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ON THE ROLE OF MEAN CURVATURE IN SOME SINGULARLY PERTURBED NEUMANN PROBLEMS*

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Abstract. We construct solutions exhibiting a single spike-layer shape around some point of the boundary as $\varepsilon \to 0$ for the problem

$$\begin{cases} \varepsilon^2 \triangle u - u + u^p = 0 \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a bounded domain with smooth boundary in R^N, $p > 1$, and $p < \frac{N+2}{N-2}$ if $N \geq 3$. Our main result states that given a *topologically nontrivial critical point* of the mean curvature function of $\partial\Omega$, for instance, a possibly degenerate local maximum, local minimum, or saddle point, there is a solution with a single local maximum, which is located at the boundary and approaches this point as $\varepsilon \to 0$ while vanishing asymptotically elsewhere.

Key words. spike layer, singular perturbations, Neumann problems

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1. Introduction.

In this paper, we are concerned with the following singularly perturbed problem:

$$\begin{cases} \varepsilon^2 \Delta u - u + u^p = 0 \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial v} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^N$ is a smooth, not necessarily bounded domain; $\varepsilon > 0$; and $1 < p < (N+2)/(N-2)$ if $N \geq 3$ and $p > 2$ if $N = 2$.

Equation (1.1) arises from various applications. For instance, it can be regarded as that satisfied by stationary solutions for the Keller–Segal system in chemotaxis (see [14], [17], [19]) and the Gierer–Meinhardt system in biological pattern formation (see [12], [21]).

In [17], Lin, Ni, and Takagi first studied the problem of existence of least-energy solutions. Subsequently, Ni and Takagi in [19] and [21] showed that the least-energy solution u_ε has a unique local maximum point P_ε, which is located on $\partial\Omega$. Moreover, $u_\varepsilon \to 0$ in $C^1_{loc}(\overline{\Omega}\backslash P_\varepsilon)$ and $u_\varepsilon(P_\varepsilon) \to \alpha > 0$ as $\varepsilon \to 0$. Such a family of solutions is usually called a *boundary spike-layer*. Moreover, they are able to locate the spike by establishing that P_ε approaches the *most curved* part of $\partial\Omega$, namely, $H(P_\varepsilon) \to$

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$\max_{P\in\partial\Omega} H(P)$, where H is the mean curvature. Later Wei studied general boundary spike solutions in [23] and showed that for any solution with single peak P_ε on $\partial\Omega$, $\nabla_{\tau_{P_\varepsilon}} H(P_\varepsilon) \to 0$, where $\nabla_{\tau_{P_\varepsilon}}$ denote the tangential gradient at $P_\varepsilon \in \partial\Omega$. On the other hand, if $P_0 \in \partial\Omega$, $\nabla_{\tau_{P_0}} H(P_0) = 0$ and the matrix $(\nabla^2_{\tau_{P_0}} H(P_0))$ is nonsingular, then there exists for ε sufficiently small, solution u_ε of (1.1) with a single peak approaching P_0. The degenerate case was left open.

In [21], Ni and Takagi constructed boundary spike solutions in the case when Ω is axially symmetric. Gui [10] has studied the case when $H(P)$ has a possibly degenerate local maximum at P_0, also constructing multiple-peak solutions at given local maximum points of $H(P)$. In the single peak case, the result in [10] states that for any set $\Lambda \subset \partial\Omega$, open relative to $\partial\Omega$, such that

$$\max_{P\in\Lambda} H(P) > \max_{P\in\partial\Lambda} H(P) \tag{1.2}$$

there exists a family of solutions with a single global maximum point which approaches a local maximum point of $H(P)$ in Λ.

In this paper, we will show that a spike-layer family indeed exists concentrating at any *topologically nontrivial critical point-region*, a variational linking notion first introduced in [5] in the framework of concentration phenomena in nonlinear Schrödinger equations.

This notion includes, for instance, the case of local maxima or local minima of the mean curvature of the boundary, in the same sense as in (1.2), and also that of a possibly degenerate *saddle-point*. More precisely, we can consider a local situation on a set $\Lambda \subset \partial\Omega$ where a change of topology of the level sets of $H(P)$ occurs. If c is the level at which this change takes place in a sense to be made precise below, then a boundary-spike family of solutions exists, with maxima $P_\varepsilon \in \Lambda$ so that $H(P_\varepsilon) \to c$.

Since we do not want to restrict ourselves to the case of a homogeneous nonlinearity, we will consider the more general semilinear Neumann problem

$$\begin{cases} \varepsilon^2 \Delta u - u + f(u) = 0 \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial v} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where ε is a small positive number. $f : R \to R$ satisfies the conditions (f1)–(f5) below:

(f1) $f \in C^1(R)$, $f(t) \equiv 0$ for $t \le 0$, and $f(t) \to +\infty$ as $t \to +\infty$.

(f2) For $t \ge 0$, f admits the decomposition in $C^1(R)$

$$f(t) = f_1(t) - f_2(t),$$

where (i) $f_1(t) \ge 0$, $f_2(t) \ge 0$ with $f_1(0) = f_1'(0) = f_2(0) = f_2'(0) = 0$; and (ii) there is a $q \ge 1$ such that $\frac{f_1(t)}{t^q}$ is nondecreasing in $t > 0$, where as $\frac{f_2(t)}{t^q}$ is nonincreasing in $t > 0$.

(f3) $|f'(t)| \le a_1 + a_2 t^{p-1}$ for some positive constants a_1, a_2 and $1 < p < (\frac{N+2}{N-2})_+$.

(f4) There exists $\eta \in (0, \frac{1}{2})$ such that $F(t) \le \eta t f(t)$, $t \ge 0$, where $F(t) = \int_0^t f(s)ds$.

To state the last condition, as in [20], we consider the problem in the whole space

$$\begin{cases} \Delta w - w + f(w) = 0, w > 0 \text{ in } R^N, \\ w(0) = \max\limits_{x\in R^N} w(x) \text{ and } w(x) \to 0 \text{ as } |x| \to +\infty. \end{cases} \tag{1.4}$$

It is well known that (1.4) has a solution w, and w is radial and unique (see [13], [4], [15]). The last condition is stated in (f5).

(f5) $L = \Delta - 1 + f'(w)$ is invertible over $H_r^2(R^N) = \{u \in H^2 : u(x) = u(|x|)\}$.

We note that the function

$$f(t) = t^p - at^q \text{ for } t \geq 0, 1 < q < p$$

with p subcritical and $a \geq 0$ satisfies all the assumptions (see [20]).

Let $H(P)$ be the mean curvature function at $P \in \partial\Omega$. In what follows, we state precisely our assumption on Ω and H. We assume that Ω is a smooth, not necessarily bounded domain in R^N, and that there is an open and bounded set $\Lambda \subset \partial\Omega$ with smooth boundary $\partial\Lambda$ and closed subsets of Λ, B, B_0 such that B is connected and $B_0 \subset B$. Let Γ be the class of all continuous functions $\phi : B \to \Lambda$ with the property that $\phi(y) = y$ for all $y \in B_0$. Assume that the max-min value

$$c = \sup_{\phi \in \Gamma} \min_{y \in B} H(\phi(y)) \tag{1.5}$$

is well defined and additionally that

(H1)

$$\min_{y \in B_0} H(y) > c.$$

(H2) For all $y \in \partial\Lambda$ such that $H(y) = c$, there exists a direction $\hat{T}$, tangent to $\partial\Lambda$ at y so that

$$\nabla H(y) \cdot \hat{T} \neq 0.$$

Note that $\partial\Lambda \subset \partial\Omega$ is an $(N-2)$-dimensional set.

