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Semi-classical States for Nonlinear Schrödinger Equations

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We consider existence and asymptotic behavior of solutions for an equation of the form

$$\varepsilon^2 \Delta u - V(x)\, u + f(u) = 0, \qquad u > 0, \qquad u \in H_0^1(\Omega), \tag{$*$}$$

where Ω is a smooth domain in $\mathbb{R}^N$, not necessarily bounded. We assume that the potential V is positive and that it possesses a *topologically nontrivial* critical value c, characterized through a min–max scheme. The function f is assumed to be locally Hölder continuous having a subcritical, superlinear growth. Further we assume that f is such that the corresponding *limiting equation in* R^N has a unique solution, up to translations.

We prove that there exists ε_0 so that for all $0 < \varepsilon < \varepsilon_0$, Eq. ($*$) possesses a solution having exactly one maximum point $x_\varepsilon \in \Omega$, such that $V(x_\varepsilon) \to c$ and $\nabla V(x_\varepsilon) \to 0$ as $\varepsilon \to 0$.

0. INTRODUCTION

Let Ω be a domain in $\mathbb{R}^N$, not necessarily bounded, with smooth or empty boundary. This work deals with the problem of finding nontrivial, finite energy solutions to an equation of the form

$$\varepsilon^2 \Delta u - V(x)\, u + u^p = 0, \qquad u \in H_0^1(\Omega), \tag{0.1}$$

where $1 < p < (N+2)/(N-2)$. Equations of this form arise in different physical and biological models, where the presence of a small diffusion

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parameter ε becomes natural, in particular, in the study of *standing waves* of the nonlinear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V(x)\, \psi - \gamma\, |\psi|^{p-1}\, \psi. \tag{0.2}$$

Namely solutions of the form $\psi(x, t) = \exp(-iEt/\hbar)\, v(x)$ reduce to an equation like (0.1). See [11], [20].

Here we are concerned with the problem of finding a family of solutions u_ε which exhibits *concentration behavior* around a special point, namely, solutions with a spike shape, a single maximum point converging to a point located around a prescribed region, while vanishing as $\varepsilon \to 0$ everywhere else in Ω.

The study of single and multiple spike solutions to this and related problems has attracted considerable attention in recent years.

The first result in this line for the Schrödinger equation when $\Omega = \mathbb{R}^N$ seems due to Floer and Weinstein [11]. These authors construct such a concentrating family in the one-dimensional case via a Lyapunov-Schmidt reduction, around any nondegenerate critical point of the potential $V(x)$. Later Oh [20, 21, 22] extended this result to higher dimensions when $1 < p < (N+2)/(N-2)$, with potentials which exhibit "mild behavior at infinity," also constructing multiple-peaked solutions. Very recently, Ambrosetti, Badiale and Cingolani [1] partially lifted the nondegeneracy assumption, obtaining existence of a single peak solution when the potential has a local minimum or maximum with nondegenerate mth-derivative. The first result for equation (0.1) in $\mathbb{R}^N$ in the possibly degenerate setting seems due to Rabinowitz [23], see also Ding and Ni [10] for an independent related result. In [23] it was shown that if $\inf_{\mathbb{R}^N} V < \liminf_{|x| \to \infty} V(x)$, then the mountain-pass value for the associated energy functional provides a solution for all small ε. This solution indeed concentrates around a global minimum of V as $\varepsilon \to 0$, as shown later by X. Wang in [26]. Moreover, Wang observed that concentration of any family of solutions with uniformly bounded energy may occur only at critical points of V. See also a recent work by Wang and Zeng [27] where these ideas are extended to the case of competing potentials.

The work by the authors [5] seems to be the first attempt to attack the degenerate case in (0.1) in a local setting. Here the authors devised a penalization approach which permitted to find *local mountain passes* around a local minimum of V with arbitrary degeneracy. More precisely, given a bounded open set Λ such that

$$\inf_{\Lambda} V < \inf_{\partial \Lambda} V, \tag{0.3}$$

a family u_ε exhibiting a single spike in Λ, at a point x_ε such that $V(x_\varepsilon) \to \inf_\Lambda V$, is constructed.

In [8], see also [12], this approach was extended to the construction of a family of solutions with several spikes located around any prescribed finite set of local minima of V in the sense of (0.3).

