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# Multi-peak solutions for some singular perturbation problems

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**Abstract.** We consider the problem

$$\begin{cases} \varepsilon^2 \Delta u - u + f(u) = 0 \ \text{ in } \ \Omega \\ \quad u > 0 \ \text{ in } \Omega, \ u = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where $\Omega$ is a smooth domain in $R^N$, not necessarily bounded, $\varepsilon > 0$ is a small parameter and $f$ is a superlinear, subcritical nonlinearity. It is known that this equation possesses a solution that concentrates, as $\varepsilon$ approaches zero, at a maximum of the function $d(x) = d(\cdot, \partial\Omega)$, *the distance to the boundary*. We obtain multi-peak solutions of the equation given above when the domain $\Omega$ presents a distance function to its boundary $d$ with multiple local maxima. We find solutions exhibiting concentration at any prescribed finite set of local maxima, possibly degenerate, of $d$. The proof relies on variational arguments, where a penalization-type method is used together with sharp estimates of the critical values of the appropriate functional. Our main theorem extends earlier results, including the single peak case. We allow a degenerate distance function and a more general nonlinearity.

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## 0 Introduction

Let us consider the problem

$$\begin{cases} \varepsilon^2 \Delta u - u + u^p = 0 \quad \text{in} \quad \Omega \\ \\ u > 0 \ \text{ in } \Omega, \ u = 0 \ \text{ on } \quad \partial\Omega, \end{cases} \tag{0.1}$$

where $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $\Omega$ is a smooth domain in $R^N$, not necessarily bounded, with boundary $\partial\Omega$, $N \geq 1$ , $\varepsilon > 0$ is a constant and $p$ satisfies $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$, and $1 < p < \infty$ for $N = 1, 2$.

A solution of problem (0.1) can be interpreted as a steady state of the corresponding reaction-diffusion equation $u_t = \varepsilon^2 \Delta u - u + u^p$, which arises in a number of areas, such as biological population and pattern formation theories and chemical reactor theory. In order to investigate the long time behavior of the solutions of the later equation, good understanding of the properties of the steady state solutions is very important.

Problem (0.1) has been recently studied by Ni and Wei in [12]. They proved that for $\varepsilon$ sufficiently small problem (0.1) has a least-energy solution which possesses a single spike-layer with its unique peak in the interior of $\Omega$. Moreover this unique peak must be situated near the “most-centered” part of $\Omega$, i.e. where the distance function $d(P, \partial\Omega)$, $P \in \Omega$, assumes its global maximum. This is in contrast with earlier results for the corresponding Neumann problem, obtained by Ni and Takagi in [10] and [11], where it was shown that for $\varepsilon$ sufficiently small, the problem has a least-energy solution which possesses a single spike-layer with its unique peak on the boundary $\partial\Omega$. This unique peak must be situated near the “most curved ” part of $\partial\Omega$, i.e. where the mean curvature of the boundary assumes its global maximum.

The results in [12] follow from an asymptotic expansion of the critical value associated to the least energy solution. This expansion requires sharp estimates of exponentially small error terms that are obtained by using a “vanishing viscosity method”.

A natural problem to study for further insight into the rich and complex structure of the solution set of this equation for small $\varepsilon$, is that of determining existence of other solutions which exhibit concentration behavior like the one described above, at one or several distinct points of the domain.

In this direction, Wei showed in [13] a local version of the above result, namely that for every strict local maximum point of the distance function, say $P$, there exists a family of solutions with a single global maximum point that approaches the given point $P$.

In a recent article [1], Cao, Dancer, Noussair and Yan addressed the problem of constructing a family of solution with multiple concentration points. In fact they found a two-peak solution for (0.1), with peaks near strict local maxima of the the distance function, say $P_1$ and $P_2$, satisfying additionally

i) $|P_1 - P_2| > \max\{1/(p-2), 2\} d(P_1, \partial\Omega)$ and
ii) $d(P_1, \partial\Omega) = d(P_2, \partial\Omega)$.

Several interesting cases are not covered by these results. For example, degeneracy of the considered maxima may arise in very natural ways. This is the case of a domain consisting of a chain of rectangles of unequal sides with narrow junctions, where local maxima of the distance to the boundary are achieved on entire segments. The methods in [13] and [1] do not seem to give account of the construction in such cases. Moreover, they do not seem to extend directly to the case where the nonlinearity is not homogeneous. On the other hand, in the 2-peak case, condition ii) above imposes a strong symmetry requirement, which we will lift. In fact, as a particular case of our main result, we will see that it is enough to assume $|P_1 - P_2| > 2d(P_1, \partial\Omega)$, which is a weaker requirement than i). We should remark that our method *does not need $\Omega$ to be bounded*.

Thus, our purpose is to construct multi-peak solutions of (0.1) under much weaker assumptions on the domain and replacing $u^p$ by a more general non-homogeneous nonlinearity. More precisely, we consider the problem

$$\begin{cases} \varepsilon^2 \Delta u - u + f(u) = 0 \ \text{ in } \Omega, \\ \\ u > 0 \ \text{ in } \Omega \text{ and } u = 0 \ \text{ in } \partial\Omega. \end{cases} \tag{0.2}$$

We will assume that $f : R^+ \to R$ is of class $C^{1+\sigma}$ and satisfies the following conditions

(f1) $f(t) \equiv 0$ for $t \le 0$ and $f(t) \to +\infty$ as $t \to \infty$.
(f2) For $t \ge 0$, $f$ admits the decomposition in $C^{1+\sigma}(R)$

$$f(t) = f_1(t) - f_2(t),$$

where $f_1$ and $f_2$ satisfy
(i) $f_1(t) \ge 0$ and $f_2(t) \ge 0$ with $f_1(0) = f_1'(0) = 0$, and
(ii) there is a $q \ge 1$ such that $f_1(t)/t^q$ is nondecreasing in $t > 0$, whereas $f_2(t)/t^q$ is nonincreasing in $t > 0$, and in case $q = 1$ we require further that the above monotonicity condition for $f_1(t)/t$ is strict.
(f3) $f(t) = O(t^p)$ as $t \to +\infty$ where $1 < p < \frac{N+2}{N-2}$ if $N \ge 3$ and $1 < p < \infty$ if $N = 2$.

