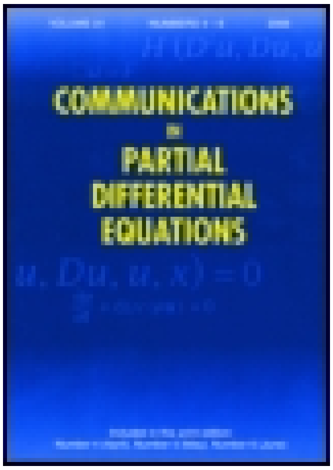

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Manuel del Pino [a], Patricio L. Felmer [a] & Juncheng Wei [b]

[a] Departamento do Ingeniería Matemática F.C.F.M, Universidad de Chile, Santiago, CHILE

[b] Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong

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ON THE ROLE OF DISTANCE FUNCTION IN SOME SINGULAR PERTURBATION PROBLEMS

Manuel del Pino[1], Patricio L. Felmer [1], Juncheng Wei[2]

[1]Departamento de Ingeniería Matemática F.C.F.M.
Universidad de Chile
Casilla 170 Correo 3, Santiago, CHILE

[2]Department of Mathematics
Chinese University of Hong Kong
Shatin, HONG KONG

Abstract

We consider the problem

$$\begin{cases} \varepsilon^2 \Delta u - u + f(u) = 0 & \text{in} \quad \Omega \\ u > 0 \ \text{ in } \Omega, \ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a smooth domain in R^N, not necessarily bounded, $\varepsilon > 0$ is a small parameter and f is a superlinear, subcritical nonlinearity. It is known that this equation possesses a solution that concentrates, as ε approaches zero, at a maximum of the function $d(x, \partial\Omega)$, *the distance to the boundary.*

We obtain single-peaked solutions associated to any *topologically nontrivial* critical point of the distance function such as for instance a local, possibly

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degenerate, saddle point. The construction relies on a variational localization argument to control a certain minmax value for an associated modified energy functional as well as on a precise asymptotic estimate for this energy level.

0 Introduction

Let us consider the problem

$$\begin{cases} \varepsilon^2 \Delta u - u + u^p = 0 & \text{in} \quad \Omega \\ u > 0 \ \text{ in } \Omega, \ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{0.1}$$

where $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, Ω is a smooth domain in R^N, not necessarily bounded, with boundary $\partial\Omega$, $N \geq 1$, $\varepsilon > 0$ is a constant and p satisfies $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$, and $1 < p < \infty$ for $N = 1, 2$.

Problem (0.1) and related ones have been widely considered in the literature of nonlinear elliptic problems in recent years, as they arise as the steady state equation of time dependent problems appearing in a number of biological and physical models.

A very interesting feature of (0.1) is the presence of families of solutions exhibiting a *spike-layer pattern* as $\varepsilon \to 0$. By this we mean solutions exhibiting a finite set of local maxima concentrating around certain special points of the domain, while vanishing at an exponential rate in $1/\varepsilon$ elsewhere.

In [12] Ni and Wei studied the behavior as $\varepsilon \to 0$ of a least energy solution to problem (0.1), characterized variationally as a mountain pass of the associated energy functional. They proved that for ε sufficiently small a least-energy solution possesses a single *spike-layer* with its unique peak in the interior of Ω. Moreover this unique peak must be situated near the *most-centered* part of Ω, that is where the distance function $d(P, \partial\Omega)$, $P \in \Omega$, assumes its global maximum. This is in contrast with earlier results for the corresponding Neumann problem, obtained by Ni and Takagi in [10] and [11], where it was shown that for ε sufficiently small, a least-energy solution possesses a single spike-layer with its unique peak located on the boundary $\partial\Omega$, which furthermore must be located near the *most curved* part of $\partial\Omega$, i.e. where the mean curvature of the boundary assumes its global maximum.

The results in [12] follow from an asymptotic expansion of the critical value associated to the least energy solution. This expansion requires sharp estimates of exponentially small error terms that are obtained by using a *vanishing viscosity method.*

A natural problem to study for further insight into the rich and complex structure of the solution set of this equation for small ε, is that of determining the role of distance function in the existence of other solutions which exhibit concentration behavior like the one described above. In this direction, Wei showed in [13] a local version of the above result, namely that for every strict local maximum point of the distance function, say P, there exists a family of solutions with a single global maximum point that approaches the given point P. In [5] this result is generalized to multiple-peaked case at several distinct possibly degenerate local maximum points of $d(P, \partial\Omega)$. For the single-peaked case, the result in [5] says that for any set $\Lambda \subset \Omega$ such that

$$\max_{P\in\Lambda} d(P, \partial\Omega) > \max_{P\in\partial\Lambda} d(P, \partial\Omega) \tag{0.2}$$

there exists a family of solutions with a single global maximum point which approaches a maximum point of $d(P, \partial\Omega)$ in Λ.

At this point we should mention a very recent work of Li and Nirenberg [9]. Assuming that for an open bounded subset Λ of Ω on whose boundary $d(P, \partial\Omega)$ is continuously differentiable

$$deg(\nabla d(\cdot, \partial\Omega), \Lambda) \neq 0 \tag{0.3}$$

they construct a family of solutions with single maximum point that belongs to Λ, however no further statement is given on the nature of the limiting points of these maxima. In [9] a more precise result is obtained when assumption (0.2) is considered. They are able to show that the constructed family has concentration around a point of maximum distance to the boundary, as also shown in [5].

In view of the above mentioned results, a natural question is whether there exist such single-peaked families concentrating around other kinds of critical points of the distance function $d(P, \partial\Omega)$. One may ask, for instance, whether there is a single-spike family concentrating around the center of the neck in the "flower-pot" domain of Figure 1, where a saddle point of the distance function appears. Our purpose in this paper is to show that associated to this point, and more generally, to any *topologically nontrivial critical point* of $d(P, \partial\Omega)$ there is a family of single-peaked solutions.

We remark that the results in [9] do not apply in a saddle point situation like that in Figure 1, since condition (0.3) cannot be fulfilled for the distance function fails to be differentiable on the entire transversal axis of the neck.

Next we describe our results in precise terms. We consider the problem

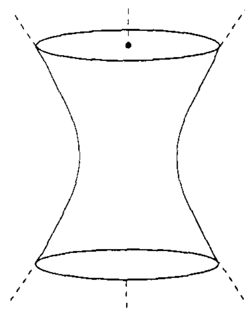

FIG. 1.

$$\begin{cases} \varepsilon^2\Delta u - u + f(u) = 0 \;\text{ in } \Omega, \\ u > 0 \;\text{ in } \Omega \text{ and } u = 0 \;\text{ in } \partial\Omega, \end{cases} \tag{0.4}$$

where we assume that $f : R^+ \to R$ is of class $C^{1+\sigma}$ and satisfies the conditions (f1)-(f5) below.

(f1) $f(t) \equiv 0$ for $t \le 0$ and $f(t) \to +\infty$ as $t \to \infty$.

(f2) The function $t \to f(t)/t$ is strictly increasing.

(f3) $f(t) = O(t^p)$ as $t \to +\infty$ where $1 < p < \frac{N+2}{N-2}$ if $N \ge 3$ and $1 < p < \infty$ if $N = 2$.

