eigenvalue Problems Related to the Onset of Superconductivity

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Abstract: We examine the asymptotic behavior of the eigenvalue $\mu(h)$ and corresponding eigenfunction associated with the variational problem

$$\mu(h) \equiv \inf_{\psi \in H^1(\Omega;\mathbf{C})} \frac{\int_\Omega |(i\nabla + h\mathbf{A})\psi|^2 \, dx\, dy}{\int_\Omega |\psi|^2 \, dx\, dy}$$

in the regime $h >> 1$. Here $\mathbf{A}$ is any vector field with curl equal to 1. The problem arises within the Ginzburg–Landau model for superconductivity with the function $\mu(h)$ yielding the relationship between the critical temperature vs. applied magnetic field strength in the transition from normal to superconducting state in a thin mesoscopic sample with cross-section $\Omega \subset \mathbf{R}^2$. We first carry out a rigorous analysis of the associated problem on a half-plane and then rigorously justify some of the formal arguments of [BS], obtaining an expansion for μ while also proving that the first eigenfunction decays to zero somewhere along the sample boundary $\partial\Omega$ when Ω is not a disc. For interior decay, we demonstrate that the rate is exponential.

1. Introduction

When a superconducting sample is subjected to a large applied magnetic field it is well-known that the effect is to drive down the critical temperature below which one first detects the presence of a supercurrent. Above this critical value, the sample is said to be in its normal state, characterized by the lack of a supercurrent and the complete permeation of the sample by the applied field. Mathematically, this relation between critical temperature and applied field can be characterized as an eigenvalue problem

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using the Ginzburg–Landau model ([GL,DGP]) and it is this eigenvalue problem that is the subject of the present study.

For the particular phenomenon under investigation, the Ginzburg–Landau theory is widely viewed as an effective model. We consider the case of a thin sample of constant cross-section, immersed in an insulating medium and subjected to a constant applied magnetic field of magnitude h directed normal to the cross-section. In experiments a typical domain radius R for these samples is very small, on the order of 1 to 5 μm (cf. [BGRW, BRPVM, MGSJQVB] and the references therein). Non-dimensionalizing with respect to the lengthscale R, and using the 2-d Ginzburg–Landau energy to model the problem, the energy can be written as

$$G(\Psi, \tilde{\mathbf{A}}) = \int_\Omega \frac{1}{2}\left|(i\nabla + \tilde{\mathbf{A}})\Psi\right|^2 + \frac{\mu}{4}(|\Psi|^2 - 1)^2\,dx + \frac{\kappa^2}{2\mu}\int_{\mathbf{R}^2}\left|\nabla \times \tilde{\mathbf{A}} - h\hat{\mathbf{z}}\right|^2\,dx\,dy. \tag{1.1}$$

Here $\Omega \subset \mathbf{R}^2$ represents the cross-section of the sample (scaled to be order 1 by $\frac{1}{R}$.) The function $\Psi : \Omega \to \mathbf{C}$ is an order parameter with $|\Psi|^2$ measuring the density of superconducting electron pairs and $\tilde{\mathbf{A}} : \mathbf{R}^2 \to \mathbf{R}^2$ is the (dimensionless) induced magnetic potential (whose curl is then the induced magnetic field). The parameter κ is the dimensionless Ginzburg–Landau parameter (not to be confused with curvature which is denoted by κ later in the paper) and μ is given by

$$\mu = \frac{R^2(T_c - T)}{\xi_0^2 T_c}, \tag{1.2}$$

where T is temperature, T_c is the critical temperature in the absence of any applied field and ξ_0 is a material dependent lengthscale ([BR1]).

Physically realizable states within this theory are then given by the stable critical points of G, where, for example, positivity of the second variation of G about a critical point can be used as a criterion for stability. The afore-mentioned normal state corresponds in Ginzburg–Landau theory to the critical point $\Psi = 0$, $\tilde{\mathbf{A}} = h\mathbf{A}$, where $\mathbf{A}$ is any vector field satisfying the condition

$$\nabla \times \mathbf{A} \equiv (0, 0, \mathbf{A}_x^{(2)} - \mathbf{A}_y^{(1)}) = (0, 0, 1) \quad \text{in } \Omega. \tag{1.3}$$

In other words, in the normal state, the induced magnetic field exactly matches the applied magnetic field. If we calculate the second variation of G about this state we obtain the functional

$$\delta^2 G\big((0, h\mathbf{A}); \psi, \mathbf{B}\big) = \int_\Omega |(i\nabla + h\mathbf{A})\psi|^2 - \mu\,|\psi|^2\,dx\,dy + \frac{\kappa^2}{\mu}\int_{\mathbf{R}^2}|\nabla \times \mathbf{B}|^2\,dx\,dy.$$

We then see that the normal state first loses stability when the temperature-related parameter μ drops below the value

$$\mu(h) \equiv \inf_{\psi \in H^1(\Omega)} \frac{\int_\Omega |(i\nabla + h\mathbf{A})\psi|^2\,dx\,dy}{\int_\Omega |\psi|^2\,dx\,dy}. \tag{1.4}$$

The variational problem (1.4) is the primary focus of this paper. Before describing our work on this problem let us mention some of the earlier results on the subject.

Saint-James and de Gennes [SD] considered the case where Ω is a half-plane or an infinite slab. Their formal calculation revealed a first eigenfunction for (1.4) which concentrated along the boundary with an exponentially small tail within the interior of the domain. This phenomenon is seen in experiments on the critical temperature/applied field relationship and is commonly known as “surface superconductivity”. More recently, Chapman ([C1, C2]) carried out a more detailed formal mathematical treatment of the half-plane problem as part of a general analysis of onset for decreasing fields, starting from a perturbation theory developed by Millman and Keller [MK]. Subsequently, Lu and Pan [LP1] carried out a rigorous analysis of (1.4) in all of $\mathbf{R}^2$ and in a half-plane and we will make use of several of their observations in our analysis.

An important advance was made by Bauman, Phillips and Tang ([BPT]) who analyzed the full nonlinear problem when Ω is a disc from the standpoint of bifurcation theory. By separating variables and using a highly nontrivial O.D.E. analysis, they rigorously showed that for a disc, the value of $\mu(h)$ is lowered below the half-plane value of Saint James and de Gennes by a term of order $h^{1/2}$ proportional to the curvature κ of the disc (not to be confused with the Ginzburg–Landau parameter). That is, they prove

$$\mu(h) = \lambda_1 h - \frac{\kappa}{3I_0} h^{1/2} + o(h^{1/2}) \quad \text{as } h \to \infty, \tag{1.5}$$

where λ_1 is the eigenvalue corresponding to the half-plane (see Proposition 2.2 for its definition) and where I_0 is a universal constant (see Lemma 2.3 below). Note that in light of (1.2), a smaller value of μ corresponds to a higher critical temperature. As in the case of a half-plane, they found that a first eigenfunction associated with (1.4) concentrates along the entire boundary of the disc while decaying exponentially in the interior. We should note here that in [BPT] as well as in most of the other studies quoted above, the authors non-dimensionalized the Ginzburg–Landau energy with respect to a lengthscale given by the penetration depth rather than a characteristic domain radius. Hence these results must be appropriately recast in order to make a comparison with this paper.

When Ω is not a disc, this problem has been recently studied by Bernoff and the third author in [BS]. Through the method of matched asymptotics, they formally establish that unlike the case of a disc, a first eigenfunction associated with (1.4) *does not concentrate along the entire boundary*, but rather does so near points of maximum curvature, tailing off exponentially away from these points.

The sensitivity of the concentration behavior of the first eigenfunction to even the slightest perturbation of the domain from a disc suggested by the analysis in [BS] makes the whole phenomenon quite subtle. The techniques and results in this paper, largely motivated by the above conjecture, constitute an effort towards the ultimate goal of a complete and rigorous description of the concentration behavior. In particular, our main result establishes the fact that when the domain is not a disc, the first eigenfunction *does not concentrate along the entire boundary*. It must decay to zero with large h somewhere along the boundary, while simultaneously decaying at an exponential rate inside the domain.

Theorem 1.1. *Let $\Omega \subset R^2$ be a bounded, open, simply connected domain with $\partial\Omega \in C^{3,\alpha_0}$ for some $\alpha_0 \in (0, 1)$. If $\{\Psi^h\}$ denotes a sequence of eigenfunctions corresponding to the first eigenvalue $\mu(h)$ given by* (1.4)*, normalized so that $\left\|\Psi^h\right\|_{L^\infty(\Omega)} = 1$, then there exists an $h_0 > 0$ such that for all $h \geq h_0$ we have*

$$\left|\Psi^h(z)\right| \leq c_1 e^{-c_2 h^{1/2}\,\mathrm{dist}(z,\partial\Omega)} \quad \textit{for all } z = (x, y) \in \Omega, \tag{1.6}$$

for constants c_1 and c_2 independent of h. Moreover, if Ω is not a disc, then we have

$$\lim_{h\to\infty}\left(\min_{z\in\partial\Omega}\left|\Psi^h(z)\right|\right)=0. \tag{1.7}$$

Property (1.6) is proved in Theorem 4.3 below, while property (1.7) is proved in Theorem 4.5. To get a feel for the significance of (1.7), one might compare the case of a disc to the case of a domain Ω which is nearly a disc in the sense that $\partial\Omega$ agrees with a circle except along a small arc.

A crucial step in our analysis is a complete characterization of the first eigenfunction for the associated problem on a half-plane (cf. Theorem 3.2). This result, in turn, relies heavily on an a priori exponential decay result, in the same vein as (1.6), for any eigenfunction in the half-plane (cf. Theorem 3.1).

As a by-product of our arguments we recover the expansion (1.5) for a disc and we obtain, in the case of a general domain Ω, the following sharp upper bound for $\mu(h)$:

$$\mu(h)\le\lambda_1 h-\frac{\kappa_{\max}}{3I_0}h^{1/2}+o(h^{1/2})\quad\text{as } h\to\infty, \tag{1.8}$$

(cf. Proposition 4.1) where $\kappa_{\max}$ denotes the maximum curvature of $\partial\Omega$.

In [BS] a higher order expansion for $\mu(h)$ was obtained by the method of formal matched asymptotics, namely

$$\mu(h)\sim\lambda_1 h-\frac{\kappa_{\max}}{3I_0}h^{1/2}+C_2h^{1/4},$$

where $C_2=C_2(\partial\Omega)$ is a positive constant depending on the second derivative of the curvature at the maximal point, a value assumed to be strictly negative (see Remark 4.2). The higher order term in the expansion seems to be related to the decay of the eigenfunction along the boundary. Intriguingly enough, the expected decay away from points of maximum curvature has a considerably slower rate than the decay towards the interior of Ω as given by (1.6). In particular, it is a rate which strongly depends on the geometry of the domain, as is clearly indicated by the case of a disc where no decay on the boundary takes place at all. We believe it is precisely upon this point that the subtlety of the phenomenon rests.

We should also mention some recent work by Lu and Pan ([LP2, LP3]) in which the authors consider a general smooth, bounded domain Ω (and variable magnetic field) and among other things, show that for a constant applied field, one has

$$\lim_{h\to\infty}\frac{\mu(h)}{h}=\lambda_1. \tag{1.9}$$

(Our analysis recovers this result as well, cf. Theorem 4.5.) They also show that the first eigenfunction tends to zero inside Ω for large h, though they do not obtain the exponential decay as in (1.6), nor do they capture any decay along the boundary. (See also [B, BH, BR2, GP] and [O].)

One might also ask about the effect on the critical temperature/applied field relationship when a domain is not smooth so that a maximum of curvature does not exist. That is the subject of ongoing research, but see [JRS] for a preliminary investigation.

We organize the paper as follows. In Sect. 2 we establish some preliminary lemmas. In Sect. 3 we treat the case of a half-plane, and in Sect. 4 we treat a general smooth bounded domain. The various results of Theorem 1.1 are proven here in Theorems 4.1, 4.3 and 4.5.

2. Preliminary Lemmas and Notation

2.1. Sturm–Liouville operators with a quadratic potential on a half-line. Throughout this paper, a crucial role will be played by the family of ordinary differential operators

$$L_\beta[u] \equiv -u'' + (x-\beta)^2 \quad \text{for } \beta \in \mathbf{R}. \tag{2.1}$$

We begin with a summary of some known results on L_β.

Proposition 2.1. *For any $\beta \in \mathbf{R}$ the spectrum of the operator L_β on $L^2([0,\infty))$ consists of a sequence of eigenvalues $0 < \lambda_1^\beta < \lambda_2^\beta < \ldots$ with $\lambda_k^\beta \to \infty$ as $k \to \infty$. The corresponding orthogonal sequence of eigenfunctions $\{\psi_k^\beta\}$, satisfying*

$$L_\beta[\psi_k^\beta] = \lambda_k^\beta \psi_k^\beta \quad \textit{for} \quad 0 < x < \infty, \quad (\psi_k^\beta)'(0) = 0,$$

forms a basis for $L^2([0,\infty))$.

Proof. Properties of the spectrum and the completeness of the eigenfunctions follow from the general theory of Sturm–Liouville operators on a half-line with an unbounded potential (cf. [LS], Sect. 4.7]). □

For any $\beta \in \mathbf{R}$, the eigenvalue λ_1^β can be characterized variationally through an associated Rayleigh quotient, namely

$$\lambda_1^\beta = \inf_{\phi \in H^1([0,\infty))} \frac{\int_0^\infty (\phi')^2 + (x-\beta)^2 \phi^2 \, dx}{\int_0^\infty \phi^2 \, dx}. \tag{2.2}$$

We can then consider the minimization of λ_1^β over all β. We summarize the results about this problem in the following proposition and lemma.

Proposition 2.2. *There exists a unique number β^* satisfying*

$$\lambda_1^{\beta^*} = \inf_{\beta \in \mathbf{R}} \lambda_1^\beta. \tag{2.3}$$

One finds $0 < \beta^* = \sqrt{\lambda_1^{\beta^*}}$, where $\lambda_1^{\beta^*} \approx .59$.

Proof. See [DH] and [BH]. The numerical approximation of $\lambda_1^{\beta^*}$ is discussed in [JRS].

Notation. We will henceforth denote the eigenvalues $\{\lambda_k^{\beta^*}\}$ associated with the operator L_{β^*} simply by $\{\lambda_k\}$ and the corresponding eigenfunctions $\{\psi_k^{\beta^*}\}$ will be written simply as $\{\psi_k\}$.