(f4) There exists a constant $\theta > 2$ such that $\theta F(t) \leq tf(t)$ for $t \geq 0$, in which

$$F(t) = \int_0^t f(s)ds. \tag{0.3}$$

To state the last condition on $f$ we need some preparations. Consider the problem in the whole space

$$\begin{cases} \Delta w - w + f(w) = 0 \ \text{ and } w > 0 \text{ in } R^N, \\ w(0) = \max_{z \in R^N} w(z) \text{ and } w(z) \to 0 \text{ as } |z| \to +\infty. \end{cases} \tag{0.4}$$

It is known that that any solution to (0.4) needs to be spherically symmetric about the origin and strictly decreasing in $r = |z|$, see [6]. A solution $w$ to (1.5) is said to be *nondegenerate* if the linearized operator

$$L = \Delta - 1 + f^{'}(w) \tag{0.5}$$

on $L^2(R^N)$, with domain $W^{2,2}(R^N)$, has a bounded inverse when it is restricted to the subspace $L^2_r(R^N) := \left\{u \in L^2(R^N) | u(z) = u(|z|)\right\}$.

Now condition (f5) is stated as follows:

(f5) Problem (0.4) has a unique solution $w$, and it is nondegenerate.

The unique solution in (f5) will be denoted by $w$ in the rest of this paper.

We note that the function

$$f(t) = t^p - at^q \ \text{ for } t \geq 0,$$

with a constant $a \geq 0$, satisfies all the assumptions (f1)-(f4) if $1 < q < p < \frac{N+2}{N-2}$. Furthermore, there is a unique solution $w$ to problem (0.4) (see [2] and [8]). The nondegeneracy condition (f5) can be derived from the uniqueness argument (see Appendix C in [11] ).

Next we describe our assumptions on the domain. We assume $\Omega$ is a smooth domain in $R^N$, not necessarily bounded, and there are $K$ smooth bounded subdomains of $\Omega$, $\Lambda_1, \ldots, \Lambda_K$, compactly contained in $\Omega$, satisfying

(H1) $\max\limits_{x \in \Lambda_i} d(x, \partial\Omega) > \max\limits_{x \in \partial\Lambda_i} d(x, \partial\Omega), i = 1, 2, \ldots, K,$

(H2) $\min\limits_{i \neq k} d(\Lambda_i, \Lambda_k) > 2 \max\limits_{i} \max\limits_{x \in \Lambda_i} d(x, \partial\Omega)$

where $d(\Lambda_i, \Lambda_k)$ is the distance between $\Lambda_i$ and $\Lambda_k$, i.e.

$$d(\Lambda_i, \Lambda_k) = \inf_{x \in \Lambda_i, y \in \Lambda_k} d(x, y).$$

Note that on each $\Lambda_i$ the distance function has a local maximum, but no assumption is made on the set where this maximum is achieved. In particular these maxima do not need to be isolated.

Next we state our main result in this paper.

**Theorem 0.1** *Assume $\Omega$ is a domain, not necessarily bounded, which satisfies assumptions (H1) and (H2). Let $f$ satisfy assumptions (f1)-(f5). Then for $\varepsilon$ sufficiently small problem (0.2) has a solution $u_\varepsilon$ which possesses exactly $K$ local maximum points $x_{\varepsilon,1}, \ldots, x_{\varepsilon,K}$ with $x_{\varepsilon,i} \in \Lambda_i$. Moreover $d(x_{\varepsilon,i}, \partial\Omega) \to \max_{x\in\Lambda_i} d(x, \partial\Omega)$, as $\varepsilon \to 0$, for all $i = 1, \ldots, K$ .*

As we mentioned before, this result largely extends those in [13] and [1].

We prove Theorem 0.1 using variational techniques developed in [4] and [5] for the construction of single and multi-peak solutions for a nonlinear Schrödinger equation. In this process, sharp estimates of critical values of the associated energy involving the distance function obtained in [12] and [13] are crucial.

Our approach consists of modifying adequately the nonlinearity and adding a penalization term, so that a new *penalized energy functional* is obtained. Then we find critical points to this functional by means of a min-max scheme on a class of maps defined on a finite dimensional set. As $\varepsilon$ approaches zero, one then shows that the critical points so obtained are solutions of the original problem. In doing so a central role is played by the estimates for least-energy solutions found in [12].

It would be interesting to determine which phenomena arises if assumption (H2) is violated. Our proof makes strong use of the validity of strict inequality. In fact, (H2) is what makes negligible the interaction between "peaks" located in different $\Lambda_i's$, when compared with the contribution to the energy due to the effect of the distance to the boundary.

This paper is organized as follows. In §1 we consider the single peak case, namely $K = 1$. In Sect. 2 we treat the general multi-peak case and conclude the proof of Theorem 0.1.

