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The Fredholm Alternative at the First Eigenvalue for the One Dimensional p-Laplacian*

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In this work we study the range of the operator $u \mapsto (|u'|^{p-2}u')' + \lambda_1 |u|^{p-2} u$, $u(0) = u(T) = 0$, $p > 1$. We prove that all functions $h \in C^1[0, T]$ satisfying $\int_0^T h(t) \sin_p(\pi_p t/T)\, dt = 0$ lie in the range, but that if $p \neq 2$ and $h \not\equiv 0$ the solution set is bounded. Here $\sin(\pi_p t/T)$ is a first eigenfunction associated to λ_1. We also show that in this case the associated energy functional $u \mapsto (1/p) \int_0^T |u'|^p - (\lambda_1/p) \int_0^T |u|^p + \int_0^T hu$ is unbounded from below if $1 < p < 2$ and bounded from below (with a global minimizer) if $p > 2$, on $W_0^{1,p}(0, T)$ (λ_1 corresponds precisely to the best constant in the L^p-Poincaré inequality). Moreover, we show that unlike the linear case $p = 2$, for $p \neq 2$ the range contains a nonempty open set in $L^\infty(0, T)$. © 1999 Academic Press

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1. INTRODUCTION

This paper is concerned with the question of solvability of a boundary value problem of the form

$$(|u'|^{p-2}u')' + \lambda |u|^{p-2}u = h(t) \qquad \text{in } (0, T), \tag{1.1}$$

$$u(0) = u(T) = 0, \tag{1.2}$$

where $p > 1$, $T > 0$, λ are given real numbers and $h \in L^\infty(0, T)$.

We observe that $(|u'|^{p-2}u')'$ is the one dimensional version of the p-Laplacian operator $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$, largely dealt with in the literature.

By a solution of problem (1.1)–(1.2) we mean a real-valued function $u \in C^1[0, T]$ satisfying (1.2) such that $|u'|^{p-2}u'$ is absolutely continuous and Eq. (1.1) holds almost everywhere in $(0, T)$. Problem (1.1)–(1.2) is a natural object of study since it corresponds to the Euler–Lagrange equation associated to the functional defined on $W_0^{1,p}(0, T)$ as

$$E(u) = \frac{1}{p}\int_0^T |u'|^p - \frac{\lambda}{p}\int_0^T |u|^p + \int_0^T hu. \tag{1.3}$$

When $p = 2$, (1.1)–(1.2) reduces to the linear problem

$$u'' + \lambda u = h(t) \qquad \text{in } (0, T), \tag{1.4}$$

$$u(0) = u(T) = 0, \tag{1.5}$$

whose solvability is fully described, for instance, by the classical linear Fredholm alternative. In fact this problem is solvable if and only if h is L^2-orthogonal to the kernel of the linear operator $u'' + \lambda u$ with zero Dirichlet boundary conditions. More precisely, problem (1.4)–(1.5) is uniquely solvable for any h if λ is not an eigenvalue of the linear operator while if $\lambda = \lambda_k = (k\pi/T)^2$, for some integer $k \geqslant 1$, then (1.4)–(1,5) is solvable if and only if

$$\int_0^T h(t) \sin\left(\frac{k\pi t}{T}\right) dt = 0. \tag{1.6}$$

In such a case, the solution set is a continuum constituted by a one dimensional linear manifold. Needless to say, such a nice characterization uses the linear structure of the problem in an essential way.

A long-standing question is that of finding analogues of the Fredholm alternative in nonlinear settings, in particular in problems involving nonlinear operators of p-Laplacian type. The relationship between solvability and *spectrum* of the p-Laplacian in the *nonresonant case* has been known for a long time. In fact, if λ is not an eigenvalue of the problem

$$(|u'|^{p-2}u')' + \lambda\,|u|^{p-2}u = 0 \qquad \text{in } (0, T), \tag{1.7}$$

$$u(0) = u(T) = 0, \tag{1.8}$$

namely if no nontrivial solution of (1.7)–(1.8) exists, then for any $h \in L^\infty(0, T)$, problem (1.1)–(1.2) has at least one solution. This fact, as well as its corresponding analogue in higher dimensions, follows from a general result contained in Chapter II of a classical monograph by Fučík *et al.* [FNSS].

However the case in which λ *is an eigenvalue* of (1.7)–(1.8) has remained widely open. Many works have appeared in recent years concerning solvability of nonlinear boundary value problems involving the p-Laplacian in one and higher dimensions via topological and variational methods, but there has been relatively little focus on this basic but, as we shall see, subtle issue.

Is has been shown, for instance, in [DEM] that the set of all eigenvalues of (1.7)–(1.8) is given by the sequence of positive numbers

$$\lambda_k = (p-1)\left(\frac{k\pi_p}{T}\right)^p \qquad \text{for} \quad k = 1, 2, ...,$$

where

$$\pi_p = 2\int_0^1 \frac{ds}{(1-s^p)^{1/p}} = \frac{2\pi}{p\sin(\pi/p)}. \tag{1.9}$$

Note that for $T = \pi_p$, $\lambda_1 = p-1$.

The set of associated eigenfunctions for $\lambda = \lambda_1$ corresponds precisely to that of constant multiples of the function $\sin_p(\pi_p t/T)$, where $\sin_p t$ is the solution of the initial value problem

$$(|u'|^{p-2}u')' + (p-1)\,|u|^{p-2}u = 0, \qquad t \in \mathbb{R},$$

$$u(0) = 0, \qquad u'(0) = 1,$$

which for $t \in [0, \pi_p/2]$ can be described implicitly by the formula

$$t = \int_0^{\sin_p t} \frac{ds}{(1-s^p)^{1/p}}. \tag{1.10}$$

In this paper we will concentrate on the study of the solvability issue in the *resonant case at the first eigenvalue*, $\lambda = \lambda_1$, of (1.7)–(1.8). It is of course natural to ask, perhaps naïvely, what the role, if any, of the corresponding analogue of (1.6) is for general p at λ_1, namely

$$\int_0^T h(t)\sin_p\left(\frac{\pi_p t}{T}\right) dt = 0. \tag{1.11}$$

Some restrictions on h are indeed needed for the solvability of (1.1)–(1.2) at $\lambda = \lambda_1$. In fact, for instance, it is shown in [FGTT] that no solution exists if h does not change sign and is not identically zero, a situation in which of course condition (1.11) is violated. (This result also holds in the higher dimensional case.) However, in [BDH_1] an example which shows that (1.11) is not necessary for existence is constructed.

Our first result below, initially surprising for us, states that if $h \in C^1[0, T]$ then the orthogonality condition (1.11), linear in nature, is actually *sufficient for existence* for any $p > 1$. Hence the set of h's for which (1.1)–(1.2) is solvable contains at least the vector space of all C^1 functions satisfying (1.11).

THEOREM 1.1. *Let us assume that* $h \in C^1[0, T]$, $h \not\equiv 0$, *satisfies condition* (1.11). *Then problem* (1.1)–(1.2) *with* $\lambda = \lambda_1$ *has at least one solution. Moreover, if* $p \neq 2$, *then the set of all possible solutions is bounded in* $C^1[0, T]$.

We observe that this result also reveals a striking difference between the case $p \neq 2$ and $p = 2$, since in the latter the solution set is an unbounded continuum. It would actually be natural t expect the number of solutions under condition (1.11) be *generically finite* if $p \neq 2$.

A by-product of the proof of Theorem 1.1 is the fact that the degree of the associated fixed-point operator in a large ball of $C^1[0, T]$ becomes $+1$ if $p > 2$ while it equals -1 if $p < 2$. This degree is of course undefined if $p = 2$.

An issue that arises naturally in light of the above result is that of analyzing the behavior of the functional E in (1.3) for $\lambda = \lambda_1$. This issue is related to the fine structure underlying the L^p-Poincaré inequality. In fact, we recall the well-known characterization of λ_1 as the *best constant* in the L^p-Poincaré inequality

$$\int_0^T |u'|^p \geqslant C \int_0^T |u|^p \qquad \text{for all} \quad u \in W_0^{1,p}(0, T). \tag{1.12}$$

$C = \lambda_1$ is precisely the largest $C > 0$ for which (1.12) holds true. Then $\int_0^T |u'|^p - \lambda_1 \int_0^T |u|^p \geqslant 0$ for all $u \in W_0^{1,p}(0, T)$ while it minimizes and equals 0 exactly on the ray generated by the first eigenfunction $\sin_p(\pi_p t/T)$. Now we consider the following question: What is the sensitivity of this optimal Poincaré inequality under a linear perturbation? We consider then the functional $E_1\colon W_0^{1,p}(0,T)\to\mathbb{R}$ given by

$$E_1(u)=\frac{1}{p}\int_0^T |u'|^p-\frac{\lambda_1}{p}\int_0^T |u|^p+\int_0^T hu \tag{1.13}$$

and ask whether E_1 is *bounded from below*. It is easy to see that a necessary condition for this to be the case is that h satisfy the orthogonality condition (1.11) for otherwise E_1 is unbounded below along the ray generated by the first eigenfunction. Take then an h satisfying (1.11). If $p=2$, an L^2-orthogonal expansion in Fourier series yields the fact that this condition is also sufficient for the boundedness from below of the functional. This approach seems however of no use when $p\neq 2$. Under the additional assumption $h\in C^1[0,T]$, the answer is provided by the following result, some conclusions of which are already implied by Theorem 1.1.

THEOREM 1.2 *Assume that* $h\in C^1[0,T]$, $h\not\equiv 0$, *and that* (1.11) *holds. Then*

(i) *for* $1<p<2$ *the functional* E_1 *is unbounded from below. The set of its critical points is nonempty and bounded.*

(ii) *For* $p>2$ *the functional* E_1 *is bounded from below and has a global minimizer. The set of its critical points is bounded; however,* E_1 *does not satisfy the Palais–Smale condition at the level* 0.

It is interesting to see that changing p from $p>2$ to $p<2$ shifts the structure of this functional from a global minimum to a saddle-point geometry for its level sets. If $p=2$ the functional is convex with a whole ray of minimizers. This result seems to open an interesting issue concerning the geometry of L^p-spaces where the absence of a good orthogonality notion makes the structure of Poincaré-type inequalities fairly subtle.

On the other hand, the last statement in the above result sets a word of warning in the use of min–max schemes based on the PS condition in resonant problems involving the p-Laplacian. Here there arises a very natural example of an equation with a priori estimates for the solutions but for which PS does not hold in the associated action functional.

As we mentioned above, condition (1.11) is not necessary for existence if $p\neq 2$.

Our next result states in particular another interesting difference with $p=2$. If $p\neq 2$, then the set of h's for which (1.1)–(1.2) is solvable has *nonempty interior* in $L^\infty(0,T)$. More precisely we have

THEOREM 1.3. *Let $p \neq 2$. Then there exists an open cone $\mathscr{C} \subset L^\infty(0, T)$ such that for all $h \in \mathscr{C}$ problem* (1.1)–(1.2), *for $\lambda = \lambda_1$, has at least two solutions. Moreover,*

$$\int_0^T h(t) \sin_p\left(\frac{\pi_p t}{T}\right) dt \neq 0, \tag{1.14}$$

for all $h \in \mathscr{C}$.