Lemma 2.3. *Define I_k as the k^{th} moment of the first eigenfunction ψ_1 of L_{β^*}:*

$$I_k \equiv \int_0^\infty x^k \psi_1^2 \, dx,$$

where we choose the normalization $\psi_1(0) = 1$. *Then for every positive integer* k, *one can express* I_k *in terms of* β^* *and* I_0. *In particular,*

$$I_1 = \beta^* I_0,$$
$$I_2 = \frac{3}{2}(\beta^*)^2 I_0 \ \Big(= \frac{3}{2}\lambda_1 I_0\Big), \quad \textit{and}$$
$$I_3 = \frac{1}{6} + \frac{5}{2}(\beta^*)^3 I_0.$$

This is proved in the appendix to [BS] with a slightly different scaling so we omit the proof here.

We will also need the following lemma regarding a related inhomogeneous problem.

Lemma 2.4. *For any number* $\lambda \le \lambda_1$, *for any function* $f \in L^2([0,\infty))$ *and any* $t \in \mathbf{R}$, *consider the problem*

$$\begin{aligned}\big(L_{(\beta^*-t)} - \lambda\big)[\phi] &\equiv -\phi''(x) + (x - \beta^* + t)^2\phi(x) - \lambda\phi(x) \\ &= f(x) \quad \textit{for } 0 < x < \infty,\end{aligned} \tag{2.4}$$

$$\phi'(0) = 0, \ \phi(\infty) = 0. \tag{2.5}$$

For $\lambda = \lambda_1$, *and any* $t \neq 0$, *a unique solution* $g \in C^\infty(\mathbf{R}_+ \times \mathbf{R} \setminus \{0\})$ *exists and satisfies*

$$\int_0^\infty |h(x,t)|^2 + \left|\frac{\partial}{\partial x} h(x,t)\right|^2 + \left|\frac{\partial^2}{\partial x^2} h(x,t)\right|^2 dx \le C(t)\, \|f\|^2_{L^2(\mathbf{R}_+)} \tag{2.6}$$

for $h = g$ *or* $h = \frac{\partial g}{\partial t}$, *where*

$$\sup_{|t|>a} C(t) < \infty \ \textit{for each } a > 0.$$

For any $\lambda < \lambda_1$ *and any* $t \in \mathbf{R}$, *a unique solution* $g \in C^\infty(\mathbf{R}_+ \times \mathbf{R})$ *exists and satisfies* (2.6) *with* $C(t)$ *replaced by a constant* C *independent of* t.

Proof. We will present the case $\lambda = \lambda_1$. The case $\lambda < \lambda_1$ follows along similar lines. This result is a fairly standard consequence of Fredholm theory. However, for the sake of completeness, we present the argument. Existence follows from Proposition 2.2 since λ_1 is not in the spectrum of $L_{(\beta^*-t)}$ for any nonzero t. To see an existence argument more explicitly and to establish (2.6), consider first the following variational problem on a finite interval $[0, N]$, where N is a positive integer:

$$\lambda_1^N(t) \equiv \inf_{\phi \in H^1([0,N]),\, \phi(N)=0} \frac{\int_0^N (\phi')^2 + (x - \beta^* + t)^2\phi^2\, dx}{\int_0^N \phi^2\, dx}. \tag{2.7}$$

By extending any admissible ϕ in (2.7) to be zero for $x > N$ we immediately see that

$$\lambda_1^N(t) \ge \lambda_1^{(\beta^*-t)} \ge \lambda_1, \tag{2.8}$$

where the last inequality is strict for $t \neq 0$. Now define

$$J_N(\phi) = \int_0^N (\phi')^2 + [(x - \beta^* + t)^2 - \lambda_1]\phi^2 - 2f\phi\, dx.$$

From (2.8) we easily find that J_N is bounded from below within the class $\{\phi \in H^1([0,N]) : \phi(N) = 0\}$, and then we can apply the Direct Method to obtain a minimizer u_N for each positive integer N. Standard regularity theory and continuous dependence theory show that $u_N = u_N(x,t)$ will be a smooth function of x and t. Furthermore, the variational characterization of u_N implies that

$$(\lambda_1^N(t) - \lambda_1) \int_0^N u_N^2\, dx - 2\int_0^N f u_N\, dx \le J_N(u_N) \le J_N(0) = 0,$$

so that by Cauchy–Schwartz and (2.8) we find

$$\|u_N\|_{L^2([0,N])} \le \frac{2}{\lambda_1^{(\beta^*-t)} - \lambda_1} \|f\|_{L^2([0,N])}. \tag{2.9}$$

Applying the condition $J_N(u_N) \le 0$ then immediately leads to a bound on the L^2-norm of u_N':

$$\begin{aligned}\int_0^N (u_N')^2\, dx &\le \int_0^N (1+\lambda)u_N^2 + f^2\, dx \\ &\le \frac{C}{(\lambda_1^{(\beta^*-t)} - \lambda_1)^2} \int_0^N f^2\, dx\end{aligned} \tag{2.10}$$

for some constant C independent of t and N.

Letting $\dot{u}_N = \frac{\partial u_N}{\partial t}$, we then note that $\dot{u}_N$ satisfies

$$L_{(\beta^*-t)}[\dot{u}_N] = -2(x - \beta^* + t)u_N, \quad (\dot{u}_N)'(0,t) = 0,\ \dot{u}_N(N,t) = 0. \tag{2.11}$$

We can bound the L^2-norm of the right-hand side of this O.D.E. by again using that $J_N(u_N) \le 0$ so that

$$\begin{aligned}\int_0^N 4(x-\beta^*+t)^2 u_N^2\, dx &\le 4\int_0^N \lambda_1 u_N^2 + 2 f u_N\, dx \\ &\le 4\int_0^N (1+\lambda_1)u_N^2 + f^2\, dx \\ &\le \frac{C}{(\lambda_1^{(\beta^*-t)} - \lambda_1)^2} \int_0^N f^2\, dx.\end{aligned} \tag{2.12}$$

After multiplying (2.11) by $\dot{u}_N$ and integrating by parts, we immediately obtain bounds on $\dot{u}_N$ and $(\dot{u}_N)'$ in the same manner as we did for u_N and u_N'.

In order to bound u_N'' in L^2 we note that

$$\begin{aligned}\int_0^N (u_N'')^2 &= \int_0^N (x-\beta^*+t)^2 u_N u_N'' - \lambda_1 u_N u_N'' - f u_N''\, dx \\ &= -\int_0^N (x-\beta^*+t)^2 (u_N')^2 + 2(x-\beta^*+t)u_N u_N' + \lambda_1 u_N u_N'' + f u_N''\, dx.\end{aligned}$$

Hence,

$$\|u_N''\|^2_{L^2([0,N])} \leq \int_0^N (x-\beta^*-t)^2 u_N^2 + (u_N')^2\,dx$$
$$+\lambda_1 \|u_N\|_{L^2([0,N])} \|u_N''\|_{L^2([0,N])} + \|f\|_{L^2([0,N])} \|u_N''\|_{L^2([0,N])}\,.$$

Using the inequality $ab \leq \frac{1}{\varepsilon^2}a^2 + \frac{\varepsilon^2}{4}b^2$ on the last two terms it then follows from (2.9), (2.10) and (2.12) that

$$\|u_N''\|_{L^2([0,N])} \leq \frac{C}{\lambda_1^{(\beta^*-t)} - \lambda_1} \|f\|_{L^2([0,N])}\,. \tag{2.13}$$

Such a bound on $\|(\dot{u}_N)''\|_{L^2([0,N])}$ follows in a similar manner. Consequently, we find that $u_N, u_N', u_N'', \dot{u}_N, (\dot{u}_N)'$ and $(\dot{u}_N)''$ all are bounded in $L^2([0,N])$ by the quantity

$$\frac{C}{\lambda_1^{(\beta^*-t)} - \lambda_1} \|f\|_{L^2([0,\infty))}$$

for some C independent of t and N.

Using a diagonalization argument, we then obtain a subsequential limit of $\{u_N\}$ which converges on compact subsets of $[0,\infty)$ and satisfies (2.5) as well as (2.6) in view of Proposition 2.2. □

2.2. Gauge invariance and preliminary results for plane, half-plane and disc. Much of this paper is devoted to the study of the functional

$$J_{\mathbf{A}}(\psi) = \frac{\int_\Omega |(i\nabla + \mathbf{A})\psi|^2\,dx\,dy}{\int_\Omega |\psi|^2\,dx\,dy},$$

where Ω is an open, simply connected subset of $\mathbf{R}^2$, $\mathbf{A} : \Omega \to \mathbf{R}^2$ and $\psi : \Omega \to \mathbf{C}$. As was noted in the introduction, this functional arises as the second variation of the full Ginzburg–Landau energy computed about the normal state. In the lemma below we record the gauge invariance property that $J_{\mathbf{A}}$ inherits from the full energy (1.1).

Lemma 2.5. *Given any $\psi \in H^1(\Omega;\mathbf{C})$, $\mathbf{A} \in L^2(\Omega;\mathbf{R}^2)$ and $\phi \in H^1(\Omega;\mathbf{R})$, we have*

$$J_{(\mathbf{A}+\nabla\phi)}(\psi e^{i\phi}) = J_{\mathbf{A}}(\psi).$$

Furthermore, if ψ, $\mathbf{A}$ and ϕ are smooth and ψ satisfies the equation

$$(i\nabla + \mathbf{A})^2\psi = \lambda\psi \text{ in } \Omega \quad \text{for some } \lambda \in \mathbf{R},$$

then the function $\tilde{\psi} = \psi e^{i\phi}$ satisfies the equation

$$(i\nabla + \mathbf{A} + \nabla\phi)^2\tilde{\psi} = \lambda\tilde{\psi} \text{ in } \Omega.$$

Both statements are easily proved by direct calculation.

Corollary 2.6. *Suppose* $\mathbf{A}$, $\mathbf{B} \in H^1(\Omega; \mathbf{R}^2)$ *satisfy*

$$\nabla \times \mathbf{A} \equiv (0, 0, A_x^{(2)} - A_y^{(1)}) = \nabla \times \mathbf{B} \quad in\ \Omega.$$

Then

$$\inf_{\|\psi\|_{L^2(\Omega)}=1} J_{\mathbf{A}}(\psi) = \inf_{\|\psi\|_{L^2(\Omega)}=1} J_{\mathbf{B}}(\psi).$$

We next present some results on the eigenvalue problem associated with the whole plane. These will be useful ingredients in our subsequent analysis of the problem on a half-plane and a general bounded domain.

Proposition 2.7. *Let* $\mathbf{A} : \mathbf{R}^2 \to \mathbf{R}^2$ *be any vector field satisfying* (1.3). *Then*

$$\inf_{\Psi \in H^1(\mathbf{R}^2;\mathbf{C})} \frac{\int_{\mathbf{R}^2} |(i\nabla + \mathbf{A})\Psi|^2 \, dx\, dy}{\int_{\mathbf{R}^2} |\Psi|^2 \, dx\, dy} = 1. \tag{2.14}$$

Remark 2.8. This result is well-known in the physics literature and in some form goes back to Landau. The proof we present below is in the general spirit of the proof of the same result to be found in [LP1]. Unfortunately, we found this latter proof to contain numerous errors, necessitating the presentation below.

Proof. Invoking Corollary 2.6, we fix $\mathbf{A} = \frac{1}{2}(-y, x)$. Insertion of the test function $\Psi = e^{-\frac{(x^2+y^2)}{4}}$ into the Rayleigh quotient (2.14) yields the claimed infimum of one, establishing an upper bound. Indeed, it will then follow from the lower bound of one that this choice is in fact a first eigenfunction.

To establish the lower bound, it suffices to consider any function $\Psi \in C_0^\infty(\mathbf{R}^2; \mathbf{C})$. Converting to polar coordinates, we expand $\tilde{\Psi}(r, \theta) \equiv \Psi(r\cos\theta, r\sin\theta)$ in a Fourier series as

$$\tilde{\Psi}(r, \theta) = \sum_{k=-\infty}^{\infty} u_k(r) e^{ik\theta},$$

where the smooth functions $u_k : [0, \infty) \to \mathbf{C}$ are given by

$$u_k(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\Psi}(r, \theta) e^{-ik\theta} \, d\theta. \tag{2.15}$$

Note in particular that

$$u_k(0) = 0 \quad \text{for } k \neq 0. \tag{2.16}$$

Inserting this expansion into the numerator of the Rayleigh quotient gives

$$\begin{aligned}
&\int_{\mathbf{R}^2} |(i\nabla + \mathbf{A})\Psi|^2 \, dx\, dy \\
&\quad = 2\pi \sum_{k=-\infty}^{\infty} \int_0^\infty \left(\left|u_k'(r)\right|^2 + (\frac{k}{r} - \frac{r}{2})^2 \, |u_k(r)|^2 \right) r\, dr \\
&\quad = 2\pi \sum_{k=-\infty}^{\infty} \left(\frac{\int_0^\infty \left(\left|u_k'(r)\right|^2 + (\frac{k}{r} - \frac{r}{2})^2 \, |u_k(r)|^2 \right) r\, dr}{\int_0^\infty |u_k(r)|^2 \, r\, dr} \right) \int_0^\infty |u_k(r)|^2 \, r\, dr \\
&\quad \equiv 2\pi \sum_{k=-\infty}^{\infty} J_k(u_k) \int_0^\infty |u_k(r)|^2 \, r\, dr.
\end{aligned}$$

Denote by $\mathcal{A}_k = \{u \in C_0^1([0,\infty);\mathbf{R}) : u(0) = 0 \text{ if } k \neq 0\}$. Since all coefficients in J_k are real we invoke (2.16) to find

$$J_k(u_k) \geq \inf_{\mathcal{A}_k} \left\{ \frac{\int_0^\infty \left(|u_k'(r)|^2 + (\frac{k}{r} - \frac{r}{2})^2 |u_k(r)|^2 \right) r\,dr}{\int_0^\infty |u_k(r)|^2 r\,dr} \right\}$$
$$\geq \inf_{\mathcal{A}_k} \left\{ \frac{2\int_0^\infty (\frac{k}{r} - \frac{r}{2}) u(r) u'(r) r\,dr}{\int_0^\infty |u(r)|^2 r\,dr} \right\} = 1,$$

where the last equality results from an integration by parts. Consequently,

$$\int_{\mathbf{R}^2} |(i\nabla + \mathbf{A})\Psi|^2\,dx\,dy \geq 2\pi \sum_{k=-\infty}^{\infty} \int_0^\infty |u_k(r)|^2 r\,dr = \int_{\mathbf{R}^2} |\Psi|^2\,dx\,dy.$$

We conclude that the infimum of the Rayleigh quotient in (2.14) is greater than or equal to one; hence it equals one, completing the proof. □

Proposition 2.9 (Cf. [LP1], Prop. 2.3**).** *Let* $\mathbf{A} : \mathbf{R}^2 \to \mathbf{R}^2$ *be any vector field satisfying* (1.3). *Let* $\lambda < 1$. *Then the only bounded,* C^2 *solution to*

$$(i\nabla + \mathbf{A})^2\Psi = \lambda\Psi \text{ on } \mathbf{R}^2$$

is $\Psi \equiv 0$.