Standard deformation arguments show that these assumptions ensure that the max-min value c is a critical value for $H(P)$ in Λ, which is topologically nontrivial (therefore, our results cover that of [10] in the single peak case). In fact, assumption (H2) "seals" Λ so that the local linking structure described indeed provides critical points at the level c in Λ, possibly admitting full degeneracy.

It is not hard to check that all these assumptions are satisfied in a general local maximum, local minimum, or saddle-point situation, not necessarily nondegenerate or isolated. Our main result asserts that there is a family of solutions to problem (1.1) concentrating around a critical point at the level c of H in Λ.

THEOREM 1.1. *Suppose f satisfies* (f1)–(f5) *and the mean curvature function H satisfied* (H1) *and* (H2). *Then there exists $\varepsilon_0 > 0$ such that when $\varepsilon \leq \varepsilon_0$, problem* (1.3) *has a solution u_ε with the property that*

(i) *u_ε has exactly one local maximum point x_ε and $x_\varepsilon \in \Lambda$;*

(ii) $\lim_{\varepsilon \to 0} H(x_\varepsilon) = c$;

(iii) $\lim_{\varepsilon \to 0} u_\varepsilon(x_\varepsilon + \varepsilon x) = w(x)$ *and there exist positive constants c, δ such that*

$$0 < u_\varepsilon(x) \leq c \ \exp\left(-\frac{\delta|x - x_\varepsilon|}{\varepsilon}\right), \quad x \in \overline{\Omega}.$$

Here w is the unique solution of (1.4).

The proof of this result makes use of ideas developed in [20] and [23] and a variational scheme similar to that in [5], where it is constructed as a bound state for the *semiclassical Schrödinger equation*

$$\varepsilon^2 \Delta u - V(x)u + u^p = 0 \quad \text{in} \quad R^N,$$

exhibiting concentration near topologically nontrivial critical points of $V(x)$; see also the work of the authors in [9]. Related results in this direction can be found in [6] and [7].

We have recently learned that Li [16] has considered, in the case of a bounded domain, a different notion of *nontriviality* not variational in nature. This notion is implied by our assumptions (H1)–(H2) in case the curvature is C^1. Thus, in case $f(s) = u^p$, with p superlinear and subcritical, and for a bounded domain, our result is a consequence of the results in [16]. However, Li's method, relying on a finite-dimensional Lyapunov–Schmidt reduction, is very different from ours.

On the other hand, our method is also applicable to obtain partial localization results even in case H is not C^1.

Finally, we remark that when $p = \frac{N+2}{N-2}$, problem (1.1) has been studied in [1], [2], [3], [11], [18], and [22], among others.

The rest of this paper will be devoted to the proof of Theorem 1.1. In section 2, we define a modified functional which satisfies the Palais–Smale (P.S.) condition and, roughly speaking, permits us to restrict ourselves to what happens in Λ. We then define a min-max value and by using assumption (H1) we prove that there is a critical point for the modified functional with this value. In section 3 by using assumption (H2) we prove that the critical point so found is actually a critical point of the original functional and we conclude the proof of Theorem 1.1.

2. Preliminary results and set-up of a min-max scheme. In this section, we first define a modified functional and state some preliminary results. We then set up a variational scheme and obtain a critical point for the modified functional.

Let $f : R \to R$ satisfying (f1)–(f5). We first define an "energy" functional

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega \varepsilon^2|\nabla u|^2 + u^2 - \int_\Omega F(u),$$

where $u \in H^1(\Omega)$, $F(u) = \int_0^u f(s)ds$.

As in [5], we now define a modification of this functional which satisfies the P.S. condition and for which we find a critical point via an appropriate min-max scheme.

Let $\mu = \frac{1}{\eta}$, where η is defined by (f4). Let $R > \frac{\mu}{\mu-2}$. Let $a > 0$ be the value at which $f(a)/a = 1/R$. Set

$$\bar{f}(s) = \begin{cases} f(s) & \text{if } s \le a, \\ \dfrac{1}{R}s & \text{if } s > a. \end{cases}$$

The following technical lemma is stated in [10] and can be proved by using local coordinate systems for $\partial\Lambda$.

LEMMA 2.1. *There exists a subdomain $\partial\Omega_0 \subset \Omega$ such that $\partial\Omega_0 \cap \partial\Omega = \overline{\Lambda}$ and $\partial\Omega_0^+ := \overline{\partial\Omega_0 \backslash \partial\Omega}$ is smooth and orthogonal to $\partial\Omega$ at $\partial\Lambda$.*

We now define

$$g(\cdot, s) = \chi_{\Omega_0} f(s) + (1 - \chi_{\Omega_0})\bar{f}(s) \quad \text{and} \quad G(x, \xi) = \int_0^\xi g(x, \tau)d\tau,$$

where χ_{Ω_0} denotes the characteristic function of Ω_0.

First we note that g is a Carathéodory function. In addition one can check that (f1)–(f4) implies that g satisfies the following conditions:

(g1) $g(x,t) = 0$ for $t \le 0$ and $g(x,t) \to \infty$ as $t \to \infty$.
(g2) $g(x,t) = o(t)$ near $t = 0$ uniformly in $x \in \Omega$.
(g3) $g(x,t) = O(t^p)$ as $t \to \infty$ for $1 < p < \frac{N+2}{N-2}$ if $N \ge 3$ and no restriction on p if $N = 1, 2$.
(g4) (i) $G(x,t) \le \mu g(x,t)t \quad \forall x \in \Omega_0, t > 0$

and

(ii) $2G(x,t) \le g(x,t)t \le \frac{1}{R}t^2 \quad \forall t \in R^+, x \notin \Omega_0$.

Consider the modified functional

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega \varepsilon^2|\nabla u|^2 + \frac{1}{2}\int_\Omega u^2 - \int_\Omega G(x,u), \qquad u \in H^1(\Omega),$$

whose critical points correspond to solutions of the equation

$$\begin{cases} \varepsilon^2\Delta u - u + g(u,x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial v} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

As in [5], J_ε satisfies the P.S. condition whether Ω is bounded or not. We observe that a solution to (2.1) which satisfies that $u \le a$ on $\overline{\Omega}\backslash\Omega_0$ will also be a solution of (1.3). We will define a min-max quantity for J_ε which will yield a solution to (2.1) which turns out to be a solution for (1.3) and thus will be the solution announced by Theorem 1.1.

To this end, we consider the solution manifold of (2.1) defined as

$$M_\varepsilon = \left\{ u \in H^1(\Omega)\backslash\{0\} | \int_\Omega (\varepsilon^2|\nabla u|^2 + u^2) = \int_\Omega g(x,u)u \right\}. \tag{2.2}$$

All nonzero critical points of J_ε of course lie on M_ε. Reciprocally, it is standard to check that critical points of J_ε constrained to this manifold are critical points of J_ε on $H^1(\Omega)$.