A by-product of the proof of this theorem is the following general fact. For any $h \in L^\infty(0, T)$ such that (1.14) holds, the set of all possible solutions of (1.1)–(1.2) is bounded and the degree of the associated fixed point operator is 0. Combining this and Theorem 1.1 yields in particular the fact that for any $h \not\equiv 0$ of class C^1 and $p \neq 2$, there are *a priori estimates* for the solution set.

Now some words about our basic methodology. We observe first that after an appropriate scaling, we may assume $T = \pi_p$ in all of the above results. Thus for the rest of the paper we will set $T = \pi_p$ in (1.1)–(1.2). We recall that in this case, the first eigenvalue λ_1 of (1.7)–(1.8) is given by $\lambda_1 = p - 1$.

The proofs of our results are based on the analysis of the initial value problem

$$(|u'|^{p-2} u')' + (p-1) |u|^{p-2} u = h(t), \qquad t \in [0, \infty),$$
$$u(0) = 0, \qquad u'(0) = \alpha,$$

with $h \in L^\infty_{\text{loc}}[0, \infty)$. We prove in Appendix I that there is a globally defined solution to this problem and that for α sufficiently large (positive or negative) a first zero $t_1^\alpha > 0$ exists. Moreover, $t_1^\alpha \to \pi_p$ as $|\alpha| \to \infty$. The key matter is to analyze the relative location of t_1^α with respect to π_p for large $|\alpha|$. Of course one has a solution of (1.1)–(1.2) for $\lambda = \lambda_1$, whenever t_1^α hits exactly π_p. This analysis is not trivial and requires the development of refined asymptotics for t_1^α, which are carried out in Section 2.

In particular, these asymptotic expansions yield, under the assumptions of Theorem 1.1,

$$t_1^\alpha < \pi_p \quad \text{if} \quad p < 2 \qquad \text{and} \qquad t_1^\alpha > \pi_p \quad \text{if} \quad p > 2,$$

whenever $|\alpha|$ is sufficiently large. From these facts, a proof of Theorem 1.1, based on degree theoretical arguments, is devised and presented in Section 3. These asymptotics are also key points in the proofs of Theorems 1.2 and 1.3, which are carried out in Section 4 and 5, respectively.

We do not know whether the C^1 assumption on h in Theorems 1.1 and 1.2, can be weakened to just L^∞ since this condition is used in the asymptotic analysis of t_1^α. On the other hand it seems to be a challenging question whether Theorem 1.1 can be extended to the higher dimensional case.

To end this introduction we would like to point out that the uniqueness/multiplicity question in (1.1)–(1.2) is delicate, and it is probably hard to obtain very general results. In this regard we point out that uniqueness holds for $\lambda \leqslant 0$, but that for $\lambda > 0$ and $p \neq 2$ one can always find an h for which (1.1)–(1.2) has at least two solutions, as established in [DT]. See also [DEM] and [FHTT] for earlier results. On the other hand, it is shown in [DM] that for $p \neq 2$ and $h \equiv 1$, the number of solutions of (1.1)–(1.2) tends to infinity as λ does. We should also mention that results related to Theorem 1.3 had already been found for other boundary conditions not including Dirichlet and for higher Dirichlet eigenvalues, respectively, in [BDH$_2$] and [DT].

2. KEY ESTIMATES

This section deals with the initial value problem

$$(\varphi_p(u'))' + (p-1)\,\varphi_p(u) = \frac{1}{p^*}\,h(t), \qquad t > 0, \tag{2.1}$$

$$u(0) = 0, \qquad u'(0) = \alpha. \tag{2.2}$$

Here and in what follows we denote $\varphi_p(s) = |s|^{p-2}\,s$, $s \neq 0$, $\varphi_p(0) = 0$, and $p^* = p/(p-1)$, α is a real number, and $h \in L^\infty_{\text{loc}}[0, \infty)$. The factor $1/p^*$ in (2.1) is introduced for convenience.

By a solution of (2.1) we understand a function $u \in C^1[0, \infty)$ such that $\varphi_p(u')$ is absolutely continuous, which satisfies Eq. (2.1) in $[0, \infty)$.

A globally defined solution of (2.1)–(2.2) indeed exists, as shown in Appendix I. Moreover, the following basic property holds true.

LEMMA 2.1. *Let u_α be a solution of* (2.1)–(2.2). *Then $u_\alpha(t)/\alpha \to \sin_p t$ as $|\alpha| \to \infty$, in the $C^1[0, K]$-sense, for any $K > 0$. In particular, for large $|\alpha|$, u_α has a first* (*finite*) *zero $t_1^\alpha > 0$ and so does u'_α at a point $t(\alpha) > 0$. Moreover, for $\alpha > 0$ ($\alpha < 0$), u_α is strictly increasing* (*strictly decreasing*) *in $(0, t(\alpha))$ and strictly decreasing* (*strictly increasing*) *in $(t(\alpha), t_1^\alpha)$, and $t(\alpha) \to \pi_p/2$, $t_1^\alpha \to \pi_p$ as $|\alpha| \to \infty$. For fixed $M > 0$, all these convergences are uniform in h with $\|h\|_{L^\infty[0, K]} \leqslant M$.*

We postpone the proof of this lemma to Appendix I and proceed in the next two propositions to the statement and proof of two finer estimates of the way in which t_1^α approaches π_p. These estimates are crucial in the proofs of Theorems 1.3, 1.1, and 1.2, respectively presented in the following sections.

For $h \in L^{\infty}[0, 2\pi_p]$ we denote

$$I_h = \int_0^{\pi_p} h(t) \sin_p t \, dt. \tag{2.3}$$

PROPOSITION 2.1. *Assume $h \in L^{\infty}_{\text{loc}}[0, 2\pi_p)$. Then*

$$t_1^{\alpha} = \pi_p + \frac{1}{p} I_h \operatorname{sgn} \alpha \, |\alpha|^{1-p} + o(|\alpha|^{1-p}) \tag{2.4}$$

as $|\alpha| \to \infty$, where $o(|\alpha|^{1-p})$ is uniform with respect to all h such that $\|h\|_{L^{\infty}(0, 2\pi_p)} < H$, for a fixed constant $H > 0$.

When h is of class C^1, the above estimate can ben refined as follows.

PROPOSITION 2.2. *Assume $h \in C^1[0, 2\pi_p)$ and $I_h = 0$. Then*

$$t_1^{\alpha} = \pi_p + |\alpha|^{2(1-p)} (p-2) J_h + o(|\alpha|^{2(1-p)}), \tag{2.5}$$

where $o(|\alpha|^{2(1-p)}$ is uniform with respect to all h with $\|h\|_{C^1[0, 2\pi_p]} < H$, for some fixed positive constant H. Here

$$J_h = \frac{1}{2p^3} \int_0^{\pi_p/2} \frac{(\int_t^{\pi_p/2} h(y) \cos_p y \, dy)^2 + (\int_t^{\pi_2} h(\pi_p - y) \cos_p y \, dy)^2}{\cos_p^p t} \, dt.$$

In particular, we have

(i) $1 < p < 2 \Rightarrow t_1^{\alpha} < \pi_p$ *for any* $|\alpha| \gg 1$,

(ii) $p > 2 \Rightarrow t_1^{\alpha} > \pi_p$ *for any* $|\alpha| \gg 1$.

Before proceeding to the proofs of these propositions we point out that the definitions of π_p and $\sin_p$ given in (1.9) and (1.10), respectively, differ slightly from those given in [DEM]. The reason for this change is that we wish to obtain some p-trigonometric identities more suitable for our purposes than similar ones derives in [L]. Thus defining $\cos_p t := \sin_p' t$, $\tan_p t := \sin_p t / \cos_p t$, and $\operatorname{arc} \sin_p s = \sin_p^{-1} s$, we have the validity of the formulas

$$\sin_p^p t + \cos_p^p t = 1, \qquad \cos_p' t = -\tan_p^{p-1} t \cos_p t, \tag{2.6}$$

$$\tan_p' t = 1 + \tan_p^p t = \frac{1}{\cos_p^p t}, \qquad \operatorname{arc} \sin_p' s = \frac{1}{(1 - s^p)^{1/p}}, \tag{2.7}$$

for all $t \in [0, \pi_p/2)$ and all $s \in [0, 1)$.

Proof of Proposition 2.1. Let us choose $\alpha \gg 1$ so large that Lemma 2.1 applies. Then, for $t \in [0, t(\alpha)]$, the initial value problem (2.1)–(2.2) is equivalent to its first integral

$$|u'_\alpha(t)|^p + |u_\alpha(t)|^p = \alpha^p + W_\alpha(u_\alpha), \tag{2.8}$$

where

$$W_\alpha(s) := \int_0^s h(\tau_\alpha(u))\, du \qquad \text{and} \qquad \tau_\alpha = u_\alpha^{-1}. \tag{2.9}$$

Hence from (2.8), we have

$$\frac{u'_\alpha(t)}{(\alpha^p + W_\alpha(u_\alpha(t)) - u_\alpha^p(t))^{1/p}} = 1, \qquad \text{for} \quad t \in [0, t(\alpha)),$$

and by integration

$$t = \int_0^{u_\alpha(t)} \frac{dw}{(\alpha^p + W_\alpha(w) - p^p)^{1/p}}. \tag{2.10}$$

In particular, for $t = t(\alpha)$, we obtain

$$t(\alpha) = \int_0^{u_\alpha(t(\alpha))} \frac{dw}{(\alpha^p + W_\alpha(w) - w^p)^{1/p}}. \tag{2.11}$$

Let us set $q := u_\alpha(t(\alpha))$ and write $W = W_\alpha$. The change of variables $w = qt$ in (2.11) yields

$$t(\alpha) = \int_0^1 \frac{d\tau}{[(\alpha/p)^p + W(\tau q)/q^p - \tau^p]^{1/p}}. \tag{2.12}$$

Substituting $t = t(\alpha)$ into (2.8), we find

$$\left(\frac{\alpha}{q}\right)^p = 1 - \frac{W(q)}{q^p}. \tag{2.13}$$

From this expression and (2.12), we obtain

$$t(\alpha) = \int_0^1 (1 - \tau^p)^{-1/p} \left[1 + \frac{W(\tau q) - W(q)}{(1 - \tau^p)\, q^p} \right]^{-1/p} d\tau. \tag{2.14}$$