The proof of Proposition 2.9 is an elementary contradiction argument utilizing the product of Ψ and a smooth cut-off function in the Rayeigh quotient. Taking the support of the cut-off function to be larger and larger, one contradicts Proposition 2.7. We also mention here the following result relating the one-dimensional eigenvalue problem on a half-line to the two-dimensional problem on a half-plane.

Proposition 2.10 (Cf. [LP1], Theorem 5.3, Step 1**).** *Let* $\mathbf{A} : \mathbf{R}^2_+ \to \mathbf{R}^2$ *satisfy* $\nabla \times \mathbf{A} = (0,0,1)$. *Then*

$$\lambda_1 = \inf_{\Psi \in H^1(\mathbf{R}^2_+;\mathbf{C})} \frac{\int_{\mathbf{R}^2_+} |(i\nabla + \mathbf{A})\Psi|^2\,dx\,dy}{\int_{\mathbf{R}^2_+} |\Psi|^2\,dx\,dy}, \tag{2.17}$$

where λ_1 $(= \lambda_1^{\beta^*})$ *is the value arising in* (2.3).

The proof that the infimum above is less than or equal to λ_1 follows easily from using $\rho_k(y)\psi_1(x)$ as a test function in the Rayleigh quotient where $\{\rho_k\}$ is a sequence of smooth cut-off functions with expanding support. The lower bound follows by computing the value of the Rayleigh quotient for any $\Psi \in C_0^\infty(\mathbf{R}^2_+)$ in terms of the Fourier transform of Ψ in the y variable.

As a final result in this subsection, we present a description of the first eigenspace for our problem when the domain is a disc. The analysis of the disc was previously carried out in detail in [BPT] (see introduction) and the result below can be found in the opening of that paper. The proof follows readily by expressing any competitor in the eigenvalue problem as a Fourier series.

Proposition 2.11 (Cf. [BPT], Lemma 2.3**).** *Let $D \subset \mathbf{R}^2$ be a disc of radius $R > 0$. Let $\mathbf{A} : D \to \mathbf{R}^2$ be any vector field satisfying* (1.3). *Then for every positive number h, there exists a finite set of integers $\{k_{h_1}, k_{h_2}, \ldots, k_{h_N}\}$, where $N = N(h)$ such that the set of minimizers of the variational problem*

$$\inf_{\Psi \in H^1(D;\mathbf{C})} \frac{\int_D |(i\nabla + h\mathbf{A})\Psi|^2 \, dx\, dy}{\int_D |\Psi|^2 \, dx\, dy}$$

consists of the span of the set

$$\{\zeta_{k_{h_1}}(r)e^{ik_{h_1}\theta}, \zeta_{k_{h_2}}(r)e^{ik_{h_2}\theta}, \ldots, \zeta_{k_{h_N}}(r)e^{ik_{h_N}\theta}\},$$

where $\{\zeta_{k_{h_1}}, \zeta_{k_{h_2}}, \ldots, \zeta_{k_{h_N}}\}$ is the set of solutions to the $1-d$ variational problem,

$$\inf_{\zeta \in H^1([0,R])} \frac{\int_0^R \left(\left|\zeta'(r)\right|^2 + (\frac{k}{r} - \frac{hr}{2})^2 |\zeta(r)|^2\right) r\, dr}{\int_0^R |\zeta(r)|^2 r\, dr}.$$

Remark 2.12. In fact, in [BPT] it is shown that off of a discrete set of h-values, the eigenspace is one-dimensional; that is, $N(h) = 1$, but we will not need this information for our purposes.

2.3. Local coordinates near the boundary and a local representation of the magnetic potential. In Sect. 4 we will take Ω to be a bounded, simply connected domain in $\mathbf{R}^2$ with $\partial\Omega \in C^{3,\alpha_0}$ for some $\alpha_0 \in (0, 1)$. We will frequently need to work in a local coordinate system valid near $\partial\Omega$. For this purpose we let s denote arclength along the boundary with some point $z_0 \in \partial\Omega$ chosen to correspond to $s = 0$. We let η denote distance from a point $z \in \overline{\Omega}$ to $\partial\Omega$. We will generally denote the curvature of the boundary by $\kappa(s)$ though occasionally, where no confusion can result, we will also write $\kappa = \kappa(z)$ for $z \in \partial\Omega$.

This local coordinate system will be well-defined in the rectangle

$$S \equiv \{(s, \eta) : -L/2 < s < L/2,\ 0 < \eta < \delta\},$$

where L denotes the arclength of the boundary, and $\delta < \frac{1}{\kappa_{\max}}$ is a positive constant depending on Ω. We adopt the convention that for Ω a disc, κ is positive. We denote by $\mathbf{t} = \mathbf{t}(s)$ a unit tangent vector to $\partial\Omega$ and we let $\mathbf{n} = \mathbf{n}(s)$ denote the inner unit normal vector. Thus, in particular, any vector field $\mathbf{F}$ defined in a neighborhood of $\partial\Omega$ can be expressed as $\mathbf{F}(s, \eta) = F_1(s, \eta)\mathbf{t}(s) + F_2(s, \eta)\mathbf{n}(s)$.

As we will be computing various derivatives in these new coordinates, it will be helpful to record here the following identities. For any scalar-valued function $f = f(s, \eta)$ one has the identities:

$$\nabla f = \partial_\eta f \mathbf{n} + \frac{1}{1 - \kappa\eta} \partial_s f \mathbf{t}, \tag{2.18}$$

$$\Delta f = \partial_{\eta\eta} f - \frac{\kappa}{1 - \kappa\eta} \partial_\eta f + \frac{1}{1 - \kappa\eta} \partial_s \Big(\frac{1}{1 - \kappa\eta}\Big) \partial_s f, \tag{2.19}$$

and for any vector field $\mathbf{F} = F_1(s, \eta)\mathbf{t} + F_2(s, \eta)\mathbf{n}$ we have

$$\operatorname{div} \mathbf{F} = \frac{1}{1-\kappa\eta}\partial_s F_1 + \frac{1}{1-\kappa\eta}\partial_\eta[(1-\kappa\eta)F_2], \tag{2.20}$$

$$\nabla \times F = \operatorname{div}(\mathrm{F}_2\mathbf{t} - \mathrm{F}_1\mathbf{n})\hat{\mathbf{z}}. \tag{2.21}$$

For much of our analysis it will be useful to find a solution to (1.3), say $\mathbf{q} = \mathbf{q}(s, \eta)$, defined on S, such that $\mathbf{q}$ satisfies the additional properties

$$\mathbf{q} \cdot \mathbf{n} = 0 \quad \text{in } S, \quad \mathbf{q}(s, 0) = 0 \quad \text{for } -\frac{L}{2} \le s \le \frac{L}{2}. \tag{2.22}$$

Seeking a solution to (1.3) and (2.22) in the form $\mathbf{q} = q(s, \eta)\mathbf{t}$, we can solve for the scalar q through the use of (2.20) and (2.21). We find that q must satisfy the first order differential equation

$$\frac{-1}{1-\kappa\eta}\partial_\eta\big((1-\kappa\eta)q\big) = 1, \quad q(s, 0) = 0,$$

so that

$$\mathbf{q}(s, \eta) = q(s, \eta)\mathbf{t}(s) = -\eta\Big(\frac{1-\kappa(s)\eta/2}{1-\kappa(s)\eta}\Big)\mathbf{t}(s). \tag{2.23}$$

We should emphasize that $\mathbf{q}$ is only locally defined near $\partial\Omega$. No such vector field could exist throughout Ω for it would violate Stokes Theorem.

3. Analysis in a Half-Plane

We denote by $\mathbf{R}^2_+$ the set $\{(x, y) : x > 0\}$. Our first goal will be to establish exponential decay of certain solutions to the eigenvalue problem

$$\Psi_{xx} + \Psi_{yy} - 2ix\Psi_y - x^2\Psi + \lambda\Psi = 0 \quad \text{in } \mathbf{R}^2_+, \tag{3.1}$$

$$\Psi_x(0, y) = 0 \quad \text{for } y \in \mathbf{R}. \tag{3.2}$$

In light of Lemma 2.5 we have taken $\mathbf{A} = (0, x)$ in writing down the equation $(i\nabla + \mathbf{A})^2\Psi = \lambda\Psi$ to obtain (3.1) .

Theorem 3.1. *For any* $\lambda < 1$, *let* Ψ *be any bounded* C^2 *solution to the problem* (3.1)–(3.2). *Then for every multi-index* α, *there exist positive constants* a_α *and* b_α *such that for all* $(x, y) \in \mathbf{R}^2_+$ *one has*

$$\left|D^\alpha\Psi(x, y)\right| \le a_\alpha e^{-b_\alpha x}. \tag{3.3}$$

Proof. Without loss of generality, we normalize so that $\|\Psi\|_{L^\infty(\mathbf{R}^2_+)} = 1$. First note that by standard elliptic theory, any C^2 solution to (3.1) is necessarily in C^∞ for $x > 0$. For any $R > 0$ and positive integer k, denote $\Omega_R^k = \{(x, y) : x > kR\}$. We will obtain exponential decay by establishing the following claim:

There exists an $R_0 > 0$ such that

$$\|\Psi\|_{L^\infty(\Omega_R^{k+1})} < \frac{1}{2}\,\|\Psi\|_{L^\infty(\Omega_R^k)} \tag{3.4}$$

for all $R \geq R_0$ and all positive integers k.

We proceed by contradiction. If claim (3.4) fails then there exists a sequence $R_j \to \infty$ and a sequence of positive integers $\{k_j\}$ such that

$$\|\Psi\|_{L^\infty(\Omega_{R_j}^{k_j+1})} \geq \frac{1}{2}\|\Psi\|_{L^\infty(\Omega_{R_j}^{k_j})}.$$

Let

$$\tilde{\Psi}_j \equiv \frac{\Psi}{\|\Psi\|_{L^\infty(\Omega_{R_j}^{k_j})}},$$

so that $\left\|\tilde{\Psi}_j\right\|_{L^\infty(\Omega_{R_j}^{k_j})} = 1$ and we can find a sequence of points (x_j, y_j) with $x_j > (k_j+1)R_j$ such that

$$\left|\tilde{\Psi}_j(x_j, y_j)\right| \geq \frac{1}{2}.$$

Now define $f^j \in C^2(B(0, R_j))$ by the formula

$$f^j(x, y) = \tilde{\Psi}_j(x_j + x, y_j + y)e^{-ix_j y}.$$

Note that

$$\left|f^j(0,0)\right| \geq \frac{1}{2}, \text{ while } \left\|f^j\right\|_{L^\infty(B(0,R_j))} \leq 1, \tag{3.5}$$

and that f^j satisfies the equation:

$$f^j_{xx} + f^j_{yy} - 2ixf^j_y - x^2 f^j + \lambda f^j = 0. \tag{3.6}$$

With an eye towards establishing compactness of the sequence $\{f^j\}$, we now fix any $\rho > 0$ and consider a smooth cut-off function $\chi \in C_0^\infty(\mathbf{R}^2)$ such that $\chi \equiv 1$ on $B(0, \rho)$, $\chi \equiv 0$ in $\mathbf{R}^2 - B(0, \rho + 1)$ and $|\nabla\chi| \leq 2$. If one multiplies (3.6) by $\chi^2\overline{f^j}$ (where $\bar{\cdot}$ denotes complex conjugation) and integrates over $B(0, \rho + 1)$, then an integration by parts yields

$$\begin{aligned}\int_{B(0,\rho+1)} \chi^2\left|\nabla f^j\right|^2 + 2\overline{f^j}\chi\nabla\chi\cdot\nabla f^j + \chi^2(2ix\overline{f^j}f^j_y + x^2\left|f^j\right|^2 \\ - \lambda\left|f^j\right|^2)\,dx\,dy = 0.\end{aligned} \tag{3.7}$$

Applying the Cauchy–Schwartz inequality to the second and third terms, and using the uniform L^∞ bound on the sequence $\{f^j\}$, we conclude that for each $\rho > 0$:

$$\int_{B(0,\rho)} \left|\nabla f^j\right|^2 dx\,dy \leq C_\rho. \tag{3.8}$$

If one then writes f^j in terms of its real and imaginary parts, $f^j = u^j + iv^j$, (3.6) becomes the uniformly elliptic system

$$\Delta u^j = -2xv^j_y + (x^2 - \lambda)u^j,$$
$$\Delta v^j = 2xu^j_y + (x^2 - \lambda)v^j,$$

and the L^2 control of the right-hand sides leads, via standard interior elliptic estimates, bootstrapping and Sobolev embedding, to an estimate of the form

$$\left\| f^j \right\|_{C^{2,\gamma}(B(0,\rho))} \leq C_\rho \tag{3.9}$$

for some $\gamma \in (0, 1)$.

In light of estimate (3.9), one can extract a subsequence $\{f^{j_k}\}$ which converges in C^2 on compact subsets of $\mathbf{R}^2$ to a limit which we denote by g. In view of (3.5), g must be a bounded, nontrivial solution to Eq. (3.1) on all of $\mathbf{R}^2$, contradicting Proposition 2.9 since $\lambda < 1$. This establishes claim (3.4).