## 1 The case of a single peak

This section is devoted to the study of the case of a single peak. The results in this section will serve as a basis for the multi-peak case, however they are interesting by themselves and correspond to Theorem 0.1 for $K = 1$.

In this section we will assume the set $\Omega$ is a smooth domain in $R^N$, not necessarily bounded. We also assume that there is a smooth bounded domain $\Lambda$, compactly contained in $\Omega$ satisfying hypothesis (H1) with $K = 1$. With respect to $f$ we consider the hypotheses (f1)-(f5).

Following the idea introduced in [4], we modify the function $f$ penalizing concentration outside the set $\Lambda$. Then we set up the mountain pass scheme in order to obtain critical points of the penalized functional and, using the results in [12], we provide the necessary estimates to discard the penalization.

Associated to equation (0.2) we have the "energy" functional

$$I_\varepsilon(u) = \frac{1}{2}\int\limits_{\Omega} \varepsilon^2|\nabla u|^2 + u^2 - \int\limits_{\Omega} F(u), \tag{1.1}$$

defined in $H_0^1(\Omega)$. In a similar way, associated to (0.4) we have the 'limiting functional' $I : H^1(R^N) \to R$ defined as

$$I(u) = \frac{1}{2}\int\limits_{R^N} |\nabla u|^2 + u^2 - \int\limits_{R^N} F(u). \tag{1.2}$$

Under the hypotheses on $f$ and $\Omega$, it is standard to check that the nontrivial critical points of $I_\varepsilon$ and $I$ correspond exactly to the positive classical solutions of equation (0.2) in $H_0^1(\Omega)$ and of equation (0.4) in $H^1(R^N)$, respectively. From the assumption (f5) the functional $I$ has a unique positive critical point, up to translations, we call $w$. The critical value of $w$ (of a mountain pass nature) will be denoted by $c$.

Next we modify the function $f$ as in [4]. Let $\theta$ be a number as given by (f4), and let us choose $k > 0$ such that $k > \frac{\theta}{\theta-2}$. Let $a > 0$ be a value at which $f'(a) = \frac{1}{k}$ and $f(t) \geq f(a) + f'(a)(t-a)$ for all $t \geq a$. Let us set

$$\tilde{f}(t) = \begin{cases} f(t) & \text{if } t \leq a \\ f(a) + f'(a)(t-a) & \text{if } t > a, \end{cases} \tag{1.3}$$

and define

$$g(\cdot, t) = \chi_\Lambda f(t) + (1-\chi_\Lambda)\tilde{f}(t), \tag{1.4}$$

where $\chi_\Lambda$ denotes the characteristic function of $\Lambda$. First we note that $g$ is a Caratheodory function, moreover it is of class $C^{1+\sigma}$ in the variable $t$. This fact is crucial in using the methods of [12]. In addition one can check that (f1)-(f4) implies that $g$ satisfies the following assumptions:

(g1) $g(x,t) = 0$ for $t \leq 0$ and $g(x,t) \to \infty$ as $t \to \infty$.
(g2) $g(x,t) = o(t)$ near $t = 0$ uniformly in $x \in \Omega$.
(g3) $g(x,t) = O(t^p)$ as $t \to \infty$ for $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$ and no restriction on $p$ if $N = 1, 2$.
(g4) (i) $\theta G(z,t) \leq g(x,t)t$ for all $x \in \Lambda, t > 0$.
and
(ii) $2G(x,t) \leq g(x,t)t \leq \frac{1}{k}t^2$ for all $t \in R^+, x \notin \Lambda$ with the number $k$ satisfying $k > \theta/(\theta-2)$.

Here we have denoted $G(z,\xi) = \int_0^\xi g(z,\tau)d\tau$.

Next we introduce the penalized functional $J_\varepsilon : H_0^1(\Omega) \to R$ as

$$J_\varepsilon(u) = \frac{1}{2}\int\limits_\Omega \varepsilon^2|\nabla u|^2 + u^2 - \int\limits_\Omega G(x,u) \quad , \quad u \in H_0^1(\Omega). \tag{1.5}$$

The functional $J_\varepsilon$ is of class $C^1$ in $H_0^1(\Omega)$ and its nontrivial critical points are the positive solutions of the equation

$$\varepsilon^2 \Delta u - u + g(x,u) = 0 \qquad \text{in} \quad \Omega. \tag{1.6}$$

The functional $J_\varepsilon$ is of class $C^1$ and it satisfies the Palais Smale condition, see Lemma 1.1 in [4]. Moreover, it can be easily seen from hypotheses (g1)-(g4) that $J_\varepsilon$ has the mountain pass structure. Thus the Mountain Pass Theorem can be applied to $J_\varepsilon$.

We define the mountain pass value

$$c_\varepsilon = \inf_{\gamma \in G} \max_{0 \le t \le 1} J_\varepsilon(\gamma(t)),$$

where $G = \{\gamma \in C([0,1], H_0^1(\Omega) \mid \gamma(0) = 0,\ \gamma(1) = e\}$ and $e \neq 0$ is such that $J_\varepsilon(e) < 0$. On the other hand we can define

$$\tilde{c}_\varepsilon = \inf_{u \in M} J_\varepsilon(u),$$

where

$$M = \{u \in H_0^1(\Omega) \mid \int_\Omega \varepsilon^2|\nabla u|^2 + u^2 = \int_\Omega ug(x,u)dx\}.$$

By the assumptions (f1)-(f4) and the definition of $g$, we can prove, slightly modifying the proof of Lemma B.1 in [11] and Lemma 1.2 in [4], that $c_\varepsilon = \tilde{c}_\varepsilon$.