Let w be the unique solution of (1.4) and let us consider its energy

$$I(w) = \frac{1}{2}\int_{R^N} (|\nabla w|^2 + w^2) - \int_{R^N} F(w). \tag{2.3}$$

For $P \in \partial\Omega$, we define w_ε^P as

$$w_\varepsilon^P = t_{\varepsilon,P}\, w\left(\frac{x-P}{\varepsilon}\right) \in M_\varepsilon,$$

with $t_{\varepsilon,P} > 0$. Let us consider the center of mass of a function $u \in L^2(\Omega)$ defined as

$$\beta(u) = \frac{\int_{\Omega_0} xu^2dx}{\int_\Omega u^2dx}. \tag{2.4}$$

For $P \in B$, it is easy to see that $\beta(w_\varepsilon^P) = P + O(\varepsilon)$. Hence, there exists a continuous function $\tau_\varepsilon(P)$ such that $\tau_\varepsilon(P) = P + O(\varepsilon)$ and $\beta(w_\varepsilon^{\tau_\varepsilon(P)}) = P$ for $P \in B$. We now define

$$w_{\varepsilon,P} = w_\varepsilon^{\tau_\varepsilon(P)}.$$

Hence we have $\beta(w_{\varepsilon,P}) = P \; \forall \; P \in B$, and by similar arguments as in Proposition 3.2 in [19] we find that, $\forall \; P \in B$,

$$J_\varepsilon(w_{\varepsilon,P}) = \varepsilon^N \left\{ \frac{1}{2} I(w) - \gamma\varepsilon(N-1)H(P) + o(\varepsilon) \right\}, \tag{2.5}$$

where

$$\gamma := \frac{1}{N+1} \int_{R^N_+} w'(y)^2 y_N dy. \tag{2.6}$$

We now consider the class Γ_ε of all continuous maps $\varphi : B \to M_\varepsilon$ such that

$$\varphi(y) = w_{\varepsilon,y} \qquad \forall y \in B_0,$$

and we define the min-max value S_ε as follows:

$$S_\varepsilon = \inf_{\varphi \in \Gamma_\varepsilon} \sup_{y \in B} J_\varepsilon(\varphi(y)). \tag{2.7}$$

We note that

$$S_\varepsilon \geq \sup_{y \in B_0} J_\varepsilon(w_{\varepsilon,y}) \tag{2.8}$$

and

$$S_\varepsilon = \inf_{\varphi \in \Gamma_\varepsilon} \sup_{y \in B} J_\varepsilon(\varphi(y)) \leq \sup_{y \in B} J_\varepsilon(w_{\varepsilon,y}). \tag{2.9}$$

Hence by (2.5), (2.8), and (2.9), we have

$$\lim_{\varepsilon \to 0} \varepsilon^{-N} S_\varepsilon = \frac{1}{2} I(w). \tag{2.10}$$

The following is the key result of this section. It implies that S_ε is a critical value for J_ε.

LEMMA 2.2. *For ε sufficiently small, we have*

$$S_\varepsilon > \sup_{y \in B_0} J_\varepsilon(w_{\varepsilon,y}). \tag{2.11}$$

In the rest of this section, we prove Lemma 2.2. To this end we will first prove a version of a result of Ni and Takagi for the modified functional J_ε (see Proposition 2.1 in [20]).

LEMMA 2.3. *Let $\Omega_1 \subset \overline{\Omega}$ be a subdomain such that $\partial\Omega_1 \cap \partial\Omega = \Lambda_1$ is open relative to $\partial\Omega$ and $\partial\Omega_1^+ := \overline{\partial\Omega_1 \backslash \partial\Omega}$ is smooth and orthogonal to $\partial\Omega$ at $\partial\Lambda_1$. We define*

$$g_{\Omega_1}(x,u) = \chi_{\Omega_1} f(u) + (1 - \chi_{\Omega_1})\bar{f}(u), \qquad G_{\Omega_1}(x,u) = \int_0^u g_{\Omega_1}(x,s)ds,$$

and

$$J_{\varepsilon,\Omega_1}(u) = \frac{1}{2}\int_\Omega \varepsilon^2 |\nabla u|^2 + \frac{1}{2}\int_\Omega u^2 - \int_\Omega G_{\Omega_1}(x,u).$$

Suppose that u_ε *is a solution of*

$$\begin{cases} \varepsilon^2 \Delta u - u + g_{\Omega_1}(x,u) = 0 \ \text{in} \ \Omega, \\ u > 0 \ \text{in} \ \Omega, \\ \dfrac{\partial u}{\partial v} = 0 \ \text{on} \ \partial\Omega, \end{cases} \tag{2.12}$$

such that

$$\varepsilon^{-N} J_{\varepsilon,\Omega_1}(u_\varepsilon) \to \frac{1}{2} I(w). \tag{2.13}$$

Then we have

$$J_{\varepsilon,\Omega_1}(u_\varepsilon) = \varepsilon^N \left\{ \frac{1}{2} I(w) - \gamma\varepsilon(N-1)H(x_\varepsilon) + o(\varepsilon) \right\}, \tag{2.14}$$

where $x_\varepsilon \in \partial\Omega_1 \cap \partial\Omega$ *is the maximum point of* u_ε *and* γ *is defined by* (2.6). *In particular,*

$$J_{\varepsilon,\Omega_1}(u_\varepsilon) \geq \varepsilon^N \left\{ \frac{1}{2} I(w) - \varepsilon\gamma \max_{x \in \partial\Omega_1 \cap \partial\Omega} (N-1)H(x) + o(\varepsilon) \right\}. \tag{2.15}$$

Before going into the proof of Lemma 2.3 we state and prove a corollary that will be useful later.

COROLLARY 2.1. *Let* $\varepsilon = \varepsilon_k \to 0$ *and* $u_\varepsilon \in M_{\varepsilon,\Omega_1}$ *be a family of functions such that*

$$\limsup_{\varepsilon \to 0} \varepsilon^{-N} J_{\varepsilon,\Omega_1}(u_\varepsilon) \leq \frac{1}{2} I(w), \tag{2.16}$$

where

$$M_{\varepsilon,\Omega_1} = \left\{ u \in H^1(\Omega) \backslash \{0\} | \int_\Omega (\varepsilon^2 |\nabla u|^2 + u^2) = \int_\Omega g_{\Omega_1}(x,u)u \right\}.$$

Let $x_\varepsilon = \beta(u_\varepsilon)$ *be the center of mass of* u_ε; *then* $x_\varepsilon \to \partial\Omega$, *and if* $\bar{x}$ *is an accumulation point of* $\{x_\varepsilon\}$, *the following estimate holds:*

$$J_{\varepsilon,\Omega_1}(u_\varepsilon) \geq \varepsilon^N \left\{ \frac{1}{2} I(w) - \gamma\varepsilon(N-1)H(\bar{x}) + o(\varepsilon) \right\}. \tag{2.17}$$

Proof. Passing to a subsequence we can assume that $x_\varepsilon \to \bar{x}$. Let us consider the modified center of mass defined as

$$\bar{\beta}(u) = \frac{\int_{B_\delta(\bar{x})} x u^2}{\varepsilon^N \int_{R^N} w^2}.$$

Given $\delta > 0$ we then have that

$$\bar{\beta}(u_\varepsilon) \in B_\delta(\bar{x}) \tag{2.18}$$

$\forall$ small ε. In fact, using a concentration-compactness-type argument similar to the one given in Lemma 1.1 in [5], we find $R > 0$, a subsequence $\varepsilon \to 0$, and $y_\varepsilon \in \Omega_\varepsilon = \varepsilon^{-1}\Omega$ such that

$$\int_{B_R(y_\varepsilon)} v_\varepsilon^2 \geq \sigma > 0,$$

where $v_\varepsilon(x) = v_\varepsilon(\varepsilon x)$.