From (3.4) we readily conclude that there exist positive a and b such that

$$|\Psi(x, y)| \leq ae^{-bx} \text{ for } x > 0. \tag{3.10}$$

It remains to establish (3.3) for multi-indices $\alpha \neq 0$. This is a consequence of manipulations similar to those used above in obtaining (3.9). Specifically, using an identity analogous to (3.7), but applied to Ψ in any ball $B \subset \mathbf{R}^2_+$ of radius 1 centered at a point (x_0, y_0), we find through the use of (3.10) that

$$\int_B |\nabla\Psi|^2 \; dx\, dy \leq C x_0^2 e^{-2bx_0}.$$

Hence, we obtain that for any multi-index α:

$$\sup_B \left| D^\alpha \Psi \right| \leq a_\alpha e^{-b_\alpha x}$$

for some positive constants a_α and b_α, using the same reasoning that led to (3.9). □

We now recall that from Proposition 2.10, we have the relation

$$\lambda_1 = \inf_{H^1(\mathbf{R}^2_+)} \frac{\int_{\mathbf{R}^2_+} |(i\nabla + \mathbf{A})\psi|^2 \; dx\, dy}{\int_{\mathbf{R}^2_+} |\psi|^2 \; dx\, dy}, \tag{3.11}$$

where, as before, $\mathbf{A} : \mathbf{R}^2_+ \to \mathbf{R}^2$ is any vector field satisfying (1.3). It is a result of [LP1] that no $L^2(\mathbf{R}^2_+)$ eigenfunction can exist corresponding to the eigenvalue λ_1. However, the analysis of the next chapter will require a complete understanding of any *bounded* solution to the associated P.D.E. To this end, we now establish

Theorem 3.2. *Let $\tilde{\Psi} \in C^2(\mathbf{R}^2_+)$ be a bounded solution to* (3.1)–(3.2). *If $\lambda = \lambda_1$, then $\tilde{\Psi}$ must take the form $\tilde{\Psi}(x, y) = c\psi_1(x)e^{i\beta^* y}$ for some complex number c, where ψ_1 is the first eigenfunction of the operator L_{β^*}. If $\lambda < \lambda_1$, then $\tilde{\Psi} \equiv 0$.*

Remark 3.3. In the preprint [LP1] one can find the same claim. However, the proof contains many gaps. As we will crucially need this result, we present below our own argument which follows very different lines.

Proof. We will first consider the case $\lambda = \lambda_1$. Let $\tilde{\Psi}$ be a smooth bounded solution to (3.1)–(3.2). Define Ψ via the gauge transformation $\Psi = \tilde{\Psi} e^{-i\beta^* y}$. This has the effect of replacing the choice $\mathbf{A} = (0, x)$ by $(0, x - \beta^*)$ so that Ψ satisfies the problem

$$-\Delta\Psi + 2i(x - \beta^*)\Psi_y + (x - \beta^*)^2\Psi = \lambda_1\Psi \quad \text{in } \mathbf{R}^2_+, \tag{3.12}$$

$$\Psi_x(0, y) = 0 \quad \text{for } y \in \mathbf{R}. \tag{3.13}$$

In light of Theorem 3.1, there exists a positive constant M such that any bounded, nontrivial smooth solution Ψ to (3.12)–(3.13) satisfies the condition

$$\left(\int_0^\infty |\Psi(x, y)|^2 \, dx \right)^{1/2} \leq M \quad \text{for each } y \in \mathbf{R}.$$

Now we will express Ψ in terms of the basis of eigenfunctions ψ_k associated with the operator L_{β^*} (cf. Sect. 2.1). Thus, we write Ψ as

$$\Psi(x, y) = \sum_{k=1}^{\infty} w_k(y)\psi_k(x), \tag{3.14}$$

where for this proof we will take each ψ_k to have L^2-norm 1. (In other parts of this paper we favor the normalization $\psi_1(0) = 1$.)

Then the smooth functions $w_k : \mathbf{R} \to \mathbf{R}$ are given by

$$w_k(y) = \int_0^\infty \Psi(x, y)\psi_k(x) \, dx \tag{3.15}$$

and by Cauchy–Schwartz we have

$$|w_k(y)| \leq M \quad \text{for all } y \in \mathbf{R} \quad \text{and all positive integers } k. \tag{3.16}$$

As a consequence of (3.16), each w_k defines a tempered distribution on $\mathbf{R}$ and as such we can take its Fourier transform, $\hat{w}$. The main content of the proof is the following claim:

Claim 1.

$$\operatorname{supp} \hat{w}_k \subset \{0\} \quad \text{for each } k.$$

We delay for a moment the proof of this claim and demonstrate how the proof of the theorem is completed once Claim 1 is established. It follows from elementary distribution theory that for each k,

$$\hat{w}_k = \sum_{i=1}^{N_k} c_i^k \delta_0^{(i)}$$

for some positive integer N_k and constants c_i^k (where $\delta_0^{(i)}$ denotes the i^{th} derivative of the Dirac distribution with support $\{0\}$). But this implies that each w_k is a polynomial of degree N_k and so as a consequence of (3.16), we find that for each k,

$$w_k \equiv d_k$$

for some constant d_k. In particular, we see that Ψ is independent of y. But then $\Psi = \Psi(x)$ is necessarily a first eigenfunction of the operator L_{β^*} and so by the results of [DH] we conclude that $\Psi = c\psi_1$.

We turn now to the proof of Claim 1. To this end, we fix any positive integer k and let $\phi \in C_0^\infty(\mathbf{R})$ be an arbitrary test function such that

$$0 \notin \operatorname{supp}(\phi). \tag{3.17}$$

To establish the claim we must show

$$\langle \hat{w}_k, \phi \rangle \equiv \langle w_k, \hat{\phi} \rangle = 0. \tag{3.18}$$

We first invoke Lemma 2.4 and denote by $g_k = g_k(x,t)$ the solution to

$$\big(L_{(\beta^*-t)} - \lambda_1\big)(g_k) = \psi_k \quad \text{for } t \neq 0, \quad 0 < x < \infty, \tag{3.19}$$

$$(g_k)_x(0,t) = 0, \quad g_k(\infty,t) = 0 \tag{3.20}$$

for $t \neq 0$. We also define Φ_k by the relation

$$\Phi_k(x,t) = \phi(t) g_k(x,t) \quad \text{for } t \neq 0, \quad x \in \mathbf{R}_+, \tag{3.21}$$

so that by linearity Φ_k satisfies (3.19)-(3.20) with ψ_k replaced by $\phi\psi_k$. From (3.17) it follows that we can extend Φ_k smoothly to all $(x,t) \in \mathbf{R}_+ \times \mathbf{R}$ by defining $\Phi_k(x,0) = 0$ for all $x \in \mathbf{R}_+$. As k is fixed throughout this argument, we will now suppress the dependence of Φ_k upon k and write simply Φ. Clearly, Φ enjoys the integrability properties guaranteed by (2.6); thus, there exists a constant $C > 0$ such that

$$\int_0^\infty |h(x,t)|^2 + \left|\frac{\partial}{\partial x} h(x,t)\right|^2 + \left|\frac{\partial^2}{\partial x^2} h(x,t)\right|^2 dx < C \tag{3.22}$$

for $h = \Phi$ or $h = \frac{\partial \Phi}{\partial t}$. We also note that since Φ is smooth and compactly supported in t we can define for each $x \in \mathbf{R}_+$ its (partial) Fourier transform $\hat{\Phi} = \hat{\Phi}(x,y)$ with respect to t.

The next claim is crucial to our analysis.

Claim 2. The following integrals are all well-defined and the corresponding equalities hold:

$$\int_{\mathbf{R}_+ \times \mathbf{R}} \Psi \frac{\partial^2 \hat{\Phi}}{\partial x^2} = \int_{\mathbf{R}_+ \times \mathbf{R}} \frac{\partial^2 \Psi}{\partial x^2} \hat{\Phi} \tag{3.23}$$

$$\int_{\mathbf{R}_+ \times \mathbf{R}} (x - \beta^*) \Psi \frac{\partial \hat{\Phi}}{\partial y} = -\int_{\mathbf{R}_+ \times \mathbf{R}} (x - \beta^*) \frac{\partial \Psi}{\partial y} \hat{\Phi} \tag{3.24}$$

$$\int_{\mathbf{R}_+ \times \mathbf{R}} \Psi \frac{\partial^2 \hat{\Phi}}{\partial y^2} = \int_{\mathbf{R}_+ \times \mathbf{R}} \frac{\partial^2 \Psi}{\partial y^2} \hat{\Phi} \tag{3.25}$$

$$\int_{\mathbf{R}_+ \times \mathbf{R}} (x - \beta^*)^2 \Psi \hat{\Phi} \quad \text{and} \int_{\mathbf{R}_+ \times \mathbf{R}} \Psi \hat{\Phi} \quad \text{are well-defined.} \tag{3.26}$$

We first show that the left-hand side of (3.23) is well-defined. Through an appeal to Theorem 3.1 we find that

$$\int_0^\infty \int_{-\infty}^\infty |\Psi(x,y)| \left|\frac{\partial^2 \hat{\Phi}}{\partial x^2}(x,y)\right| dy\, dx \leq \int_0^\infty \int_{-\infty}^\infty a e^{-b|x|} \left|\frac{\partial^2 \hat{\Phi}}{\partial x^2}\right| dy\, dx.$$

However, the Cauchy–Schwartz inequality, Parseval's identity and (3.21) imply that

$$\begin{aligned}\int_{-\infty}^{\infty}\left|\frac{\partial^2\hat{\Phi}}{\partial x^2}\right| dy &= \int_{-\infty}^{\infty}\sqrt{1+y^2}\left|\frac{\partial^2\hat{\Phi}}{\partial x^2}\right|\frac{1}{\sqrt{1+y^2}}\,dy\\ &\le \frac{1}{2}\int_{-\infty}^{\infty}(1+y^2)\left|\frac{\partial^2\hat{\Phi}}{\partial x^2}\right|^2 dy + C\\ &= \frac{1}{2}\int_{-\infty}^{\infty}\left|\frac{\partial^2\Phi}{\partial x^2}\right|^2 + \left|\frac{\partial^2}{\partial x^2}\Big(\frac{\partial\Phi}{\partial t}\Big)\right|^2 dt + C\\ &= \frac{1}{2}\int_{\operatorname{supp}\phi}\phi^2(t)\left|\frac{\partial^2 g_k}{\partial x^2}(x,t)\right|^2 + \left|\frac{\partial}{\partial t}\Big(\phi(t)\frac{\partial^2 g_k}{\partial x^2}(x,t)\Big)\right|^2 dt + C.\end{aligned}$$

Then as a consequence of (3.22), $\int_{\mathbf{R}_+\times\mathbf{R}}\Psi\frac{\partial^2\hat{\Phi}}{\partial x^2}$ is finite. In a similar manner one finds that $\frac{\partial^2\Psi}{\partial x^2}\hat{\Phi}\in L^1(\mathbf{R}_+\times\mathbf{R})$. The equivalence (3.23) then follows after two integrations by parts where the boundary terms all vanish in light of Theorem 3.1, (3.13), (3.20) and (3.21).

Essentially the same approach works on identity (3.24). One invokes Theorem 3.1 to obtain

$$\int_0^{\infty}\int_{-\infty}^{\infty}\left|x-\beta^*\right||\Psi(x,y)|\left|\frac{\partial\hat{\Phi}}{\partial y}(x,y)\right| dy\,dx \le \int_0^{\infty}\int_{-\infty}^{\infty} ae^{-b|x|}\left|\frac{\partial\hat{\Phi}}{\partial y}\right| dy\,dx.$$

Then we observe that

$$\begin{aligned}\int_{-\infty}^{\infty}\left|\frac{\partial\hat{\Phi}}{\partial y}\right| dy &= \int_{-\infty}^{\infty}\sqrt{1+y^2}\left|\frac{\partial\hat{\Phi}}{\partial y}\right|\frac{1}{\sqrt{1+y^2}}\,dy\\ &\le \frac{1}{2}\int_{-\infty}^{\infty}(1+y^2)\left|\frac{\partial\hat{\Phi}}{\partial y}\right|^2 dy + C\\ &= \frac{1}{2}\int_{-\infty}^{\infty}|t\Phi|^2 + \left|\frac{\partial}{\partial t}(t\Phi)\right|^2 dt + C.\end{aligned}$$

Hence, the integrals in (3.24) are well-defined in light of (3.22) and their equivalence then follows from Fubini's Theorem and integration by parts. Properties (3.25) and (3.26) of Claim 2 are handled similarly.

We are now prepared to establish (3.18). To this end, note that by (3.15) we have

$$\langle\hat{w}_k,\phi\rangle \equiv \langle w_k,\hat{\phi}\rangle = \int_{-\infty}^{\infty}\int_0^{\infty}\Psi(x,y)\psi_k(x)\hat{\phi}(y)\,dx\,dy.$$

Now recall that Φ_k solves

$$\begin{aligned}&-\frac{\partial^2\Phi_k}{\partial x^2}(x,t) + (x-\beta^*)^2\Phi_k(x,t) + 2(x-\beta^*)t\Phi_k(x,t) + (t^2-\lambda_1)\Phi_k(x,t)\\ &\quad = \phi(t)\psi_k(x)\end{aligned} \tag{3.27}$$

for $t \in \mathbf{R}$ and $x > 0$. Since both ϕ and $t \to \Phi_k(x,t)$ are C^∞ compactly supported functions of t, we can take the (partial) Fourier transform of (3.27) to obtain

$$-\frac{\partial^2 \hat\Phi_k}{\partial x^2}(x,y) + (x-\beta^*)^2 \hat\Phi_k(x,y) - 2i(x-\beta^*)\frac{\partial \hat\Phi_k}{\partial y}(x,y) - \frac{\partial^2 \hat\Phi_k}{\partial y^2}(x,y) \\ - \lambda_1 \hat\Phi_k(x,y) = \hat\phi(y)\psi_k(x). \tag{3.28}$$

From (3.23)–(3.26) we have that the integral of Ψ against each term on the left-hand side of (3.28) over the set $\mathbf{R}_+ \times \mathbf{R}$ is well-defined. Using the identities of Claim 2, (3.28), (3.12) and (3.15) we reach the conclusion

$$\begin{aligned}\langle \hat w_k, \phi\rangle &= \int_{-\infty}^{\infty}\int_0^{\infty} \Psi(x,y)\psi_k(x)\hat\phi(y)\,dx\,dy \\ &= \int_{-\infty}^{\infty}\int_0^{\infty}\Big(-\Delta\Psi + 2i(x-\beta^*)\frac{\partial\Psi}{\partial y} + (x-\beta^*)^2\Psi - \lambda_1\Psi\Big)\hat\Phi_k\,dx\,dy = 0\end{aligned}$$

and Claim 1 is established.

The case $\lambda < \lambda_1$ is handled similarly. The only difference is that Claim 1 changes to the statement supp $\hat w_k = \emptyset$ for each k. This follows since for $\lambda < \lambda_1$ we no longer need the stipulation (3.17) for ϕ. □

4. Analysis in a Bounded Domain

We now consider the eigenvalue problem on a bounded domain associated with the onset of superconductivity in the presence of high magnetic fields. Let $\Omega \subset \mathbf{R}^2$ be a bounded, simply connected domain with $\partial\Omega \in C^{3,\alpha_0}$, $\alpha_0 \in (0,1)$. Then recall that for any $h \in \mathbf{R}$, the value $\mu(h)$ is given by the infimum:

$$\mu(h) = \inf_{\psi \in H^1(\Omega)} J_h(\psi) \equiv \inf_{\psi\in H^1(\Omega)} \frac{\int_\Omega |(i\nabla + h\mathbf{A})\psi|^2\,dx\,dy}{\int_\Omega |\psi|^2\,dx\,dy}, \tag{4.1}$$

where $\mathbf{A} : \Omega \to \mathbf{R}^2$ is any smooth vector field satisfying (1.3).