Let $u_\varepsilon \in H_0^1(\Omega)$ be a critical point of $J_\varepsilon$ satisfying $J_\varepsilon(u_\varepsilon) = c_\varepsilon$, that is $u_\varepsilon$ is a least energy critical point of $J_\varepsilon$. In the rest of this section we will obtain estimates for $c_\varepsilon$ and use them for studying the limiting behavior of $u_\varepsilon$. Certainly $u_\varepsilon$ satisfies (1.6), but we will prove that for $\varepsilon$ small enough $u_\varepsilon$ satisfies (0.2) also.

Given $v \in H^1(R^N)$ we define the *projection of $v$ into the domain $D$*, and denote it by $P_D v$, as the unique solution of the problem

$$\begin{cases} \varepsilon^2 \Delta u - u + f(v(x)) = 0 & \text{in } D \\ \\ u = 0 & \text{on } \partial D. \end{cases} \tag{1.7}$$

Let $w$ be the unique solution of (0.4) and define $w_{\varepsilon,P}(x) = w(\frac{x-P}{\varepsilon})$ for $P \in \Omega$.

Following [12] we define the function

$$\psi_\varepsilon(x,P) = -\varepsilon \log\{w_{\varepsilon,P}(x) - P_\Omega w_{\varepsilon,P}(x)\}. \tag{1.8}$$

The behavior of $\psi_\varepsilon$ when $\varepsilon$ converges to zero is studied in [12], where the next result is proved (see Lemmas 4.2 and 4.3)

**Lemma 1.1** *(i)* $\|\nabla\psi_\varepsilon(x,P)\|_{L^\infty(\Omega')} \leq C$ *for any compact set* $\Omega' \subset \Omega$.

*(ii)* $\psi_\varepsilon(\cdot,P)$ *converges, as* $\varepsilon \to 0$, *uniformly to a function* $\psi(\cdot,P) \in W^{1,\infty}(\Omega)$ *which can be explicitly written as*

$$\psi(x,P) = \inf_{z\in\partial\Omega}\{|z-P| + L(z,x)\}, \tag{1.9}$$

*where* $L(x,y)$ *is the infimum of* $T$ *such that there exists* $\xi \in C^{0,1}([0,T],\overline{\Omega})$, *with* $\xi(0) = x, \xi(T) = P$, *and* $|\frac{d\xi}{ds}| \leq 1$ $a.e.$ *in* $[0,T]$. *In particular,* $\psi(P,P) = 2d(P,\partial\Omega)$.

In what follows we will make two notational simplifications. We will write $\beta = 1/\varepsilon$ and $\psi_\varepsilon(x) = \psi_\varepsilon(x,x)$.

Now we state the main result of the section, that can be seen as a single-peak version of Theorem 0.1. It also contains the asymptotic expansion of $c_\varepsilon$.

**Theorem 1.1** *Assume* $\Omega$ *satisfies (H1) and* $f$ *satisfies (f1) - (f5). Then for* $\varepsilon$ *sufficiently small problem (0.2) has a solution* $u_\varepsilon$ *which possesses exactly one local maximum points* $x_\varepsilon \in \Lambda$ *and* $d(x_\varepsilon,\partial\Omega) \to \max_{x\in\Lambda} d(x,\partial\Omega)$, *as* $\varepsilon \to 0$.

*Moreover, the following estimate holds*

$$c_\varepsilon = \varepsilon^N\{c + Ae^{-\beta\psi_\varepsilon(x_\varepsilon)} + o(e^{-\beta\psi_\varepsilon(x_\varepsilon)})\}, \tag{1.10}$$

*where*

$$\psi_\varepsilon(x_\varepsilon) \to 2\max_{x\in\Lambda} d(x,\partial\Omega)$$

*and*

$$A = \int_{R^N} f(w)u_* > 0. \tag{1.11}$$

*Here* $u_*$ *is the unique radial solution of*

$$\left.\begin{aligned} \Delta u - u = 0 \text{ in } R^N \\ u(0) = 1, u > 0 \text{ in } R^N. \end{aligned}\right\} \tag{1.12}$$

In order to prove this theorem, we obtain first the rough behavior of the family $\{u_\varepsilon\}$ of least energy critical points of $J_\varepsilon$.

**Lemma 1.2** *Let $u_\varepsilon$ as defined above. Then, for $\varepsilon$ small enough $u_\varepsilon$ possesses a unique maximum point $x_\varepsilon \in \Lambda$ and the family $v_\varepsilon$ defined as $v_\varepsilon(y) = u_\varepsilon(x_\varepsilon + \varepsilon y)$ converges in the $C^1_{loc}(R^N)$ and $H^1(R^N)$ sense to the solution $w$ of (0.4).*

*Proof.* We just give a very brief sketch. After obtaining estimates of $c_\varepsilon$ and the $H^1(\Omega)$ norm of $u_\varepsilon$, one can take weak limits of $v_\varepsilon$. Calling $v$ a limit point of $v_\varepsilon$ we find that it satisfies an equation of the form

$$\Delta v - v + \bar{g}(x,s) = 0 \qquad \text{in } R^N, \tag{1.13}$$

where

$$\bar{g}(x,s) = \chi(x) f(s) + (1-\chi(x))\tilde{f}(s), \tag{1.14}$$

and $\chi$ is the characteristic function of an hyperplane in $R^N$. It is shown in [5] Lemma 2.3, that $v$ is actually a solution of (0.4), proving the last sentence in the lemma. This fact is crucial for later arguments.

