C. R. Acad. Sci. Paris, t. 327, Série I, p. 461-465, 1998
Équations aux dérivées partielles/*Partial Differential Equations*

The Fredholm alternative at the first eigenvalue for the one-dimensional p-Laplacian

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(Reçu et accepté le 8 juillet 1998)

Abstract. In this work we study the range of the operator

$$u \mapsto (|u'|^{p-2}u')' + \lambda_1 |u|^{p-2}u, \quad u(0) = u(T) = 0,$$

$p > 1$. We prove that all functions $h \in C^1[0,T]$ satisfying $\int_0^T h(t) \sin_p \frac{\pi_p t}{T} \mathrm{d}t = 0$ lie in the range, but that if $p \not\equiv 2$ and $h \equiv 0$, the solution set is bounded. Here $\sin_p \frac{\pi_p t}{T}$ is a first eigenfunction associated to λ_1. © Académie des Sciences/Elsevier, Paris

***L'alternative de Fredholm à la première valeur propre pour le p-laplacien en dimension* 1**

Résumé. *On étudie l'image de l'opérateur*

$$u \mapsto (|u'|^{p-2}u')' + \lambda_1 |u|^{p-2}u, \quad u(0) = u(T) = 0,$$

$p>1$. *On montre que toutes les fonctions* $h \in C^1[0,T]$ *vérifiant* $\int_0^T h(t) \sin_p \frac{\pi_p t}{T} \mathrm{d}t = 0$, *sont dans l'image, mais que si* $p \not\equiv 2$ *et* $h \equiv 0$, *l'ensemble des solutions est borné. Ici* $\sin_p \frac{\pi_p t}{T}$ *désigne une fonction propre associée à la première valeur propre* λ_1.
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Dans cette Note nous étudions les propriétés de l'image de l'opérateur :

$$u \mapsto (|u'|^{p-2}u')' + \lambda_1 |u|^{p-2}u, \quad u(0) = u(T) = 0, \tag{1}$$

Note présentée par Haïm BRÉZIS.

où $T > 0$, $p > 1$ et $\lambda_1 > 0$ est la première valeur propre du p-laplacien avec les conditions aux limites de Dirichlet homogènes en 0 et T. On note $\sin_p\left(\frac{\pi_p t}{T}\right)$ la première fonction propre positive associée à λ_1. Nous prouvons que l'image de l'opérateur (1) contient tous les éléments $h \in \mathrm{C}^1[0,T]$ satisfaisant

$$\int_0^T h(t)\sin_p\left(\frac{\pi_p t}{T}\right)\mathrm{d}t = 0, \tag{2}$$

et également un cône ouvert non vide $C \subset \mathrm{L}^\infty(0,T)$ tel que, pour tout $h \in C$, on ait

$$\int_0^T h(t)\sin_p\left(\frac{\pi_p t}{T}\right)\mathrm{d}t \neq 0, \tag{3}$$

D'autre part, il existe $h \in \mathrm{C}^\infty[0,T]$ qui n'appartient pas à l'image de l'opérateur (1). En particulier, notre résultat implique que toutes les solutions possibles du problème aux limites :

$$\begin{cases} (|u'|^{p-2}u')' + \lambda_1|u|^{p-2}u = h \text{ sur } (0,T), \\ u(0) = 0 = u(T), \end{cases} \tag{4}$$

sont a priori bornées (si $h \in \mathrm{C}^1[0,T]$ et (2) est vérifiée ou si $h \in \mathrm{L}^\infty(0,T)$ et (3) est vérifiée). Considérons la fonctionnelle d'énergie $J : \mathrm{W}_0^{1,p}(0,T) \to \mathbf{R}$ associée à (4) :

$$J(u) = \frac{1}{p}\int_0^T |u'|^p - \frac{\lambda_1}{p}\int_0^T |u|^p - \int_0^T hu.$$

Nous montrons que, pour $1 < p < 2$, la fonctionnelle J est non bornée inférieurement, tandis que, pour $p > 2$, elle a un minimiseur global sur $\mathrm{W}_0^{1,p}(0,T)$. Dans le cas où $p > 2$, nous donnons un exemple de suite de Palais–Smale non bornée, i.e. $\|u_n\| \to \infty$, $J(u_n) \to 0$ et $J'(u_n) \to 0$.