First we establish an upper bound on $\mu(h)$.

Proposition 4.1. *The eigenvalue $\mu(h)$ satisfies the asymptotic upper bound*

$$\limsup_{h\to\infty} \frac{\mu(h) - \lambda_1 h}{h^{1/2}} \le -\frac{\kappa_{\max}}{3I_0},$$

where λ_1 is the first eigenvalue introduced in Proposition 2.2, I_0 is the first moment of the corresponding eigenfunction and $\kappa_{\max}$ is the maximum of curvature of $\partial\Omega$.

Remark 4.2. If one makes the further assumption that $\partial\Omega$ achieves a maximum of curvature at a unique point z_0 and that this maximum is strict in the sense that $\kappa_{ss}(0) < 0$ (with $s = 0$ corresponding to z_0), then one can capture another term in an upper bound for $\mu(h)$ following the construction in [BS]. This involves more careful consideration of the tangential variation (i.e. s-dependence) of the amplitude and is accomplished by replacing the factor $e^{-h^{1/4}s^2}$ in definition (4.4) below by a factor $e^{-\alpha h^{1/4}s^2}$, where α is a positive constant depending on $\kappa_{ss}(0)$. One then obtains the bound

$$\mu(h) \le \lambda_1 h - \frac{\kappa_{\max}}{3I_0}h^{1/2} + \Big(\frac{-\lambda_1^{1/2}\kappa_{ss}(0)}{6}\Big)^{1/2} h^{1/4} + o(h^{1/4})$$

which we believe to be sharp.

Proof. This result is based upon the use of the approximate first eigenfunction derived in [BS] as a test function in the energy J_h defined in (4.1).

Here we recall the local coordinates (s, η) valid in a neighborhood of $\partial\Omega$ that were introduced in Sect. 2.3. We choose the point z_0 on $\partial\Omega$ corresponding to $s = 0$ to be a point where the curvature is maximized.

Fix any vector field $\mathbf{A}$ satisfying (1.3). Recalling the definition of the vector field $\mathbf{q}$ given in (2.23), we then define a vector field $\mathbf{p} : S \to \mathbf{R}^2$ by the relation $\tilde{\mathbf{A}} = \mathbf{p} + \mathbf{q}$, where $\tilde{\mathbf{A}}(s, \eta) \equiv \mathbf{A}(x(s, \eta), y(s, \eta))$. Note in particular that $\mathbf{p}$ will then be L-periodic and conservative:

$$\nabla \times \mathbf{p} = 0 \quad \text{in} \quad S. \tag{4.2}$$

Motivated by the gauge invariance (cf. Lemma 2.5), and utilizing (4.2), we now introduce a phase Φ on the rectangle S through the relation $\nabla\Phi = \mathbf{p}$. Hence, for any $(s, \eta) \in S$, we let

$$\Phi(s, \eta) = \int_\gamma \mathbf{p} \cdot d\mathbf{r}, \tag{4.3}$$

where γ is any path in S joining $(0, 0)$ to (s, η).

We are now ready to define a sequence of test functions $\{\Psi^h\}$ for the energy J_h given by (4.1). First we define rectangles $\mathcal{N}_h^1$ and $\mathcal{N}_h^2$ in terms of $s - \eta$ coordinates by

$$\mathcal{N}_h^1 = \{(s, \eta) : -\frac{1}{h^{1/16}} < s < \frac{1}{h^{1/16}},\ 0 \le \eta < \frac{1}{h^{1/4}})\},$$
$$\mathcal{N}_h^2 = \{(s, \eta) : -\frac{2}{h^{1/16}} < s < \frac{2}{h^{1/16}},\ 0 \le \eta < \frac{2}{h^{1/4}})\}.$$

We choose Ψ^h to take the form

$$\Psi^h = \begin{cases} \psi^h e^{ih\Phi} e^{-ih^{1/2}\beta^* s} & \text{in } \mathcal{N}_h^2, \\ 0 & \text{elsewhere.} \end{cases}$$

We take $\psi^h = \psi^h(s, \eta)$ to be a smooth real-valued function vanishing outside $\mathcal{N}_h^2$ and given by

$$\psi^h(s, \eta) = \psi_1(h^{1/2}\eta) e^{-h^{1/4}s^2} \quad \text{in } \mathcal{N}_h^1. \tag{4.4}$$

Here ψ_1 denotes the first eigenfunction of the operator L_{β^*}, with ψ_1 normalized so that $\psi_1(0) = 1$. In light of the exponential decay of both ψ_1 as a function of η and $e^{-h^{1/4}s^2}$ as a function of s, we note that the smooth transition to zero outside of $\mathcal{N}_h^2$ can be accomplished with only an exponentially small contribution to the number $J_h(\Psi^h)$. Invoking Lemma 2.5 and (2.18) we then find that for some $\gamma > 0$ we have

$$J_h(\Psi^h) = \frac{\int_{\mathcal{N}_h^1} \left|(i\nabla - V_h(s, \eta)\mathbf{t})\psi^h\right|^2 (1 - \kappa(s)\eta)\, ds\, d\eta}{\int_{\mathcal{N}_h^1} \left|\psi^h\right|^2 (1 - \kappa(s)\eta)\, ds\, d\eta} + \mathcal{O}(e^{-h^\gamma})$$
$$= \frac{\int_{\mathcal{N}_h^1} \left((\psi_\eta^h)^2 + \frac{(\psi_s^h)^2}{(1-\kappa(s)\eta)^2} + V_h^2(s, \eta)(\psi^h)^2\right)(1 - \kappa(s)\eta)\, d\eta\, ds}{\int_{\mathcal{N}_h^1} \left|\psi^h\right|^2 (1 - \kappa(s)\eta)\, d\eta\, ds} + \mathcal{O}(e^{-h^\gamma}),$$

where

$$V_h(s,\eta) \equiv h\big(\frac{\eta(1-\kappa(s)\eta/2) - h^{-1/2}\beta^*}{1-\kappa(s)\eta}\big)$$

and the factor of $1-\kappa(s)\eta$ in the numerator and denominator of (3.23) represents the Jacobian associated with the change of variables $(x, y) \to (s, \eta)$.

We now make one further change of variables and introduce

$$\tau = h^{1/8}s \quad \text{and} \quad \xi = h^{1/2}\eta. \tag{4.5}$$

For τ-ξ values corresponding to $(s,\eta) \in \mathcal{N}_h^2$, a brief calculation yields that $\tilde{V}_h(\tau,\xi) \equiv V_h(\tau/h^{1/8}, \xi/h^{1/2})$ satisfies

$$\tilde{V}_h(\tau,\xi) = h^{1/2}(\xi - \beta^*) + \kappa_{\max}(\frac{1}{2}\xi^2 - \beta^*\xi) + \mathcal{O}(h^{-1/4}).$$

Here we have used the smoothness of $\partial\Omega$ to Taylor expand the curvature as a function of τ about $\tau = 0$ and we have used that $\kappa_s(0) = 0$ since curvature is maximized at $s = 0$. Consequently, we obtain

$$J_h(\Psi^h) = \frac{Ah + Bh^{1/2} + \mathcal{O}(h^{1/4})}{C - Dh^{-1/2} + \mathcal{O}(h^{-3/4})} = \frac{A}{C}h + \big(\frac{B}{C} - \frac{AD}{C^2}\big)h^{1/2} + o(h^{1/2}),$$

where

$$A = \int_0^{h^{1/4}} [(\psi_1)_\xi]^2 + (\xi - \beta^*)^2\psi_1^2\, d\xi,$$

$$B = \kappa_{\max}\int_0^{h^{1/4}} \Big((\xi^2 - 2\beta^*\xi)(\xi-\beta^*)\psi_1^2 - \xi[(\psi_1)_\xi]^2 - \xi(\xi-\beta^*)^2\psi_1^2\Big)\, d\xi,$$

$$C = \int_0^{h^{1/4}} \psi_1^2\, d\xi, \quad \text{and}$$

$$D = \kappa_{\max}\int_0^{h^{1/4}} \xi\psi_1^2\, d\xi.$$

We note that up to order $h^{1/2}$, the τ dependence only enters each term as $\int e^{-2\tau^2}\,d\tau$ and so it cancels out of the computation. In light of the exponential decay of ψ_1 and its derivative, we may replace the domain of integration in each of the integrals above with $\int_0^\infty$ and only introduce an exponentially small error. Then invoking the moment identities of Proposition 2.3, a tedious but straightforward calculation yields the desired result, namely

$$\mu(h) \le J_h(\Psi^h) = \lambda_1 h - \frac{\kappa_{\max}}{3I_0}h^{1/2} + o(h^{1/2}). \qquad \square$$

We will now invoke methods similar to those in the proof of Theorem 3.1 to establish:

Theorem 4.3. *Let* $\{\Psi^h\}$ *be any sequence of eigenfunctions solving the minimization problem* (4.1)*, normalized so that* $\|\Psi^h\|_{L^\infty(\Omega)} = 1$*. Then there exists a constant* $h_0 > 0$ *and for every multi-index* α *with* $|\alpha| \le 2$*, there exist positive constants* c_1^α *and* c_2^α *independent of* h *such that*

$$\left|D^\alpha\Psi^h(z)\right| \le h^{\frac{1}{2}|\alpha|}c_1^\alpha e^{-c_2^\alpha h^{1/2}\operatorname{dist}(z,\partial\Omega)} \quad \text{for all } z = (x,y) \in \Omega \tag{4.6}$$

provided $h \ge h_0$.

Remark 4.4. Note that we do not assert the uniqueness of eigenfunctions here. Indeed, it was shown in [BPT] that for Ω a disc, there exists a sequence of values $\{h_j\} \to \infty$ such that $\mu(h_j)$ is a double eigenvalue. (See Remark 2.12.)

Proof. The estimates up to the boundary contained in (4.6) will follow from a standard "flattening of the boundary". As this type of formulation and estimate is carried out in the proof of Theorem 4.5, we omit it here and focus on the interior decay. Since the argument follows along the same lines as the one used to prove Theorem 3.1, we only sketch the main idea here. It will be convenient to take $\mathbf{A} = 1/2(-y, x)$. Once the estimates (4.6) are demonstrated for this choice, it will follow for all others since a different gauge will only alter the values of $\left|D^\alpha \Psi^h(z)\right|$ by an h-independent constant.

Note that a minimizer Ψ^h to the problem (4.1) will satisfy the equation

$$(i\nabla + h\mathbf{A})^2 \Psi^h = \mu(h)\Psi^h \quad \text{in } \Omega. \tag{4.7}$$

Now let $\Omega(k, h, R) = \{z \in \Omega : \text{dist}(z, \partial\Omega) \geq \frac{k}{h^{1/2}} R\}$ for any positive integer k and any $h > 0$ and $R > 0$. Decay follows from the claim:

There exists an $h_0 > 0$ and an $R_0 > 0$ such that

$$\left\| \Psi^h \right\|_{L^\infty(\Omega(k+1,h,R))} < \frac{1}{2} \left\| \Psi^h \right\|_{L^\infty(\Omega(k,h,R))} \tag{4.8}$$

for all $h \geq h_0$, all $R \geq R_0$ and all positive integers k.

Proceeding by contradiction, we note that if claim (4.8) fails then there exist sequences $h_j \to \infty$ and $R_j \to \infty$ and a sequence of positive integers $\{k_j\}$ such that

$$\left\| \Psi^{h_j} \right\|_{L^\infty(\Omega(k_j+1,h_j,R_j))} \geq \frac{1}{2} \left\| \Psi^{h_j} \right\|_{L^\infty(\Omega(k_j,h_j,R_j))} \equiv \frac{1}{2} m_j. \tag{4.9}$$

Then define $\tilde{\Psi}^{h_j}$ by the formula

$$\tilde{\Psi}^{h_j}(z) = \frac{\Psi^{h_j} e^{i h_j \mathbf{A}(z_j)\cdot z}}{m_j},$$

where the sequence of points $\{z_j\}$, each lying in the set $\Omega(k_j + 1, h_j, R_j)$, are chosen so that

$$\left| \Psi^{h_j}(z_j) \right| \geq \frac{1}{2} \left\| \Psi^{h_j} \right\|_{L^\infty(\Omega(k_j,h_j,R_j))}.$$

Hence,

$$\left| \tilde{\Psi}^{h_j}(z_j) \right| \geq \frac{1}{2} \quad \text{while} \quad \left\| \tilde{\Psi}^{h_j} \right\|_{L^\infty(\Omega(k_j,h_j,R_j))} = 1. \tag{4.10}$$

Now we introduce $f_j : B(0, R_j) \to \mathbf{C}$ by the relation

$$f_j(z) = \tilde{\Psi}^{h_j}(z_j + \frac{z}{\sqrt{h_j}}).$$

In view of Lemma 2.5 and (4.7), we easily find that f_j satisfies the P.D.E.

$$(i\nabla + \mathbf{A})^2 f_j = \frac{\mu(h_j)}{h_j} f_j \text{ on } B(0, R_j).$$

Note that by Proposition 4.1, we know

$$\frac{\mu(h_j)}{h_j} \leq \lambda_1 < 1 \tag{4.11}$$

for h_j sufficiently large. Invoking the same elliptic theory as in the proof of Theorem 3.1, we can then extract a subsequence of $\{f_j\}$ which converges in $C^2_{\text{loc}}(\mathbf{R}^2)$ to a limit f_0 satisfying

$$(i\nabla + \mathbf{A})^2 f_0 = \mu^* f_0 \quad \text{on } \mathbf{R}^2,$$

where $\mu^* < 1$ arises as a subsequential limit of $\{\frac{\mu(h_j)}{h_j}\}$. Since $f_0(0) \geq \frac{1}{2}$ in light of (4.10), we reach a contradiction of Proposition 2.9; hence Claim (4.8) is established. The exponential decay of $|\Psi^h|$ follows immediately.