(f4) There exists a constant $\theta > 2$ such that $\theta F(t) \le tf(t)$ for $t \ge 0$, in which

$$F(t) = \int_0^t f(s)ds. \tag{0.5}$$

To state the last condition on f we need some preparations. Consider the problem in the whole space

$$\begin{cases} \Delta w - w + f(w) = 0 \;\text{ and } w > 0 \text{ in } R^N, \\ w(0) = \max_{z\in R^N} w(z) \text{ and } w(z) \to 0 \text{ as } |z| \to +\infty. \end{cases} \tag{0.6}$$

It is known that that any solution to (0.6) needs to be spherically symmetric about the origin and strictly decreasing in $r = |z|$, see [6]. A solution w to (1.5) is said to be *nondegenerate* if the linearized operator

$$L = \Delta - 1 + f'(w) \tag{0.7}$$

on $L^2(R^N)$, with domain $W^{2,2}(R^N)$, has a bounded inverse when it is restricted to the subspace $L^2_r(R^N) := \left\{ u \in L^2(R^N) | u(z) = u(|z|) \right\}$.

Now condition (f5) is stated as follows:

(f5) Problem (0.6) has a unique solution w, and it is nondegenerate.

The unique solution in (f5) will be denoted by w in the rest of this paper.

In what follows, we state precisely our assumption on $d(P, \partial\Omega)$. We consider Ω a smooth domain in R^N, not necessarily bounded. We assume that there is an open and bounded set Λ with smooth boundary such that $\overline{\Lambda} \subset \Omega$ and closed subsets of Λ, B, B_0 such that B is connected and $B_0 \subset B$. Let Γ be the class of all continuous functions $\phi : B \to \Lambda$ with the property that $\phi(y) = y$ for all $y \in B_0$. Assume the maxmin value

$$c = \sup_{\phi \in \Gamma} \min_{y \in B} d(\phi(y), \partial\Omega) \tag{0.8}$$

is well defined and additionally that

(H1) $\min_{y \in B_0} d(y, \partial\Omega) > c$.

(H2) At any point $y \in \partial\Lambda$ such that $d(y, \partial\Omega) = c$, there is a direction $\hat{T}$, tangent to $\partial\Lambda$ at y, such that $\hat{T} \cdot \tau \neq 0$ for any $\tau \in conv(S(y) - y)$. Here $S(y) = \partial\Omega \cap \overline{B}_{d(y,\partial\Omega)}(y)$, and $conv$ detones the convex hull.

In the standard language of calculus of variations, we see that the sets B_0, B, $\{d(y, \partial\Omega) \leq c\}$ *"link"* in Λ.

It is not hard to check that all these assumptions are satisfied in a general local saddle point situation. Note that these maxima or saddle points do not need to be isolated.

We now state our main result in this paper.

Theorem 0.1 *Assume Ω is a domain, not necessarily bounded, which satisfies assumptions (H1) and (H2). Let f satisfy assumptions (f1)-(f5). Then for ε sufficiently small problem (0.4) has a solution u_ε which possesses exactly one local maximum points x_ε with $x_\varepsilon \in \Lambda$. Moreover $d(x_\varepsilon, \partial\Omega) \to c$, as $\varepsilon \to 0$. Moreover, we have*

$$u_\varepsilon(x) \leq \alpha exp(-\frac{\beta|x - x_\varepsilon|}{\varepsilon}) \tag{0.9}$$

for certain positive constants α, β.

Remark. We note that the function $f(t) = t^p$ satisfies all the assumptions (f1)-(f5) if $1 < p < \frac{N+2}{N-2}$. See [7] and Appendix C in [11].

Our proof of Theorem 0.1 can be extended, with minor changes, to cover more general nonlinearities. Namely we can replace hypothesis (f2) by

(f2') For $t \geq 0$, f admits the decomposition in $C^{1+\sigma}(R)$

$$f(t) = f_1(t) - f_2(t),$$

where f_1 and f_2 satisfy

(i) $f_1(t) \geq 0$ and $f_2(t) \geq 0$ with $f_1(0) = f_1'(0) = 0$. and

(ii) There is a $q \geq 1$ such that $f_1(t)/t^q$ is nondecreasing in $t > 0$, whereas $f_2(t)/t^q$ is nonincreasing in $t > 0$, and in case $q = 1$ we require further that the above monotonicity condition for $f_1(t)/t$ is strict.

The function

$$f(t) = t^p - at^q \quad \text{for } t \geq 0,$$

with a constant $a \geq 0$, satisfies assumptions (f1), (f2') and (f3) if $1 < q < p < \frac{N+2}{N-2}$. Furthermore (f5) is proved in [1], [7] and [8]. And the nondegeneracy condition (f5) can be derived from the uniqueness arguments in [7], [8] or [1], (see Appendix C in [11]).

Theorem 0.1 implies that *topologically " nontrivial critical points"* of the distance function (note that the distance function is not differentiable everywhere, so we can only speak about generalized derivatives) have associated single-peaked solutions. This is much in line with the results of [3], in which it is shown the existence of concentrated bound states for the following nonlinear Schrödinger equation

$$\hbar^2 \Delta u - V(x)u + u^p = 0 \text{ in } R^N. \tag{0.10}$$

It is shown in [3] that at any topologically nontrivial critical points of $V(x)$ in a similar sense as that above for the distance function there exist concentrated bound states. In fact, we prove Theorem 0.1 by using variational techniques developed in [2] and [4]. In this process, sharp estimates of critical values of the associated energy involving the distance function obtained in [12] and [13] are crucial.

Our approach consists of modifying adequately the nonlinearity, so to penalize concentration outside Λ. This gives rise to a new *penalized energy functional.* Then we use the *linking* condition (H1)-(H2) to find critical points of this functional by means of a minmax scheme on a class of maps defined on a finite dimensional set. As ε approaches zero, one then shows that the critical points so obtained are solutions of the original problem. In this process condition (H2) is crucial.

This paper is organized as follows. In §1, we modify the energy functional and set up a min-max procedure. Then we use (H1) to show that a critical point exists for the modified functional. In §2, we show that the critical point is indeed a solution to (0.3) and has all the properties stated in Theorem 0.1.

Throughout this paper, we use $C, C_0, c, etc.$ to denote various generic constants. The symbols $O(A), o(A)$ mean that $|O(A)| \leq C|A|, o(A)/|A| \to 0$ respectively.

1 The Min-Max setting

Following the idea introduced in [2], we modify the function f penalizing concentration outside the set Λ. Then we set up the mountain pass scheme in order to obtain critical points of the penalized functional and, using the results in [12], we provide the necessary estimates to discard the penalization.

Associated to equation (0.4) we have the "energy" functional

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega \varepsilon^2|\nabla u|^2 + u^2 - \int_\Omega F(u), \tag{1.1}$$

defined in $H_0^1(\Omega)$. In a similar way, associated to (0.6) we have the *limiting functional* $I : H^1(R^N) \to R$ defined as

$$I(u) = \frac{1}{2}\int_{R^N} |\nabla u|^2 + u^2 - \int_{R^N} F(u). \tag{1.2}$$

Under the hypotheses on f and Ω, it is standard to check that the nontrivial critical points of I_ε and I correspond exactly to the positive classical solutions of equation (0.4) in $H_0^1(\Omega)$ and of equation (0.6) in $H^1(R^N)$, respectively. From assumption (f5), the functional I has a unique positive critical point, up to translations, denoted by w. The critical value of w (of a mountain pass nature) will be denoted by $c_* \equiv I(w)$.