1. Main results

Let us consider the following boundary value problem:

$$(\varphi_p(u'))' + \lambda_1\varphi_p(u) = h \text{ in } (0,T), \quad u(0) = u(T) = 0, \tag{1.1}$$

where $\varphi_p(s) := |s|^{p-2}s$ for $p \in (1,\infty)$, so that the differential operator $(\varphi_p(u'))'$ is the one-dimensional version of the p-Laplacian. We assume that $h \in \mathrm{L}^\infty(0,T)$ and that $\lambda_1 > 0$ is the first eigenvalue of the homogeneous problem:

$$(\varphi_p(u'))' + \lambda\varphi_p(u) = 0 \text{ in } (0,T), \quad u(0) = u(T) = 0. \tag{1.2}$$

It is well known (*see* for instance [4]) that this first eigenvalue is explicitly given by

$$\lambda_1 = \left(\frac{\pi_p t}{T}\right)^p, \text{ where } \pi_p = 2(p-1)^{\frac{1}{p}}\int_0^T (1-s^p)^{-\frac{1}{p}}\mathrm{d}s.$$

By a solution of problem (1.1) we understand a real-valued fiction $u \in \mathrm{C}^1[0,T]$ such that $u(0) = u(T) = 0$, $\varphi_p(u')$ is absolutely continuous, and (1.2) holds almost everywhere in $(0,T)$.

In the case $p = 2$, the classical linear Fredholm alternative provides a transparent necessary and sufficient condition for the solvability of (1.1), namely,

$$\int_0^T h(t)\sin\left(\frac{\pi t}{T}\right)\mathrm{d}t = 0. \tag{1.3}$$

Needless to say, the proof of this result uses in essentially way the linearity of the equation. On the other hand, a number of works have dealt with finding nonlinear analogues to the linear Fredholm Alternative. For instance, the general theory in [7] yields that, if λ_1 in the above equation is replaced by a number λ which is not an eigenvalue of the homogeneous problem, then the existence of a solution is guaranteed (this result also holds in the higher dimensional case). The situation for the resonant case we treat here is much more subtle. We refer the reader to [1], [2], [4], [6] for some results related to this issue. For instance, it follows from [6] that if the sign of h does not change, then no solution exists.

Our results in this work demonstrate that for $p \neq 2$, rather surprisingly, the exact analogue of (1.3) is sufficient for the solvability of problem (1.1). On the other hand, we show the presence of striking differences in the structure of the set of right hand sides h for which (1.1) is solvable. Last but nit least, we point out strong differences in the qualitative behaviour of the energy functional associated with (1.1) if $1 < p < 2$ and if $p > 2$. Let $\sin_p\left(\frac{\pi_p t}{T}\right)$ be the positive normalized eigenfunction associated with λ_1. Our main results are stated in the following three theorems.

THEOREM 1. – *Let us assume that* $h \in \mathrm{C}^1[0,T]$ *and*

$$\int_0^T h(t)\sin_p\left(\frac{\pi_p t}{T}\right)\mathrm{d}t = 0. \tag{1.4}$$

Then problem (1.1) *has at least one solution. Moreover, for any* $h \in \mathrm{C}^1[0,T]$, $h \not\equiv 0$, *any possible solution of* (1.1) *is a priori bounded.*