Let us assume first that $dist(y_\varepsilon, \partial\Omega_\varepsilon) \to \infty$. Since v_ε is bounded in $H^1(\Omega_\varepsilon)$, given $\delta > 0$ there exists $r > 0$ such that

$$\int_{B_{r+1}(0)\setminus B_r(0)} |\nabla u_\varepsilon|^2 + u_\varepsilon^2 \leq \delta.$$

Then we choose an appropriate cut-off function ψ so that $\psi = 1$ on $B_r(0)$ and $\psi = 0$ on $B_{r+1}(0)$ and we find

$$u_\varepsilon = \psi u_\varepsilon + (1-\psi)u_\varepsilon = w_\varepsilon + v_\varepsilon.$$

If we choose δ small enough, we find that for both v_ε and w_ε we can find t_ε^1, t_ε^2 very close to 1 so that $\tilde{w}_\varepsilon = t_\varepsilon^1 w_\varepsilon$ and $\tilde{v}_\varepsilon = t_\varepsilon^2 v_\varepsilon$ are in M_{ε,Ω_1}. But this implies that lim inf $J_{\varepsilon,\Omega_1}(u_\varepsilon) \geq I(w)$, contradicting the hypothesis.

Therefore, we must have that $dist(y_\varepsilon, \partial\Omega_\varepsilon) \leq C$. We can assume that $y_\varepsilon \in \partial\Omega_\varepsilon$. By the argument given above, taking a sequence $\delta_n \to 0$ and using (2.16) we find a subsequence $u_\varepsilon = v_\varepsilon + w_\varepsilon$ with $w_\varepsilon \to 0$.

Finally, using the minimizing character of this sequence u_ε and Ekeland's variational principle we find that $u_\varepsilon(x_\varepsilon + \varepsilon y)$ converges in H^1-sense to a least energy critical point w of the limiting functional I given in (2.3) in the half space. We certainly have that $x_\varepsilon + \varepsilon y_\varepsilon \to x \in \partial\Omega$, thus proving (2.18).

Then we have

$$J_{\varepsilon,\Omega_1}(u_\varepsilon) \geq \inf\{J_{\varepsilon,\Omega_1}(u) \mid u \in M_{\varepsilon,\Omega_1},\ \bar{\beta}(u) \in B_\delta(\bar{x})\}.$$

Since the functional J_{ε,Ω_1} satisfies the P.S. condition, it follows that the latter number is attained at some function $\bar{u}_\varepsilon \in H^1(\Omega)$. Working out a first variation with test functions supported outside $B_\delta(\bar{x})$, we see that $\bar{u}_\varepsilon$ satisfies the equation

$$\varepsilon^2 \Delta \bar{u}_\varepsilon - \bar{u}_\varepsilon + g_{\Omega_1}(x, \bar{u}_\varepsilon) = 0 \text{ in } \Omega \setminus B_\delta(\bar{x}).$$

Again, if we set $v_\varepsilon(y) = \bar{u}_\varepsilon(\bar{x}_\varepsilon + \varepsilon y)$ with $\bar{x}_\varepsilon = \beta(\bar{u}_\varepsilon)$, then v_ε converges in the H^1-sense to w in the half space. In particular, elliptic estimates applied to the above equation imply that $\bar{u}_\varepsilon$ goes to zero uniformly, away from the ball $B_\delta(\bar{x})$. Thus we have that

$$J_{\varepsilon,\Omega_1}(\bar{u}_\varepsilon) = J_{\varepsilon,\Omega_1 \cap B_{2\delta}(\bar{x})}(\bar{u}_\varepsilon)$$

and also $\bar{u}_\varepsilon \in M_{\varepsilon,\Omega_1 \cap B_{2\delta}(\bar{x})}$. Let us consider a set Ω_δ so that $\Omega_1 \cap B_{2\delta}(\bar{x}) \subset \Omega_\delta \subset \Omega_1 \cap B_{3\delta}(\bar{x})$, satisfying the hypotheses of Lemma 2.3. Then we obtain

$$J_{\varepsilon,\Omega_1}(\bar{u}_\varepsilon) \geq \inf_{u \in M_{\varepsilon,\Omega_\delta}} J_{\varepsilon,\Omega_\delta}(u).$$

However the latter number can be estimated from below using Lemma 2.3. Doing so we have

$$J_{\varepsilon,\Omega_1}(\bar{u}_\varepsilon) \geq \varepsilon^N \left\{ \frac{1}{2} I(w) - \varepsilon\gamma \max_{x \in \partial\Omega_\delta \cap \partial\Omega} (N-1)H(x) + o(\varepsilon) \right\}.$$

To obtain (2.17), we first use the continuity of H to choose δ and then we choose ε small enough, according to (2.15). This finishes the proof. □

Now we will give a proof of Lemma 2.3. We start with some preliminaries.

Proof of Lemma 2.3. Since u_ε satisfies (2.12) and $\varepsilon^{-N} J_{\varepsilon,\Omega_1}(u_\varepsilon)$ is bounded, u_ε converges locally in the H^1 sense to a solution of the limiting equation. Then a concentration-compactness argument gives that $\|\tilde{u}_\varepsilon - w\|_{H^1(\Omega_{\varepsilon,z_\varepsilon})} \to 0$ for some $z_\varepsilon \in \overline{\Omega}$, where

$$\Omega_{\varepsilon,P} = \{y | \varepsilon y + P \in \overline{\Omega}\}, \qquad P \in \overline{\Omega},$$

and $\tilde{u}_\varepsilon(y) = u_\varepsilon(\varepsilon y + z_\varepsilon)$. Moreover, because of (2.13) we have that $\frac{d(z_\varepsilon,\partial\Omega)}{\varepsilon} \le C$ and $z_\varepsilon \in \Omega_1$ (otherwise, the energy of u_ε will be at least of the order of $\varepsilon^N I(w)$; see Lemma 1.1 in [5]). Observe that u_ε satisfies

$$\varepsilon^2 \Delta u_\varepsilon - u_\varepsilon + f(u_\varepsilon) + h_\varepsilon = 0, \tag{2.19}$$

where $h_\varepsilon = (1 - \chi_{\Omega_1})(\bar{f}(u_\varepsilon) - f(u_\varepsilon))$. Hence $h_\varepsilon = o(1)$ uniformly and $\tilde{u}_\varepsilon \to w$ in a C^1_{loc} sense. Furthermore, there exist constants α, $\beta > 0$ such that

$$\tilde{u}_\varepsilon(y) \le \alpha \ \exp(-\beta|y|).$$

Next, an argument given in [19] shows that u_ε has only one local maximum point x_ε and $x_\varepsilon \in \partial\Omega_1 \cap \partial\Omega$.

We now consider two cases. Let $b > 0$ so that $w(b) = a$.

Case 1. If $\liminf_{\varepsilon\to 0} d(x_\varepsilon, \partial\Omega_1^+)/\varepsilon > b$, then u_ε satisfies

$$\varepsilon^2 \Delta u_\varepsilon - u_\varepsilon + f(u_\varepsilon) = 0,$$

and then, by Proposition 2.1 in [20], we have that

$$J_{\varepsilon,\Omega_1}(u_\varepsilon) = \varepsilon^N \left\{ \frac{1}{2} I(w) - \gamma\varepsilon(N-1)H(x_\varepsilon) + o(\varepsilon) \right\},$$

finishing the proof of the lemma.

Case 2. $\liminf_{\varepsilon\to 0} d(x_\varepsilon, \partial\Omega_1^+)/\varepsilon \le b$. We see first that we can assume that $\liminf_{\varepsilon\to 0} d(x_\varepsilon, \partial\Omega_1^+)/\varepsilon = b$ since the contrary, together with the convergence of $\tilde{u}_\varepsilon$ to w, implies a contradiction with (2.13).