The phenomena described above is connected with other concentration phenomena known in the literature for related elliptic equations. For instance, Ni and Takagi [17], [18] have characterized the mountain pass, or least energy solution to (0.1) in a bounded domain under Neumann boundary conditions and $V \equiv 1$, as a single spike located at the boundary, concentrating around a point where its mean curvature maximizes. Ni and Wei in [19] have considered the Dirichlet problem in a bounded domain when $V \equiv 1$ and found that the least energy solution concentrates around a global maximum of the distance to the boundary. Reciprocally, a strict local maximum of this function yields a concentrating family, see [28].

Another problem considered in the literature, that yields concentration behavior is the following

$$\Delta u + u^{(N+2)/(N-2)-\varepsilon} = 0, \qquad u \in H_0^1(\Omega)$$

with Ω bounded. From Rey [24], Han [13], and Wei [29], it is known that the least energy positive solution to this problem blows up as $\varepsilon \to 0$ at a single point which minimizes the Robin's function, that is, the diagonal of the regular part of the Green's function in Ω. Reciprocally, from the reduction method developed by Rey in [24], it follows that any non-degenerate critical point of that function determines a blowing-up family of solutions as $\varepsilon \to 0$. This reduction method has been extended by Bahri, Li and Rey in [2] to a functional which determines solutions with multiple blow-up.

Another related example is the Ginzburg-Landau equation in a bounded domain in $\mathbb{R}^2$,

$$\begin{aligned} \varepsilon^2 \Delta u + (1 - |u|^2)\, u &= 0 \qquad \text{in } \Omega \\ u &= g \qquad \text{on } \partial\Omega \end{aligned}$$

where $g\colon \partial\Omega \to S^1$ has degree $d > 0$. It was proven by Bethuel, Brézis and Hélein in [3], that if Ω is star-shaped, then the global minimizer of the associated energy converges smoothly to a harmonic map from Ω into S^1, away from d points, its singularities, all of them with degree 1. These d points happen to minimize globally a certain finite-dimensional functional called the *renormalized energy*. The star-shapeness assumption was lifted by Struwe in [25].

By a careful study of the associated heat flow, F. H. Lin in [14] proved that a nondegenerate local minimizer of this functional determines a family of solutions exhibiting asymptotic singularities at the corresponding points. The variational penalization method in [5], [8], was extended in [7], to show that actually at a possibly degenerate local minimizer of the renormalized energy, in the same sense as in (0.3), the same answer is true, with the additional information that the associated solutions are local minimizers of the energy. Recently F. H. Lin and T. C. Lin [15] have used the heat flow method to cover the case of a nondegenerate critical point of the energy.

In all examples quoted above, for which complete account of existing bibliography would be impossible here, a common pattern clearly shows: An *underlying finite dimensional energy* (potential V, mean curvature, distance to the boundary, Robin's function, renormalized energy) resembles or determines near some of its critical points the structure of the full energy functional near a concentrating family of solutions.

It seems that fully degenerate cases treated with global variational methods have only covered solutions that are in some sense locally least-energy, with the exception of [9]. On the other hand, local reduction methods like those originally developed by Floer and Weinstein and by Rey in different settings, rely in important ways on nondegeneracy of the finite-dimensional critical points, while capturing very precisely the features of the solutions in such cases. In all examples above, however, it is easy to produce situations where full degeneracy appears at, say, local saddle points of the underlying finite dimensional energy.

Of all above examples, it seems that the technically simplest case is the nonlinear Schrödinger equation, where the underlying energy appears explicitly as the potential V. This may be regarded in some sense as a model situation for the others, where more subtle finite dimensional objects are the key.

Our purpose in this paper is to show that the penalization method developed in [5], [6], [8], can be adapted to capture, via a global variational technique, families of solutions around any *topologically nontrivial* critical point of the potential V, with arbitrary degeneracy, a situation where the local reductions apparently do not apply directly.

To motivate what we mean by topological nontriviality, we observe that arbitrary critical points of V are not all candidates for concentration. For example it was shown by Wang in [26] that no solution of (0.1) exists if $\Omega = \mathbb{R}^N$ and V is nondecreasing and not identically constant in one direction, a situation which allows for many, in a sense *topologically trivial*, critical points. The type of critical points we will deal with are those that can be captured in a general way with a local min–max characterization.