Since $\|h\|_{L^\infty(0, 2\pi_p)} < H$, from (2.9), it follows that $|(W(\tau q) - W(q))/(1 - \tau^p)\, q|$ $\leqslant$ const, for $\tau \in [0, 1]$, i.e., $(W(\tau q) - W(q))/(1 - \tau^p)\, q = o(q^{1-p})$ as $q \to \infty$, holds uniformly with respect to $\tau \in [0, 1]$ and h. Hence using a first order Taylor expansion of the second bracket in (2.14) yields

$$t(\alpha) = \frac{\pi_2}{2} - \frac{1}{p} \int_0^1 \frac{1}{q^{p-1}(1-\tau^p)^{1+1/p}} \frac{W(\tau q) - W(q)}{q} \, d\tau + o(q^{17-p}), \tag{2.15}$$

where $o(q^{1-p})$ holds uniformly with respect to h. Note that all asymptotic expressions in the forthcoming formulas are understood either for $q \to \infty$ or $\alpha \to \infty$. Again using (2.9), $\|h\|_{L^\infty(0, 2\pi_p} < H$, and (2.13), it follows that

$$\alpha = q + o(q), \tag{2.16}$$

and so Lemma 2.1 together with (2.16) yields

$$u_\alpha(t) = \alpha \sin_p t + o(\alpha) = q \sin_p t + o(q). \tag{2.17}$$

Next, setting $s = \tau q$ in (2.9), and using the change of variables $u = \tau q v$ and $\tau v = \sin_p s$, we find

$$\begin{aligned} \frac{W(\tau q)}{q} &= \frac{1}{q} \int_0^{\tau q} h(\tau_\alpha(u)) \, du \\ &= \tau \int_0^1 h(\tau_\alpha(\tau q v)) \, dv \\ &= \tau \int_0^1 h(\text{arc} \sin_p \tau v) \, dv + o(1) \\ &= \int_0^{\text{arc} \sin_p \tau} h(s) \cos_p s \, ds + o(1). \end{aligned} \tag{2.18}$$

Hence from (2.15), we obtain

$$t(\alpha) = \frac{\pi_p}{2} + \frac{q^{1-p}}{p} \int_0^1 (1-\tau^p)^{-1-1/p} \int_{\text{arc} \sin_p \tau}^{\pi_p/2} h(s) \cos_p s \, ds \, d\tau + o(q^{1-p}). \tag{2.19}$$

Our next step is to estimate $t_1^\alpha - t(\alpha)$. To do this we consider the initial value problem

$$\begin{gathered} (\varphi_p(u'))' + (p-1)\, \varphi_p(u) = \frac{1}{p^*} h, \\ u(t_1^\alpha) = 0, \qquad u'(t_1^\alpha) = -\beta \end{gathered} \tag{2.20}$$

for $t < t_1^\alpha$ and $\beta \gg 1$. Since this problem is equivalent to its "reflexion"

$$(\varphi_p(u_\beta'))' + (p-1)\,\varphi_p(u_\beta) = \frac{1}{p^*}\,\tilde{h},$$
$$u_\beta(0) = 0, \qquad u_\beta'(0) = \beta, \tag{2.21}$$

where $\tilde{h}(t) := h(t_1^\alpha - t)$, by an argument similar to that used before we arrive at

$$\tilde{t}(\beta) := t_1^\alpha - t(\alpha)$$
$$= \frac{\pi_p}{2} + \frac{q^{1-p}}{p} \int_0^1 (1-\tau^p)^{-1-1/p} \int_{\text{arc sin}_p \tau}^{\pi_p/2} \tilde{h}(s) \cos_p s \, ds \, d\tau + o(q^{1-p}). \tag{2.22}$$

Because $\pi_p - t_1^\alpha = o(1)$, and $\|h\|_{L^\infty(0,\,2\pi_p)} < H$, it follows that

$$\int_{\text{arc sin}_p \tau}^{\pi_p/2} \tilde{h}(s) \cos_p s \, ds = \int_{\text{arc sin}_p \tau}^{\pi_p/2} h(\pi_p - s) \cos_p s \, ds + o(1)$$

and so from (2.22), we get

$$\tilde{t}(\beta) = \frac{\pi_p}{2} + \frac{q^{1-p}}{p} \int_0^1 (1-\tau^p)^{-1-1/p} \int_{\text{arc sin}_p \tau}^{\pi_p/2} h(\pi_p - s) \cos_p s \, ds \, d\tau + o(q^{1-p}). \tag{2.23}$$

We prove in Appendix II that

$$\int_0^1 (1-\tau^p)^{-1-1/p} \int_{\text{arc sin}_p \tau}^{\pi_p/2} [h(s) + h(\pi_p - s)] \cos_p s \, ds \, d\tau = I_h \tag{2.24}$$

(I_h is given by (2.3)). Then, from (2.19), (2.23), and (2.24), the expansion

$$t_1^\alpha = t(\alpha) + \tilde{t}(\beta) = \pi_p + I_h \frac{q^{1-p}}{p} + o(q^{1-p}) \tag{2.25}$$

follows, which thanks to (2.16) is equivalent to (2.4).

The assertion for $\alpha < 0$ is a consequence of the symmetry of the equation (2.1) and of the transformation $h \mapsto -h$. ∎

Proof of Proposition 2.2. Let $\alpha \gg 1$ and consider first the initial value problem (2.1)–(2.2) on $[0, t(\alpha)]$. Making the change of variables $w = qs$ in (2.10) and eliminating α by means of (2.13) in the resultant expression yield

$$t = \int^{u_\alpha(t)/q} (1-s^p)^{-1/p} \left[1 + \frac{\Gamma(s, q)}{(1-s^p)\, q^{p-1}} \right]^{-1/p} ds, \tag{2.26}$$

where $\Gamma(s,q) := (W(sq) - W(q))/q$. By performing a second order Taylor expansion of the second bracket in (2.26), we get

$$t = \int_0^{u_\alpha(t)/q} \frac{1}{(1-s^p)^{1/p}} \left\{ 1 - \frac{1}{p}\frac{1}{q^{p-1}}\frac{\Gamma(s,q)}{1-s^p} + \frac{(1/p)((1/p)+1)}{2}\frac{1}{q^{2(p-1)}} \right.$$

$$\left. \times \frac{\Gamma^2(s,q)}{(1-s^p)^2} \right\} ds + o(q^{2(1-p)}), \tag{2.27}$$

where $o(q^{2(1-p)})$ is uniform with respect to $\|h\|_{L^\infty(0,2\pi)} < H$.

Then from (2.17) we can obtain the first order approximation of u_α^{-1},

$$u_\alpha^{-1}(\tau q) = \text{arc sin}_p\, \tau + o(1), \tag{2.28}$$

so that Γ is given by

$$\Gamma(s,q) = -\int_s^1 h(u_\alpha^{-1}(\tau q))\, d\tau = \int_s^1 h(\text{arc sin}_p\, \tau + o(1))\, d\tau. \tag{2.29}$$

Substituting (2.29) into (2.27) we obtain an approximation for u_α^{-1} better than (2.28), given by

$$u_\alpha^{-1}(\tau q) = \text{arc sin}_p\, \tau + \frac{1}{pq^{p-1}} \int_0^\tau \frac{1}{(1-s^p)^{1+1/p}} \int_s^1 h(\text{arc sin}_p\, x + o(1))\, dx\, ds$$

$$= \text{arc sin}_p\, \tau + \frac{1}{q^{p-1}}\, \Theta(\tau) + o(q^{1-p}), \tag{2.30}$$

where

$$\Theta(\tau) := \frac{1}{p} \int_0^\tau \frac{\int_s^1 h(\text{arc sin}_p\, x)\, dx}{(1-s^p)^{1+1/p}}\, ds \tag{2.31}$$

and $o(q^{1-p})$ is uniform with respect to $\|h\|_{C^1[0,2\pi_p]} < H$.

With this at hand we can obtain an approximation for Γ better than (2.29), given by

$$\Gamma(s,q) = -\int_s^1 h\left(\text{arc sin}_p\, \tau + \frac{1}{q^{p-1}}\, \Theta(\tau) + o(q^{1-p}) \right) d\tau. \tag{2.32}$$

Expanding h in (2.32) (recall that h is C^1), we find

$$\Gamma(s,q) = -\int_s^1 h(\arcsin_p \tau)\, d\tau - \int_s^1 h'(\arcsin_p \tau) \times\left(\frac{\Theta(\tau)}{q^{p-1}} + o\left(\frac{1}{q^{p-1}}\right)\right) d\tau + \int_s^1 o\left(\frac{\Theta(\tau)}{q^{p-1}} + o\left(\frac{1}{q^{p-1}}\right)\right) d\tau. \tag{2.33}$$

Then, $\|h\|_{C^1[0,2\pi_p]} < H$ and (2.33) yield

$$\Gamma(s,q) = -\int_s^1 h(\arcsin_p \tau)\, d\tau - \frac{1}{q^{p-1}}\int_s^1 h'(\arcsin_p \tau)\,\Theta(\tau)\, d\tau + o(q^{1-p}). \tag{2.34}$$

Now substituting (2.34) into (2.27) (with $t = t(\alpha)$), we obtain

$$\begin{aligned} t(\alpha) &= \frac{\pi_p}{2} + \frac{1}{q^{p-1}}\frac{1}{p}\int_0^1 \frac{\int_s^1 h(\arcsin_p \tau)\, d\tau}{(1-s^p)^{1+1/p}}\, ds \\ &\quad + \frac{1}{q^{2(p-1)}}\frac{1}{p}\left[\int_0^1 \frac{\int_s^1 h'(\arcsin_p \tau)\,\Theta(\tau)\, d\tau}{(1-s^p)^{1+1/p}}\, ds\right. \\ &\quad \left. + \frac{p+1}{2p}\int_0^1 \frac{\left(\int_s^1 h(\arcsin_p \tau)\, d\tau\right)^2}{(1-s^p)^{2+1/p}}\, ds\right] + o(q^{2(1-p)}), \end{aligned} \tag{2.35}$$

where $o(q^{2(1-p)})$ is uniform with respect to $\|h\|_{C^1[0,2\pi_p]} < H$.