To prove the lemma in this case we need some work. We next consider some notation. Let $\bar{x}_\varepsilon \in \partial\Omega_1^+$ be such that $d(x_\varepsilon, \partial\Omega_1^+) = |x_\varepsilon - \bar{x}_\varepsilon|$. Then since $\partial\Omega_1^+$ is orthogonal to $\partial\Omega$ at Λ_1, we have that the projection of $\bar{x}_\varepsilon$ onto $\partial\Lambda_1$, which we call $\bar{x}_\varepsilon^p$, satisfies

$$\frac{|x_\varepsilon - \bar{x}_\varepsilon^p|}{\varepsilon} \to b \qquad \text{and} \qquad \frac{|\bar{x}_\varepsilon - \bar{x}_\varepsilon^p|}{\varepsilon} \to 0. \tag{2.20}$$

Without loss of generality, we can assume that $\nu_{x_\varepsilon} = -e_N$, where ν_{x_ε} denotes the exterior normal at x_ε and that $\bar{x}_\varepsilon = d(x_\varepsilon, \partial\Omega_1^+)e_1^\varepsilon$, where $e_1^\varepsilon \to e_1$ as $\varepsilon \to 0$.

Set $x = x_\varepsilon + \varepsilon y$, $\Omega_\varepsilon = \{y : x_\varepsilon + \varepsilon y \in \Omega\}$. For notational convenience in the rest of the paper, given a function $p : \Omega \to R$, we denote by $\tilde{p}$ the function defined on Ω_ε as $\tilde{p}(y) = p(x)$. We observe that support of the function $\tilde{h}_\varepsilon$ is contained in $B_{\delta_\varepsilon}((\bar{x}_\varepsilon - x_\varepsilon)/\varepsilon) \cap \overline{\Omega}_\varepsilon$, where $\delta_\varepsilon \to 0$. This fact follows from the uniform convergence of $\tilde{u}_\varepsilon$ to w and the exponential decay of w at infinity.

Now we will study the asymptotic behavior of u_ε. First we define the function ϕ_ε as

$$u_\varepsilon(x) = w_\varepsilon(x) + \varepsilon\phi_\varepsilon, \qquad x \in \Omega, \tag{2.21}$$

where $w_\varepsilon(x) = w(\frac{x-x_\varepsilon}{\varepsilon})$. It is our goal to study the behavior of the function ϕ_ε. The next lemma provides an important estimate.

LEMMA 2.4. *For ε sufficiently small, we have*

$$\|\tilde{h}_\varepsilon\|_{L^1(\Omega_\varepsilon)} \leq o(\varepsilon). \tag{2.22}$$

Proof. We multiply the equation satisfied by $\tilde{u}_\varepsilon$ (see (2.19)) by $\frac{\partial \tilde{u}_\varepsilon}{\partial y_1}$ and integrate by parts to obtain

$$\int_{\Omega_\varepsilon} \tilde{h}_\varepsilon \frac{\partial \tilde{u}_\varepsilon}{\partial y_1} = \int_{\partial\Omega_\varepsilon} \left\{ F(\tilde{u}_\varepsilon) - \frac{1}{2}\tilde{u}_\varepsilon^2 \right\} \nu_1 dy,$$

where ν_1 is the first component of the normal vector. To estimate the right-hand side of the above equality we give a local representation of the boundary near the origin and find that $\nu_1 = \varepsilon \sum_{i=1}^{N-1} \alpha_i y_i + O(\varepsilon^2)$. On the other hand, from the radial symmetry of w we have that

$$\int_{\partial R_+^N} \left\{ F(w) - \frac{1}{2} w^2 \right\} y_i dy = 0 \qquad \text{for} \quad i = 1, \ldots, N-1. \tag{2.23}$$

Then

$$\int_{\partial\Omega_\varepsilon} \left\{ F(\tilde{u}_\varepsilon) - \frac{1}{2}\tilde{u}_\varepsilon^2 \right\} \nu_1 dy = o(\varepsilon). \tag{2.24}$$

To finish we observe that since $\mathrm{supp}(\tilde{h}_\varepsilon) \subset B_{2\delta_\varepsilon}(be_1)$, for small ε, we have that $\frac{\partial \tilde{u}_\varepsilon}{\partial u_1} \to \frac{\partial w}{\partial y_1}(be_1) \neq 0$ for all $y \in \mathrm{supp}(\tilde{h})$ and hence

$$\int_{\Omega_\varepsilon} \tilde{h}_\varepsilon = o(\varepsilon),$$

proving (2.22). □

Next we study the behavior of the function $\tilde{\phi}_\varepsilon$. We see that $\tilde{\phi}_\varepsilon$ satisfies the equation

$$\begin{cases} \Delta\tilde{\phi}_\varepsilon - (1 + d_\varepsilon)\tilde{\phi}_\varepsilon + f'(w)\tilde{\phi}_\varepsilon + \dfrac{\tilde{h}_\varepsilon}{\varepsilon} = 0 \text{ in } \Omega_\varepsilon, \\ \dfrac{\partial \tilde{\phi}_\varepsilon}{\partial \nu} = -\dfrac{1}{\varepsilon}\dfrac{\partial w}{\partial \nu} \text{ on } \partial\Omega_\varepsilon, \end{cases} \tag{2.25}$$

where

$$d_\varepsilon = \frac{1}{\varepsilon\tilde{\phi}_\varepsilon}(f(\tilde{u}_\varepsilon) - f(w)) - f'(w).$$

We observe that $d_\varepsilon \to 0$ uniformly and we note that $\tilde{w}_\varepsilon = w$.

A local representation of Ω near x_ε is considered next. There is $R > 0$ and a neighborhood N_ε of x_ε so that $(y', y_N) \in N_\varepsilon \cap \Omega$ if and only if $y_N > \rho_\varepsilon(y')$,

$y' \in B(0,R)$, $\rho_\varepsilon(0) = x_\varepsilon$, and $\nabla\rho_\varepsilon(0) = 0$. We observe that if $x_\varepsilon \to x_0$ as $\varepsilon \to 0$, then $\rho_\varepsilon \to \rho$ in C^3 uniformly, where ρ is a local representation of the boundary centered at x_0.

Now we get an asymptotic formula for the normal derivative of w. We find, for $y \in B(0, \frac{R}{\varepsilon})$, that

$$\frac{\partial w}{\partial \nu}(y, \tilde{\rho}_\varepsilon(y)) = \frac{\varepsilon w'(|y|)}{2|y|}(\rho_\varepsilon)_{ij} y_i y_j + o(\varepsilon), \tag{2.26}$$

where $(\rho_\varepsilon)_{ij}$ denotes the partial derivatives of ρ_ε at 0. Here and in what follows we use the Einstein convention for summations.

In studying the behavior of $\tilde{\phi}_\varepsilon$ we need the limiting equation

$$\begin{cases} \Delta\phi - \phi + f'(w)\phi = 0 \text{ in } R^N_+, \\ \dfrac{\partial\hat{\phi}}{\partial y_N} = -\dfrac{w'(|y|)}{2|y|}\rho_{ij} y_i y_j \quad \text{on } \partial R^N_+. \end{cases} \tag{2.27}$$

We have the following lemma.

LEMMA 2.5. *There is $1 < q < N/(N-1)$ so that $\|\tilde{\phi}_\varepsilon\|_{L^q(\Omega_\varepsilon)}$ is bounded and there are constants α, β, $R_0 > 0$ so that*

$$|\tilde{\phi}_\varepsilon(y)| \le \alpha \ \exp(-\beta|y|) \qquad \textit{for} \quad |y| > R_0. \tag{2.28}$$

Moreover,

$$\|\tilde{\phi}_\varepsilon - \tilde{\phi}_0\|_{L^q(\Omega_\varepsilon)} \to 0, \tag{2.29}$$

where $\tilde{\phi}_0 \in H^1(R^N_+)$ is the solution to (2.27).