In order to obtain stronger convergence, we use concentration compactness argument to obtain a unique maximum of $v_\varepsilon$. Then the maximum principle together with elliptic estimates will complete the lemma. See [4], [5] and also [12] for further details. □

*Proof of Theorem 1.1* First we find an upper bound of $c_\varepsilon$. Following Sect. 5 in [12], we consider the test function $P_\Omega w_{\epsilon,P}(x)$ where $d(P,\partial\Omega) = \max_{z\in\Lambda} d(z,\partial\Omega)$, and find

$$c_\varepsilon \le \varepsilon^N \left\{ c + Ae^{-\beta\psi_\varepsilon(P)} + o(e^{-\beta\psi_\varepsilon(P)}) \right\}. \tag{1.15}$$

We observe that $J_\varepsilon(P_\Omega w_{\varepsilon,P}) = I_\varepsilon(P_\Omega w_{\varepsilon,P})$ for small $\varepsilon$.

Next we show that $u_\varepsilon$ is actually a solution of the original equation, when $\varepsilon$ is small enough. For this purpose we assume the contrary, that is, there is a sequence $\varepsilon_j$ converging to 0 such that $x_{\varepsilon_j}$ converges to a point $\bar{x} \in \Lambda$, where $x_{\varepsilon_j}$ is the maximum point of the function $u_{\varepsilon_j}$.

Let us choose a number $d > 0$ such that

$$d(\bar{x},\partial\Omega) < d < \max_{x\in\Omega} d(x,\partial\Omega) \equiv d_0,$$

and consider the domain $\Omega^d$ defined as the connected component of $\Omega \cap B(\bar{x},d)$ containing $\bar{x}$. Then we have

$$d' \equiv \max_{x\in\Omega^d} d(x,\partial\Omega^d) < d,$$

since $B(\bar{x},d) \not\subset \Omega$. Now we choose $d''$ such that $d' < d'' < d$.

We define on $H_0^1(\Omega^d)$ the functional

$$I_\varepsilon^d(u) = \frac{1}{2}\int_{\Omega^d} \varepsilon^2|\nabla u|^2 + u^2 - \int_{\Omega^d} F(u). \tag{1.16}$$

Let $\eta \in C_0^\infty(R^N)$ be a cut-off function with $\eta(x) = 1$ if $|x - \bar{x}| \leq d''$ and $\eta(x) = 0$ if $|x - \bar{x}| \geq d$. Then consider $\tilde{u}_\varepsilon = \eta(x)u_\varepsilon(x) \in H_0^1(\Omega^d)$ and let $t_\varepsilon > 0$ be such that

$$I_\varepsilon^d(t_\varepsilon u_\varepsilon) = \max_{t\geq 0} I_\varepsilon^d(tu_\varepsilon).$$

We claim that $t_\varepsilon \to 1$. In fact, if we set $\tilde{v}_\varepsilon(y) = \tilde{u}_\varepsilon(x_\varepsilon + \varepsilon y)$, then from Lemma 1.2 $\tilde{v}_\varepsilon \to w$ in $H^1(R^N)$. Then, by definition of $t_\varepsilon$,

$$\int_{R^N} \varepsilon^2|\nabla \tilde{v}_\varepsilon|^2 + \tilde{v}_\varepsilon^2 = \int_{R^N} \frac{f(t_\varepsilon \tilde{v}_\varepsilon)}{t_\varepsilon}\tilde{v}_\varepsilon, \tag{1.17}$$

from where the claim follows.

Next we find a lower estimate for $c_{\varepsilon_j}$ under our contradiction hypothesis. For notational convenience we will replace, when no confusion arises, $\varepsilon_j$ by $j$ in the rest of the proof.

By definition of $c_j$ and the functionals we have

$$c_j = J_j(u_j) \geq J_j(t_j u_j) \geq I_j(t_j u_j). \tag{1.18}$$

Now we claim that

$$I_j(t_j u_j) \geq I_j^d(t_j \tilde{u}_j) - ce^{-2\beta_j(d''-\delta)}, \tag{1.19}$$

where $c$ is a positive constant and $\delta > 0$ is chosen so that $d'' - \delta > d'$. In fact, from the Maximum Principle and elliptic estimates we have

$$\max_{x\in\Omega}\{|u_j|, |\nabla u_j|\} \leq Ce^{-\beta_j(d''-\delta)}, \tag{1.20}$$

so that, from the definition of $\tilde{u}_j$, the estimate (1.19) clearly follows.

Finally we have

$$I_j^d(t_j\tilde{u}_j) \geq \inf_{u\neq 0, u\in H_0^1(\Omega^d)} \sup_{t\geq 0} I_j^d(tu) \equiv c_j^d. \tag{1.21}$$

Now we are in a position to apply the main estimate in [12], Proposition 6.3, to obtain

$$c_j^d \geq \varepsilon_j^N \left\{c + Ae^{-\beta_j\psi_j^d} + o(e^{-\beta_j\psi_j^d})\right\} \tag{1.22}$$

where the $\psi_j^d$'s are certain numbers with

$$\psi_j^d \to 2\max_{x\in\Omega^d} d(x, \partial\Omega^d), \tag{1.23}$$

and $A$ is given by (1.11). Therefore, for large $j$ we have

$$c_j^d \geq \varepsilon_j^N \left\{ c + \frac{A}{2} e^{-\beta_j 2d'} \right\} \tag{1.24}$$

Then, using (1.18), (1.19), (1.21), (1.22), and the fact that $d' < d'' - \delta$, we find that

$$c_j \geq \varepsilon_j^N \left\{ c + \frac{A}{3} e^{-2\beta_j d'} \right\}. \tag{1.25}$$

Since $d(P, \Omega) > d'$ and estimate (1.15) holds, it follows that (1.25) is not possible. Thus we have a contradiction which shows that $u_\varepsilon$ is a solution of the original equation for all sufficiently small $\varepsilon$. We observe also that using the same argument yields $d(x_\varepsilon, \partial\Omega) \to \max_{x \in \Lambda} d(x, \partial\Omega)$, as $\varepsilon \to 0$.