We shall prove in Appendix III the identity

$$\begin{aligned} &\int_0^1 \frac{\int_s^1 h'(\arcsin_p \tau)\,\Theta(\tau)\, d\tau}{(1-s^p)^{1+1/p}}\, ds + \frac{p+1}{2p}\int_0^1 \frac{\left(\int_s^1 h(\arcsin_p \tau)\, d\tau\right)^2}{(1-s^p)^{2+1/p}}\, ds \\ &= \frac{1}{p}\, h\left(\frac{\pi_p}{2}\right)\int_0^{\pi_p/2} h(y)\sin_p y\, dy - \frac{2-p}{2p}\int_0^{\pi_p/2} \frac{\left(\int_t^{\pi_p/2} h(y)\cos_p y\, dy\right)^2}{\cos_p^p t}\, dt. \end{aligned} \tag{2.36}$$

So (2.35) and (2.36) imply the following approximation of $t(\alpha)$,

$$\begin{aligned} t(\alpha) &= \frac{\pi_p}{2} + \frac{1}{q^{p-1}p}\int_0^1 \frac{\int_s^1 h(\arcsin_p \tau)\, d\tau}{(1-s^p)^{1+1/p}}\, ds \\ &\quad + \frac{1}{q^{2(p-1)}p}\left[h\left(\frac{\pi_p}{2}\right)\int_0^{\pi_p/2} h(y)\sin_p y\, dy\right. \\ &\quad \left. - \frac{2-p}{2p}\int_0^{\pi_p/2} \frac{\left(\int_t^{\pi_p/2} h(y)\cos_p y\, dy\right)^2}{\cos_p^p t}\, dt\right] + o(q^{2(1-p)}). \end{aligned} \tag{2.37}$$

We now estimate $\tilde{t}(\beta) = t_1^\alpha - t(\alpha)$. As before, we consider the initial value problem (2.20), via its reflexion form (2.21). Again in a similar form, we obtain

$$\tilde{t}(\beta) = \frac{\pi_p}{2} + \frac{1}{q^{p-1}p} \int_0^1 \frac{\int_s^1 \tilde{h}(\operatorname{arc\,sin}_p \tau)\, d\tau}{(1-s^p)^{1+1/p}}\, ds$$
$$+ \frac{1}{q^{2(p-1)}p^2} \left[\tilde{h}\left(\frac{\pi_2}{2}\right) \int_0^{\pi_p/2} \tilde{h}(y) \sin_p y\, dy \right.$$
$$\left. - \frac{2-p}{2p} \int_0^{\pi_p/2} \frac{\left(\int_t^{\pi_p/2} \tilde{h}(y) \cos_p y\, dy\right)^2}{\cos_p^p t}\, dt \right] + o(q^{2(1-p)}). \qquad (2.38)$$

From Proposition 2.1 it follows that if $I_h = 0$ holds, then $|t_1^\alpha - \pi_p| = o(|q|^{1-p})$. Hence for $\|h\|_{C^1[0,2\pi_p]} < H$, we have the validity of the relationships

$$\int_0^1 \frac{\int_s^1 \tilde{h}(\operatorname{arc\,sin}_p \tau)\, d\tau}{(1-s^p)^{1+1/p}}\, ds = \int_0^1 \frac{\int_s^1 h(\pi_p - \operatorname{arc\,sin}_p \tau)\, d\tau}{(1-s^p)^{1+1/p}}\, ds + o(q^{1-p}), \qquad (2.39)$$

$$\tilde{h}\left(\frac{\pi_p}{2}\right) \int_0^{\pi_p/2} \tilde{h}(y) \sin_p y\, dy = h\left(\frac{\pi_p}{2}\right) \int_0^{\pi_p/2} h(\pi_p - y) \sin_p y\, dy + o(q^{1-p}), \qquad (2.40)$$

and

$$\int_0^{\pi_p/2} \frac{\left(\int_t^{\pi_p/2} \tilde{h}(y) \cos_p y\, dy\right)^2}{\cos_p^p t}\, dt$$
$$= \int_0^{\pi_p/2} \frac{\left(\int_t^{\pi_p/2} h(\pi_p - y) \cos_p y\, dy\right)^2}{\cos_p^p t}\, dt + o(q^{1-p}). \qquad (2.41)$$

So, from (2.37) to (2.41), we obtain

$$\tilde{t}(\beta) = \frac{\pi_p}{2} + \frac{1}{q^{p-1}p} \int_0^1 \frac{\int_s^1 h(\pi_p - \operatorname{arc\,sin}_p \tau)\, d\tau}{(1-s^p)^{1+1/p}}\, ds$$
$$+ \frac{1}{q^{2(p-1)}p^2} \left[h\left(\frac{\pi_p}{2}\right) \int_0^{\pi_p/2} h(\pi_p - y) \sin_p y\, dy \right.$$
$$\left. - \frac{2-p}{2p} \int_0^{\pi_p/2} \frac{\left(\int_2^{\pi_p/2} h(\pi_p - y) \cos_p y\, dy\right)^2}{\cos_p^p t}\, dt \right] + o(q^{2(1-p)}). \qquad (2.42)$$

Since $t_1^\alpha = t(\alpha) + \tilde{t}(\beta)$, from (2.37) and (2.42), we find that

$$t_1^\alpha = \pi_p + \frac{p-2}{2p^3} \frac{1}{q^{2(p-1)}} \times \int_0^{\pi_p/2} \frac{(\int_t^{\pi_p/2} h(y) \cos_p y \, dy)^2 + (\int_t^{\pi_p} h(\pi_p - y) \cos_p y \, dy)^2}{\cos_p^p t} \, dt + o(q^{2(1-p)}). \tag{2.43}$$

But thanks to (2.16), the estimate (2.43) is equivalent to (2.5).

The corresponding assertion for $\alpha < 0$ is again a consequence of the symmetry of Eq. (2.1) and the transformation $h \mapsto -h$. ∎

3. PROOF OF THEOREM 1.1.

Let us begin this section by recalling that the boundary value problem (1.1)–(1.2) for $T = \pi_p$ and $\lambda = \lambda_1$ is given by

$$(\varphi_p(u'))' + (p-1)\, \varphi_p(u) = \frac{1}{p^*} h, \qquad u(0) = u(\pi_p) = 0. \tag{3.1}$$

Set $X := C_0^1[0, \pi_p] = \{u \in C^1[0, \pi_p];\, u(0) = u(\pi_p) = 0\}$, and let $\Lambda = [0, +\infty)$.

For $h \in L^\infty(0, \pi_p)$ and $\lambda \in \Lambda$ define an operator $T_{\lambda, h}\colon X \to X$ by $T_{\lambda, h}(v) = u$ if and only if

$$(\varphi_p(u'))' = \lambda \left[\frac{1}{p^*} h(\lambda^{1/p} t) - (p-1)\, \varphi_p(v) \right], \tag{3.2}$$

$$u(0) = u(\pi_p) = 0. \tag{3.3}$$

Standard arguments based on the Arzela–Ascoli theorem imply that $R_{\lambda, h}$ is a well-defined operator which is compact from X into X^*. Moreover, $T_{\lambda, h}$ depends continuously (in the operator norm) on the perturbations of $h \in L^\infty(0, \pi_p)$ and $\lambda \in \mathbb{R}$.

A formula for the change of the index for $T_{\lambda, 0}$, when the spectral parameter $\lambda \in \mathbb{R}$ crosses the first eigenvalue $\lambda_1 = 1$ can be found in [DEM, Theorem 4.1] or [D, Theorem 14.9]. Adapting that result to our case, we have, for small $\varepsilon > 0$, and any $R > 0$,

$$\deg[I - T_{1-\varepsilon, 0};\, B_R(0), 0] = 1, \tag{3.4}$$

$$\deg[I - T_{1+\varepsilon, 0};\, B_R(0), 0] = -1, \tag{3.5}$$

where $B_R(0) := \{u \in X;\ \|u\|_X < \mathbb{R}\}$. Using the homogeneity in Eq. (3.2) and the boundary conditions (3.3) we see that for fixed $h \in L^\infty(0, \pi_p)$ we can take $R > 0$ so large that (3.4), (3.5) extend to

$$\deg[I - T_{1-\varepsilon, h};\ B_R(0), 0] = 1, \tag{3.6}$$

$$\deg[I - T_{1+\varepsilon, h};\ B_R(0), 0] = -1. \tag{3.7}$$

We distinguish between the two cases $1 < p < 2$ and $p > 2$.

Case $1 < p < 2$. Let $h \in C^1[0, \pi_p]$ be such that

$$\int_0^{\pi_p} h(t) \sin_p t\, dt = 0. \tag{3.8}$$

For $t \geqslant \pi_p$, let us extend h to $[0, +\infty)$ as a C^1-function (e.g., as a linear function $h(t) = h'(\pi_p)\, t + h(\pi_p)$).

We claim that there exists a constant $R > 0$ such that for any $\lambda \in [1, 2^{p-1}]$ the boundary value problem

$$\begin{gathered} (\varphi_p(u'))' + \lambda(p-1)\, \varphi_p(u) = \frac{\lambda}{p^*}\, h(\lambda^{1/p} t), \\ u(0) = u(\pi_p) = 0, \end{gathered} \tag{3.9}$$

has no solution with $\|u\|_{C^1[0, \pi_p]} \geqslant R$.

To prove this claim we argue by contradiction. Thus we suppose that there exist sequences $\{u_n\}_{n=1}^\infty \subset C^1[0, \pi_p]$, $\{\lambda_n\}_{n=1}^\infty \subset [1, 2^{p-1}]$, such that $\lambda_n \to \bar{\lambda} \in [1, 2^{p-1}]$, and $\|u_n\|_{C^1[0, \pi_p]} \to \infty$, and u_n, λ_n satisfy (3.9). By Lemma 2.1 it is not difficult to see that $|\alpha_n| \to \infty$, where as before, $\alpha_n = u_n'(0)$. Assume that $\alpha_n \to \infty$ (the other case is similar). Then u_n, $n \in \mathbb{N}$ is the solution of the initial value problem

$$(\varphi_p(u_n'))' + \lambda_n(p-1)\, \varphi_p(u_n) = \frac{\lambda_n}{p^*}\, h(\lambda_n^{1/p} t),$$

$$u_n(0) = 0, \qquad u_n'(0) = \alpha_n$$

on $[0, \infty)$, and hence $v_n(t) := u_n(t\lambda_n^{-1/p})$ solves the initial value problem

$$(\varphi_p(v_n'))' + (p-1)\, \varphi_p(v_n) = \frac{1}{p^*}\, h(t),$$

$$v_n(0) = 0, \qquad v_n'(0) = \tilde{\alpha}_n,$$

where $\tilde{\alpha}_n = \alpha_n \lambda_n^{1/p} \to \infty$. By Lemma 2.1 the first positive zero point $t_1^{\tilde{\alpha}_n}$ of v_n satisfies $t_1^{\tilde{\alpha}_n} \to \pi_p$ as $n \to \infty$ and similarly the second positive zero point approaches $2\pi_p$. Then condition (3.8) and Proposition 2.2 imply that $t_1^{\tilde{\alpha}_n} < \pi_p$ for n large enough. But this contradicts the fact that $0 = u_n(\pi_p) = v_n(\pi_p \lambda_n^{1/P})$ because $1 \leqslant \lambda_n \leqslant 2^{p-1}$ for any $n \in \mathbb{N}$. Thus the claim is proved.

From this claim we see that for $\varepsilon > 0$ small the homotopy $\mathscr{H}: [1, 1+\varepsilon] \times X \to X$ defined by $\mathscr{H}(u, \lambda) = u - T_{\lambda, h_\lambda}(u)$, where $h_\lambda(t) = h(\lambda^{1/p} t)$, satisfies $\mathscr{H}(u, \lambda) \neq 0$ for all $\lambda \in [1, 1+\varepsilon]$ and $\|u\|_{C^1[0, \pi_p]} \geqslant R$. Thus, from the homotopy invariance property of the Leray–Schauder degree, we obtain

$$\deg[I - T_{1,h}; B_R(0), 0] = \deg[I - T_{1+\varepsilon, h_{1+\varepsilon}}; B_R(0), 0] = -1,$$

by (3.7). This proves that for given $h \in C^1[0, \pi_p]$ satisfying (3.8) the boundary value problem (3.1) has at least one solution. Moreover, it follows from our considerations that all possible solutions of (3.1) are a priori bounded in the $C^1[0, \pi_p]$ norm.