THEOREM 2. – *There exists an open cone* $C \subset \mathrm{L}^\infty(0,T)$ *such that for any* $h \in C$ *the b.v.p.* (1.1) *has at least two distinct solutions. Moreover,*

$$I = \int_0^T h(t)\sin_p\left(\frac{\pi_p t}{T}\right)\mathrm{d}t \neq 0 \tag{1.5}$$

holds for any $h \in C$. *On the other hand, there exists* $h \in \mathrm{C}^\infty[0,T]$ *satisfying* (1.5), *for which the b.v.p.* (1.1) *has no solution.*

Let us consider the energy functional associated to problem (1.1), $J : \mathrm{W}_0^{1,p}(0,T) \to \mathbf{R}$, defined as:

$$J(u) = \frac{1}{p}\int_0^T |u'|^p - \frac{\lambda_1}{p}\int_0^T |u|^p - \int_0^T hu.$$

It is well known that λ_1 corresponds precisely to the best constant in the p-Poincaré's inequality, which tells us that the functional is nonnegative if $h \equiv 0$, but vanishing along the ray generated by the first eigenfunction.

It is easily seen that a necessary condition for the functional to be bounded from below is that the orthogonality condition (1.4) holds, and one may ask whether this condition is also sufficient. For $p = 2$ this is the case, as it is readily checked via Fourier series expansions. The full answer, provided by the next result in case that h is C^1, shows an interesting change of topological type in the level sets of the functional once we shift p from $p < 2$ to $p > 2$.

THEOREM 3. – *Let us assume that* $h \in \mathrm{C}^1[0,T]$, $h \not\equiv 0$ *and* (1.4) *holds. Then the set of critical points of* J *is nonempty and bounded if* $p \neq 2$. *Moreover:*

(i) *for* $1 < p < 2$ *the functional* J *is unbounded from below;*

(ii) *for* $p > 2$ *the functional* J *is bounded from below and it has a global minimizer on* $\mathrm{W}_0^{1,p}(0,T)$.

2. Discussion of the results and proofs

Let us consider the following initial value problem associated with (1.1):

$$(\varphi_p(u'))' + \lambda_1 \varphi_p(u) = h \text{ in } (0,\infty), \quad u(0) = \beta,\ u'(0) = \alpha, \tag{2.1}$$

where $h \in L^\infty_{\mathrm{loc}}(0,\infty)$, $\lambda > 0$ and $\alpha,\ \beta \in \mathbf{R}$. By a solution of the i.v.p. we understand a (real-valued) function $u \in C^1[0,\infty)$ satisfying initial conditions, such that $\varphi_p(u')$ is absolutely continuous and (2.1) holds a.e. in $[0,\infty)$. We prove that for $\beta = 0$ and $|\alpha| \gg 1$, the i.v.p. has a global oscillatory solution $u = u(t)$. Let t_1^α denote the first positive zero point of u. Then the following two assertions play the crucial role in the proofs of our three theorems.

PROPOSITION 2.1. – *Let $h \in L^\infty_{\mathrm{loc}}(0,\infty)$ and assume* (1.5) *holds. Then*

$$t_1^\alpha = \pi_p + I \operatorname{sgn}\alpha|\alpha|^{1-p} + o(|\alpha|^{1-p}) \text{ as } |\alpha| \to \infty, \tag{2.2}$$

where $o(|\alpha|^{1-p})$ is uniform with respect to all h satisfying $\|h\|_{L^\infty(0,2\pi_p)} < H$ for some fixed $H > 0$. In particular:
(i) $I > 0 \Rightarrow t_1^\alpha > \pi_p$ *for* $\alpha \gg 1$ *and* $t_1^\alpha < \pi_p$ *for* $-\alpha \gg 1$,
(ii) $I < 0 \Rightarrow t_1^\alpha < \pi_p$ *for* $\alpha \gg 1$ *and* $t_1^\alpha < \pi_p$ *for* $-\alpha \gg 1$.