Next we modify the function f as in [2]. Let θ be a number as given by (f4), and let us choose $k > 0$ such that $k > \frac{\theta}{\theta-2}$. Let $a > 0$ be the value at which $\frac{f(a)}{a} = \frac{1}{k}$. Let us set

$$\tilde{f}(s) = \begin{cases} f(s) & \text{if } s \leq a \\ \frac{1}{k}s & \text{if } s > a, \end{cases}$$

and define

$$g(\cdot, s) = \chi_\Lambda f(s) + (1 - \chi_\Lambda)\tilde{f}(s),$$

where Λ is a bounded domain as in the assumptions of Theorem 0.1, and χ_Λ denotes its characteristic function. Let us denote $G(x, \xi) = \int_0^\xi g(x, \tau)d\tau$, and consider the modified functional introduced in [2], defined on $H_0^1(\Omega)$ as

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega \varepsilon^2|\nabla u|^2 + u^2 - \int_\Omega G(x, u) \quad ,$$

The functional J_ε is of class C^1 in $H_0^1(\Omega)$ and its nontrivial critical points are precisely the positive solutions of the equation

$$\varepsilon^2 \Delta u - u + g(x, u) = 0 \qquad \text{in} \quad \Omega. \tag{1.3}$$

The functional J_ε is of class C^1 and satisfies the Palais Smale condition no matter whether Ω is bounded or not, see Lemma 1.1 in [2].

Let us also set

$$M_\varepsilon = \{u \in H_0^1(\Omega) \setminus \{0\} \mid \int_\Omega \varepsilon^2|\nabla u|^2 + u^2 = \int_\Omega ug(x, u)dx\}.$$

Then, one can show, similarly to Lemma B in [11], that $u \in M_\varepsilon$ if and only if $J_\varepsilon(u) = sup_{t>0} J_\varepsilon(tu)$. Moreover,

$$c_\varepsilon = \inf_{u \in M_\varepsilon} J_\varepsilon(u)$$

is a critical value of J_ε thanks to the P.S. condition.

Next we will define a min-max quantity for the functional J_ε. Given $v \in H^1(R^N)$ we define the *projection of* v *into the domain* D, and denote it by $Q_D v$, as the unique solution of the problem

$$\begin{cases} \varepsilon^2 \Delta u - u + f(v(x)) = 0 & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases} \tag{1.4}$$

We define $w_{\varepsilon,P}(x) = w(\frac{x-P}{\varepsilon})$ for $P \in \Omega$, where w is the unique solution of (0.6).

For each $P \in \Omega$, we define

$$w_\varepsilon^P(x) = t_{\varepsilon,P} Q_{B_{d(P,\partial\Omega)}(P)} w_{\varepsilon,P}$$

where $t_{\varepsilon,P}$ is such that $w_\varepsilon^P \in M_\varepsilon$. Note that w_ε^P is radial.

Let Λ, B, B_0 be the sets in assumptions (H1)-(H2) and Γ_ε be the set of all continuous maps $\phi : B \to M_\varepsilon$ with the property that

$$\phi(y) = w_\varepsilon^y, \forall y \in B_0.$$

Then we define a min-max value S_ε for the functional J_ε as follows

$$S_\varepsilon = \inf_{\phi \in \Gamma_\varepsilon} \sup_{y \in B} J_\varepsilon(\phi(y)). \tag{1.5}$$

We will show that S_ε is a critical value of J_ε. From standard deformation arguments, this is a consequence of the following result.

Lemma 1.1 *For ε sufficiently small, we have $S_\varepsilon > \sup_{y\in B_0} J_\varepsilon(\phi(y)), \forall \phi \in \Gamma_\varepsilon$.*

To prove Lemma 1.1, the following is the key estimate we need.

Lemma 1.2 *Let Λ_0 be an open bounded set such that $\bar{\Lambda}_0 \subset \Omega$. Define*

$$J_{\varepsilon,\Lambda_0}(u) = \frac{\varepsilon^2}{2}\int_\Omega |\nabla u|^2 + \int_\Omega u^2 - \int_\Omega G_{\Lambda_0}(x,u),$$

$$M_{\varepsilon,\Lambda_0} = \{u \in H_0^1(\Omega)\backslash\{0\} | \varepsilon^2 \int_\Omega |\nabla u|^2 + \int_\Omega u^2 = \int_\Omega u g_{\Lambda_0}(x,u)\},$$

where

$$g_{\Lambda_0}(x,u) := \chi_{\Lambda_0} f(u) + (1-\chi_{\Lambda_0})\tilde{f}(u), G_{\Lambda_0}(x,u) = \int_0^u g_{\Lambda_0}(x,t)dt.$$

Let u_ε be a nontrivial critical point of $J_{\varepsilon,\Lambda_0}$ such that

$$\limsup_{\varepsilon\to 0} \varepsilon^{-N} J_{\varepsilon,\Lambda_0}(u_\varepsilon) \le c_*.$$

Then, for all sufficiently small ε, u_ε has a single local maximum point x_ε, which is located in Λ_0. We also have the estimate

$$J_{\varepsilon,\Lambda_0}(u_\varepsilon) = \varepsilon^N\{c_* + e^{-\frac{2}{\varepsilon}(d(x_\varepsilon,\partial\Omega)+o(1))}\}, \tag{1.6}$$

In particular,

$$\inf_{u\in M_{\varepsilon,\Lambda_0}} J_{\varepsilon,\Lambda_0}(u) \ge \varepsilon^N\{c_* + e^{-2(d_{\Lambda_0}+o(1))/\varepsilon}\}, \tag{1.7}$$

where

$$d_{\Lambda_0} = \sup_{x\in\Lambda_0} d(x,\partial\Omega).$$

For a given function $u \in H_0^1(\Omega) \setminus \{0\}$ we define its center of mass as

$$\beta(u) = \frac{\int_{\Omega\cap B_R(0)} x u^2 dx}{\int_\Omega u^2 dx}, \tag{1.8}$$

where $R > 0$ is such that $\Lambda_0 \subset B_{R/2}(0)$. When Ω is bounded we can avoid the intersection with $B_R(0)$ in the integral of the numerator.

Corollary 1.1 *Let $\varepsilon = \varepsilon_k \to 0$ and $u_\varepsilon \in M_{\varepsilon,\Lambda_0}$ be a family of functions such that*

$$\limsup_{\varepsilon\to 0} \varepsilon^{-N} J_{\varepsilon,\Lambda_0}(u_\varepsilon) \le c_*.$$

Then the following estimate holds

$$J_{\varepsilon,\Lambda_0}(u_\varepsilon) \ge \varepsilon^N \{c_* + e^{-\frac{2}{\varepsilon}(d(x_\varepsilon,\partial\Omega)+o(1))}\},$$

where $x_\varepsilon = \beta(u_\varepsilon)$.

Proof of Corollary 1.1. Passing to a subsequence, assume that $x_\varepsilon \to \bar{x}$. Then given $\delta > 0$, for all small ε one has that

$$\tilde{\beta}(u_\varepsilon) \in B_\delta(\bar{x}),$$

where

$$\tilde{\beta}(u) = \frac{\int_{B_\delta(\bar{x})} xu^2}{\varepsilon^N \int_{R^N} w^2}.$$

In fact, a standard concentration-compactness type argument together with the minimizing character of the sequence u_ε and the Ekeland variational principle give that $u_\varepsilon(x_\varepsilon+\varepsilon y)$ converges in H^1-sense the a least energy critical point w of the limiting functional I in (1.2).