Case $p > 2$. Let h be as in the previous case. We claim now that there exists a constant $R > 0$ such that for any $\lambda \in [\frac{1}{2}, 1]$ the boundary value problem (3.9) has no solution with $\|u\|_{C^1[0, \pi_p] \geqslant R}$.

The proof of this claim follows the same steps as those in the previous case, and we now obtain

$$\deg[I - T_{1,h}; B_R(0), 0] = \deg[I - T_{1-\varepsilon, h_{1-\varepsilon}}; B_R(0), 0] = 1,$$

by (3.6). Thus the proof of Theorem 1.1 is completed.

4. PROOF OF THEOREM 1.2

Let us consider the energy functional $E: W_0^{1,p}(0, \pi_p) \to \mathbb{R}$ associated with the boundary value problem (3.1),

$$E(u) = \frac{1}{p} \int_0^{\pi_p} |u'|^p - \frac{p-1}{p} \int_0^{\pi_p} |u|^p + \frac{1}{p^*} \int_0^{\pi_p} hu, \tag{4.1}$$

$h \in C^1[0, \pi_p]$, h satisfies (3.8). In this case we will also distinguish between $1 < p < 2$ and $p > 2$.

Case $1 < p < 2$. For $\alpha \gg 1$, say $\alpha_n \to \infty$, $n \in \mathbb{N}$, consider the solutions to the initial value problem (2.1)–(2.2) given by $u_n(t) = u_{\alpha_n}(t)$ for $t \in [0, t_1^{\alpha_n})$, $u_n(t) = 0$ for $t \in [t_1^{\alpha_n}, \pi_p]$ (recall that $t_1^{\alpha_n} < \pi_p$ by Proposition 2.2). Clearly $u_n \in W_0^{1,p}(0, \pi_p)$, $n \in \mathbb{N}$.

By Proposition 2.2, it follows that

$$\delta_n := \pi_p - t_1^{\alpha_n} = \frac{(2-p)\, J_h}{\alpha_n^{2(p-1)}} + o(\alpha_n^{2(1-p)}) \qquad \text{as} \quad n \to \infty. \tag{4.2}$$

We shall prove that the energy functional E defined in (4.1) satisfies

$$\lim_{n\to\infty} E(u_n) = -\infty. \tag{4.3}$$

By definition

$$E(u_n) = \frac{1}{p}\int_0^{\pi_p-\delta_n} |u_n'|^p - \frac{p-1}{p}\int_0^{\pi_p-\delta_n} |u_n|^p + \frac{1}{p^*}\int_0^{\pi_p-\delta_n} hu_n. \tag{4.4}$$

Multiplying $(\varphi_p(u'_{\alpha_n}))' + (p-1)\,\varphi_p(_{\alpha_n}) = (1/p^*)\,h$ by u_{α_n} and integrating over $[0, \pi_p - \delta_n]$, we find

$$-\int_0^{\pi_p-\delta_n} |u_n'|^p + (p-1)\int_0^{\pi_p-\delta_n} |u_n|^p = \frac{1}{p^*}\int_0^{\pi_p-\delta_n} hu_n. \tag{4.5}$$

Then, from (4.4) and (4.5),

$$E(u_n) = -\frac{1}{p^*}\left[\int_0^{\pi_p-\delta_n} |u_n'|^p - (p-1)\int_0^{\pi_p-\delta_n} |u_n|^p\right]. \tag{4.6}$$

On the other hand from the Poincaré inequality, we have

$$\int_0^{\pi_p-\delta_n} |u_n'|^p \geqslant (p-1)\left(\frac{\pi_p}{\pi_p-\delta_n}\right)^p \int_0^{\pi_p-\delta_n} |u_n|^p,$$

and then, from (4.6)

$$E(u_n) \leqslant -p\left[\left(1-\frac{\delta_n}{\pi_p}\right)^{-p} - 1\right]\int_0^{\pi_p-\delta_n} |u_n|^p. \tag{4.7}$$

Now since by (4.2),

$$\begin{aligned}\left[\left(1-\frac{\delta_n}{\pi_p}\right)^{-p} - 1\right] &= \frac{p\delta_n}{\pi_p} + o(\delta_n)\\ &= \frac{p(2-p)\,J_h}{\pi_p}\,\alpha_n^{2(1-p)} + o(\alpha_n^{2(1-p)}),\end{aligned}$$

as $n\to\infty$, and by (2.17), $u_n(t) = \alpha_n \sin_p t + o(\alpha_n)$ for $t \in [0, \pi_p - \delta_n]$, it follows from (4.7) that

$$\begin{aligned}
E(u_n) &\leqslant -\frac{p^2(2-p)\,J_h}{\pi_p}\,\alpha_n^{2(1-p)}\left[\alpha_n^p\int_0^{\pi_p-\delta_n}\sin_p^p t\,dt+o(\alpha_n^p)\right]\\
&\quad+o(\alpha^{2(1-p)})\left[\alpha_n^p\int_0^{\pi_p-\delta_n}\sin_p^p t\,dt+o(\alpha_n^p)\right]\\
&=-\frac{p^2(2-p)\,J_h}{\pi_p}\,\alpha_n^{2-p}\int_0^{\pi_p}\sin_p^p t\,dt+o(\alpha_n^{2-p}) \qquad (4.8)
\end{aligned}$$

for $n\to\infty$. Thus (4.3) follows from (4.8).

Case $p>2$. For a large positive number α, let us consider the solutions u_α and $u_{-\alpha}$ of the initial value problem (2.1)–(2.2). Then from Proposition 2.2, $u_{-\alpha}$ and u_α are respectively lower and upper solutions (see Appendix IV for precise definitions) of the boundary value problem (3.1). Now, from Proposition A.1 in Appendix IV, we obtain that E attains its minimum on the set of functions between $u_{-\alpha}$ and U_α, at a (global) critical point of E. The set of such global critical points is compact, from Theorem 1.1. Let $-K$ be the minimum value of E on this set. Since, given any $\psi\in C_0^\infty(0,\pi_p)$ and sufficiently large α, ψ lies between $u_{-\alpha}$ and u_α, so that we get $E(\psi)\geqslant -K$. Finally, by the density of $C_0^\infty(0,\pi_p)$ in $W_0^{1,p}(0,\pi_p)$ we get $E(u)\geqslant -K$ for any $u\in W_0^{1,p}(0,\pi_p)$. Moreover, E minimizes precisely on the (nonempty) set of its critical points.

Finally, we will exhibit an example which shows that the *Palais–Smale condition fails* for $p>2$ at the level zero.

Let $h\in C^1[0,\pi_p]$ such that (3.8) holds for $p>2$. Consider the solutions $u_n=u_n(t)$ of the initial value problem (2.1)–(2.2) with $u_n'(0)=\alpha_n\to\infty$. Then, from (2.9) and (2.13), $\alpha=q+O(q^{2-p})$ and thus for $t\in[0,2\pi_p]$, we have

$$u_n(t)=\alpha_n\sin_p t+O(\alpha_n^{2-p}) \qquad \text{as} \quad n\to\infty. \qquad (4.9)$$

Since u_n solves the initial value problem (2.1)–(2.2), the function $v_n(t):=u_n((t_1^{\alpha_n}/\pi_p)\,t)$ solves in turn the boundary value problem

$$\begin{gathered}
(\varphi_p(v_n'))'+(p-1)\left(\frac{\pi_p}{t_1^{\alpha_n}}\right)^p\varphi_p(v_n)=\frac{1}{p^*}\left(\frac{\pi_p}{t_1^{\alpha_n}}\right)^p\tilde{h},\\
v_n(0)=v_n(\pi_p)=0,
\end{gathered} \qquad (4.10)$$

where $\tilde{h}(t)=h((t_1^{\alpha_n}/\pi_p)\,t)$.

From (4.9) and $\|h\|_{C^1[0,2\pi_p]}<H$, it follows that

$$v_n(t)=\alpha_n\sin_p t+O(\alpha_n^{2-p}) \qquad \text{as} \quad n\to\infty, \qquad (4.11)$$

and

$$\tilde{h}(t) = h(t) + o(\alpha_n^{2(1-p)}) \qquad \text{as} \quad n \to \infty. \tag{4.12}$$

Also, from Hölder, inequality and (4.10), we have

$$\begin{aligned}
&\sup_{\|\psi\|_{W_0^{1,p}} \leqslant 1} |\langle E'(v_n), \psi \rangle| \\
&\quad = \sup_{\|\psi\|_{W_0^{1,p}} \leqslant 1} \left| \int_0^{\pi_p} \varphi_p(v_n')\, \varphi' - (p-1) \int_0^{\pi_p} \varphi_p(v_n)\, \psi + \frac{1}{p^*} \int_0^{\pi_p} h\psi \right| \\
&\quad \leqslant (p-1) \left| \left(\frac{\pi_p}{t_1^{\alpha_n}} \right)^p - 1 \right| \sup_{\|\psi\|_{W_0^{1,p}} \leqslant 1} \int_0^{\pi_p} \varphi_p(v_n)\, \psi \\
&\qquad + \frac{1}{p^*} \int_0^{\pi_p} h\psi - \frac{1}{p^*} \left(\frac{\pi_p}{t_1^{\alpha_n}} \right)^p \int_0^{\pi_p} \tilde{h}\psi \\
&\quad \leqslant (p-1) \left| \left(\frac{\pi_p}{t_1^{\alpha_n}} \right)^p - 1 \right| \left(\int_0^{\pi_p} |v_n|^p \right)^{1/p^*} + \frac{1}{p^*} \left(\int_0^{\pi_p} |h - \tilde{h}|^{p^*} \right)^{1/p^*} \\
&\qquad + \frac{1}{p^*} \left| 1 - \left(\frac{\pi_p}{t_1^{\alpha_n}} \right)^p \right| \left(\int_0^{\pi_p} |\tilde{h}|^{p^*} \right)^{1/p^*}.
\end{aligned} \tag{4.13}$$

By (2.5),

$$\left| \left(\frac{\pi_p}{t_1^{\alpha_n}} \right)^p - 1 \right| = \frac{p(p-2)}{\pi_p} J_h \alpha_n^{2(1-p)} + o(\alpha_n^{2(1-p)}), \tag{4.14}$$

and by (4.12),

$$|h(t) - \tilde{h}(t)| = o(\alpha_n^{2(1-p)}), \tag{4.15}$$

as $n \to \infty$; then from (4.14) and (4.15), we find that

$$\sup_{\|\psi\|_{W_0^{1,p}} \leqslant 1} |\langle E'(v_n), \psi \rangle| \leqslant \frac{p(p-1)(p-2)}{\pi_p} J_h \alpha_n^{1-p} + o(\alpha_n^{1-p})$$

as $n \to \infty$, i.e., $\lim_{n \to \infty} E'(v_n) = 0$.