Proof. Let us assume that $\|\tilde{\phi}_\varepsilon\|_{L^q(\Omega_\varepsilon)}$ is not bounded and define the function $\hat{\phi}_\varepsilon = \tilde{\phi}_\varepsilon / \|\tilde{\phi}_\varepsilon\|_{L^q(\Omega)_\varepsilon)}$. Then $\hat{\phi}_\varepsilon$ satisfies

$$\begin{cases} \Delta\hat{\phi}_\varepsilon - (1 + d_\varepsilon)\hat{\phi}_\varepsilon + f'(w)\hat{\phi}_\varepsilon + \hat{h}_\varepsilon = 0 \text{ in } \Omega_\varepsilon, \\ \dfrac{\partial\hat{\phi}_\varepsilon}{\partial\nu} = n_\varepsilon \text{ on } \partial\Omega_\varepsilon, \end{cases} \tag{2.30}$$

where $\hat{h}_\varepsilon = \tilde{h}_\varepsilon / \varepsilon\|\tilde{\phi}_\varepsilon\|_{L^q(\Omega_\varepsilon)} \to 0$ in the L^1 sense and

$$n_\varepsilon = -\frac{1}{\varepsilon}\frac{\partial w}{\partial\nu} / \|\tilde{\phi}_\varepsilon\|_{L^q(\Omega_\varepsilon)}.$$

We observe that $n_\varepsilon \to 0$ uniformly and that it satisfies an estimate of the form

$$|n_\varepsilon(y)| \le \alpha_\varepsilon \ \exp(-\bar{\beta}|y|) \qquad \text{for} \quad y \in \partial\Omega_\varepsilon \tag{2.31}$$

for some constants α_ε, $\bar{\beta} > 0$, and $\alpha_\varepsilon \to 0$.

We recall that $\text{supp}(\tilde{h}_\varepsilon) \subset B_{2\delta_\varepsilon}(be_1)$, with $\delta_\varepsilon \to 0$. Thus, standard elliptic estimates and comparison arguments, using the facts just mentioned and that $\|\hat{h}_\varepsilon\|_{L^q(\Omega_\varepsilon)}$ is bounded, yield the existence of constants R_0, α, $\beta > 0$ such that

$$|\hat{\phi}_\varepsilon(y)| \le \alpha \ \exp(-\beta|y|) \qquad \text{for} \quad |y| > R_0. \tag{2.32}$$

Since $\|\Delta\hat{\phi}_\varepsilon\|_{L^1(\Omega_\varepsilon)} \le C$, a well-known elliptic estimate yields that

$$\|\hat{\phi}_\varepsilon\|_{W^{1,q}(\Omega_\varepsilon \cap B_{R_0}(0))} \le C_{R_0}. \tag{2.33}$$

By the boundedness of $\hat{\phi}_\varepsilon$ in L^q we have that for a subsequence $\hat{\phi}_\varepsilon \rightharpoonup \hat{\phi}$ weakly in L^q. Now, (2.32) and (2.33) implies that this convergence is strong in L^q, in particular, $\hat{\phi} \ne 0$. Moreover, $\hat{\phi} \in W^{1,q}(R^N_+)$, it satisfies

$$\begin{cases} \Delta\hat{\phi} - \hat{\phi} + f'(w)\hat{\phi} = 0 \text{ in } R^N_+, \\ \dfrac{\partial\hat{\phi}}{\partial y_N} = 0 \text{ on } \partial R^N_+ \end{cases} \tag{2.34}$$

and

$$|\hat{\phi}(y)| \le \alpha \ \exp(-\beta|y|) \qquad \text{for} \quad |y| \quad \text{large.} \tag{2.35}$$

We observe that $\nabla w(0) = 0$ and that $\nabla u_\varepsilon(x_\varepsilon) = 0$; then $\nabla\hat{\phi}_\varepsilon(0) \to \nabla\hat{\phi}(0) = 0$. Thus hypothesis (f5) and the argument given in the proof of Lemma 4.6 of Ni and Takagi [20] imply that $\hat{\phi} \equiv 0$, which is a contradiction.

Next we can give a similar argument to obtain that the family $\tilde{\phi}_\varepsilon$ satisfies (2.28) and that, if $\tilde{\phi}_0$ is the solution of (2.27), then

$$\|\tilde{\phi}_\varepsilon - \tilde{\phi}_0\|_{L^q(\Omega_\varepsilon)} \to 0, \tag{2.36}$$

finishing the proof of Lemma 2.5. □

Proof of Lemma 2.3 (*finished*). We have

$$\begin{aligned} \varepsilon^{-N} J_{\varepsilon,\Omega_1}(u_\varepsilon) &= \int_{\Omega_\varepsilon} \frac{1}{2}(|\nabla\tilde{u}_\varepsilon|^2 + \tilde{u}_\varepsilon^2) - F(\tilde{u}_\varepsilon) - F(\tilde{u}_\varepsilon) + \int_{\Omega_\varepsilon\setminus\Omega_{1\varepsilon}} \bar{F}(\tilde{u}_\varepsilon) - F(\tilde{u}_\varepsilon). \\ &= I_1 + I_2 \end{aligned}$$

We first estimate integral I_2. It follows from hypothesis (f5) and Lemma 2.4 that

$$\begin{aligned} |I_2| &= \int_{\Omega_\varepsilon} (1 - \chi_{\Omega_{1\varepsilon}})(F(\tilde{u}_\varepsilon) - \bar{F}(\tilde{u}_\varepsilon)) \\ &= \int_{\Omega_\varepsilon} (1 - \chi_{\Omega_{1\varepsilon}}) \int_0^{\tilde{u}_\varepsilon} (f(s) - \bar{f}(s))ds \\ &\le \int_{\Omega_\varepsilon} (1 - \chi_{\Omega_{1\varepsilon}}) \frac{f(\tilde{u}_\varepsilon) - \bar{f}(\tilde{u}_\varepsilon)}{\tilde{u}_\varepsilon} \frac{\tilde{u}_\varepsilon^2}{2} = o(\varepsilon). \end{aligned} \tag{2.37}$$

Next we study I_1; for that purpose, we write

$$\begin{aligned} I_1 = &\int_{\Omega_\varepsilon} \frac{1}{2}(|\nabla w|^2 + w^2) - F(w) + \\ &+ \varepsilon \int_{\Omega_\varepsilon} \{\nabla w \cdot \nabla\tilde{\phi}_\varepsilon + w\tilde{\phi}_\varepsilon - f(w)\tilde{\phi}_\varepsilon\} + E_\varepsilon = I_1' + I_2' + E_\varepsilon. \end{aligned} \tag{2.38}$$

A direct computation using the properties of w yields

$$I_1' = \frac{1}{2}I(w) - \gamma\varepsilon(N-1)H(x_\varepsilon) + o(\varepsilon). \tag{2.39}$$

Using integration by parts and the equation satisfied by w we find

$$I_2' = \varepsilon \int_{\partial\Omega_\varepsilon} \frac{\partial w}{\partial \nu} \tilde{\phi}_\varepsilon = o(\varepsilon), \tag{2.40}$$

where the last equality follows from (2.26) and the fact that $\varepsilon\tilde{\phi}_\varepsilon \to 0$ uniformly. Finally we consider E_ε: using Taylor expansion we have

$$E_\varepsilon = \varepsilon^2 \left\{ \int_0^1 (1-t) \left(\int_{\Omega_\varepsilon} |\nabla \tilde{\phi}_\varepsilon|^2 + \tilde{\phi}_\varepsilon^2 - f'(w + t\varepsilon\tilde{\phi}_\varepsilon)\tilde{\phi}_\varepsilon^2 \right) dt \right\}. \tag{2.41}$$