Then we have

$$J_{\varepsilon,\Lambda_0}(u_\varepsilon) \ge \inf\{J_{\varepsilon,\Lambda_0}(u) \mid u \in M_{\varepsilon,\Lambda_0}, \tilde{\beta}(u) \in \overline{B_\delta}(\bar{x})\}.$$

Since the functional satisfies P.S, it follows that the latter number is attained at some function $\tilde{u}_\varepsilon$. Working out a first variation with test functions supported outside $B_\delta(\bar{x})$, we see that $\tilde{u}_\varepsilon$ satisfies the equation

$$\varepsilon^2\Delta\tilde{u}_\varepsilon - \tilde{u}_\varepsilon + g(x,\tilde{u}_\varepsilon) = 0 \text{ in } \Omega\backslash B_\delta(\overline{x}).$$

Again, if we set $v_\varepsilon(y) = \tilde{u}_\varepsilon(\tilde{x}_\varepsilon + \varepsilon y)$, with $\tilde{x}_\varepsilon = \beta(\tilde{u}_\varepsilon)$ then v_ε converges in the $H^1(R^N)$-sense to w, the least energy critical point of the functional I in (1.2). In particular, elliptic estimates applied to the above equation imply that $\tilde{u}_\varepsilon$ goes to zero uniformly, away from the ball $B_\delta(\bar{x})$. In particular, we have that

$$J_{\varepsilon,\Lambda_0}(\tilde{u}_\varepsilon) = J_{\varepsilon,\Lambda_0\cap B_{2\delta}(\bar{x})}(\tilde{u}_\varepsilon),$$

and also $\tilde{u}_\varepsilon \in M_{\varepsilon,\Lambda_0\cap B_{2\delta}(\bar{x})}$. Let us set $\tilde{\Lambda} = \Lambda_0 \cap B_{2\delta}(\bar{x})$. Then we obtain

$$J_{\varepsilon,\Lambda_0}(\tilde{u}_\varepsilon) \ge \inf_{u\in M_{\varepsilon,\tilde{\Lambda}}} J_{\varepsilon,\tilde{\Lambda}}(u).$$

But the latter number is estimated from below by the estimate given by Lemma 1.2. Hence

$$J_{\varepsilon,\Lambda_0}(\tilde{u}_\varepsilon) \geq \varepsilon^N\{c_* + e^{-2(d_{\tilde{\Lambda}}+o(1))/\varepsilon}\},$$

where

$$d_{\tilde{\Lambda}} = \sup_{x\in\tilde{\Lambda}} d(x,\partial\Omega) \leq d(\bar{x},\partial\Omega) + 2\delta = d(x_\varepsilon,\partial\Omega) + 2\delta + o(1).$$

Since δ can be chosen arbitrarily small, the result of the corollary follows. □

The proof of Lemma 1.2 is postponed until the end of this section. Let us now use this corollary to prove Lemma 1.1. Suppose that Lemma 1.1 is not true, namely that there exists $\varepsilon_k \to 0$ such that

$$S_{\varepsilon_k} \leq \sup_{y\in B_0} J_{\varepsilon_k}(\phi(y)) \quad \forall\phi \in \Gamma_{\varepsilon_k}.$$

Hence, given $L > 0$ there exists $\phi_k \in \Gamma_{\varepsilon_k}$ such that

$$\sup_{y\in B} J_{\varepsilon_k}(\phi_k(y)) \leq \sup_{y\in B_0} J_{\varepsilon_k}(\phi_k(y)) + e^{-L/\varepsilon_k}.$$

Using concentration-compactness arguments as those provided in Lemma 1.1 in [3] we find that for large k

$$\beta(\phi_k(y)) \in \tilde{\Lambda} = \{x | d(x,\Lambda) \leq \frac{\delta}{8}\} \qquad \forall y \in B,$$

where $\delta > 0$ is a small positive number such that $d(y,\partial\Omega) \geq c+\delta$ for $y \in B_0$, which can be chosen thanks to assumption (H1).

Now, the linking assumptions (H1)-(H2) applied to a slight modification of $\psi_k(y) = \beta(\phi_k(y))$ yields the existence of a $y_k \in B$ so that

$$d(\beta(\phi_k(y_k)),\partial\Omega) \leq c + \frac{\delta}{2}. \tag{1.9}$$

We will show that this is impossible. Indeed, let us denote $u_k = \phi_k(y_k)$. Then Corollary 1.1. applies to the sequence u_k to yield the estimate

$$J_{\varepsilon_k}(u_k) \geq \varepsilon_k^N\{c_* + e^{-\frac{2}{\varepsilon_k}(d(\beta(u_k),\partial\Omega)+o(1))}\}. \tag{1.10}$$

On the other hand, from our assumption

$$J_{\varepsilon_k}(u_k) \leq \sup_{y\in B} J_{\varepsilon_k}(\phi_k(y)) \leq \max_{y\in B_0} J_{\varepsilon_k}(w^y_{\varepsilon_k}) + e^{-L/\varepsilon_k}.$$

Now, estimating the right hand side of the above inequality using the estimates in [12], Section 5, and the fact that for $y \in B_0$ we have $d(y, \partial\Omega) \geq c+\delta$, we see that

$$J_{\varepsilon_k}(u_k) \leq \varepsilon_k^N\{c_* + e^{-2(c+\delta+o(1))/\varepsilon_k}\}. \tag{1.11}$$

Thus, combining estimates (1.10) and (1.11) one gets

$$d(\beta(u_k), \partial\Omega) \geq c + \delta.$$

This immediately contradicts estimate (1.9), and the proof is thus concluded. □

Thus Lemma 1.1 holds true. Since the P.S. condition is satisfied for J_ε, see [3], we have from a standard deformation argument that S_ε is a critical value of J_ε.

It remains to prove Lemma 1.2.

Proof of Lemma 1.2: Let us set $\Omega_\varepsilon = \varepsilon^{-1}(\Omega - x_\varepsilon)$ and $\Lambda_\varepsilon = \varepsilon^{-1}(\Lambda_0 - x_\varepsilon)$ and $v_\varepsilon(y) = u_\varepsilon(x_\varepsilon + \varepsilon y)$. Let w be the radially symmetric least energy critical point of I, and let $z = P_{\Omega_\varepsilon} w$ be the unique solution of

$$-\Delta z + z = f(w) \quad \text{in } \Omega_\varepsilon,$$

$$z = 0 \quad \text{on } \partial\Omega_\varepsilon.$$

Let us also set $\phi_\varepsilon = v_\varepsilon - P_{\Omega_\varepsilon} w$. Similarly to the first part of the proof of Corollary 1.1, we have that v_ε has just a local maximum point, and that $v_\varepsilon \to w$ uniformly and in the H^1-sense. See also the proof of Proposition 2.1 in [3]. To continue the analysis let us consider the function

$$h_\varepsilon = (1 - \chi_{\Lambda_\varepsilon})(f(v_\varepsilon) - \tilde{f}(v_\varepsilon)).$$

We observe that the desired energy estimate corresponds exactly to that shown in [12] in case that $h_\varepsilon \equiv 0$, so we shall assume otherwise. In such case we must have that $h_\varepsilon \to 0$ uniformly with its support shrinking to a point $\hat{b}$ so that $w(\hat{b}) = a$ with a as in the definition of $\tilde{f}$ (otherwise after scaling we would end up in the limit with an energy higher than the upper a priori estimate c^*).