From (3.8), (4.10), (4.11), (4.13), (4.14), and (4.15), we obtain

$$\begin{aligned}|E(v_n)| &= \left|\frac{1}{p}\int_0^{\pi_p} |v_n'|^p - \frac{p-1}{p}\int_0^{\pi_p} |v_n|^p + \frac{1}{p^*}\int_0^{\pi_p} hv_n\right| \\ &\leqslant \frac{p-1}{p}\left|\left(\frac{\pi_p}{t_1^{\alpha_n}}\right)^p - 1\right| \int_0^{\pi_p} |v_n|^p \\ &\quad + \frac{1}{p^*}\left|\int_0^{\pi_p} hv_n\right| + \frac{1}{pp^*}\left(\frac{\pi_p}{t_1^{\alpha_n}}\right)^p \left|\int_0^{\pi_p} \tilde{h}v_n\right| \\ &\leqslant \frac{(p-1)(p-2)}{\pi_p} J_h \alpha_n^{2-p} + o(\alpha_n^{2-p}) \qquad \text{as} \quad n \to \infty,\end{aligned}$$

i.e., $\lim_{n\to\infty} E(v_n) = 0$. Hence $\{v_n\}_{n=1}^{\infty} \subset W_0^{1,p}(0, \pi_p)$ is an unbounded Palais–Smale sequence.

5. PROOF OF THEOREM 1.3

With the notation of Section 3 let us define $T_h: X \to X$ by $T_h = T_{1,h}$. Thus for each $v \in X$ and $h \in L^\infty(0, \pi_p)$, $u = T_h(v)$ satisfies

$$(\varphi_p(u'))' = \frac{1}{p^*} h - (p-1)\, \varphi_p(v), \qquad u(0) = u(\pi_p) = 0.$$

A mentioned before, T_h is a well-defined compact operator and depends continuously (in the operator norm) on the perturbations of the (parameter) function h (with respect to the $L^\infty(0, \pi_p)$-norm).

The following assertion follows from Proposition 2.1.

PROPOSITION 5.1. *Assume that for $h \in L^\infty(0, \pi_p)$ relation $I_h \neq 0$ holds. Then solutions to the boundary value problem* (3.1) *are a priori bounded and there exists $R > 0$ such that*

$$\deg[I - T_h; B_r(0), 0] = 0. \tag{5.1}$$

Proof. Let us assume, that

$$\int_0^{\pi_p} h(t) \sin_p t \, dt > 0,$$

and define the homotopy $\mathscr{H}: X\times[0,1]\to X$ by $\mathscr{H}(u,\sigma)=u-T_{h_\sigma}(u)$, where $h_\sigma(t):=1-\sigma+\sigma h(t)$. Now assume that there are sequences $\{u_n\}_{n=1}^{\infty}\subset X$ and $\{\sigma_n\}_{n=1}^{\infty}\subset[0,1]$, with $\|u_n\|_X\to\infty$ such that

$$\mathscr{H}(u_n,\sigma_n)=u_n-T_{h_{\sigma_n}}(u_n)=0.$$

Dividing this last expression by $\|u_n\|_X$ and letting $n\to\infty$ we find that $u_n(t)/\|u_n\|_X\to\pm\sin_p t$ in X. Hence, if $\alpha_n:=u_n'(0)$, we have $|\alpha_n|\to\infty$.

Assume $\alpha_n\to\infty$ (the other case is treated similarly) and $\sigma_n\to\sigma\in[0,1]$ (after passing to suitable subsequences). Extending $h_n(t):=1-\sigma_n+\sigma_n h(t)$, $n\in\mathbb{N}$, e.g., by zero for $t>\pi_p$, we can apply Lemma 2.1 to the initial value problem

$$(\varphi_p(u_n'))'+(p-1)\,\varphi_p(u_n)=\frac{1}{p^*}\,h_n,$$

$$u_n(0)=0,\qquad u_n'(0)=\alpha_n.$$

Notice that there exist constants $\delta>0$ and $H>0$, independent of n, and such that

$$\int_0^{\pi_p} h_n(t)\sin_p t\,dt\geqslant\delta\qquad\text{and}\qquad\|h_n\|_{L^\infty(0,\pi_p)}<H.$$

Hence, from Lemma 2.1, it follows that there exists $n_0\in\mathbb{N}$ such that for any $n\geqslant n_0$ we have $u_n(\pi_p)>0$. This proves the existence of $R>0$ such that

$$\deg[I-T_h;B_R(0),0]=\deg[I-T_1;B_R(0),0].\tag{5.2}$$

For the case $h\equiv1$ it was shown in [DM, Theorem 2.1] that the boundary value problem (3.1) has no solution. In particular, this implies

$$\deg[I-T_1;B_R(0),0]=0.\tag{5.3}$$

Thus assertion (5.1) follows from (5.2) and (5.3).

In the case $\int_0^{\pi_p}h(t)\sin_p t\,dt<0$, by constructing a similar homotopy we obtain

$$\deg[I-T_h;B_R(0),0]=\deg[I-T_{-1};B_R(0),0]=0.\quad\blacksquare$$

We begin here the proof of Theorem 3.1.

Let us construct first an auxiliary function $h_0\in C^2[0,\pi_p]$ for which the boundary value problem (3.1) (with $h=h_0$) has a solution and, moreover

$$\int_0^{\pi_p}h_0(t)\sin_p t\,dt\neq0.$$

Actually, our h_0 is a "smooth refinement" of an analogous function constructed in [BDH$_1$]. For $0<\varepsilon\ll 1$, set

$$u_\varepsilon(t)=\begin{cases} -\dfrac{p-1}{p+1}\dfrac{(\varepsilon-t)^{(p+1)/(p-1)}}{\varepsilon^{(p+1)/(p-1)}}+\dfrac{p-1}{p+1} & \text{for } t\in[0,\varepsilon),\\[2ex] \dfrac{p-1}{p+1} & \text{for } t\in\left[\varepsilon,\dfrac{\pi_p}{2}\right],\\[2ex] u_\varepsilon(\pi_p-t) & \text{for } t\in\left(\dfrac{\pi_p}{2},\pi_p\right]. \end{cases}$$

Let us define $h_\varepsilon := p^*[(\varphi_p(u_\varepsilon'))'+(p-1)\,\varphi_p(u_\varepsilon)]$. Straightforward calculation yields $h_\varepsilon\in C^2[0,\pi_p]$ and, by definition, $u_\varepsilon\in X$ is a positive solution of the boundary value problem (3.1) with $h=h_\varepsilon$.

On the other hand, the following asymptotic estimates for $\varepsilon\to 0_+$ hold,

$$\begin{aligned}\frac{1}{p^*}I_\varepsilon &:= \frac{1}{p^*}\int_0^{\pi_p} h_\varepsilon(t)\sin_p t\,dt\\ &=2\int_0^{\pi_p/2}(\varphi_p(u_\varepsilon'))'\sin_p t\,dt+2(p-1)\int_0^{\pi_p/2}\varphi(u_\varepsilon)\sin_p t\,dt\\ &=-2\int_0^{\pi_p/2}|u_\varepsilon'|^{p-2}u_\varepsilon'\cos_p t\,dt+2(p-1)\int_0^{\pi_p/2}|u_\varepsilon|^{p-2}u_\varepsilon\sin_p t\,dt\\ &=-2\int_0^{\varepsilon}\frac{(\varepsilon-t)^2}{\varepsilon^{p+1}}\cos_p t\,dt+2(p-1)\int_0^{\varepsilon}|u_\varepsilon|^{p-2}u_\varepsilon\sin_p t\,dt\\ &\quad+2(p-1)\int_\varepsilon^{\pi_p/2}|u_\varepsilon|^{p-2}u_\varepsilon\sin_p t\,dt.\end{aligned}$$

Using the facts that $\sin_p\varepsilon=\varepsilon+o(\varepsilon)$ and $\cos_p\varepsilon=1+o(1)$, we obtain

$$\begin{aligned}\frac{1}{p^*}I_\varepsilon &= -2\int_0^{\varepsilon}\frac{(\varepsilon-t)^2}{\varepsilon^{p+1}}(1+o(1))\,dt+2O(\varepsilon^2)\\ &\quad+\left(\frac{p-1}{p+1}\right)^{p-1}(p-1)\int_0^{\pi_p}\sin_p t\,dt+o(1)\\ &=-\frac{2}{3}\varepsilon^{2-p}+\left(\frac{p-1}{p+1}\right)^{p-1}(p-1)\int_0^{\pi_p}\sin_p t\,dt+o(1).\end{aligned}$$

Hence, for $1<p<2$, we have $I_\varepsilon>0$ while for $p>2$, we have $I_\varepsilon<0$, if $0<\varepsilon\leqslant\varepsilon_0$ with ε_0 small enough. So we can take $h_0:=h_{\varepsilon_0}$. We must distinguish between the case $1<p<2$ and $p>2$.

Case $1<p<2$. In this case we have

$$\int_0^{\pi_p} h_0(t)\sin_p t\,dt>0, \tag{5.4}$$

and the boundary value problem (3.1) with $h=h_0$ has a positive solution $u_0:=u_{\varepsilon_0}\in X$. By (5.4) there exists $\delta>0$ so small that for $u_{-\delta}(t):=u_0(t)-\delta$ and $h_{-\delta}:=p^*[(\varphi_p(u'_{-\delta}))'+(p-1)\,\varphi_p(u_{-\delta)}]$ we also have

$$\int_0^{\pi_p} h_{-\delta}(t)\sin_p t\,dt>0. \tag{5.5}$$

Fix such a δ. Clearly we can choose a small number $\rho>0$ such that for any h with $\|h-h_{-\delta}\|_\infty<\rho$ one has $h<h_{-\delta/2}$ on $(0,\pi_p)$, and also $\int_0^{\pi_p} h(t)\sin_p t\,dt>0$. Let us fix such an h and extend it, e.g., by zero for $t>\pi_p$. Then, from Lemma 2.1, it follows that for α sufficiently large and positive, the solution u_α of the initial value problem

$$(\varphi_p(u'_\alpha))'+(p-1)\,\varphi_p(u_\alpha)=\frac{1}{p^*}\,h,$$

$$u_\alpha(0)=0, \qquad u'_\alpha(0)=\alpha$$

satisfies $u_\alpha\geqslant u_{-\delta/2}$, $u_\alpha(\pi_p)>0$. Since $h<h_{-\delta/2}$ it follows that $u_{-\delta/2}$ and u_α are respectively lower and upper solutions of the boundary value problem (3.1) with this h.