For a given large R, we obtain

$$\begin{aligned} \varepsilon \int_{\Omega_\varepsilon} |\nabla \tilde{\phi}_\varepsilon|^2 = \varepsilon \int_{\Omega_\varepsilon \cap B_R(0)} |\nabla \tilde{\phi}_\varepsilon|^2 + \varepsilon \int_{\partial(\Omega_\varepsilon \cap B_R(0))} \nabla \tilde{\phi}_\varepsilon \cdot \nu \tilde{\phi}_\varepsilon \\ - \varepsilon \int_{\Omega_\varepsilon \cap B_R(0)^c} \Delta \tilde{\phi}_\varepsilon \tilde{\phi}_\varepsilon. \end{aligned} \tag{2.42}$$

The first and second term on the right-hand side above go to zero because $\varepsilon\tilde{\phi}_\varepsilon \to 0$ in C^1_{loc} and $\tilde{\phi}_\varepsilon \in W^{1,q}_{loc}(\Omega_\varepsilon)$. Next, using the equation for $\tilde{\phi}_\varepsilon$ and (2.28) we find that the third term also converges to 0, so we conclude that

$$\varepsilon \int_{\Omega_\varepsilon} |\nabla \tilde{\phi}_\varepsilon|^2 = o(1). \tag{2.43}$$

Using similar arguments we treat the other terms appearing in (2.41). Thus we finally obtain that $E_\varepsilon = o(\varepsilon)$, finishing the proof. ∎

Proof of Lemma 2.2. Suppose (2.11) is not true; then

$$S_\varepsilon = \sup_{y \in B_0} J_\varepsilon(w_{\varepsilon,y}). \tag{2.44}$$

Hence

$$\begin{aligned} S_\varepsilon &= \varepsilon^N \left\{ \frac{1}{2} I(w) - \gamma\varepsilon \min_{y \in B_0} (N-1) H(y) + o(\varepsilon) \right\} \\ &\le \varepsilon^N \left\{ \frac{1}{2} I(w) - \gamma\varepsilon(c+\delta) + o(\varepsilon) \right\}, \end{aligned}$$

where $c + \delta \le \min_{y \in B_0} H(y)$ for some $\delta > 0$ (by assumption (H1)). Then, by definition of S_ε there exists $\varphi_\varepsilon \in \Gamma_\varepsilon$ such that

$$J_\varepsilon(\varphi_\varepsilon(y)) \le \varepsilon^N \left\{ \frac{1}{2} I(w) - \gamma\varepsilon \left(c + \frac{\delta}{2} \right) + o(\varepsilon) \right\} \quad \forall\, y \in B. \tag{2.45}$$

Take a sequence $\varepsilon_n \to 0$ and denote $\varphi_{\varepsilon_n} = \varphi_n$. Let Λ^+ be a small fixed neighborhood of Λ and $\pi : \Lambda^+ \to \Lambda$ a continuous map which equals the identity on Λ. Define $\phi_n(y) = \pi(\beta(\varphi_n(y)))$ for $y \in B$, where β is the center of mass defined in (2.4). We claim that for large n we have

$$\beta(\varphi_n(y)) \in \Lambda^+ \quad \text{and} \quad H(\phi_n(y)) \ge c + \frac{\delta}{4} \quad \forall\, y \in B. \tag{2.46}$$

This immediately yields the desired contradiction. In fact, since $\varphi_n(y) = w_{\varepsilon_n,y}$ for $y \in B_0$, it follows that $\phi_n(y) = y$ for $y \in B_0$. Hence $\phi_n \in \Gamma$ and by definition of c, we have

$$c \geq \min_{y\in B} H(\phi_n(y)), \tag{2.47}$$

which is impossible in view of (2.46).

We now prove (2.46). The fact that $\beta(\varphi_n(y)) \in \Lambda^+$ is obtained by slightly modifying the arguments in [5, Lemma 1.1]. Thus, we just need to prove the second statement in (2.46). Suppose it is not true; then there exists $y_n \in B$ such that

$$H(\phi_n(y_n)) \leq c + \frac{\delta}{4}.$$

We can assume that $\phi_n(y_n) \to x_0 \in \bar{\Lambda}$ and then $H(x_0) \leq c + \frac{\delta}{4}$.

Next we apply Corollary 2.1 to the family of functions $u_n = \varphi_n(y_n)$ and obtain that

$$J_\varepsilon(u_n) \geq \varepsilon_n^N \left\{ \frac{1}{2} I(w) - \gamma\varepsilon \left(c + \frac{\delta}{4} \right) + o(\varepsilon) \right\}. \tag{2.48}$$

Comparing (2.45) and (2.48) we get a contradiction and thus Lemma 2.2 is proved. ∎

By Lemma 2.2, we have by a standard deformation argument the main result of this section, namely, the following proposition.

PROPOSITION 2.6. *The number defined by* (2.8) *is a critical value of* J_ε. *That is, there is a solution* $u_\varepsilon \in H^1$ *to* (2.1) *such that* $J_\varepsilon(u_\varepsilon) = S_\varepsilon$ $\forall$ ε *sufficiently small.*

3. Proof of Theorem 1.1. In this section, we show that the solution u_ε to (2.1) constructed in Proposition 2.6 is a solution of (1.3). The key step is the following proposition.

PROPOSITION 3.1. *If* m_ε *is given by* $m_\varepsilon = \max_{x\in\partial\Omega_0} u_\varepsilon(x)$, *then*

$$\lim_{\varepsilon\to 0} m_\varepsilon = 0. \tag{3.1}$$

Before we prove the above proposition, we need the following lemma.

LEMMA 3.2. *Let* x_ε *be the maximum point of* u_ε; *then we have*

$$\lim_{\varepsilon\to 0} H(x_\varepsilon) \to c,$$

where c is the max-min value defined in (1.5).

Proof. By Lemma 2.3, we have

$$J_\varepsilon(u_\varepsilon) = \varepsilon^N \left\{ \frac{1}{2} I(w) - \gamma\varepsilon(N-1) H(x_\varepsilon) + o(\varepsilon) \right\} \tag{3.2}$$

and then

$$\limsup_{\varepsilon\to 0} H(x_\varepsilon) \leq c. \tag{3.3}$$

In fact, assuming the contrary we have $H(x_\varepsilon) \geq c + \frac{\delta}{2}$ for ε and δ small and then we have a similar situation as in (2.45), so that following the arguments given from there we get a contradiction.

On the other hand, let $\delta > 0$ and $\phi_0 \in \Gamma$ be such that

$$\min_{y\in B} H(\phi_0(y)) \geq c - \delta.$$

Then, by (2.5) and the definition of $S_\varepsilon = J_\varepsilon(u_\varepsilon)$, we have

$$\begin{aligned} J_\varepsilon(u_\varepsilon) &\leq \sup_{y\in B} J_\varepsilon(w_{\varepsilon,\phi_0(y)}) \\ &\leq \varepsilon^n \left\{ \frac{1}{2} I(w) - \varepsilon\gamma(N-1) \min_{y\in B} H(\phi_0(y)) + o(\varepsilon) \right\} \\ &\leq \varepsilon^n \left\{ \frac{1}{2} I(w) - \varepsilon\gamma(N-1)(c-\delta) + o(\varepsilon) \right\}. \end{aligned} \tag{3.4}$$

From here and (3.2) we obtain

$$H(x_\varepsilon) \geq c - \delta + o(1).$$

Since δ is arbitrary using (3.3) we then conclude with the proof. □

We are now in a position to prove Proposition 3.1.