The function ϕ_ε satisfies the equation

$$\Delta\phi_\varepsilon - (1+\theta_\varepsilon)\phi_\varepsilon + f'(P_{\Omega_\varepsilon} w)\phi_\varepsilon = h_\varepsilon + f(w) - f(P_{\Omega_\varepsilon} w) \quad \text{in } \Omega_\varepsilon, \tag{1.12}$$

where

$$\theta_\varepsilon = -\frac{f(v_\varepsilon) - f(P_{\Omega_\varepsilon} w)}{\phi_\varepsilon} + f'(P_{\Omega_\varepsilon} w) \to 0,$$

uniformly, as $\varepsilon \to 0$. We write

$$E_\varepsilon(v) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla v|^2 + v^2 - \int_{\Omega_\varepsilon} F(v).$$

Hence we have

$$\varepsilon^{-N} J_{\varepsilon,\Lambda_0}(u_\varepsilon) = E_\varepsilon(v_\varepsilon) + \int_{\Omega_\varepsilon\setminus\Lambda_\varepsilon} (F(v_\varepsilon) - \tilde{F}(v_\varepsilon)).$$

Then we write, using the mean value theorem,

$$E_\varepsilon(v_\varepsilon) = E_\varepsilon(P_{\Omega_\varepsilon} w + \phi_\varepsilon) =$$

$$= E_\varepsilon(P_{\Omega_\varepsilon} w) + < E'_\varepsilon(P_{\Omega_\varepsilon} w), \phi_\varepsilon > + \frac{1}{2}\int_0^1 < E''_\varepsilon(P_{\Omega_\varepsilon} w + t\phi_\varepsilon)\phi_\varepsilon, \phi_\varepsilon > dt.$$

In other words, using that ϕ_ε satisfies equation (1.12) we obtain after expanding the above terms,

$$\varepsilon^{-N} J_{\varepsilon,\Lambda_0}(u_\varepsilon) = E_\varepsilon(P_{\Omega_\varepsilon} w) - \frac{1}{2}\int_{\Omega_\varepsilon} h_\varepsilon \phi_\varepsilon +$$

$$+\frac{1}{2}\int_{\Omega_\varepsilon} (f(w) - f(P_{\Omega_\varepsilon} w))\phi_\varepsilon + \frac{1}{2}\int_{\Omega_\varepsilon} \tilde{\theta}_\varepsilon \phi_\varepsilon^2 + \int_{\Omega_\varepsilon\setminus\Lambda_\varepsilon} (F(v_\varepsilon) - \tilde{F}(v_\varepsilon)) =$$

$$= E_\varepsilon(P_{\Omega_\varepsilon} w) + I + II + III + IV, \tag{1.13}$$

where

$$\tilde{\theta}_\varepsilon = -\theta_\varepsilon + \int_0^1 f'(P_{\Omega_\varepsilon} w) - f'(P_{\Omega_\varepsilon} w + t\phi_\varepsilon)dt$$

is a uniformly bounded function. From the results of [12], Section 5, we have the validity of the estimate

$$E_\varepsilon(P_{\Omega_\varepsilon} w) = c_* + e^{-\frac{2}{\varepsilon}(d_\varepsilon + o(1))},$$

where $d_\varepsilon = d(x_\varepsilon, \partial\Omega)$. Hence to prove the desired estimate it suffices to establish that all other terms in the expansion are of an exponential size smaller than $e^{-2d_\varepsilon/\varepsilon}$. To this end, the following claims constitute central steps.

CLAIM 1. *There exists $\eta > 0$ such that for all small ε*

$$\left(\int_{\Omega_\varepsilon} |\phi_\varepsilon|^2\right)^{1/2} \le e^{-(1+\eta)d_\varepsilon/\varepsilon}.$$

CLAIM 2.

$$\int_{\Omega_\varepsilon} h_\varepsilon \le C d_\varepsilon^{N-1} e^{-2\frac{d_\varepsilon}{\varepsilon}}.$$

CLAIM 3. *There is a $\rho > 0$ such that for each $p > 1$ one has*

$$\int_{\Omega_\varepsilon} |f(w) - f(P_{\Omega_\varepsilon} w)|^p \leq e^{-p(1+\rho)(d_\varepsilon + o(1))/\varepsilon}.$$

First we will establish Claim 2. To this end we assert that

$$\int_{\Omega_\varepsilon} h_\varepsilon \leq C \int_{\partial\Omega_\varepsilon} |\nabla v_\varepsilon|^2 d\sigma. \tag{1.14}$$

In fact, let us assume that the direction x_1 is normal to Λ_ε at the closest point in $\partial\Lambda_\varepsilon$ from the origin. Then using $\frac{\partial v_\varepsilon}{\partial x_1}$ as a test function in the equation satisfied by v_ε we obtain

$$\int_{\Omega_\varepsilon \setminus \Lambda_\varepsilon} h_\varepsilon \frac{\partial v_\varepsilon}{\partial x_1} dx = \int_{\partial\Omega_\varepsilon} |\nabla v_\varepsilon|^2 \nu_1 d\sigma.$$

Since the support of h_ε is shrinking to $\hat{b}$ and $\frac{\partial v_\varepsilon}{\partial x_1}(\hat{b}) \to \frac{\partial w}{\partial x_1}(\hat{b}) \neq 0$, the assertion readily follows.

Thus we just need to estimate $|\nabla v_\varepsilon|$ on $\partial\Omega_\varepsilon$. We claim that

$$v_\varepsilon(y) \leq Ce^{-|y|}|y|^{-(N-1)/2} \tag{1.15}$$

for all y in a neighborhood of $\partial\Omega_\varepsilon$. We observe that such an estimate also holds for w as shown in [6]. We prove the claim. A standard comparison argument gives that for any given $\rho > 0$ there is a C such that

$$v_\varepsilon(y) \leq Ce^{-(1-\rho)|y|},$$

and hence

$$f(v_\varepsilon(y)) \leq Ce^{-(1-\rho)\theta|y|},$$

for all y in a neighborhood of $\partial\Omega_\varepsilon$. Here $\theta > 1$ is the number in assumption (f4). Thus we may choose ρ so that $\theta(1-\rho) = 1 + \delta$ with $\delta > 0$. Let $k(|y|)$ be the fundamental solution of

$$-\Delta k + k = \delta_0.$$

It is well known that

$$k(y) \leq C|y|^{-(N-1)/2} e^{-|y|}.$$

Then we set

$$z(y) = Mk(|y|) - \alpha e^{-(1+\delta/2)|y|}.$$

It is then easy to check that for a sufficiently large R_0 and a suitable choice of α one gets

$$-\Delta z + z \geq f(v_\varepsilon), \quad |y| > R_0,$$

so that choosing M large enough the maximum principle yields $v_\varepsilon(y) \leq z(y)$ for all $|y| > R_0$ and the claim (1.15) thus follows. Now using this fact and local elliptic estimates at the boundary one obtains

$$|\nabla v_\varepsilon(z)| \leq C\left(\frac{d_\varepsilon}{\varepsilon}\right)^{-(N-1)/2} e^{-\frac{d_\varepsilon}{\varepsilon}},$$

uniformly on $z \in \partial\Omega_\varepsilon$. Finally this and (1.14) yields

$$\int_{\Omega_\varepsilon} h_\varepsilon \leq C d_\varepsilon^{-(N-1)} e^{-2\frac{d_\varepsilon}{\varepsilon}}$$

as desired, and the proof of Claim 2 is concluded. □

Now we prove Claim 3. Set $z = w - P_{\Omega_\varepsilon} w$. Then z satisfies

$$-\Delta z + z = 0 \quad \text{in } \Omega_\varepsilon,$$

$$z = w \quad \text{on } \partial\Omega_\varepsilon.$$

Then z is positive and it maximizes on $\partial\Omega$. Now, the largest value of w on $\partial\Omega$ is

$$w(d_\varepsilon/\varepsilon) = e^{-(d_\varepsilon + o(1))/\varepsilon}.$$

Let B_ε be the ball centered at the origin with radius $d_\varepsilon/\varepsilon$. Let $\delta > 0$ be an arbitrarily small number and set $\tilde{z}(y) = A(\cosh((1-\delta)|y|) + K)$. It is easy to see that if K is chosen large enough (dependent on δ), then $\tilde{z}$ satisfies $-\Delta\tilde{z} + \tilde{z} \geq 0$. Then, if one chooses the constant $A = e^{-(2-2\delta)d_\varepsilon/\varepsilon}$, one gets that for small ε, $\tilde{z} \geq z$ on ∂B_ε, therefore the inequality holds in the entire ball B_ε. It follows that

$$z(y) \leq C e^{-(2-2\delta)d_\varepsilon/\varepsilon} e^{|y|}, \quad \text{for } |y| < d_\varepsilon/\varepsilon.$$