Hence setting $\underline{u}=u_{-\delta/2}$ and $\bar{u}=u_\alpha$ in Proposition A.1 we obtain the existence of at least one solution u, which lies between $\underline{u}$ and $\bar{u}$. We claim that there exists at least a second solution. Assume the opposite, namely that only one solution exists. Then, from Proposition A.1, it follows that for a certain bounded open set Ω in $C^1[0,\pi_p]$ which contains u, we have

$$\deg[I-T_h;\Omega,0]=1. \tag{5.6}$$

On the other hand, Proposition 5.1 guarantees that for $R>0$ so large that $\Omega\subset B_R(0)$, we have

$$\deg[I-T_h;B_R(0),0]=0. \tag{5.7}$$

Now, (5.6), (5.7), and the additivity of the Leray–Schauder degree yield that the boundary value problem (3.1) has a second solution in $B_R(0)\backslash\Omega$. Hence the boundary value problem (3.1) has at least two distinct solutions for any $h\in B_\rho(h_{-\delta})$. The existence of an open cone C with the desired property is now a consequence of the homogeneity of the boundary value problem (3.1).

Case $p>2$. Now we have

$$\int_0^{\pi_p} h_0(t)\sin_p t\,dt<0, \tag{5.8}$$

and the boundary value problem (3.1) with $h=h_0$ has a positive solution u_0. By (5.8) there exists $\delta>0$ so small that for $u_\delta(t):=u_0(t)+\delta$ and $h_\delta:=p^*[(\varphi_p(u'_\delta))'+(p-1)\,\varphi_p(u_\delta)]$ we also have

$$\int_0^{\pi_p} h_\delta(t)\sin_p t\,dt<0. \tag{5.9}$$

It follows from (5.9) and Lemma 2.1 that for large and positive α the solution of the initial value problem

$$(\varphi_p(u'_{-\alpha}))'+(p-1)\,\varphi_p(u_{-\alpha})=\frac{1}{p^*}h_\delta,$$

$$u_{-\alpha}(0)=0, \qquad u'_{-\alpha}(0)=-\alpha$$

satisfies $u_{-\alpha}\leqslant u_\delta$, $u_{-\alpha}(\pi_p)<0$. Then we are in a position to proceed symmetrically to the previous case. In this form we have completed the proof of Theorem 3.1.

APPENDIX I

In this appendix we prove some properties of the *initial value problem*

$$(\varphi_p(u'))'+\lambda\varphi_p(u)=h \text{ in } [0,\infty), \qquad u(0)=u_0, \quad u'(0)=u_1, \tag{A.1}$$

where $h\in L^\infty_{\mathrm{loc}}[0,\infty)$ and $\lambda>0$.

For $h\equiv 0$ the initial value problem (A.1) has a unique solution (see, e.g., del Pino, Elgueta, and Manásevich [DEM, Sect. 3]). The nonautonomous case $h\neq 0$ is more complicated and uniqueness does not hold in general (see, e.g., Reichel and Walter [RW, Theorem 4]). However, we have the following global existence result.

LEMMA A.1 *Let* $h\in L^\infty_{\mathrm{loc}}[0,\infty)$, $\lambda>0$, $u_0, u_1\in\mathbb{R}$. *Then a global solution of the initial value problem* (A.1) *exists. Moreover, any local solution of* (A.1) *can be continued to all of* $[0,\infty)$.

Proof. Local existence follows from a standard application of the Schauder fixed point theorem. Let $p^* = p/p-1$ denote the conjugate exponent to p. Then the first order system

$$u' = \varphi_{p^*}(v),\ v' = \lambda\varphi_p(u) + h, \qquad u(0) = u_0,\ v(0) = u_1 \tag{A.2}$$

is equivalent to (A.1). Integrating (A.2) over a compact interval $[0, K] \subset [0, \infty)$ and using Hölder's inequality we obtain

$$|u(t)|^p + |v(t)|^{p^*} \leqslant c_3(K) + c_4(K) \int_0^t [|u(\tau)|^p + |v(\tau)|^{p^*}]\, d\tau. \tag{A.3}$$

It follows from (A.3) and Gronwall's lemma that the solution can be extended to $[0, K]$. Since $K > 0$ is arbitrary the assertion for the lemma follows. ∎

Proof of Lemma 2.1. The function $t \mapsto \sin_p t$ is the unique solution of the initial value problem

$$(\varphi_p(u'))' + (p-1)\,\varphi_p(u) = 0, \qquad u(0) = 0,\ u'(0) = 1.$$

Assume, for instance, that α is large and positive and let u_α be a globally defined solution of (2.1)–(2.2), as predicted by the previous lemma. Observe that $w_\alpha = u_\alpha/\alpha$ satisfies

$$(\varphi_p(w_\alpha'))' + (p-1)\,\varphi_p(w_\alpha) = \frac{1}{p^*\alpha^{p-1}}\, h, \qquad w_\alpha(0) = 0,\ w_\alpha'(0) = 1.$$

Since the right hand side tends to zero uniformly as $\alpha \to \infty$, a standard argument after, for instance, expressing the equation as a first order system like (A.2), implies that $w_\alpha \to \sin_p$ in the local C^1-sense. From here the fact that $t_1^\alpha \to \pi_p$ as $\alpha \to \infty$ immediately follows. Moreover, from the fact that $u_\alpha(t)/\alpha \to \sin_p t$ uniformly on compacts and the validity of Eq. (2.1), we easily see that for large α (dependent on the L^∞ norm of h on $[0, \pi_p]$), u_α is strictly concave on any given compact subinterval of $(0, \pi_p)$. From here the rest of the assertions of Lemma 2.1 follow easily. ∎

APPENDIX II

Let us prove (2.24). The change of variables $x = \operatorname{arc}\sin_p \tau$ and integration by parts yield

$$\int_0^1 (1-\tau^p)^{-1-1/p} \int_{\operatorname{arc\,sin}_p \tau}^{\pi_p/2} [h(s)+h(\pi_p - s)] \cos_p s \, ds \, d\tau$$

$$= \int_0^{\pi_p/2} \frac{1}{\cos_p^p x} \int_x^{\pi_p/2} [h(s)+h(\pi_p - s)] \cos_p s \, ds \, dx$$

$$= \int_0^{\pi_p/2} [h(x)+h(\pi_p - x)] \sin_p x \, dx$$

$$= \int_0^{\pi_p} h(x) \sin_p x \, dx.$$

Here we used (2.7) and the fact

$$\lim_{x \to \pi_p/2} \tan_p x \int_x^{\pi_p/2} [h(s)+h(\pi_p - s)] \cos_p s \, ds = 0,$$

which holds due to $\|h\|_{L^\infty(0, 2\pi_p)} < H$.

APPENDIX III

Let us prove (2.36). Denote

$$I_1 := \int_0^1 \frac{\int_s^1 h'(\operatorname{arc\,sin}_p \tau) \, \Theta(\tau) \, d\tau}{(1-s^p)^{1+1/p}} \, ds,$$

$$I_2 := \frac{p+1}{2p} \int_0^1 \frac{(\int_s^1 h(\operatorname{arc\,sin}_p \tau) \, d\tau)^2}{(1-s^p)^{2+1/p}} \, ds,$$

where

$$\Theta(\tau) := \frac{1}{p} \int_0^\tau \frac{\int_w^1 h(\operatorname{arc\,sin}_p \sigma) \, d\sigma}{(1-w^p)^{1+1/p}} \, dw.$$

First, let us develop $\Theta(\tau)$, using the substitution $y = \operatorname{arc\,sin}_p \sigma$,

$$\Theta(\tau) = \frac{1}{p} \int_0^{\operatorname{arc\,sin}_p \tau} \frac{\int_{\sin_p s}^1 h(\operatorname{arc\,sin}_p \sigma) \, d\sigma}{\cos_p^p z} \, dz$$

$$= \frac{1}{p} \int_0^{\operatorname{arc\,sin}_p \tau} \frac{\int_z^{\pi_p/2} h(y) \cos_p y \, dy}{\cos_p^p z} \, dz.$$

Setting $\tau = \sin_p x$ and integrating by parts we get

$$\Theta(\sin_p x) = \frac{1}{p} \int_0^x \frac{\int_z^{\pi_p/2} h(y) \cos_p y \, dy}{\cos_p^p z} \, dz$$

$$= \frac{1}{p} \left[\tan_p x \int_x^{\pi_p/2} h(y) \cos_p y \, dy + \int_0^x h(y) \sin_p y \, dy \right]. \tag{A.4}$$

Developing I_1 using substitutions $s = \sin_p t$ and $\tau = \sin_p x$ yields

$$I_1 = \int_0^{\pi_p/2} \frac{\int_{\sin_p t}^1 h'(\text{arc} \sin_p \tau) \, \Theta(\tau) \, d\tau}{\cos_p^p t} \, dt$$

$$= \int_0^{\pi_p/2} \frac{\int_t^{\pi_p/2} h'(x) \, \Theta(\sin_p x) \cos_p x \, dx}{\cos_p^p t} \, dt.$$

Using (A.4) we arrive at

$$I_1 = \frac{1}{p} (II_1 + II_2), \tag{A.5}$$

where

$$II_1 := \int_0^{\pi_p/2} \frac{\int_t^{\pi_p/2} h'(x) \sin_p x \int_x^{\pi_p/2} h(y) \cos_p y \, dy \, dx}{\cos_p^p t} \, dt,$$

$$II_2 := \int_0^{\pi_p/2} \frac{\int_t^{\pi_p/2} h'(x) \cos_p x \int_0^x h(y) \sin_p y \, dy \, dx}{\cos_p^p t} \, dt.$$

Let us develop II_1 integrating by parts. We use

$$\lim_{t \to \pi_p/2-} \tan_p t \int_t^{\pi_p/2} h'(x) \sin_p x \int_x^{\pi_p/2} h(y) \cos_p y \, dy \, dx = 0$$

because $\tan_p t \sim |\pi_p/2 - t|^{1/(p_1)}$ and

$$\int_t^{\pi_p/2} h'(x) \sin_p x \int_x^{\pi_p/2} h(y) \cos_p y \, dy \, dx \sim \left| \frac{\pi_p}{2} - t \right|^{1+p^*}$$

as $t \to \pi_p/2$ due to $\|h\|_{C^1[0,2\pi_p]} < H$, to get

$$II_1 = \int_0^{\pi_p/2} h'(t) \sin_p t \tan_p t \left(\int_t^{\pi_p/2} h(y) \cos_p y \, dy \right) dt.$$

Integrating by parts once again and using formulas (2.6) and (2.7) and the fact that

$$\lim_{t \to \pi_p/2-} h(t) \sin_p t \tan_p t \left(\int_t^{\pi_p/2} h(y) \cos_p y \, dy \right) dt = 0,$$

we get

$$\begin{aligned} II_1 &= -\int_0^{\pi_p/2} h(t) \frac{d}{dt} \left[\sin_p t \tan_p t \int_t^{\pi_p/2} h(y) \cos_p y \, dy \right] dt \\ &= \int_0^{\pi_p/2} h^2(t) \sin_p^2 t \, dt - \int_0^{\pi_p/2} h(t) \sin_p t \left(\int_t^{\pi_p/2} h(y) \cos_p y \, dy \right) dt \\ &\quad - \int_0^{\pi_p/2} h(t) \tan_p^p t \sin_p t \left(\int_t^{\pi_p/2} h(y) \cos_p y \, dy \right) dt \\ &\quad - \int_0^{\pi_p/2} h(t) \tan_p t \cos_p t \left(\int_t^{\pi_p/2} h(y) \cos_p y \, dy \right) dt. \end{aligned} \tag{A.6}$$