In what follows we state precisely our assumption on V. First we state our only global assumption

(H0) V is of class C^1 and there exists $\alpha > 0$ such that

$$V(x) \geqslant \alpha \qquad \forall x \in \Omega. \tag{0.4}$$

Locally we consider the following setting. We assume that there is an open and bounded set Λ with smooth boundary such that $\bar{\Lambda} \subset \Omega$, and closed subsets of Λ, B, B_0 such that B is connected and $B_0 \subset B$. Let Γ be the class of all continuous functions $\phi: B \to \Lambda$ with the property that $\phi(y) = y$ for all $y \in B_0$. Define the min-max value c as

$$c = \inf_{\phi \in \Gamma} \sup_{y \in B} V(\phi(y)), \tag{0.5}$$

and assume additionally

(H1)

$$\sup_{y \in B_0} V(y) < c.$$

(H2) For all $\phi \in \Gamma$, $\phi(B) \cap \{y \in \Lambda \mid V(y) \geqslant c\} \neq \emptyset$.

We observe that in the standard language of calculus of variations, the sets B_0, B, $\{V \geqslant c\}$ *link* in Λ.

(H3) For all $y \in \partial\Lambda$ such that $V(y) = c$, one has $\partial_\tau V(y) \neq 0$, where ∂_τ denotes tangential derivative.

Standard deformation arguments show that these assumptions ensure that the min-max value c is a critical value for V in Λ, which is *topologically nontrivial*. In fact, assumption (H3) "seals" Λ so that the local linking structure described indeed provides critical points at the level c in Λ, possibly admitting full degeneracy.

It is not hard to check that all these assumptions are satisfied in a general local maximum, local minimum or saddle point situation. Our main result asserts that there is a family of solutions to problem (0.1) concentrating around a critical point at the level c in Λ.

In fact, with the aid of the penalization method developed in [5], we will find that the above min-max quantity for V inherits a min-max value for the energy associated to (0.1) which provides the desired solutions.

We should remark that existence of solutions for small ε in the case $\Omega = \mathbb{R}^N$, in a global saddle-point situation for V, has been recently considered by O. Miyagaki and the authors in [9].

Since we do not want to restrict ourselves to a situation where the nonlinearity in (0.1) is homogeneous, we will consider the more general problem

$$\begin{gathered} \varepsilon^2 \Delta u - V(x)\, u + f(u) = 0, \\ u > 0 \qquad \text{in } \Omega, \qquad u \in H_0^1(\Omega). \end{gathered} \tag{0.6}$$

We will assume that $f\colon \mathbb{R}^+ \to \mathbb{R}$ is locally Hölder continuous and satisfies the following conditions.

(f1) $f(\xi) = o(\xi)$ near $\xi \geqslant 0$.

(f2) $\lim_{\xi \to \infty} f(\xi)/\xi^p = 0$ for some $1 < p < (N+2)/(N-2)$.

(f3) For some $2 < \mu \leqslant p+1$ we have

$$0 < \mu F(\xi) \leqslant f(\xi)\, \xi \qquad \text{for all} \quad \xi > 0, \tag{0.7}$$

where $F(\xi) = \int_0^\xi f(\tau)\, d\tau$.

(f4) The function $\xi \to f(\xi)/\xi$ is increasing.

(f5) The *limiting functional* I_a, $a > 0$, defined as

$$I_a(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + av^2 - \int_{\mathbb{R}^N} F(v), \qquad v \in H^1(\mathbb{R}^N), \tag{0.8}$$

possesses a unique critical point, with critical value denoted by b^a.

Our main result for equation (0.6) is the following.

THEOREM 0.1. *Assume that hypotheses* (H0)–(H3) *and* (f1)–(f5) *hold. Then there is an* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$ *a positive solution* $u_\varepsilon \in H_0^1(\Omega)$ *to problem* (0.6) *exists. Moreover,* u_ε *possesses just one local* (*hence global*) *maximum point* x_ε, *which is in* Λ. *We also have that* $V(x_\varepsilon) \to c$, $\nabla V(x_\varepsilon) \to 0$, *where* c *is the min-max quantity given by* (0.5) *and*

$$u_\varepsilon(x) \leqslant \alpha \exp\left(-\frac{\beta}{\varepsilon} |x - x_\varepsilon| \right), \tag{0.9}$$

for certain constants α, β.

The rest of this paper will be devoted to the proof of this result. In Section 1 we define a modified functional which satisfies P.S. and, roughly speaking, permits us to restrict ourselves to what happens in Λ. We also define a min-max value and prove some preliminary lemmas. In Section 2 we conclude the proof of Theorem 0.1.