Finally, estimate (1.10) can be obtained from the results of [12]. In fact Proposition 6.3 applies to our situation, since $u_\varepsilon$ is a solution of (0.2) and $x_\varepsilon$ is its maximum point. This finishes the proof of the theorem. □

## 2 The case of multiple peaks

In this section we prove Theorem 0.1. For this purpose we introduce an appropriate penalization so that the concentration outside the sets $\Lambda_i$ is avoided. We set up the minimax scheme in order to obtain critical points of the penalized functional and then we provide some estimates in order to show that these critical points are solutions of the original equation.

We consider, as in Sect. 1, the functional $I_\varepsilon$ defined in (1.1). We set $\Lambda = \cup_{i=1}^K \Lambda_i$, with the bounded domains $\Lambda_i$ as in the assumptions of Theorem 0.1, and define

$$g(\cdot, t) = \chi_\Lambda f(t) + (1 - \chi_\Lambda)\tilde{f}(t), \tag{2.1}$$

where $\chi_\Lambda$ denoting is the characteristic function of $\Lambda$. Then we consider the functional $J_\varepsilon$ as in (1.5); its critical points are the solutions of (1.6).

Next we define in $H_0^1(\Omega)$ a ‘local’ version of $J_\varepsilon$ as

$$J_\varepsilon^i(u) = \frac{1}{2} \int_{\Omega_i} \varepsilon^2 |\nabla u|^2 + u^2 - \int_{\Omega_i} G(x, u), \qquad u \in H_0^1(\Omega). \tag{2.2}$$

We also introduce a further penalization term, similar to that considered in [5], defined as

$$P_\varepsilon(u) = M \sum_{i=1}^K \left\{ \left(J_\varepsilon^i(u)_+\right)^{\frac{1}{2}} - (\frac{3}{2} c\varepsilon^N)^{\frac{1}{2}} \right\}_+^2, \tag{2.3}$$

where $M$ is a constant to be chosen later, and the sets $\Omega_i$ are defined as

$$\Omega_i = \{x \in \Omega \mid d(x, \Lambda_i) < \frac{1}{2} \min_{k \neq i} d(\Lambda_i, \Lambda_k)\}, \qquad i = 1, \ldots, K.$$

We note that the sets $\Omega_i$ contain $\Lambda_i$ compactly and they are mutually disjoint. Finally we define the penalized functional $E_\varepsilon : H_0^1(\Omega) \to R$ as

$$E_\varepsilon(u) = J_\varepsilon(u) + P_\varepsilon(u). \tag{2.4}$$

The functionals $J_\varepsilon$ and $P_\varepsilon$ are of class $C^1$ hence so is $E_\varepsilon$. Moreover, the functional $E_\varepsilon$ satisfies the Palais-Smale condition, see Lemma 1.1 in [5].

The previous considerations make possible to use Critical Point Theory to find critical points of the functional $E_\varepsilon$. We formulate now an appropriate minimax problem for $E_\varepsilon$. Set

$$\hat{u}_i = P_{\Omega_i} w_{\varepsilon, P_i},$$

where $P_i \in \Lambda_i$ is such that $d(P_i, \partial\Omega) = \max\{d(P, \partial\Omega) \mid P \in \Lambda_i\}$, $i = 1, \ldots, K$. Define the class $\Gamma$ as

$$\Gamma = \Big\{\gamma \in C([0,T]^K, H_0^1) \,/\, \gamma(t) = \sum_{i=1}^{K} t_i \hat{u}_i,$$
$$\forall\, t = (t_1, \ldots, t_K) \in \partial[0,T]^K\Big\}.$$

The number $T$ is chosen so large that $I(Tw) < 0$. Since $w$ is exponentially decaying, it is easy to show that for any $\gamma \in \Gamma$ we have

$$E_\varepsilon(\gamma(t)) < \varepsilon^N (Kc + o(1)) \tag{2.5}$$

for all $t \in \partial[0,T]^K$, if $\varepsilon$ is small enough. Here and in what follows we denote by $o(1)$ a quantity approaching zero as $\varepsilon \to 0$. We can define the minimax value associated to the class $\Gamma$ as follows

$$C_\varepsilon = \inf_{\gamma \in \Gamma} \sup_{t \in [0,T]^K} E_\varepsilon(\gamma(t)).$$

The following lemma provides a first estimate on the minimax value $C_\varepsilon$.

**Lemma 2.1**

$$C_\varepsilon = \varepsilon^N (Kc + o(1)). \tag{2.6}$$

The proof of this lemma is similar to that of Lemma 1.2 in [5], hence we omit it.

It follows from estimates (2.5) and (2.6) that there exits a critical point $u_\varepsilon \in H_0^1(\Omega)$ of $E_\varepsilon$ such that $E_\varepsilon(u_\varepsilon) = C_\varepsilon$.