Note also that $z(y) \leq w(y) \leq e^{-|y|}$ for large $|y|$. Now, using assumption (f2) and the $C^{1+\sigma}$ character of f, we find

$$0 \leq f(w) - f(P_{\Omega_\varepsilon} w) = \int_0^1 f'(P_{\Omega_\varepsilon} w + tz) z\, dt \leq C w^\sigma z \leq C e^{-\sigma|y|} z.$$

Then

$$\int_{\Omega_\varepsilon} |f(w) - f(P_{\Omega_\varepsilon} w)|^p \leq C e^{-p\{2-2\delta\}d_\varepsilon/\varepsilon} \int_{B_\varepsilon} e^{p(1-\sigma)|y|} dy + \int_{\Omega_\varepsilon\setminus B_\varepsilon} e^{-p(1+\sigma)|y|} dy \leq$$

$$\leq e^{-p(1-3\delta+\sigma)d_\varepsilon/\varepsilon} + e^{-p(1+\sigma/2)d_\varepsilon/\varepsilon}.$$

From here Claim 2 readily follows after choosing δ so small that $3\delta < \sigma$. ◻

Now we prove Claim 1. Let $1 < p < 2$. Observe that Claim 2 implies that

$$\|h_\varepsilon\|_{L^p(\Omega_\varepsilon)} \leq e^{-(1+\eta_1)d_\varepsilon/\varepsilon}.$$

Let $\eta < \min\{\eta_1, \rho\}$ where ρ is such that Claim 3 holds. We assert that Claim 1 holds for this η if p was chosen appropriately. To prove this we assume the opposite, namely that along a sequence $\varepsilon = \varepsilon_k \to 0$ one has

$$\frac{e^{-(1+\eta)d_\varepsilon/\varepsilon}}{\left(\int_{\Omega_\varepsilon} |\phi_\varepsilon|^2\right)^{1/2}} \to 0.$$

Let us set

$$\bar{\phi}_\varepsilon \equiv \frac{\phi_\varepsilon}{\left(\int_{\Omega_\varepsilon} |\phi_\varepsilon|^2\right)^{1/2}}.$$

Then $\bar{\phi}_\varepsilon$ satisfies

$$\Delta\bar{\phi}_\varepsilon - (1+\theta_\varepsilon)\bar{\phi}_\varepsilon + f'(P_{\Omega_\varepsilon}w)\bar{\phi}_\varepsilon = \bar{h}_\varepsilon \quad \text{in } \Omega_\varepsilon,$$

with $\bar{h}_\varepsilon \to 0$ in L^p. Then from a standard elliptic estimate we find that $\bar{\phi}_\varepsilon$ is bounded in $W^{1,p}$ on each compact subset of R^N. Assume also that p was chosen sufficiently close to 2 so that $2 < Np/(N-p)$. Hence passing to a subsequence we may assume from compact Sobolev's embedding that $\bar{\phi}_\varepsilon$ converges locally in L^2 to a $\bar{\phi}$. Now, this local convergence is actually in C^2-sense away from any neighborhood of $\hat{b}$, the point to which the support of h_ε is shrinking. Then, similarly to the proof of inequality (1.15) we may use a comparison function independent of ε, defined on the exterior of a large fixed ball centered at the origin and exponentially decaying, so that we obtain that $\bar{\phi}_\varepsilon$ has a uniform exponential decay. It follows that the local convergence in L^2 is also global, so that in particular $\bar{\phi}$ is in $L^2(R^N)$ and is not zero. $\bar{\phi}$ satisfies weakly, hence strongly, the equation

$$\Delta\bar{\phi} - \bar{\phi} + f'(w)\bar{\phi} = 0 \quad \text{in } R^N.$$

since $\bar{\phi}$ also decays at infinity, it follows from the nondegeneracy assumption (f5) that

$$\bar{\phi} = \sum_{i=1}^{N} \alpha_i \frac{\partial w}{\partial y_i},$$

for some constants α_i, not all zero. This implies that $\nabla\bar{\phi}(0) \neq 0$. On the other hand we recall that v_ε attained its maximum value at 0, while the maximum point of $P_{\Omega_\varepsilon} w$ approaches 0. Hence $\nabla\bar{\phi}_\varepsilon(0) \to 0$. This and the local C^1-convergence of $\bar{\phi}_\varepsilon$ to $\bar{\phi}$, near the origin, provides a contradiction which proves the validity of Claim 1. □□

Now we are ready to estimate the terms I-IV in the expansion (1.13).

$$|I| \leq \int_{\Omega_\varepsilon} h_\varepsilon|\phi_\varepsilon| \leq \|h_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\phi_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq e^{-(2+\eta)d_\varepsilon/\varepsilon},$$

from Claims 1 and 2. Now,

$$|II| \leq \int_{\Omega_\varepsilon} |f(w)-f(P_{\Omega_\varepsilon}w)||\phi_\varepsilon| \leq \|f(w)-f(P_{\Omega_\varepsilon}w)\|_{L^2}\|\phi_\varepsilon\|_{L^2} \leq e^{-(2+\eta+\rho)d_\varepsilon/\varepsilon},$$

from Claims 1 and 3. As for III

$$|III| \leq C\int_{\Omega_\varepsilon} \phi_\varepsilon^2 \leq e^{-(2+2\eta)d_\varepsilon/\varepsilon}.$$

It only remains to estimate IV. We have that

$$|IV| = \int_{\Omega_\varepsilon\setminus\Lambda_\varepsilon} (F(v_\varepsilon) - \tilde{F}(v_\varepsilon)) = \int_{\{v_\varepsilon(x)\geq a\}} dx \int_a^{v_\varepsilon(x)} \frac{f(s)-\tilde{f}(s)}{s} sds \leq$$

$$\leq \int_{\{v_\varepsilon(x)\geq a\}} (\frac{f(v_\varepsilon)-\tilde{f}(v_\varepsilon)}{v_\varepsilon})(v_\varepsilon^2 - a^2)dx \leq$$

$$\leq \frac{1}{a}\int_{\{v_\varepsilon(x)\geq a\}} h_\varepsilon(v_\varepsilon^2 - a^2)dx \leq C\int_{\Omega_\varepsilon} h_\varepsilon(v_\varepsilon - a)_+ dx.$$

Now

$$(v_\varepsilon - a)_+ \leq (w-a)_+ + |P_{\Omega_\varepsilon}w - w| + |\phi_\varepsilon|,$$

hence

$$|IV| \leq C\{\int_{\Omega_\varepsilon} h_\varepsilon(w-a)_+dx + \int_{\Omega_\varepsilon} |P_{\Omega_\varepsilon}w - w|h_\varepsilon + \int_{\Omega_\varepsilon} |\phi_\varepsilon|h_\varepsilon\}.$$

The last two terms in the right hand side of the above inequality admit estimates of high exponential order similar to those for I-III. Thus it only remains to find such an estimate for $\int_{\Omega_\varepsilon} h_\varepsilon(w-a)_+dx$. To do so, we just need to establish that $(w-a)_+ \leq e^{-\rho d_\varepsilon/\varepsilon}$ for some $\rho > 0$, uniformly on the support of h_ε.