1. PRELIMINARY RESULTS

We begin with an important observation concerning the sets Λ, B and B_0. Let $\delta>0$ be an arbitrary, but fixed small number and $0<\varepsilon<\delta$. Then, with no loss of generality we may assume

$$\Lambda \subset \{x \mid V(x) > c-\delta\}, \tag{1.1}$$

$$B_0 \subset \{x \in \Lambda \mid V(x) = c(\varepsilon)\}, \qquad B \subset \{x \in \Lambda \mid V(x) \geqslant c(\varepsilon)\}, \tag{1.2}$$

where $c-\delta<c(\varepsilon)<c$, $\lim_{\varepsilon\to 0} c(\varepsilon)=c-\delta$ and $\operatorname{dist}(B_0, \partial\Lambda)=\varepsilon^{1/2}$. Additionally we may assume that

$$\partial_\tau V(x) \neq 0, \forall x \in \partial\Lambda \cap \{x \mid c-\delta < V(x) \leqslant c\}. \tag{1.3}$$

In fact, we may redefine

$$\Lambda_\delta = \Lambda \cap \{x \in \Lambda \mid V(x) > c-\delta\},$$

$$B^{\delta,\varepsilon} = B \cap \{x \in \Lambda \mid V(x) \geqslant c(\varepsilon)\},$$

$$B_0^{\delta,\varepsilon} = B \cap \{x \in \Lambda \mid V(x) = c(\varepsilon)\},$$

where the number $c(\varepsilon)$ is chosen as

$$c(\varepsilon) = \inf\{\zeta \mid \operatorname{dist}(\{x \in \Lambda \mid V(x)=\zeta\}, \Lambda_\delta) \geqslant \varepsilon^{1/2}\}.$$

Let us observe that $B_0^{\delta,\varepsilon}$ is non-empty, thanks to the connectedness of B.

Let $\varphi\colon B^{\delta,\varepsilon} \to \Lambda_\delta$ continuous, such that $\varphi(x)=x$ on $B_0^{\delta,\varepsilon}$. Let $\tilde\varphi$ be its extension as the identity to $B\backslash B^{\delta,\varepsilon}$. Then $\tilde\varphi\colon B\to\Lambda$, and $\sup_{x\in B} V(\tilde\varphi(x)) = \sup_{x\in B^{\delta,\varepsilon}} V(\tilde\varphi(x)) \geqslant c$.

In the framework of Theorem 0.1, let us consider a function $f\colon \mathbb{R}\to\mathbb{R}$ satisfying (f1)–(f4) on $\mathbb{R}^+$ and defined as zero for negative values.

Associated to equation (0.6) is the "energy" functional

$$E_\varepsilon(x) = \frac{1}{2}\int_\Omega \varepsilon^2 |\nabla u|^2 + V(x)\, u^2 - \int_\Omega F(u), \tag{1.4}$$

which is well defined for $u \in H$ where

$$H = \left\{ u \in H_0^1(\Omega) \Big/ \int_\Omega V(x)\, u^2 < \infty \right\}.$$

H is a Hilbert space, continuously embedded in $H_0^1(\Omega)$, when endowed with the inner product

$$\langle u, v\rangle = \int_\Omega \varepsilon^2 \nabla u \cdot \nabla v + V(x)\, uv \tag{1.5}$$

whose associated norm we denote by $\|\cdot\|_H$.

Under the regularity assumptions on V and f, it is standard to check that the nontrivial critical points of E_ε correspond exactly to the positive classical solutions in $H_0^1(\Omega)$ of equation (0.6).

As in [5], we will define a modification of this functional which satisfies the P.S. condition and for which we will find a critical point via an appropriate min-max scheme. This critical point will eventually be shown to be a solution of the original equation when ε is sufficiently small.