To obtain decay of derivatives of Ψ^h, fix any point $z_0 \in \Omega$. Define F^h by the formula

$$F^h(z) = \Psi^h(z) e^{ihA(z_0)\cdot z}$$

and then change variables to $w = h^{1/2}(z - z_0)$ and introduce

$$\tilde{F}^h(w) = F^h(z_0 + \frac{w}{h^{1/2}}). \tag{4.12}$$

As in the earlier part of this proof, one finds that $\tilde{F}^h$ satisfies the P.D.E.

$$(i\nabla + \mathbf{A})^2 \tilde{F}^h = \frac{\mu(h)}{h} \tilde{F}^h \quad \text{for } w \in B(0, 1)$$

for h large, where $\mathbf{A} = \mathbf{A}(w) = \frac{1}{2}(-w_2, w_1)$. Through the use of a cut-off function and the same manipulation as in the derivation of (3.8), one obtains uniform estimates on any derivative of $\tilde{F}^h$ of the form

$$\int_{B(0,1/2)} \left|D^\alpha \tilde{F}^h(w)\right|^2 dw \leq C_\alpha \int_{B(0,1)} \left|\tilde{F}^h(w)\right|^2 dw$$

for a constant C_α independent of h. Consequently, one concludes from the embedding of $H^l(B(0, 1))$ in $C^k(B(0, 1))$ for l large that

$$\sup_{B(0,1/2)} \left|D^\alpha \tilde{F}^h(w)\right| \leq C'_\alpha \left(\int_{B(0,1)} \left|\tilde{F}^h(w)\right|^2 dw \right)^{1/2}$$

for a constant C'_α. Reverting back to the variable z and invoking the exponential decay just established for $|\Psi^h|$, one arrives at the estimates (4.6). □

We conclude with a result yielding a proof of property (1.7) of Theorem 1.1 as well as properties (1.5) and (1.9).

Theorem 4.5. *Let $\Omega \subset \mathbf{R}^2$ be a bounded, open, simply connected domain with $\partial\Omega \in C^{3,\alpha_0}$ for some $\alpha_0 \in (0, 1)$. Then the minimal eigenvalue $\mu(h)$ given by* (1.4) *satisfies the condition*

$$\lambda_1 h - o(h) \leq \mu(h) \leq \lambda_1 h - \frac{\kappa_{\max}}{3I_0} h^{1/2} + o(h^{1/2}) \quad \text{as } h \to \infty. \tag{4.13}$$

If Ω is a disc, then one has

$$\mu(h) = \lambda_1 h - \frac{\kappa}{3I_0} h^{1/2} + o(h^{1/2}) \quad \text{as } h \to \infty. \tag{4.14}$$

Furthermore, if $\{\Psi^h\}$ denotes a sequence of eigenfunctions corresponding to the eigenvalue $\mu(h)$, normalized so that $\|\Psi^h\|_{L^\infty(\Omega)} = 1$, then for any Ω that is not a disc we have

$$\lim_{h\to\infty} \left(\min_{z\in\partial\Omega} \left|\Psi^h(z)\right| \right) = 0. \tag{4.15}$$

Remark 4.6. In light of the formal results of [BS], we expect (4.14) to hold for any domain Ω (with κ replaced by $\kappa_{\max}$) and we expect

$$\lim_{h\to\infty} \left|\Psi^h(z)\right| = 0$$

for all $z \in \partial\Omega$, where $\kappa \neq \kappa_{\max}$. However, we do not yet have a proof of these stronger claims. This predicted (exponential) decay along $\partial\Omega$ seems very much related to an assumption of nondegeneracy at the point of maximum curvature, an assumption we do not make in this paper. The issue is complicated by the subtlety of the boundary concentration problem. For example, the analysis in [BS] predicts a decay rate for the first eigenfunction which is different for the tangential and normal directions. Hence the seemingly natural scaling by $h^{1/2}$ in the normal direction turns out to be an inappropriate scaling to capture tangential decay of the amplitude of the eigenfunction, a decay that we believe manifests itself on a lengthscale no shorter than $h^{-1/8}$. We are optimistic, however, that a modification of the techniques presented here will ultimately yield a rigorous confirmation of the full set of results predicted in [BS] and we are presently pursuing these questions.

Proof. Let $\{\Psi^h\}$ denote any sequence of minimizers to (4.1). Recall that in the case where Ω is a disc, Proposition 2.11 asserts the existence for each h of an eigenfunction with a radially dependent amplitude. Hence, throughout the proof when considering a disc, we will take

$$\Psi^h(z) = \left|\Psi^h\right|(r)e^{ik_h\theta}, \tag{4.16}$$

where (r, θ) are polar coordinates and k_h is an integer. Since Theorem 4.5 only involves statements about $\mu(h)$ and not about Ψ^h for the case of a disc, this assumption is justified. Consider any sequence of points $\{z_h\}$ in $\overline{\Omega}$ satisfying

$$\lim_{h\to\infty} \left|\Psi^h(z_h)\right| = 1. \tag{4.17}$$

In light of Theorem 4.3, any such sequence must satisfy

$$\mathrm{dist}(z_h, \partial\Omega) \leq \frac{C_0}{h^{1/2}} \text{ for some } C_0 > 0. \tag{4.18}$$

Supressing subsequential notation, we denote by $z_0 \in \partial\Omega$, the limit of $\{z_h\}$.

At this point we recall the discussion in Sect. 2.3 in which we introduced a local coordinate system (s, η) describing a neighborhood of $\partial\Omega$. We write $\tilde{\Psi}_h(s, \eta) \equiv$

$\Psi^h(x(s,\eta), y(s,\eta))$ and note that for each h, $\tilde{\Psi}_h$ is a smooth function defined on the rectangle

$$S \equiv \{(s,\eta) : -L/2 < s < L/2,\ 0 < \eta < \delta\},$$

where L denotes the arclength of the boundary, and δ is a positive constant depending on Ω such that the local coordinate system is well-defined for $z \in \Omega$ satisfying $\operatorname{dist}(z, \partial\Omega) < \delta$. Working in a smaller neighborhood of $\partial\Omega$ if necessary, we now assume

$$\delta < \frac{1}{2\kappa_{\max}}. \tag{4.19}$$

Without loss of generality, we take the arclength value $s = 0$ to correspond to the point z_0. Define now the sequence $\{\tilde{z}_h\} \subset \partial\Omega$ as the sequence satisfying the relation

$$|z_h - \tilde{z}_h| = \operatorname{dist}(z_h, \partial\Omega), \tag{4.20}$$

and then let $\{s_h\}$ denote the sequence of arclength values corresponding to the boundary points $\tilde{z}_h$ so that $s_h \to 0$ as $h \to \infty$. As in the proof of Proposition 4.1, we introduce the function Φ through formula (4.3) and then introduce a sequence of functions $f^h : S \to \mathbf{C}$ defined by

$$\tilde{\Psi}_h(s,\eta) = f^h(s,\eta) e^{i(h\Phi(s,\eta) - h^{1/2}\beta^* s)}. \tag{4.21}$$

Note that we are not asserting that f^h is real.

Through the use of Lemma 2.5 we find that f^h satisfies the equation

$$(i\nabla + h\mathbf{q} + h^{1/2}\beta^*\nabla s)^2 f^h = \mu(h) f^h \quad \text{in } S, \tag{4.22}$$

where $\mathbf{q}$ is given by (2.23). The functions f^h also satisfy the boundary condition

$$\frac{\partial f^h}{\partial \eta}(s, 0) = 0 \text{ for } |s| < L/2. \tag{4.23}$$

In light of the smoothness of the function $\Psi^h = \Psi^h(z)$, note that $\tilde{\Psi}_h = \tilde{\Psi}_h(s,\eta)$ is necessarily periodic in s. Thus, from (4.21) we conclude that

$$\frac{\partial^k}{\partial s^k} e^{i(h\Phi(s,\eta) - h^{1/2}\beta^* s)} f^h(s,\eta)_{|s=L/2} = \frac{\partial^k}{\partial s^k} e^{i(h\Phi(s,\eta) - h^{1/2}\beta^* s)} f^h(s,\eta)_{|s=-L/2} \tag{4.24}$$

for $k = 0, 1$ and for $0 \le \eta \le \delta$. Utilizing the fact that $\mathbf{p}$ is periodic and conservative, and that $\mathbf{p}(s, 0) = \mathbf{A}(x(s, 0), y(s, 0))$, we conclude through (4.3) that

$$\begin{aligned}\Phi(L/2,\eta) - \Phi(-L/2,\eta) &= \Phi(L/2, 0) - \Phi(-L/2, 0) \\ &= \int_{-L/2}^{L/2} \mathbf{p}(s, 0)\cdot(1, 0)\, ds = \int_{\partial\Omega} \mathbf{A}\cdot d\mathbf{t} \\ &= \int_\Omega \nabla\times\mathbf{A}\cdot\hat{\mathbf{z}}\, dx\, dy = |\Omega|\,.\end{aligned}$$

Hence, the boundary conditions (4.24) can be phrased as

$$\frac{\partial^k}{\partial s^k} f^h(s,\eta)_{|s=L/2} = \frac{\partial^k}{\partial s^k} f^h(s,\eta)_{|s=-L/2} e^{i\gamma_h} \tag{4.25}$$

for $k = 0, 1$ and for $0 \le \eta \le \delta$, where

$$\gamma_h \equiv -h\,|\Omega| + h^{1/2}\beta^* L. \tag{4.26}$$

Using the definition of $\mathbf{q}$ given in (2.23) as well as the transformation formula (2.18), we find that (4.22) takes the form

$$(i\nabla - h^{1/2}\tilde{V}^h(s,\eta)\mathbf{t})^2 f^h = \mu(h) f^h \text{ in } S, \tag{4.27}$$

where $\tilde{V}^h$ is given by

$$\tilde{V}^h(s,\eta) = \frac{h^{1/2}\eta(1-\kappa(s)\eta/2) - \beta^*}{1-\kappa(s)\eta}.$$

We now invoke a blow-up procedure about the point $(s_h, 0) \in S$ by introducing the stretched coordinates $\tau = h^{1/2}(s - s_h)$ and $\xi = h^{1/2}\eta$. Let

$$S_h = [a_h, b_h] \times [0, \delta h^{1/2}],$$

where

$$a_h \equiv -h^{1/2}(\frac{L}{2} + s_h) \quad \text{and} \quad b_h \equiv h^{1/2}(\frac{L}{2} - s_h). \tag{4.28}$$

Then define the sequence of functions $\psi^h : S_h \to \mathbf{C}$ through the formula

$$\psi^h(\tau,\xi) = f^h\big(s_h + \frac{1}{h^{1/2}}\tau, \frac{1}{h^{1/2}}\xi\big).$$

It will also be convenient to introduce the function $\kappa^h : [a_h, b_h] \to \mathbf{R}$ through the relation

$$\kappa^h(\tau) = \kappa\big(s_h + \frac{1}{h^{1/2}}\tau\big) \tag{4.29}$$

and the function $\mathcal{A}^h : S_h \to \mathbf{C}$ given by

$$\mathcal{A}^h(\tau,\xi) = \frac{1}{1 - h^{-1/2}\xi\kappa^h(\tau)}. \tag{4.30}$$

We note here that the functions $\mathcal{A}^h$ are smooth and, in light of (4.19) and the fact that $\partial\Omega \in C^{3,\alpha_0}$, they satisfy

$$\left|\mathcal{A}^h\right| \ge \frac{1}{1 + \frac{\|\kappa\|_{L^\infty}}{2\kappa_{\max}}} \text{ in } S_h, \qquad \left\|\mathcal{A}^h\right\|_{C^{1,\alpha}(S^h)} < C \tag{4.31}$$

for some C independent of h. We also define the function V^h on S_h by the formula

$$V^h(\tau,\xi) \equiv \tilde{V}^h(s_h + \frac{1}{h^{1/2}}\tau, \frac{\xi}{h^{1/2}}) = \mathcal{A}^h(\tau,\xi)\Big[\xi\big(1 - \frac{\kappa^h(\tau)\xi}{2h^{1/2}}\big) - \beta^*\Big]. \tag{4.32}$$

With an appeal to (2.18) and (2.19) we can now convert the problem (4.22)–(4.24) satisfied by f^h on S into the equation

$$-\psi^h_{\xi\xi} - \mathcal{A}^h\big(\mathcal{A}^h\psi^h_\tau\big)_\tau + \frac{1}{h^{1/2}}\kappa^h\mathcal{A}^h\psi^h_\xi - 2i\mathcal{A}^hV^h\psi^h_\tau - i\mathcal{A}^hV^h_\tau\psi^h + (V^h)^2\psi^h = \frac{\mu(h)}{h}\psi^h \quad \text{for } (\tau,\xi)\in S_h, \tag{4.33}$$

and the conditions

$$\left|\psi^h(\tau,\xi)\right| \le c_1e^{-c_2\xi} \quad \text{for } (\tau,\xi)\in S_h, \tag{4.34}$$

$$\psi^h_\xi(\tau,0) = 0 \quad \text{for } \tau\in[a_h,b_h], \tag{4.35}$$

$$\frac{\partial^k}{\partial\tau^k}\psi^h(\tau,\xi)_{|\tau=b_h} = e^{i\gamma_h}\frac{\partial^k}{\partial\tau^k}\psi^h(\tau,\xi)_{|\tau=a_h} \quad \text{for } k=0,1. \tag{4.36}$$

The condition (4.34) is simply the content of Theorem 4.3 expressed in terms of ψ^h. Note that the positive constants c_1 and c_2 appearing in (4.34) are independent of both τ and h.

We will present the remainder of the proof in a sequence of steps.

Step 1. We wish to decompose the operator on the left-hand side of (4.33) into the sum of three operators acting on ψ^h. This is a routine but tedious calculation requiring the expansion of V^h, V^h_τ, $\mathcal{A}^h$ and $\mathcal{A}^h_\tau$. Details can be found in the appendix, but the conclusion is that the Eq. (4.33) can be written as

$$\mathcal{L}_0[\psi^h] + \frac{\kappa^h}{h^{1/2}}P_h[\psi^h] + \frac{1}{h}Q_h[\psi^h] = \frac{\mu(h)}{h}\psi^h \tag{4.37}$$

where

$$\mathcal{L}_0[\psi^h] = -\psi^h_{\xi\xi} - \psi^h_{\tau\tau} - 2i(\xi-\beta^*)\psi^h_\tau + (\xi-\beta^*)^2\psi^h, \tag{4.38}$$

$$P_h[\psi^h] = \mathcal{A}^h\psi^h_\xi - 2\mathcal{A}^h\xi\psi^h_{\tau\tau} + 2i\mathcal{A}^h\xi(2\beta^* - \frac{3}{2}\xi)\psi^h_\tau + 2\mathcal{A}^h\xi(\xi-\beta^*)(\frac{\xi}{2}-\beta^*)\psi^h, \tag{4.39}$$

and

$$\begin{aligned} Q_h[\psi^h] = &-(\mathcal{A}^h)^2\xi^2(\kappa^h)^2\psi^h_{\tau\tau} - (\mathcal{A}^h)^3\xi(\kappa^h)'\psi^h_\tau \\ &+ 2i(\mathcal{A}^h)^2\xi^2(\kappa^h)^2(\beta^* - \frac{\xi}{2})\psi^h_\tau + i\xi(\mathcal{A}^h)^3(\beta^* - \frac{\xi}{2})(\kappa^h)'\psi^h \\ &+ (\mathcal{A}^h)^2\xi^2(\kappa^h)^2(\beta^* - \frac{\xi}{2})^2\psi^h. \end{aligned} \tag{4.40}$$

Here $(\kappa^h)'$ denotes the quantity $\frac{d}{ds}\kappa(s)$ evaluated at $s = s_h + \frac{\tau}{h^{1/2}}$.