PROPOSITION 2.2. – *Let $h \in C^1[0,2\pi_p]$, $h \not\equiv 0$ and assume* (1.4). *Then there exists $I_{p,h} \neq 0$, such that $I_{p,h} > 0$ for $p > 2$ and $I_{p,h} < 0$ for $1 < p < 2$, and*

$$t_1^\alpha = \pi_p + I_{p,h}|\alpha|^{2(1-p)} + o(|\alpha|^{2(1-p)}) \text{ as } \alpha \to \infty, \tag{2.3}$$

where $o(|\alpha|^{2(1-p)})$ is uniform with respect to all h satisfying (1.4) *and $\|h\|_{C^1[0,2\pi_p]} < H$ for some fixed $H > 0$. In particular:*
(i) $1 < p < 2 \Rightarrow t_1^\alpha < \pi_p$ *for* $|\alpha| \gg 1$,
(ii) $p > 2 \Rightarrow t_1^\alpha > \pi_p$ *for any* $|\alpha| \gg 1$.

The constant $I_{p,h}$ is actually explicit, and given by:

$$I_{h,p} = (p-2)\frac{1}{2p^3}\int_0^{\frac{\pi_p}{2}} \frac{\left(\int_t^{\frac{\pi_p}{2}} h(y)\cos_p y\,dy\right)^2 + \left(\int_t^{\frac{\pi_p}{2}} h(\pi_p - y)\cos_p y\,dy\right)^2}{\cos_p^p t}\,dt.$$

Here $\cos_p t = (\sin_p t)'$.

Note that Propositions 2.1 and 2.2 provide an a priori estimate for any possible solution of problem (1.1) with nonzero right hand side h. Actually, we can get (on the basis of these estimates) the following useful information about the leray–Schauder degree of the operator associated with (1.1).

Let us define an operator $T_h : C_0^1[0,\pi_p] \to C_0^1[0,\pi_p]$ by $T_h(v) = u$ if and only if

$$(\varphi_p(u'))' = h - \lambda_1\varphi_p(v), u(0) = u(\pi_p) = 0.$$

Then, under the assumptions of Proposition 2.1, we have

$$\deg[I - T_h; B_R(0), 0] = 0, \tag{2.4}$$

while the assumptions of Proposition 2.2 imply

$$\deg[I - T_h; B_R(0), 0] = \pm 1, \tag{2.5}$$

where $+1$ holds if $p > 2$ and -1 holds if $1 < p < 2$.

The assertion of Theorem 1 follows then from (2.5). The assertion of Theorem 2 follows from (2.4) and from the stability of the value of the Leray–Schauder degree of $I - T_h$ with respect to perturbations of h in $L^\infty(0,\pi_p)$ norm. The multiplicity of the solution follows from (2.4) combined with the method of lower and upper solutions. The proof of Theorem 3 for $p > 2$ relies also on the method of lower and upper solutions. In the case $1 < p < 2$ we prove that the sequence $\{u_k\}_{k=1}^\infty \subset W_0^{1,p}(0,\pi_p)$, where $u_k = u_k(t)$ is a solution of the i.v.p.

$$(\varphi_p(u'))' + \lambda_1\varphi(u) = h, u(0) = 0,\ u'(0) = k, \tag{2.6}$$

for $t \in [0, t_1^k]$ (note that $t_1^k > \pi_p$ in the case $1 < p < 2$) and $u_k \equiv 0$ on $[t_1^k, \pi_p]$, satisfies $J(u_k) \to -\infty$ as $k \to \infty$. Finally, we show that for $p > 2$ the sequence $\{v_k\}_{k=1}^\infty \subset W_0^{1,p}(0,\pi_p)$ defined as $v_k(t) := u_k\left(\frac{t_1^k}{\pi_p}t\right)$, where u_k solves (2.6), with suitably extended h, satisfies $\|v_k\|_{W_0^{1,p}} \to \infty$, $J'(v_k) \to 0$ and $J(v_k) \to 0$.

Acknowledgement. This work has been supported by FONDAP de Matematicar Aplicadar, FONDECYT GRANTS 1970332 and 1970775, CHILE, and by Grant agency Czech Republic grant 201/97/0395.

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