Since the support of h_ε is shrinking to the point $\hat{b}$, we see that

$$(w(y)-a)_+ \leq C\delta_\varepsilon,$$

where $\delta_\varepsilon = b - d(x_\varepsilon, \partial\Lambda_\varepsilon)$ and $b = |\hat{b}|$. We will show that δ_ε has the desired exponential order. Note that from Claim 2

$$\int_{\Lambda_\varepsilon^c} (f(v_\varepsilon) - \tilde{f}(v_\varepsilon)) \le Ce^{-2d_\varepsilon/\varepsilon}.$$

Now

$$\int_{\Lambda_\varepsilon^c} (f(v_\varepsilon) - \tilde{f}(v_\varepsilon)) = \int_{\Lambda_\varepsilon^c} (\frac{f(v_\varepsilon)}{v_\varepsilon} - \frac{f(a)}{a})\frac{(v_\varepsilon - a)_+}{(v_\varepsilon - a)} v_\varepsilon =$$
$$= \int_{\Lambda_\varepsilon^c} \int_0^1 \psi'(a + t(v_\varepsilon - a))dt(v_\varepsilon - a)_+ v_\varepsilon,$$

where $\psi(s) = f(s)/s$. We may assume with no loss of generality that $\psi'(a) > 0$. Thus

$$\int_{\Lambda_\varepsilon^c} (f(v_\varepsilon) - \tilde{f}(v_\varepsilon)) \ge C \int_{\Lambda_\varepsilon^c} (v_\varepsilon - a)_+ \ge$$

$$\ge C \int_{\Lambda_\varepsilon^c} (w - a)_+ - |\phi_\varepsilon| - |P_{\Omega_\varepsilon} w - w|.$$

But

$$\int_{\Lambda_\varepsilon^c} (w - a)_+ \ge C\delta_\varepsilon^{(N+1)/2}\delta_\varepsilon,$$

and

$$\int_{\Omega_\varepsilon} |\phi_\varepsilon| + |P_{\Omega_\varepsilon} w - w| \le e^{-(1+\eta)d_\varepsilon/\varepsilon},$$

for some $\eta > 0$. Combining the above inequalities we immediately obtain that $\delta_\varepsilon \le e^{-\rho d_\varepsilon/\varepsilon}$ for some $\rho > 0$, and the proof of the lemma is thus complete. ◻

2 Proof of Theorem 0.1

In this section we prove Theorem 0.1. We recall that from Section 1, we have a critical point u_ε of the functional J_ε. However, this critical point does not provide a solution to equation (0.4), unless $u_\varepsilon \le a$ over $\partial\Lambda$, so avoiding the penalization. In obtaining this last fact hypothesis (H2) is crucial. The following proposition is a key step.

Proposition 2.1 *As $\varepsilon \to 0$ we have*

$$\max_{x\in\partial\Lambda} u_\varepsilon(x) \to 0. \tag{2.1}$$

Since u_ε is a critical point of J_ε, it satisfies

$$\begin{cases} \varepsilon^2\Delta u_\varepsilon - u_\varepsilon + \chi_\Lambda f(u_\varepsilon) + (1 - \chi_\Lambda)\tilde{f}(u_\varepsilon) = 0 \text{ in } \Omega \\ u_\varepsilon = 0 \ \text{ on } \ \partial\Omega \end{cases} \tag{2.2}$$

Now, Lemma 1.2 ensures the presence of just one local maximum x_ε of u_ε which is in Λ. Furthermore the critical value of u_ε, can be estimated as

$$S_\varepsilon = J_\varepsilon(u_\varepsilon) = \varepsilon^N\{c_* + e^{-\frac{2}{\varepsilon}(d(x_\varepsilon,\partial\Omega)+o(1))}\}.$$

In order to prove Proposition 2.1 we need the following preliminary result

Lemma 2.1

$$d(x_\varepsilon, \partial\Omega) \to c, \quad as\ \varepsilon \to 0,$$

where c is the max-min value in (H1)-(H2).

Proof. First we see that given $\delta > 0$ we have that

$$d(x_\varepsilon, \partial\Omega) \geq c - \delta,$$

for all small ε. In fact, take any $\phi \in \Gamma$ where Γ is defined as in (H1)-(H2), and consider its associated $\phi_\varepsilon \in \Gamma_\varepsilon$ defined as

$$\phi_\varepsilon(y) = w_\varepsilon^{\phi(y)}.$$

See definition of S_ε in Section 1. Then

$$J_\varepsilon(u_\varepsilon) = S_\varepsilon \leq \sup_{y\in B} J_\varepsilon(\phi_\varepsilon(y)) \leq \varepsilon^N\{c_* + e^{-\frac{2}{\varepsilon}\{\inf_{y\in B} d(\phi(y),\partial\Omega)-\delta/4)\}}\}$$

for all sufficiently small ε. Choosing ϕ adequately, from here it follows that

$$J_\varepsilon(u_\varepsilon) \leq \varepsilon^N\{c_* + e^{-\frac{2}{\varepsilon}(c-\delta/2)}\}.$$

This inequality combined with the lower estimate in Lemma 1.2 gives $d_\varepsilon = d(x_\varepsilon, \partial\Omega) \geq c - \delta$ as desired. Therefore we have shown that

$$\liminf_{\varepsilon\to 0} d(x_\varepsilon, \partial\Omega) \geq c.$$

Next we establish that

$$\limsup_{\varepsilon\to 0} d(x_\varepsilon, \partial\Omega) \leq c.$$

Assume by contradiction that for a certain $\delta > 0$ we have that along a sequence $\varepsilon = \varepsilon_k \to 0$,

$$d(x_\varepsilon, \partial\Omega) \geq c + \delta.$$

We will show that this is not possible. In fact, this and (1.6) in Lemma 1.2 imply that

$$S_\varepsilon \leq \varepsilon^N\{c_* + e^{-\frac{2}{\varepsilon}(c+\delta/2)}\}. \tag{2.3}$$

Then we consider a path $\phi_\varepsilon \in \Gamma_\varepsilon$ for which

$$\sup_{y\in B} J_\varepsilon(\phi_\varepsilon(y)) \leq S_\varepsilon + e^{-L/\varepsilon},$$

for a large and fixed L. Choose now $y_\varepsilon \in B$ such that

$$d(\beta(\phi_\varepsilon(y_\varepsilon)), \partial\Omega) = \min_{y\in B} d(\beta(\phi_\varepsilon(y)), \partial\Omega), \tag{2.4}$$

where β was defined in (1.8). Set now $\tilde{u}_\varepsilon \equiv \phi_\varepsilon(y_\varepsilon)$ and observe that Corollary 1.1 applies to this sequence to yield

$$J_\varepsilon(\tilde{u}_\varepsilon) \geq \varepsilon^N\{c_* + e^{-\frac{2}{\varepsilon}(d(\beta(\tilde{u}_\varepsilon),\partial\Omega)+o(1))}\}. \tag{2.5}$$

Then combining (2.3), (2.4), (2.5) we obtain

$$\min_{y\in B} d(\beta(\phi_\varepsilon(y)), \partial\Omega) \geq c + \delta/4.$$

Now, the usual concentration-compactness argument gives that $\beta(\phi_\varepsilon(y)) \in \tilde{\Lambda}$ for all small ε, uniformly on $y \in B$, where $\tilde{\Lambda}$ is a small fixed neighborhood of Λ. Then a slight modification of $\beta(\phi_\varepsilon(y))$ provides a test path $\psi_\varepsilon(y)$ in Γ for which

$$\min_{y\in B} d(\psi_\varepsilon(y)), \partial\Omega) \geq c + \delta/8.$$

This contradicts the definition of the number c in (0.7), thus concluding the proof. □

Next we prove Proposition 2.1.

Proof of Proposition 2.1: Suppose not, namely that there exists $\overline{x}_\varepsilon \in \partial\Lambda, u_\varepsilon(\overline{x}_\varepsilon) \geq b > 0$. We may assume, after passing to a suitable subsequence, that $\bar{x}_\varepsilon \to \bar{x} \in \partial\Lambda$. Let $\hat{T}$ be a unit vector tangent to $\partial\Lambda$ at $\bar{x}$.