Step 2. Our next immediate goal is to establish compactness of the sequence $\{\psi^h\}$ by establishing h-independent $C^{2,\alpha}$ bounds. Since the procedure is very similar to one carried out earlier in the proofs of Theorems 3.1 and 4.3, we only outline the argument.

For any $R > 2C_0$, where C_0 is given by (4.18), let $B(R)$ denote the ball of radius R and center $(\tau,\xi) = (s_h, 0)$. Then let $B^+(R)$ denote the half-ball $B(R)\cap\overline{S_h}$. Note that within $B^+(R)$, all coefficients in the uniformly elliptic system (4.33) can be bounded in $C^{0,\alpha}$ by a constant $C = C(R)$ which is in particular independent of h. This follows from the C^{3,α_0} assumption on $\partial\Omega$ leading to $C^{1,\alpha}$ control of curvature and so of $\mathcal{A}^h$ and V^h as well. In particular, this will give an h-independent bound on the $L^2(B^+(R))$-norm of all terms in (4.33) involving ψ^h undifferentiated. Then, we can multiply (4.33) by $\overline{\psi^h}\chi^2$ and integrate over $B^+(R)$, where $\chi \in C_0^\infty(B(R))$ and $\chi \equiv 1$ on $B(R/2)$. Utilizing the Neumann boundary condition (4.35), we find after an integration by parts that this leads to uniform bounds on $\|\psi^h\|_{H^1(B^+(R/2))}$. Writing (4.33) as a system in terms of Re ψ^h and Im ψ^h, we apply standard elliptic theory to each equation separately to obtain h-independent bounds on $\|\psi^h\|_{H^2(B^+(R/2))}$, which by Morrey's Theorem lead to h-independent bounds on $\|\psi^h\|_{C^{0,\alpha}(B^+(R/2))}$. It then follows from Schauder theory for elliptic systems (cf. [ADN], Theorem 9.3) that there exists a positive constant $C_1(R)$ independent of h such that the sequence $\{\psi^h\}$ satisfies the uniform bound

$$\left\|\psi^h\right\|_{C^{2,\alpha}(B^+(R/2))} < C_1(R). \tag{4.41}$$

Now in light of the uniform $C^{2,\alpha}$ bounds provided by (4.41), we conclude that there exists a subsequence $\{\psi^{h_j}\}$ converging in $C^{2,\alpha}$ on compact sets in the half-plane $\{(\tau,\xi) : \xi \geq 0\}$ to a limit ψ^*. The upper bound on $\mu(h)$ provided by Theorem 4.1 implies, after perhaps passing to another subsequence, that

$$\lim_{j\to\infty} \frac{\mu(h_j)}{h_j} = \lambda \quad \text{where } \lambda \leq \lambda_1.$$

As (4.41) also implies a uniform bound on $\|P^h\|_{L^\infty(B^+(0,R))}$ and $\|Q^h\|_{L^\infty(B^+(0,R))}$ for each $R > 0$, we infer from (4.35) and (4.37), that ψ^* must satisfy the equation

$$\mathcal{L}_0[\psi^*] = \lambda\psi^* \text{ for } -\infty < \tau < \infty,\ 0 < \xi < \infty$$

and the boundary condition

$$\psi^*_\xi(\tau, 0) = 0 \text{ for all } \tau.$$

Additionally, we find through assumption (4.17) and the normalization $\|\Psi^h\|_{L^\infty(\Omega)} = 1$ that

$$0 < \|\psi^*\|_{L^\infty(\{\xi>0\})} \leq 1.$$

Through an appeal to Theorem 3.2, we then conclude that in fact $\lambda = \lambda_1$ and $\psi^* = B\psi_1$ for some nonzero $B \in \mathbf{C}$; that is,

$$\psi^{h_j} \text{ converges to } B\psi_1 \quad \text{in } C^{2,\alpha} \text{ on compact subsets of } \{(\tau,\xi) : \xi \geq 0\} \tag{4.42}$$

for some nonzero $B \in \mathbf{C}$. In particular, we have established (4.13).

We shall henceforth denote quantities indexed by h_j simply with a sub- or superscript j. In particular, we will write ψ^j for ψ^{h_j} and S_j for S_{h_j}.

Step 3. We now multiply $\mathcal{L}_0[\psi^j]$ by $\overline{\psi^j}$, the conjugate of ψ^j (cf. (4.38)), and integrate over S_j to obtain

$$\begin{aligned}&\int\int_{S_j} \mathcal{L}_0[\psi^j]\overline{\psi^j}\, d\tau\, d\xi \\ &= \int\int_{S_j} |\psi^j_\xi|^2 + \left|\psi^j_\tau\right|^2 - 2i(\xi-\beta^*)\overline{\psi^j}\psi^j_\tau + (\xi-\beta^*)^2\left|\psi^j\right|^2 d\tau\, d\xi \\ &\quad - \int_{a_j}^{b_j} \overline{\psi^j}\psi^j_\xi\Big]_{\xi=0}^{\xi=\delta h_j^{1/2}} d\tau - \int_0^{\delta h_j^{1/2}} \overline{\psi^j}\psi^j_\tau\Big]_{\tau=a_j}^{\tau=b_j} d\xi.\end{aligned}$$

Invoking (4.34) and (4.35), we then find that

$$\begin{aligned}\mathrm{Re}\int\int_{S_j} \mathcal{L}_0[\psi^j]\overline{\psi^j}\, d\tau\, d\xi &= \int\int_{S_j}\left|\left(i\nabla + (\beta^*-\xi,0)\right)\psi^j\right|^2 d\,\tau\, d\xi \\ &\quad - \mathrm{Re}\int_0^{\delta h_j^{1/2}} \overline{\psi^j}\psi^j_\tau\Big]_{\tau=a_j}^{\tau=b_j} d\xi + \mathcal{O}(e^{-c_2\delta h_j^{1/2}}).\end{aligned} \tag{4.43}$$

Note that the second term on the right vanishes in light of (4.36).

We now define an extension $\tilde{\psi}^j : [a_j, b_j]\times[0,\infty)$ of ψ^j as follows. Let

$$\tilde{\psi}^j(\tau,\xi) = \begin{cases}\psi^j & \text{for } \xi\in[0,\delta h_j^{1/2}] \\ \text{linear in } \xi & \text{for } \xi\in(\delta h_j^{1/2}, 2\delta h_j^{1/2}) \\ 0 & \text{for } \xi\geq 2\delta h_j^{1/2}\end{cases}.$$

In light of the exponential decay of ψ^j and its derivatives provided by (4.6) we find that $\tilde{\psi}^j$ will be a Lipschitz continuous function satisfying

$$\begin{aligned}&\frac{\int\int_{S_j}\left|\left(i\nabla+(\beta^*-\xi,0)\right)\psi^j\right|^2 d\,\tau\, d\xi}{\int\int_{S_j}\left|\psi^j\right|^2 d\,\tau\, d\xi} \\ &\geq \frac{\int_0^\infty\int_{a_j}^{b_j}\left|\left(i\nabla+(0,\beta^*-\xi)\right)\tilde{\psi}^j\right|^2 d\,\tau\, d\xi}{\int_0^\infty\int_{a_j}^{b_j}\left|\tilde{\psi}^j\right|^2 d\,\tau\, d\xi} - \mathcal{O}(e^{-ch_j^{1/2}})\end{aligned} \tag{4.44}$$

for some positive constant c.

Now we introduce a periodic extension $\tilde{\psi}^j_p$ of $\tilde{\psi}^j$ defined on the entire half-plane $\{(\tau,\xi) : \xi\geq 0\}$ as follows. For each integer k we denote by I^k the interval

$$[a_j + k(b_j-a_j), b_j + k(b_j-a_j)],$$

and then on each half-strip $I^k\times[0,\infty)$ we define $\tilde{\psi}^j_p$ by the formula

$$\tilde{\psi}^j_p(\tau,\xi) = e^{ik\gamma_j}\tilde{\psi}^j(\tau-k(b_j-a_j),\xi),$$

where γ_j $(=\gamma_{h_j})$ is given by (4.26). Note that $\tilde{\psi}^j_p$ will be Lipschitz continuous in view of (4.36).

For each positive integer l, we then let $\rho_l = \rho_l(\tau)$ be a smooth cut-off function satisfying

$$\rho_l(\tau) = \begin{cases} 0 & \text{for } \tau \le a_j - l(b_j - a_j) - 1 \\ 1 & \text{for } a_j - l(b_j - a_j) \le \tau \le b_j + l(b_j - a_j) \\ 0 & \text{for } \tau \ge b_j + l(b_j - a_j) + 1. \end{cases};$$

We may insert the function $\tilde{\psi}_p^j \rho_l$ into the Rayleigh quotient for the half-plane and apply Proposition 2.10 to assert that

$$\begin{aligned} \lambda_1 &\le \frac{\int\int_{\mathbf{R}^2_+} \left| \big(i\nabla + (\beta^* - \xi, 0)\big)(\tilde{\psi}_p^j \rho_l) \right|^2 d\tau\, d\xi}{\int\int_{\mathbf{R}^2_+} \left| \tilde{\psi}_p^j \rho_l \right|^2 d\tau\, d\xi} \\ &\le \frac{l \int_{a_j}^{b_j} \int_0^\infty \left| \big(i\nabla + (\beta^* - \xi, 0)\big)\tilde{\psi}^j \right|^2 d\tau\, d\xi + C_1}{l \int_{a_j}^{b_j} \int_0^\infty \left| \tilde{\psi}^j \right|^2 d\tau\, d\xi + C_2}, \end{aligned}$$

where the constants C_1 and C_2 arise from estimating the corresponding integrals over the two half-strips where $\rho_l' \ne 0$. Estimates (4.34) and (4.41) imply that both constants are independent of l and j.

Sending $l \to \infty$, the resulting inequality and (4.44) lead to the conclusion that

$$\int\int_{S_j} \left| \big(i\nabla + (\beta^* - \xi, 0)\big)\psi^j \right|^2 d\tau\, d\xi \ge \lambda_1 \int\int_{S_j} \left| \psi^j \right|^2 d\tau\, d\xi - \mathcal{O}(e^{-ch_j^{1/2}}).$$

If we combine this inequality with (4.37) and (4.43) we obtain

$$\begin{aligned} \frac{\big(\mu(h_j) - \lambda_1 h_j\big)}{h_j^{1/2}} \ge\ & \frac{\mathrm{Re} \int\int_{S_j} \kappa^j P_j[\psi^j]\overline{\psi^j}\, d\tau\, d\xi}{\int\int_{S_j} |\psi^j|^2\, d\tau\, d\xi} \\ & + \frac{1}{h_j^{1/2}} \frac{\mathrm{Re} \int\int_{S_j} Q_j[\psi^j]\overline{\psi^j}\, d\tau\, d\xi}{\int\int_{S_j} |\psi^j|^2\, d\tau\, d\xi} \\ & - \mathcal{O}(e^{-ch_j^{1/2}}). \end{aligned} \tag{4.45}$$

Step 4. We now pursue a more precise lower bound on the right-hand side of (4.45) as $h_j \to \infty$. To this end, let us first define the function $\alpha_j : [a_j, b_j] \to \mathbf{C}$ by

$$\alpha_j(\tau) = \frac{\int_0^{\delta h_j^{1/2}} P_j[\psi^j]\overline{\psi^j}\, d\xi}{\int_0^{\delta h_j^{1/2}} |\psi^j|^2\, d\xi}.$$

In light of (4.34), (4.39), (4.42) and Lemma 2.3, we note that

$$\lim_{j\to\infty} \alpha_j(\tau) = \frac{\int_0^\infty \psi_1(\psi_1)_\xi + 2\xi(\xi - \beta^*)(\frac{\xi}{2} - \beta^*)\, |\psi_1|^2\, d\xi}{\int_0^\infty |\psi_1|^2\, d\xi} = -\frac{1}{3I_0}, \tag{4.46}$$

where the convergence is uniform on compact τ-intervals.

Suppose now that Ω is a disc. One can easily check that in this case, the phase in (4.21) is linear in the tangential variable and so by (4.16), all integrals on the right-hand side of (4.45) are independent of τ and in particular, α_j is a constant. The same line of reasoning that leads to (4.46) then applies to yield

$$\lim_{j\to\infty} \frac{1}{h_j^{1/2}} \frac{\mathrm{Re} \int\int_{S_j} Q_j[\psi^j]\overline{\psi^j}\, d\tau\, d\xi}{\int\int_{S_j} \left|\psi^j\right|^2 d\tau\, d\xi} = 0. \tag{4.47}$$

As a consequence of (4.45), (4.46) and (4.47), one obtains for a disc that

$$\liminf_{j\to\infty} \frac{\big(\mu(h_j) - \lambda_1 h_j\big)}{h_j^{1/2}} \geq -\frac{\kappa}{3I_0}.$$

As this lower bound matches the upper bound provided by Proposition 4.7, we have established (4.14).