Let μ be a number as given by (f3), and let us choose $k>0$ such that $k>\mu/(\mu-2)$. Let $a>0$ be the value at which $f(a)/a=\alpha/k$, where α is as in (0.4). Let us set

$$\tilde{f}(s) = \begin{cases} f(s) & \text{if } s \leqslant a \\ \dfrac{\alpha}{k}\, s & \text{if } s > a, \end{cases} \tag{1.6}$$

and define

$$g(\cdot, s) = \chi_\Lambda f(s) + (1-\chi_\Lambda)\, \tilde{f}(s), \tag{1.7}$$

where Λ is a bounded domain as in the assumptions of Theorem 0.1, and χ_Λ denotes its characteristic function. Let us denote $G(x,\xi)=\int_0^\xi g(x,\tau)\, d\tau$, and consider the modified functional introduced in [5], defined on H as

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega \varepsilon^2\, |\nabla u|^2 + V(x)\, u^2 - \int_\Omega G(x,u), \qquad u \in H, \tag{1.8}$$

whose critical points correspond to solutions of the equation

$$\varepsilon^2 \Delta u - V(x)\, u + g(x,u) = 0 \qquad \text{in } \Omega. \tag{1.9}$$

It was shown in [5] that J_ε satisfies the Palais-Smale condition, no matter whether Ω is bounded or not. Note that this may not be the case for E_ε. We observe that a solution to (1.9) which satisfies that $u \leqslant a$ on $\Omega \setminus \Lambda$ will also be a solution of (0.1). We will define a min-max quantity for J_ε which will yield a solution to (1.9) that will eventually satisfy this property and thus will be the solution predicted by Theorem 0.1.

To this end we consider the *solution manifold* of equation (1.9) defined as

$$\mathcal{M}_\varepsilon = \left\{ u \in H \backslash \{0\} \;\middle|\; \int_\Omega \varepsilon^2 \, |\nabla u|^2 + V(x)\, u^2 = \int_\Omega g(x, u)\, u \right\}. \tag{1.10}$$

All nonzero critical points of J_ε of course lie on $\mathcal{M}_\varepsilon$; reciprocally, it is standard to check that critical points of J_ε constrained to this manifold are critical points of J_ε on H. We shall define a min-max quantity for the constrained functional. A useful fact which we will make use of, well known to be satisfied from the hypothesis (f4), is that $u \in \mathcal{M}_\varepsilon$ if and only if

$$J_\varepsilon(u) = \sup_{t>0} J_\varepsilon(tu). \tag{1.11}$$

It is useful to consider the limiting functionals I_a, defined (0.8), whose unique critical value can be characterized by

$$b^a = \inf_{v \neq 0} \sup_{t>0} I_a(tv). \tag{1.12}$$

It can be shown that b^a is a strictly increasing, continuous function of $a > 0$.

Associated to b^a there exists a radially symmetric critical point that is solution of the equation

$$\Delta w - aw + f(w) = 0, \qquad \text{in } \mathbb{R}^N. \tag{1.13}$$

Let us fix a small number $\delta_0 > 0$. For each $y \in \Omega$ with $\operatorname{dist}(y, \partial\Omega) > \delta_0$ we denote by w_ε^y the function in H given by

$$w_\varepsilon^y(x) = \eta(|x-y|/\delta_0)\, w_{V(y)}\left(\frac{y-x}{\varepsilon}\right), \tag{1.14}$$

where $\eta(s)$ is a smooth cut-off function which equals one for $0 < s < 1$ and zero for $s > 2$, and $w_{V(y)}$ is a solution of (1.13) with $a = V(y)$. Let B^ε, B_0^ε be the sets given in our assumptions, that satisfy additionally (1.1), (1.2) and (1.3), where we dropped the explicit mention δ for notational convenience. We consider the class Γ_ε of all continuous maps $\phi\colon B^\varepsilon \to \mathcal{M}_\varepsilon$ with the property that

$$\phi(y) = t(\varepsilon, y)\, w_\varepsilon^y, \qquad \forall y \in B_0^\varepsilon,$$

where $t(\varepsilon, y)>0$ is such that $t(\varepsilon, y)\, w_\varepsilon^y \in \mathcal{M}_\varepsilon$. Then we define the min-max value S_ε as follows:

$$S_\varepsilon = \inf_{\phi \in \Gamma_\varepsilon} \sup_{y \in B^\varepsilon} J_\varepsilon(\phi(y)). \tag{1.15}$$

Using an appropriate test function builded up upon w_ε^y, and taking in account that the values of V on B_0^ε equal $c(\varepsilon)$, we have from the definition of S_ε that

$$b^c \geqslant \limsup_{\varepsilon\to 0} \varepsilon^{-N} S_\varepsilon \geqslant \liminf_{\varepsilon \to 0} \varepsilon^{-N} S_\varepsilon \geqslant b^{c-\delta}. \tag{1.16}$$

The following is the key result of this section, which will lead to the fact that S_ε is a critical value for J_ε.