Proof of Proposition 3.1. Suppose, on the contrary, that $m_\varepsilon \geq \delta > 0$. Then let $u_\varepsilon(x_\varepsilon) = \max_{x\in\bar\Omega} u_\varepsilon(x)$. Then $x_\varepsilon \in \overline{\Lambda}$, $\frac{d(x_\varepsilon,\partial\Lambda)}{\varepsilon} \to b > 0$, and $w(b) = a$, and by Lemma 3.2 $H(x_\varepsilon) \to c$ as $\varepsilon \to 0$. We recall that the function $\tilde{u}_\varepsilon$ satisfies

$$\begin{cases} \Delta\tilde{u}_\varepsilon - \tilde{u}_\varepsilon + f(\tilde{u}_\varepsilon) + \tilde{h}_\varepsilon = 0 \text{ in } \Omega_\varepsilon, \\ \dfrac{\partial\tilde{u}_\varepsilon}{\partial\nu} = 0 \text{ on } \partial\Omega_\varepsilon. \end{cases} \tag{3.5}$$

Let $\hat{T}_\varepsilon$ be a direction, tangent to Λ_ε at $\bar{x}^p_\varepsilon$. We assume that $\hat{T}_\varepsilon$ converges to $\hat{T}_0$ and we observe that $\hat{T}_0 \perp e_N$, with the notational convention given in the proof of Lemma 2.3. Next we multiply (3.5) by $\nabla\tilde{u}_\varepsilon \cdot \hat{T}_\varepsilon$ and we integrate by parts to obtain

$$\int_{\partial\Omega_\varepsilon} \left\{ \frac{|\nabla\tilde{u}_\varepsilon|^2}{2} + \frac{\tilde{u}_\varepsilon^2}{2} - F(\tilde{u}_\varepsilon) \right\} \hat{T}_\varepsilon \cdot \nu = \int_{\Omega_\varepsilon} \tilde{h}_\varepsilon \frac{\partial\tilde{u}_\varepsilon}{\partial\hat{T}_\varepsilon}. \tag{3.6}$$

Using the asymptotic expansion (2.21), integrating by parts again, and using the equation for w we obtain that

$$\begin{aligned} &\int_{\partial\Omega_\varepsilon} \frac{\partial w}{\partial\nu}\frac{\partial w}{\partial\hat{T}_\varepsilon} + \varepsilon \int_{\partial\Omega_\varepsilon}\int_0^1 \left\{ \nabla\tilde{u}_\varepsilon(t)\cdot\nabla\tilde{\phi}_\varepsilon + \tilde{u}_\varepsilon(t)\tilde{\phi}_\varepsilon - f(\tilde{u}_\varepsilon(t))\tilde{\phi}_\varepsilon \right\} \hat{T}_\varepsilon \cdot \nu \, dt \\ &\qquad = \int_{\Omega_\varepsilon} \tilde{h}_\varepsilon \frac{\partial\tilde{u}_\varepsilon}{\partial\hat{T}_\varepsilon}, \end{aligned} \tag{3.7}$$

where $\tilde{u}_\varepsilon(t) = w + t\varepsilon\tilde{\phi}_\varepsilon$. For later reference, we write $I_1 + I_2 = I_3$ above. We first claim that by slightly modifying $\hat{T}_\varepsilon$ we can get $I_3 = 0$. In fact, the normal vector ν near the origin, in a ball of fixed radius $R_0 > 0$, has the form

$$\nu = 0(1 + O(\varepsilon))e_N + \varepsilon \sum_{i=1}^{N-1} \vec{\alpha}_i y_i + o(\varepsilon). \tag{3.8}$$

Thus, taking into account that the support of $\tilde{h}_\varepsilon$ shrinks to a point, that $\tilde{h}_\varepsilon \geq 0$, and that $\tilde{u}_\varepsilon$ converges to w, we perturb $\hat{T}_\varepsilon$ so that $\hat{T}_\varepsilon \perp e_N$ and $I_3 = 0$, and still keep that $\hat{T}_\varepsilon \to \hat{T}_0$.

Next we consider I_2. We observe that

$$\int_{\partial R^N_+} \left\{ \nabla w \cdot \nabla \tilde{\phi}_0 + w \tilde{\phi}_0 - f(w)\tilde{\phi}_0 \right\} y_i = 0, \tag{3.9}$$

since the function $\tilde{\phi}_0$, the solution of (2.27), is even on the boundary and so is w. From here, and taking into account (3.8), (2.28), and the convergence of $\tilde{\phi}_\varepsilon$ to ϕ_0 in $W^{1,q}_{loc}(\Omega_\varepsilon)$, we find that $I_2 = o(\varepsilon^2)$ and thus

$$\int_{\partial\Omega_\varepsilon} \frac{\partial w}{\partial \nu} \frac{\partial w}{\partial \hat{T}_\varepsilon} = o(\varepsilon^2). \tag{3.10}$$

Now we turn to study this last term. For this purpose, we obtain an expansion for derivatives of w near the origin and on the boundary of Ω_ε. A direct calculation, using Taylor expansion of the function w and the local representation the boundary, with the notation given in section 2, gives

$$w_\ell(y, \tilde{\rho}_\varepsilon(y)) = w_\ell(y, 0) + O(\varepsilon^2), \qquad 1 \leq \ell \leq N-1, \tag{3.11}$$

and

$$\frac{\partial u}{\partial \nu}(y, \tilde{\rho}_\varepsilon(y)) = \frac{\varepsilon}{2} \frac{w'}{|y|} (\rho_\varepsilon)_{ij} y_i y_j + \frac{\varepsilon^2}{3} \frac{w'}{|y|} (\rho_\varepsilon)_{ijk} y_i y_j y_k + o(\varepsilon^2). \tag{3.12}$$

Using evenness-oddness properties of these functions, we see that

$$\int_{\partial R^N_+} w_\ell(y, 0) \frac{w'}{|y|} \rho_{ij} y_i y_j = 0,$$

and then, computing the integral on $\partial\Omega_\varepsilon$, we see that for any $R > 0$ we have

$$\int_{\partial\Omega_\varepsilon \cap B(0,R)} w_i(y, 0) \frac{w'}{|y|} (\rho_\varepsilon)_{ij} y_i y_j = O(\varepsilon^2).$$

We also see that

$$\int_{\partial R^N_+} w_\ell(y, 0) \frac{w'}{|y|} \rho_{ijk} y_i y_j y_k = K \rho_{ii\ell}, \tag{3.13}$$

where K is a nonzero constant. Then we conclude that

$$\frac{1}{\varepsilon^2} I_1 = \rho_{ii\ell} \hat{T}^\ell_0 + o(1). \tag{3.14}$$

From here and (3.10), taking the limit as $\varepsilon \to 0$ we find that

$$\nabla H(\bar{x}) \cdot \hat{T}_0 = 0 \tag{3.15}$$

and this contradicts hypothesis (H2). ∎

Finally we prove Theorem 1.1.

Proof of Theorem 1.1. By (3.1), we have that

$$u_\varepsilon(x) < a \quad \forall\, x \in \partial\Omega_0.$$

Hence u_ε satisfies (1.3) since $f(u_\varepsilon) = \bar{f}(u_\varepsilon)$ for $x \notin \Omega_0$. Since $\varepsilon^{-N} J_\varepsilon(u_\varepsilon) \to \frac{1}{2} I(w)$, by [19] or [23], we have that u_ε has only one local maximum point x_ε and $x_\varepsilon \in \Lambda$. By Lemma 3.2, $\lim_{\varepsilon \to 0} H(x_\varepsilon) = c$. The rest of the proof follows from [19] and [20]. ∎

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