We define the local weight

$$\mathrm{w}_\varepsilon^i = M \left\{ \left(J_\varepsilon^i(u_\varepsilon)_+\right)^{\frac{1}{2}} - (\frac{3}{2}c\varepsilon^N)^{\frac{1}{2}} \right\}_+ \left(J_\varepsilon^i(u_\varepsilon)\right)^{-\frac{1}{2}},$$

and then the function

$$\mathrm{w}_\varepsilon = \sum_{i=1}^K \mathrm{w}_\varepsilon^i \chi_{\Omega_i}. \tag{2.7}$$

Thus the critical point $u_\varepsilon$ is a weak solution of the equation

$$\varepsilon^2 \mathrm{div}((1+\mathrm{w}_\varepsilon)\nabla u) - (1+\mathrm{w}_\varepsilon)u + (1+\mathrm{w}_\varepsilon)g(x,u) = 0 \quad \text{in} \quad \Omega, \tag{2.8}$$

and $u_\varepsilon$ satisfies

$$\varepsilon^2 \Delta u - u + g(x,u) = 0 \quad \text{in} \quad \mathcal{O}, \tag{2.9}$$

for every set $\mathcal{O} \subset \Omega$ not intersecting $\partial(\cup_{i=1}^K \Omega_i)$.

We define the sets $\Omega_\varepsilon = \{y \in R^N \,/\, \varepsilon y \in \Omega\}$ and $\Omega_i^\varepsilon = \{y \in R^N \,/\, \varepsilon y \in \Omega_i\}$. We rescale the function $u_\varepsilon$ as $v_\varepsilon(y) = u_\varepsilon(\varepsilon y)$ for $y \in \Omega_\varepsilon$. This function $v_\varepsilon$ belongs to $H_0^1(\Omega_\varepsilon)$ and then to $H^1(R^N)$, and it satisfies, in a weak sense, the equation

$$\mathrm{div}((1+\mathrm{w}_\varepsilon(\varepsilon y))\nabla u) - (1+\mathrm{w}_\varepsilon(\varepsilon y))u + (1+\mathrm{w}_\varepsilon(\varepsilon y))g(\varepsilon y, u) = 0 \quad \text{in} \quad \Omega_\varepsilon, \tag{2.10}$$

and over sets $\mathcal{O}$, subsets of $\Omega_\varepsilon$ not intersecting $\partial(\cup_{i=1}^K \Omega_i^\varepsilon)$, it satisfies

$$\Delta u - u + g(\varepsilon y, u) = 0 \qquad \text{in} \quad \mathcal{O}. \tag{2.11}$$

In order to complete the proof of Theorem 0.1 we need to show that $u_\varepsilon$ is a critical point of the original functional $I_\varepsilon$ whenever $\varepsilon$ is sufficiently small. Toward this end, the following lemma constitutes a crucial step.

**Lemma 2.2** *If in the definition of $E_\varepsilon$ in (2.3) and (2.4), $M > 0$ was chosen sufficiently large, then*

$$\lim_{\varepsilon \to 0} J_\varepsilon^i(u_\varepsilon)\varepsilon^{-N} = c, \quad \textit{for all} \quad i = 1, \ldots, K. \tag{2.12}$$

The proof of this lemma can be obtained by slightly modifying the proof of Lemmas 2.1 and 2.2 in [5] and it is thus omitted.

From Lemma 2.2, we have in particular that $u_\varepsilon$ satisfies the equation

$$\varepsilon^2 \Delta u_\varepsilon - u_\varepsilon + \chi_\Lambda f(u_\varepsilon) + (1-\chi_\Lambda)\tilde{f}(u_\varepsilon) = 0 \text{ in } \Omega, \tag{2.13}$$

where, we recall, $\Lambda = \cup_{i=1,K} \Lambda_i$. We also have that $u_\varepsilon$ has $K$ maximum points, say $x_{\varepsilon,i}$, one in each $\Lambda_i$. The next step is to show that $u_\varepsilon$ satisfies the original equation. For that purpose it will be enough to prove the following lemma.

**Lemma 2.3**

$$\sup_{\partial\Lambda_i} u_\varepsilon \to 0 \textit{ as } \varepsilon \to 0 \textit{ for all } i = 1, \dots, K. \tag{2.14}$$

*Proof.* We will use energy estimates in the spirit of Sect. 1. First we consider *local problems*, defining the mountain pass values

$$c_{\varepsilon,i} = \inf_{u \in M_i} J_\varepsilon^i(u),$$

where

$$M_i = \left\{ u \in H_0^1(\Omega_i) \ \Big|\ \frac{1}{2}\int_{\Omega_i} |\varepsilon^2 \nabla u|^2 + \int_{\Omega_i} u^2 = \int_{\Omega_i} ug(x,u)dx \right\}.$$

Theorem 1.1 applies to these local problems, so that estimates like (1.10) hold for every $i = 1, \dots, K$. Moreover, a critical point $u_{\varepsilon,i}$ of $J_\varepsilon^i$ with value $c_{\varepsilon,i}$ possesses a unique maximum point $P_{\varepsilon,i}$ and $d(P_{\varepsilon,i}, \partial\Omega_i) \to \max_{x\in\Lambda_i} d(x, \partial\Omega_i)$.