LEMMA 1.1.

$$\liminf_{\varepsilon\to 0} \varepsilon^{-N} S_\varepsilon > b^{c-\delta}. \tag{1.17}$$

Proof. Let us assume that (1.17) does not hold, i.e. that there is a sequence $\varepsilon_n \to 0$ such that

$$\varepsilon_n^{-N} S_{\varepsilon_n} \leqslant b^{c-\delta} + o(1).$$

We choose $\phi_n \in \Gamma_{\varepsilon_n}$ such that

$$\varepsilon_n^{-N} \sup_{y \in B^{\varepsilon_n}} J_{\varepsilon_n}(\phi_n(y)) \leqslant b^{c-\delta} + o(1). \tag{1.18}$$

We start by showing that $\phi_n(y)$ vanishes rapidly away from Λ in the L^2-sense, uniformly in y. More precisely, setting $\Lambda_n = \{x \mid \operatorname{dist}(x, \Lambda) < \varepsilon_n^{1/2}\}$, we are going to show that

$$\lim_{n\to\infty} \varepsilon_n^{-N} \sup_{y \in B^{\varepsilon_n}} \int_{\Omega\setminus\Lambda_n} \phi_n(y)^2 = 0. \tag{1.19}$$

Let $y_n \in B^{\varepsilon_n}$ and denote $u_n = \phi_n(y_n)$. Then since $u_n \in \mathcal{M}_{\varepsilon_n}$ we have that

$$J_{\varepsilon_n}(u_n) \geqslant J_{\varepsilon_n}(tu_n) \tag{1.20}$$

for any $t>0$. Let us set

$$E_n(v) = \tfrac{1}{2} \int_{\Lambda_n} \varepsilon^2\, |\nabla v|^2 + V(x)\, v^2 - \int_{\Lambda_n} G(x, v)\, dx.$$

Let us choose the number $t_n > 0$ so that

$$E_n(t_n u_n) = \max_{t > 0} E_n(t u_n). \tag{1.21}$$

Then from (1.18), (1.20) with $t = t_n$, and the fact that for a certain $\gamma > 0$

$$\frac{V(x)}{2} s^2 - G(s, x) \geqslant \gamma s^2 \; \forall x \in \Omega \backslash \Lambda, \; s > 0,$$

we obtain

$$E_n(t_n u_n) + \gamma t_n^2 \int_{\Omega \backslash \Lambda_n} u_n^2 \leqslant \varepsilon_n^N (b^{c-\delta} + o(1)), \tag{1.22}$$

where the term $o(1)$ goes to zero, uniformly on $\{y_n\}$. Next we claim that there is $\sigma_0 > 0$, independent of $\{y_n\}$, such that

$$t_n \geqslant \sigma_0, \qquad \text{for all} \quad n \in \mathbb{N}. \tag{1.23}$$

For this purpose we first see that there is a $C > 0$, independent of $\{y_n\}$, such that

$$\int_\Omega \varepsilon_n^2 \, |\nabla u_n|^2 + u_n^2 \leqslant C \varepsilon_n^N. \tag{1.24}$$

In fact, from assumption (f3) and the definition of g we have that $\mu G(x, s) \leqslant g(x, s) s$ with $\mu > 2$. Then, since $J_{\varepsilon_n}(u_n) \leqslant C \varepsilon_n^N$ and $u_n \in \mathscr{M}_{\varepsilon_n}$, (1.24) follows.

Next, we set $v_n(z) = t_n u_n(\varepsilon_n z)$ and $\tilde{\Lambda}_n = \varepsilon_n^{-1} \Lambda_n$. Then, the definition of t_n and hypotheses (f1) and (f2) yield that

$$\int_{\tilde{\Lambda}_n} |\nabla v_n|^2 + V(\varepsilon_n z) \, v_n^2 = \int_{\tilde{\Lambda}_n} g(\varepsilon_n z, v_n) \, v_n \, dz \leqslant \int_{\tilde{\Lambda}_n} C v_n^{p+1} + \rho v_n^2, \tag{1.25}$$

where $\rho > 0$ can be taken arbitrarily small. Now, Sobolev's embedding Theorem yields that

$$\int_{\tilde{\Lambda}_n} v_n^{p+1} \leqslant \bar{C} \left(\int_{\tilde{\Lambda}_n} |\nabla v_n|^2 + v_n^2 \right)^{(p+1)/2}.$$