Step 5. For the rest of the proof, we assume that Ω is not a disc. It remains to establish (4.15), so we suppose by way of contradiction that (4.15) fails; that is, suppose

$$\limsup_{h\to\infty} \Big(\min_{z\in\partial\Omega} \left|\Psi^h(z)\right| \Big) > 0. \tag{4.48}$$

We then claim that (4.46) holds uniformly over the entire interval $a_j \leq \tau \leq b_j$. To establish this claim, suppose by contradiction that along some sequence h_k $(= h_{j_k})$ there exists a sequence τ_k such that for all k we have

$$\left|\alpha_k(\tau_k) + \frac{1}{3I_0}\right| > \sigma \tag{4.49}$$

for some $\sigma > 0$. Let $\tau' = \tau - \tau_k$ and define $\zeta_k = \zeta_k(\tau', \xi)$ by the formula

$$\zeta_k(\tau', \xi) = \psi^{h_k}(\tau_k + \tau', \xi) = f^{h_k}\Big(s_{h_k} + \frac{1}{h_k^{1/2}}\tau_k + \frac{1}{h_k^{1/2}}\tau', \frac{1}{h_k^{1/2}}\xi\Big).$$

We can view the sequence $\{\zeta_k\}$ as being defined on $(-\frac{L}{2}h_k^{1/2}, \frac{L}{2}h_k^{1/2}) \times [0, \delta h_k^{1/2}]$ by simply shifting the origin of the original s-coordinate so that s is defined on a new interval of length L centered at $s = s_{h_k} + \frac{\tau_k}{h_k^{1/2}}$. The analysis leading to the compactness result (4.42) for the sequence $\{\psi^h\}$ then applies equally well to obtain a subsequence ζ_{k_l} and a non-zero complex number B' satisfying:

$$\zeta_{k_l} \text{ converges to } B'\psi_1 \text{ in } C^{2,\alpha} \text{ on compact subsets of } \{(\tau', \xi) : \xi \geq 0\}. \tag{4.50}$$

Note that the conclusion $|B| > 0$ in (4.42) followed from the assumption (4.17) while the analogous conclusion that $\left|B'\right| > 0$ follows from the condition (4.48). In view of (4.50) and Lemma 2.3, we reach a contradiction of (4.49) and conclude that the convergence (4.46) is indeed uniform over the entire interval $a_j \leq \tau \leq b_j$ based on the validity of the earlier contradiction hypothesis (4.48).

Step 6. We continue to assume that Ω is not a disc and pursue a contradiction under the assumption (4.48). We can now use the uniform convergence of $\{\alpha_j\}$ to evaluate the limit of the right-hand side of (4.45). We begin with the first term on the right-hand side of this inequality and write

$$\frac{\mathrm{Re}\int\int_{S_j}\kappa^j P_j[\psi^j]\overline{\psi^j}\,d\tau\,d\xi}{\int\int_{S_j}\left|\psi^j\right|^2\,d\tau\,d\xi} = \frac{\mathrm{Re}\int_{a_j}^{b_j}\left(\kappa(s_{h_j}+\frac{\tau}{h_j^{1/2}})\alpha_j(\tau)\int_0^{\delta h_j^{1/2}}\left|\psi^j\right|^2\,d\xi\right)d\tau}{\int\int_{S_j}\left|\psi^j\right|^2\,d\tau\,d\xi}$$
$$= \mathrm{Re}\int_{-L/2}^{L/2}\kappa(s)\alpha_j\left(h_j^{1/2}(s-s_j)\right)d\nu_j(s), \tag{4.51}$$

where

$$d\nu_j(s) \equiv h_j^{1/2}\frac{\int_0^{\delta h_j^{1/2}}\left|\psi^j\left(h_j^{1/2}(s-s_j),\xi\right)\right|^2\,d\xi}{\int\int_{S_j}\left|\psi^j\right|^2\,d\tau\,d\xi}\,ds. \tag{4.52}$$

Upon noting that

$$\int_{-L/2}^{L/2}d\nu_j(s) = 1 \text{ for each } j$$

we may extract a subsequence of $\{\nu_j\}$ which converges weak-$*$ to a probability measure ν. The uniform convergence of $s\to\alpha_j\left(h_j^{1/2}(s-s_j)\right)$ to $-\frac{1}{3I_0}$ established in Step 5 then yields

$$\lim_{j\to\infty}\frac{\mathrm{Re}\int\int_{S_j}\kappa^j P_j[\psi^j]\overline{\psi^j}\,d\tau\,d\xi}{\int\int_{S_j}\left|\psi^j\right|^2\,d\tau\,d\xi} = -\frac{1}{3I_0}\int_{-L/2}^{L/2}\kappa(s)\,d\nu(s). \tag{4.53}$$

The reasoning used above can be applied equally well to the second term on the right-hand side of inequality (4.45). In this case, however, the factor of $\frac{1}{h_j^{1/2}}$ leads to the result

$$\lim_{j\to\infty}\frac{1}{h_j^{1/2}}\frac{\mathrm{Re}\int\int_{S_j}Q_j[\psi^j]\overline{\psi^j}\,d\tau\,d\xi}{\int\int_{S_j}\left|\psi^j\right|^2\,d\tau\,d\xi} = 0. \tag{4.54}$$

Combining (4.53) and (4.54), we see from (4.45) that

$$\liminf_{j\to\infty}\frac{\left(\mu(h_j)-\lambda_1 h_j\right)}{h_j^{1/2}} \geq -\frac{1}{3I_0}\int_{-L/2}^{L/2}\kappa(s)\,d\nu(s). \tag{4.55}$$

We will reach a contradiction of the upper bound from Proposition 4.1 if we can show that

$$\mathrm{supp}\ \nu\cap\{s\in[-L/2,L/2]:\kappa(s)<\kappa_{\max}\}\neq\emptyset \tag{4.56}$$

To this end, let $[r_1, r_2]$ be any interval contained in the set of s-values where $\kappa < \kappa_{\max}$. Then fix any continuous, nonnegative function f supported on $[r_1, r_2]$. From (4.52) we find

$$\int_{r_1}^{r_2} f(s)\,d\nu_j(s) = \int_{r_1}^{r_2} f(s)\left\{\frac{\int_0^{\delta h_j^{1/2}} \left|\psi^j\left(h_j^{1/2}(s-s_j),\xi\right)\right|^2 d\xi}{\int_{-L/2}^{L/2}\int_0^{\delta h_j^{1/2}} \left|\psi^j\left(h_j^{1/2}(s'-s_j),\xi\right)\right|^2 d\xi\,ds'}\right\} ds.$$

Now from the uniform upper bound on $\frac{\partial}{\partial\xi}(\left|\psi^j\right|^2)$ provided by Theorem 4.3, along with (4.48), the bound $\left\|\psi^j\right\|_{L^\infty} = 1$ and the uniform exponential decay of ψ^j in ξ, it follows that there exist positive constants C_1 and C_2 satisfying

$$C_1 \le \int_0^{\delta h_j^{1/2}} \left|\psi^j(\tau,\xi)\right|^2 d\xi \le C_2 \quad \text{for all } \tau \in [a_j, b_j].$$

Hence, there exists a positive constant C_3 depending on f but not j such that

$$\int_{r_1}^{r_2} f(s)\,d\nu_j(s) \ge C_3 \text{ for all } j.$$

Consequently,

$$\int_{r_1}^{r_2} f(s)\,d\nu(s) \ge C_3,$$

yielding (4.56) and the desired contradiction. □

5. Appendix: Decomposition of Equation (4.33)

In this appendix, we give the details behind the decomposition of (4.33) given by (4.37). To this end, first note that V^h defined through (4.32) can be written as

$$V^h = (\xi - \beta^*) - \frac{1}{h^{1/2}}\xi\mathcal{A}^h\kappa^h(\beta^* - \frac{\xi}{2}). \tag{5.1}$$

Consequently,

$$\begin{aligned}(V^h)^2 = (\xi-\beta^*)^2 &- \frac{1}{h^{1/2}}(2\mathcal{A}^h\xi)(\xi-\beta^*)(\beta^*-\frac{\xi}{2})\kappa^h \\ &+ \frac{1}{h}\xi^2(\mathcal{A}^h)^2(\kappa^h)^2(\beta^*-\frac{\xi}{2})^2.\end{aligned} \tag{5.2}$$

Then noting that $\kappa^h_\tau = \frac{1}{h^{1/2}}(\kappa^h)'$ (where $' = \frac{d}{ds}$) and that

$$\mathcal{A}^h = 1 + \frac{1}{h^{1/2}}\mathcal{A}^h\xi\kappa^h, \tag{5.3}$$

$$\mathcal{A}^h_\tau = \frac{1}{h}\xi(\mathcal{A}^h)^2(\kappa^h)', \tag{5.4}$$

we calculate from (5.1), (5.3) and (5.4) that

$$\begin{aligned} V_\tau^h &= -\frac{1}{h^{1/2}}\xi(\beta^* - \frac{\xi}{2})\big[\mathcal{A}_\tau^h\kappa^h + \kappa_\tau^h\mathcal{A}^h\big] \\ &= -\frac{1}{h}\xi\mathcal{A}^h(\beta^* - \frac{\xi}{2})(\kappa^h)'\big[\frac{1}{h^{1/2}}\xi\kappa^h\mathcal{A}^h + 1\big] \\ &= -\frac{1}{h}\xi(\mathcal{A}^h)^2(\beta^* - \frac{\xi}{2})(\kappa^h)'. \end{aligned} \tag{5.5}$$

Now

$$\mathcal{A}^h\big(\mathcal{A}^h\psi_\tau^h\big)_\tau = (\mathcal{A}^h)^2\psi_{\tau\tau}^h + \mathcal{A}^h\mathcal{A}_\tau^h\psi_\tau^h,$$

so that through the use of (5.3) and (5.4) we find

$$\begin{aligned} \mathcal{A}^h\big(\mathcal{A}^h\psi_\tau^h\big)_\tau = \psi_{\tau\tau}^h &+ \frac{2}{h^{1/2}}\mathcal{A}^h\xi\kappa^h\psi_{\tau\tau}^h \\ &+ \frac{1}{h}\xi^2(\kappa^h)^2(\mathcal{A}^h)^2\psi_{\tau\tau}^h + \frac{1}{h}\xi(\mathcal{A}^h)^3(\kappa^h)'\psi_\tau^h. \end{aligned}$$

Now from (5.1) and (5.3) we calculate

$$\begin{aligned} 2i\mathcal{A}^hV^h &= 2iV^h + \frac{2i}{h^{1/2}}\xi\kappa^h\mathcal{A}^hV^h \\ &= 2i(\xi - \beta^*) + \frac{2i}{h^{1/2}}\xi\mathcal{A}^h\kappa^h(\frac{3}{2}\xi - 2\beta^*) + \frac{2i}{h}\xi^2(\mathcal{A}^h)^2(\kappa^h)^2(\frac{\xi}{2} - \beta^*). \end{aligned} \tag{5.6}$$

Then we use (5.5) to obtain

$$i\mathcal{A}^hV_\tau^h = -\frac{i}{h}\xi(\mathcal{A}^h)^3(\beta^* - \frac{\xi}{2})(\kappa^h)'. \tag{5.7}$$

Substitution of these identities into (4.33) then leads to the decompostion (4.37).

References

[ADN] Agmon, S., Douglis, A. and Nirenberg, L.: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II. C.P.A.M. **17**, 35–92 (1964)

[BPT] Bauman, P., Phillips, D. and Tang, Q.: Stable nucleation for the Ginzburg–Landau system with an applied magnetic field. Arch. Rat. Mech. Anal. **142**, 1–43 (1998)

[BR1] Berger, J. and Rubinstein, J.: Formation of topological defects in thin superconducting rings. Phil. Trans. R. Soc. Lond. A **355**, 1969–1978 (1997)

[BR2] Berger, J. and Rubinstein, J.: Bifurcation analysis for phase transitions in superconducting rings with nonuniform thickness. SIAM J. Appl. Math. **58**, 103–121 (1998)

[BS] Bernoff, A. and Sternberg, P.: Onset of superconductivity in decreasing fields for general domains. J. Math. Phys. **39**, 1272–1284 (1998)

[B] Bolley, C.: Modélisation de champ de retard à la condensation d'un supraconducteur par un problème de bifurcation. M^2AN **26**, 235–287 (1992)

[BH] Bolley, C. and Helffer, B.: An application of semi-classical analysis to the asymptotic study of the supercooling field of a superconducting material. Ann. Inst. H. Poincaré: Phys. Theor. **58**, 189–233 (1993)

[BRPVM] Bruyndoncx, V., Rodrigo, J.G., Puig, T. Van Look, L. Moshchalkov, V.V.: Giant vortex state in perforated aluminum microsquares. Preprint

[BGRW] Buisson, O., Gandit, P., Rammal, R., Wang, Y.Y. and Pannetier, B.: Magnetization oscillations of a superconducting disk. Phys. Lett. A **150**, 36–42 (1990)

[C1] Chapman, S.J.: Nucleation of superconductivity in decreasing fields, I. Euro. J. Appl. Math. **5**, 449–468 (1994)
[C2] Chapman, S.J.: Nucleation of superconductivity in decreasing fields, II. Euro. J. Appl. Math. **5**, 469–494 (1994)
[DH] Dauge, M. and Helffer, B.: Eigenvalues variation I, Neumann problem for Sturm–Liouville operators. J.D.E. **104**, 243–262 (1993)
[DGP] Du, Q., Gunzburger, M.D. and Peterson, J.S.: Analysis and approximation of the Ginzburg–Landau model of superconductivity. SIAM Review **34**, 45–81 (1992)
[GL] Ginzburg, V.L. and Landau, L.D.: On the theory of superconductivity. J.E.T.P. **20**, 1064 (1950)
[GP] Giorgi, T. and Phillips, D.: The breakdown of superconductivity due to strong fields for the Ginzburg–Landau model. SIAM J. Math. Anal. **30**, 341–359 (1999)
[JRS] Jadallah, H. Rubinstein, J. and Sternberg, P.: Phrase transition curves for mesoscopic superconducting samples. Phys. Rev. Let. **82** 14, 2935–2938 (1999)
[LS] Levitan, B.M. and Sargsjan, I.S.: *Sturm–Liouville and Dirac Operators*. Mathematics and Its Applications (Soviet Series), Vol. **59**, Amsterdam: Kluwer Academic Publications, 1991
[LP1] Lu, K. and Pan, X.B.: Gauge invariant eigenvalue problems on $\mathbf{R}^2$ and $\mathbf{R}^2_+$. Preprint
[LP2] Lu, K. and Pan, X.B.: Eigenvalue problems of Ginzburg–Landau operator in bounded domains. J. Math. Phys. **40**, 2647–670 (1999)
[LP3] Lu, K. and Pan, X.B.: Estimates of the upper critical external magnetic field for the Ginzburg–Landau equations: Physica D **127**, 73–104 (1999)
[MK] Millman, M.H. and Keller, J.B.: Perturbation theory of nonlinear boundary value problems. J. Math. Phys. **10**, 342–61 (1969)
[MGSJQVB] Moschchalkov, V.V., Gielen, L. Strunk, C., Jonckheere, R., Qiu, X., Van Haesendonck, C. and Bruynseraede, Y.: Effect of sample topology on the critical fields of mesoscopic superconductors. Nature **373**, 319–322 (January 1995)
[O] Odeh, F.: Existence and bifurcation theorems for the Ginzburg–Landau equations. J. Math. Phys. **8**, 2351–357 (1967)
[SD] Saint-James, D. and de Gennes, P.G.: Onset of superconductivity in decreasing fields. Phys. Lett. **7**, 306–308 (1963)

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