Integrating by parts and using the fact

$$\lim_{t \to \pi_p/2-} \tan_p t \int_t^{\pi_p/2} h'(x) \cos_p x \left(\int_0^x h(y) \sin_p y \, dy \right) dx = 0,$$

we develop II_2,

$$II_2 = \int_0^{\pi_p/2} h'(t) \sin_p t \left(\int_0^t h(y) \sin_p y \, dy \right) dt.$$

Integrating by parts once again and using the Fubini theorem we get

$$\begin{aligned} II_2 &= h\left(\frac{\pi_p}{2}\right) \int_0^{\pi_p/2} h(y) \sin_p y \, dy - \int_0^{\pi_p/2} h(t) \sin_p t \left(\int_t^{\pi_p/2} h(y) \cos_p y \, dy \right) dt \\ &\quad - \int_0^{\pi_p/2} h^2(t) \sin_p^2 t \, dt. \end{aligned} \tag{A.7}$$

Hence, we get from (A.5)–(A.7) that

$$pI_1 = h\left(\frac{\pi_p}{2}\right)\int_0^{\pi_p/2} h(y)\sin_p y\,dy - 2\int_0^{\pi_p/2} h(t)\sin_p t\left(\int_t^{\pi_p/2} h(y)\cos_p y\,dy\right)dt$$

$$-\int_0^{\pi_p/2} h(t)[\tan_p^p t + \tan_p t\cos_p t]\int_t^{\pi_p/2} h(y)\cos_p y\,dy\,dt$$

$$= h\left(\frac{\pi_p}{2}\right)\int_0^{\pi_p/2} h(y)\sin_p y\,dy$$

$$+\frac{1}{2}\int_0^{\pi_p/2}\frac{d}{dt}\left(\int_t^{\pi_p/2} h(y)\cos_p y\,dy\right)^2[3\tan_p t + \tan_p^{p+1} t]\,dt.$$

Integrating by parts and using the facts that

$$\lim_{t\to\pi_p/2-}(3\tan_p t + \tan_p^{p+1} t)\left(\int_t^{\pi_p/2} h(y)\cos_p y\,dy\right)^2 = 0,$$

$$\frac{d}{dt}(3\tan_p t + \tan_p^{p+1} t) = \frac{p+1}{\cos_p^{2p} t} + \frac{2-p}{\cos_p^p t},$$

we arrive at

$$I_1 = \frac{1}{p}h\left(\frac{\pi_p}{2}\right)\int_0^{\pi_p/2} h(y)\sin_p y\,dy - \frac{p+1}{2p}\int_0^{\pi_p/2}\frac{(\int_t^{\pi_p/2} h(y)\cos_p y\,dy)^2}{\cos_p^{2p} t}\,dt$$

$$-\frac{2-p}{2p}\int_0^{\pi_p/2}\frac{(\int_t^{\pi_p/2} h(y)\cos_p y\,dy)^2}{\cos_p^p t}\,dt. \tag{A.8}$$

Let us develop I_2 using the substitutions $s = \sin_p t$ and $\tau = \sin_p y$. We get

$$I_2 = \frac{p+1}{2p}\int_0^{\pi_p}\frac{(\int_{\sin_p t}^1 h(\arcsin_p \tau)\,d\tau)^2}{\cos_p^{2p} t}\,dt$$

$$= \frac{p+1}{2p}\int_0^{\pi_p/2}\frac{(\int_t^{\pi_p/2} h(y)\cos_p y\,dy)^2}{\cos_p^{2p} t}\,dt. \tag{A.9}$$

Then (2.36) now follows from (A.8) and (A.9).

APPENDIX IV

In this appendix we consider the issue of lower and upper solutions of the problem (3.1), i.e.,

$$(\varphi_p(u'))' + (p-1)\,\varphi_p(u) = \frac{1}{p^*}\,h, \qquad u(0) = u(\pi_p) = 0.$$

We shall call a function $\underline{u} \in C^1[0, \pi_p]$, with $\varphi_p(\underline{u}')$ absolutely continuous a *lower solution* of (3.1) if it is not a solution to (3.1), $\underline{u}(0) \geqslant 0$, $\underline{u}(\pi_p) \geqslant 0$, and

$$(\varphi_p(\underline{u}'))' + (p-1)\,\varphi_p(\underline{u}) \geqslant \frac{1}{p^*}\,h,$$

almost everywhere in $(0, \pi_p)$.

In an analogous way we define an *upper solution* $\bar{u}$ of (3.1).

PROPOSITION A.1 *Assume that $\underline{u}$ and $\bar{u}$ are respectively lower and upper solutions of* (3.1) *with $\underline{u} \leqslant \bar{u}$. Then the boundary value problem* (3.1) *has at least one solution u such that $\underline{u} \leqslant u \leqslant \bar{u}$. Moreover, u can be variationally characterized as*

$$E(u) = \inf\{E(v);\, v \in W_0^{1,\,p}(0, \pi_p),\, \underline{u} \leqslant v \leqslant \bar{u}\}.$$

Here E is the functional defined in (4.1).

If only one solution of (3.1) *exists, then for all sufficiently large $R > 0$, one also has*

$$\deg[I - T_h;\, B_R(0), 0] = 1,$$

where T_h is the operator defined in Section 5.

Proof. Let us write (3.1) in an equivalent form

$$\begin{aligned} (\varphi_p(u'))' - (p-1)\,\varphi_p(u) &= f(u, t), \\ u(0) = u(\pi_p) &= 0, \end{aligned} \tag{A.10}$$

where $f(u, t)$: $-2(p-1)\,\varphi_p(u) + (1/p^*)\,h(t)$. Modify f in the following way:

$$\tilde{f}(u, t) = \begin{cases} f(u, t) & \text{if} \quad \underline{u}(t) \leqslant u(t) \leqslant \bar{u}(t), \\ f(\underline{u}(t), t) & \text{if} \quad u(t) \leqslant \underline{u}(t), \\ f(\bar{u}(t), t) & \text{if} \quad u(t) \geqslant \bar{u}(t). \end{cases}$$

Then every solution of the modified boundary value problem

$$\begin{gathered}(\varphi_p(u'))' - (p-1)\,\varphi_p(u) = \tilde{f}(u,t),\\ u(0) = u(\pi_p) = 0,\end{gathered} \tag{A.11}$$

is also a solution of the original boundary value problem (A.10). Indeed, let us assume that u is a solution of (A.11) and $u(t) > \bar{u}(t)$ for some $t \in [0, \pi_p]$. Then integrating by parts it follows from (A.11) that

$$\int_0^{\pi_p} \varphi_p(u')(u-\bar{u})'_+ + (p-1)\int_0^{\pi_p} \varphi_p(u)(u-\bar{u})_+ = -\int_0^{\pi_p} f(\bar{u}(t),t)(u-\bar{u})_+. \tag{A.12}$$

The definition of the upper solution yields

$$\int_0^{\pi_p} \varphi_p(\bar{u}')(u-\bar{u})'_+ + (p-1)\int_0^{\pi_p} \varphi_p(\bar{u})(u-\bar{u})_+ \geqslant -\int_0^{\pi_p} f(\bar{u}(t),t)(u-\bar{u})_+. \tag{A.13}$$

Since $(u-\bar{u})'_+ = u' - \bar{u}'$ a.e. in $I_+ = \{t \in [0, \pi_p]; u(t) > \bar{u}(t)\}$, from (A.12) and (A.13), it follows that

$$\int_{I_+} (\varphi_p(u') - \varphi_p(\bar{u}'))(u' - \bar{u}') + (p-1)\int_{I_+} (\varphi_p(u) - \varphi_p(\bar{u}))(u-\bar{u}) \leqslant 0,$$

which implies $u - \bar{u} \equiv 0$ on I_+. The same argument proves that $u(t) \geqslant \underline{u}(t)$ for $t \in [0, \pi_p]$.

Next, we consider the following modification of the functional E,

$$\tilde{E}(u) = \frac{1}{p}\int_0^{\pi_p} |u'|^p + \frac{(p-1)}{p}\int_0^{\pi_p} |u|^p + \int_0^{\pi_p} \tilde{F}(u,t),$$

where $\tilde{F}(s,t) = \int_0^s \tilde{f}(\sigma,t)\,d\sigma$. Then $\tilde{E}$ attains its global minimum in $W_0^{1,p}(0,\pi_p)$, since it is easily checked that $\tilde{E}$ is coercive and weakly lower semicontinuous in $W_0^{1,p}(0,\pi_p)$. Let $\tilde{u}$ denote one such minimizer. Then $\tilde{u}$ solves the boundary value problem (A.11). Hence it solves also (A.10) and

$$\tilde{u} \in \Lambda = \{u \in W_0^{1,p}(0,\pi_p); \underline{u} \leqslant u \leqslant \bar{u}\}.$$

It follows that $E(\tilde{u}) = \inf_{u \in \Lambda} E(u)$, as desired.

Next, let us assume that no other solutions of (A.10) exist. It follows that there are $\rho > 0$ and $\varepsilon > 0$ such that for $\Omega := \{u \in B_\rho(0); \underline{u} - \varepsilon < u < \bar{u} + \varepsilon\}$ and for all $\tau \in [0, 1]$ the equation

$$(\varphi_p(u'))' - (p-1)\,\varphi_p(u) = f(u, t) + \tau(\tilde{f}(u, t) - f'(u, t)),$$
$$u(0) = u(\pi_p) = 0, \tag{A.14}$$

has no solution u with $u \in \partial\Omega$. Here f and $\tilde{f}$ are as above. It follows that the degree of the fixed point operator associated to (A.14) with respect to Ω is well defined and is constant in τ. Thus it suffices to show that $\deg[I - \tilde{T}_h; \Omega, 0] = 1$, where $u = \tilde{T}_h(v)$ if and only if

$$(\varphi_p(u'))' = (p-1)\,\varphi_p(v) + \tilde{f}(v, t),$$
$$u(0) = u(\pi_p) = 0.$$

Since $\tilde{f}$ is uniformly bounded, it easily follows from a direct homotopy that the degree on a large ball $B_\rho(0)$, $\deg[I - \tilde{T}_h; B_\rho(0), 0]$, equals one.

On the other hand, as we have already seen, a fixed point of $\tilde{T}_h$ lies necessarily between the upper and the lower solution. Therefore, from the excision property of the degree we find $\deg[I - \tilde{T}_h; \Omega, 0] = 1$. The proof of the proposition is thus concluded. ∎

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