Next, we want to find a good estimate for $J_\varepsilon^i(u_\varepsilon)$. For this purpose we first see that, since $\Omega_i \cap \Omega_k = \phi$, for $i \neq k$,

$$C_\varepsilon \leq \sum_{i=1}^{K} c_{\varepsilon,i}. \tag{2.15}$$

On the other hand consider the function $\tilde{u}_{\varepsilon,i}(x) = \eta_i(x) u_\varepsilon(x) t_{\varepsilon,i} \in H_0^1(\Omega_i)$ where $t_{\varepsilon,i}$ is such that $\tilde{u}_{\varepsilon,i}(x) \in M_i$ and $\eta_i \in C_0^\infty(\Omega_i)$ is a function taking the value 1, except for neighborhood of $\partial\Omega_i$ of small radius $\rho_0$. □

The next lemma provides an estimate on $t_{\varepsilon,i}$.

**Lemma 2.4**

$$t_{\varepsilon,i} = 1 + O\left( e^{-\beta(1-\delta_1) \min_{k\neq i} d(\Lambda_i, \Lambda_k)} \right), \qquad for\ 0 < \delta_1 < 1. \tag{2.16}$$

*Proof.* We have

$$\begin{aligned} &\int_{\Omega_i} \varepsilon^2 |\nabla u_\varepsilon \eta_i|^2 + (u_\varepsilon \eta_i)^2 - \int_{\Omega_i} g(x, u_\varepsilon \eta_i) u_\varepsilon \eta_i \\ &= \int_{\Omega_i} \varepsilon^2 |\nabla u_\varepsilon|^2 + u_\varepsilon^2 - \int_{\Omega_i} g(x, u_\varepsilon) u_\varepsilon + O\left( e^{-\beta(1-\delta_1) \min_{j\neq i} d(\Lambda_i, \Lambda_j)} \right) \\ &= O\left( e^{-\beta(1-\delta_1) \min_{j\neq i} d(\Lambda_i, \Lambda_j)} \right). \end{aligned} \tag{2.17}$$

Therefore

$$t_{\varepsilon,i} = 1 + O\left(e^{-\beta(1-\delta_1)\min\limits_{j\neq i} d(\Lambda_i,\Lambda_j)}\right).$$

□

Continuing with the proof of Lemma 2.3, from (2.16) and an estimate like (1.20) we obtain $J^i_\varepsilon$ we obtain

$$\begin{aligned} J^i_\varepsilon(u_\varepsilon) &= \frac{1}{2}\int_{\Omega_i} \varepsilon^2|\nabla u_\varepsilon|^2 + u_\varepsilon^2 - \int_{\Omega_i} G(x,u_\varepsilon) \\ &= \frac{1}{2}\int_{\Omega_i} \varepsilon^2|\nabla \tilde{u}_\varepsilon|^2 + \tilde{u}_\varepsilon^2 - \int_{\Omega_i} G(x,\tilde{u}_\varepsilon) \\ &+ O\left(e^{-\beta(1-\delta_1)\min\limits_{k\neq i} d(\Lambda_i,\Lambda_k)}\right) \\ &\geq c_{\varepsilon,i} + O\left(e^{-\beta(1-\delta_1)\min\limits_{k\neq i} d(\Lambda_i,\Lambda_k)}\right). \end{aligned} \tag{2.18}$$

Noticing that

$$C_\varepsilon \geq \sum_{i=1}^{K}\left(\frac{1}{2}\int_{\Omega_i} \varepsilon^2|\nabla u_\varepsilon|^2 + u_\varepsilon^2 - \int_{\Omega_i} G(x,u_\varepsilon)\right),$$

and combining (2.15) with (2.18) we obtain, for every $i = 1,\dots,K$, that

$$\begin{aligned} &\frac{1}{2}\int_{\Omega_i} \varepsilon^2|\nabla u_\varepsilon|^2 + u_\varepsilon^2 - \int_{\Omega_i} G(x,u_\varepsilon) \\ &\quad = c_{\varepsilon,i} + O\left(e^{-\beta(1-\delta_1)\min\limits_{k\neq i} d(\Lambda_i,\Lambda_k)}\right). \end{aligned} \tag{2.19}$$

Now we prove (2.14). Assume by contradiction that there exists $i \in \{1,\dots, K\}$ and a sequence $\{\varepsilon_j\}$ such that $x_{\varepsilon_j,i} \to \bar{x}_i \in \partial\Lambda_i$, where $x_{\varepsilon_j,i}$ is the maximum point of $u_{\varepsilon_j}$. We recall that we have a precise upper estimate of $c_{\varepsilon,i}$ in (2.19). On the other hand, using our contradiction assumption, and proceeding similarly as in the proof of Theorem 1.1, we obtain a lower estimate for the left hand side of (2.19) which is not compatible with the corresponding one for the right hand side. Here we may need to reduce the numbers $\delta_1$ and $\rho_0$ appropriately, making use of hypothesis (H2) to reach the contradiction. This finishes the proof of Lemma 2.3. □

*Proof of Theorem 0.1* Once (2.14) is proved we have a true solution to problem (0.1), i.e. $u_\varepsilon$ satisfies

$$\varepsilon^2\Delta u_\varepsilon - u_\varepsilon + f(u_\varepsilon) = 0 \qquad \text{in } \Omega.$$

By the same method in [12] or [5], one can show that $u_\varepsilon$ has exactly $K$ local maximum points $x_{\varepsilon,i} \in \Lambda_i,\ i = 1, \ldots, K$. Moreover the argument provided in the proof of Lemma 2.3 yields

$$d(x_{\varepsilon,i}, \partial\Omega) \to \max_{x \in \Lambda_i} d(x, \partial\Omega),\ i = 1, \ldots, K.$$

The proof is concluded. $\square$

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