Let $\hat{T}_\varepsilon$ be a sequence of unit vectors $\hat{T}_\varepsilon \to \hat{T}$ which we will choose later. We use $\nabla u_\varepsilon \cdot \hat{T}_\varepsilon$ as a test function in equation (2.2) to get

$$\frac{1}{2}\int_{\partial\Omega} |\nabla u_\varepsilon|^2 \hat{T}_\varepsilon \cdot \nu \, d\sigma = -\int_{\partial\Lambda}[F(u_\varepsilon) - \tilde{F}(u_\varepsilon)]\hat{n}\cdot \hat{T}_\varepsilon d\sigma.$$

Note that the support of $F(u_\varepsilon) - \tilde{F}(u_\varepsilon)$ shrinks to the point $\bar{x}$ and that this function is nonnegative. Then we may choose $\hat{T}_\varepsilon$ in such a way that $\int_{\partial\Lambda}[F(u_\varepsilon) - \tilde{F}(u_\varepsilon)]\hat{n}\cdot\hat{T}_\varepsilon d\sigma = 0$, so that

$$\frac{1}{2}\int_{\partial\Omega} |\nabla u_\varepsilon|^2 \hat{T}_\varepsilon \cdot \nu \, d\sigma = 0. \tag{2.6}$$

Next we estimate $|\nabla u_\varepsilon|^2$ on $\partial\Omega$. To find an upper estimate we consider z_ε, the solution of

$$\varepsilon^2\Delta z_\varepsilon - (1-\delta)^2 z_\varepsilon = 0 \quad \text{in } B(x_\varepsilon, R\varepsilon),$$

such that $z_\varepsilon = 1$ on $\partial B(x_\varepsilon, R\varepsilon)$. If R is chosen large enough, the maximum principle implies that $u_\varepsilon(x) \le z_\varepsilon(x)$ in $\Omega \setminus B(x_\varepsilon, R\varepsilon)$. But it is not hard to check that

$$z_\varepsilon(x) \le Ce^{-|x-x_\varepsilon|(1-2\delta)/\varepsilon}.$$

Then, local elliptic estimates near the boundary imply that there is an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ and all $x \in \partial\Omega$ one has

$$|\nabla u_\varepsilon(x)| \le e^{-|x-x_\varepsilon|(1-3\delta)/\varepsilon}.$$

Next we derive a lower estimate for the gradient. Let us consider

$$S = \partial\Omega \cap B(\bar{x}, d), \quad d = dist(\bar{x}, \partial\Omega),$$

and, $S_\varepsilon = \{x \in \partial\Omega \,/\, dist(x, S) < \varepsilon\}$. It is easy to see that there is a $C > 0$, so that for $x \in S_\varepsilon$, ε small, there is a point $\tilde{x}_\varepsilon \in \Omega$ such that

$$|\tilde{x}_\varepsilon - \bar{x}| < C\varepsilon, \quad |d_\varepsilon - d| < C\varepsilon \text{ and } B(\tilde{x}_\varepsilon, d_\varepsilon) \subset \Omega,$$

where $d_\varepsilon = dist(\tilde{x}_\varepsilon, \partial\Omega)$. Now we choose z_ε to be the solution of

$$\varepsilon^2\Delta z_\varepsilon - z_\varepsilon = 0 \quad \text{in } B(\tilde{x}_\varepsilon, d_\varepsilon) \setminus B(\tilde{x}_\varepsilon, R\varepsilon),$$

such that $z_\varepsilon = \eta$ on $\partial B(\tilde{x}_\varepsilon, R\varepsilon)$ and $z_\varepsilon = 0$ on $\partial B(\tilde{x}_\varepsilon, d_\varepsilon)$. Here $\eta > 0$ is chosen so small that

$$u_\varepsilon(x) \ge \eta \quad \forall x \in \partial B(\tilde{x}_\varepsilon, R\varepsilon),$$

for an appropriately chosen $R > 0$. Then, from the maximum principle we find that $u_\varepsilon \ge z_\varepsilon$, from where it follows that

$$\left|\frac{\partial u_\varepsilon}{\partial n}(x)\right| \ge \left|\frac{\partial z_\varepsilon}{\partial n}(x)\right| \quad x \in S_\varepsilon.$$

Now, a direct computation gives

$$\left|\frac{\partial z_\varepsilon}{\partial n}(x)\right| \ge e^{-\frac{1}{\varepsilon}(d+o(1))}, \quad x \in S_\varepsilon.$$

Hence the latter inequalities imply

$$\int_{\partial\Omega} |\nabla u_\varepsilon|^2 d\sigma \ge \int_{S_\varepsilon} |\nabla u_\varepsilon|^2 d\sigma \ge e^{-\frac{2}{\varepsilon}(d+o(1))}.$$

Next we consider the probability measure $d\mu_\varepsilon$ on $\partial\Omega$ given by

$$d\mu_\varepsilon(x) = \frac{|\nabla u_\varepsilon(x)|^2 d\sigma(x)}{\int_{\partial\Omega} |\nabla u_\varepsilon|^2 d\sigma}.$$

Then passing to a subsequence we have

$$d\mu_\varepsilon \rightharpoonup d\mu,$$

where $d\mu$ is a Borel probability measure on $\partial\Omega$. We claim that the support of $d\mu$ is contained in S. Indeed if $\mathcal{O}$ is an open subset of $\partial\Omega$ which does not intersect S then for some $\delta > 0$

$$\mu(\mathcal{O}) = \lim_{\varepsilon\to 0} \int_{\mathcal{O}} d\mu_\varepsilon \le \lim_{\varepsilon\to 0} \frac{\int_{\{x\in\partial\Omega \ | \ dist(x,S)\ge\delta\}} |\nabla u_\varepsilon|^2 d\sigma}{\int_{\partial\Omega} |\nabla u_\varepsilon|^2 d\sigma} \le \lim_{\varepsilon\to 0} \frac{e^{-\frac{2}{\varepsilon}(d+\delta')}}{e^{-\frac{2}{\varepsilon}(d+o(1))}},$$

for some small number $\delta' > 0$. It follows that $\mu(\mathcal{O}) = 0$ and the claim is thus proven.

Next we observe that relation (2.6) can be written as

$$\int_{\partial\Omega} \hat{T}_\varepsilon \cdot \nu d\mu_\varepsilon = 0.$$

Since $\hat{T}_\varepsilon \cdot \nu \to \hat{T} \cdot \nu$ uniformly, it follows that

$$\int_{\partial\Omega} \hat{T} \cdot \nu d\mu = \int_S \hat{T} \cdot \nu d\mu = 0.$$

Now, observe that when $x \in S$ one has $\nu(x) = (x - \bar{x})/d$, so that

$$\int_S \hat{T} \cdot (x - \bar{x}) d\mu(x) = \hat{T} \cdot \int_S (x - \bar{x}) d\mu(x) = 0.$$

We see that $\tau = \int_S (x - \bar{x}) d\mu(x)$ belongs to $conv(S - \bar{x})$. Thus, choosing $\hat{T}$ appropriately, also using that $dist(\bar{x}, \partial\Omega) = c$ by Lemma 2.1, and assumption (H2) one gets a contradiction which finishes the proof. □

Proof of Theorem 0.1: By Proposition 2.1, there exists ε_0 such that for $0 < \varepsilon < \varepsilon_0, u_\varepsilon < a$ for all $x \in \partial\Omega$. The function $u_\varepsilon \in H_0^1$ solves then equation (0.3). Then Lemma 2.1 and Lemma 1.2 provide all conclusions of the theorem. This concludes the proof. □

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