Here $\bar{C}$ may be chosen to be the same for all $\tilde{\Lambda}_n$'s. In fact, the domain Λ can be assumed, with no loss of generality, to be Lipschitz, and the constant in Sobolev's embedding depends on a uniform cone condition for the domain but not on its volume. This inequality combined with (1.25) yields that

$$\int_{\tilde{\Lambda}_n} v_n^{p+1} \geqslant \sigma > 0, \tag{1.26}$$

hence $\int_{\tilde{\Lambda}_n} |\nabla v_n|^2 + v_n^2 \geqslant \sigma > 0$, with σ independent of $\{y_n\}$, and so that

$$\left(t_n^2 \int_{\Lambda_n} \varepsilon_n^2 |\nabla u_n|^2 + u_n^2 \right) \geqslant \sigma \varepsilon_n^N.$$

This and (1.24) imply the validity of (1.23), and the claim is proved.

Now, by definition of t_n, we have that

$$E_n(t_n u_n) \geqslant \inf_{u \in H^1(\Lambda_n) \setminus \{0\}} \sup_{t>0} E_n(tu) \equiv b_n, \tag{1.27}$$

where b_n corresponds to the mountain pass value for E_n in $H^1(\Lambda_n)$. Since the least value of $V(x)$ in Λ_n approaches $c - \delta$, we have

$$\lim_{n \to \infty} b_n = b^{c-\delta}. \tag{1.28}$$

The proof of this fact follows from that of Lemma 1.3 in [8], with minor changes. Now, combining (1.28) with (1.27), (1.23) and (1.22) we obtain the validity of (1.19).

Next we consider the *center of mass* of a nonzero function u in $L^2(\Omega)$ to be the quantity

$$\beta(u) = \frac{\int_{\Lambda^+} x u^2(x)\, dx}{\int_\Omega u^2(x)\, dx},$$

where Λ^+ is an arbitrary, but fixed small neighborhood of $\bar{\Lambda}$. We assume that $\delta_0 < \operatorname{dist}(\partial \Lambda^+, \bar{\Lambda})$, where δ_0 is given in (1.14). We claim next that for all large n we have,

$$\beta(\phi_n(y)) \in \Lambda^+ \cap \left\{ x \mid V(x) \leqslant c - \frac{\delta}{2} \right\} \qquad \forall y \in B^{\varepsilon_n}. \tag{1.29}$$

This immediately yields the desired contradiction. Indeed, let $\varphi_n(y) = \pi(\beta(\phi_n(y)))$ where $\pi\colon \Lambda^+ \to \Lambda$ is a continuous map which equals the identity on Λ. Note that since $\phi_n(y) = w_{\varepsilon_n}^y$ for $y \in B_0^{\varepsilon_n}$ and this function is radially symmetric around y, it follows that $\varphi_n(y) = y$ for $y \in B_0^{\varepsilon_n}$. Thus φ_n is in the class Γ appearing in the definition of the number c in (0.5), hence (H2) implies $c \leqslant \sup_{y \in B^{\varepsilon_n}} V(\varphi_n(y))$. This contradicts (1.29) for sufficiently large n, finishing the proof.

Thus, it only remains to establish the validity of (1.29). Assume that this relation is not true, namely that, passing to a subsequence, one has the existence of $y_n \in B^{\varepsilon_n}$ such that

$$\beta(\phi_n(y_n)) \notin \Lambda^+ \cap \left\{ x \mid V(x) \leqslant c - \frac{\delta}{2} \right\}. \tag{1.30}$$

Then, setting $u_n \equiv \phi_n(y_n)$ and $v_n(z) = u_n(\varepsilon_n z)$, we have that the following inequality holds

$$\sup_{t>0} I_{c-\delta}(tv_n) \leqslant b^{c-\delta} + o(1). \tag{1.31}$$

To see this, we review first some facts about the sequence v_n. We already know that v_n is bounded in H^1-norm. Since we also have

$$\int_{\tilde{\Omega}_n} |\nabla v_n|^2 + V(\varepsilon_n z)\, v_n^2 = \int_{\tilde{\Omega}_n} g(\varepsilon_n z, v_n)\, v_n \leqslant \int_{\tilde{\Omega}_n} f(v_n)\, v_n,$$

the same argument leading to (1.26) yields that

$$\int_{\tilde{\Omega}_n} v_n^{p+1} \geqslant \sigma > 0. \tag{1.32}$$

Here $\tilde{\Omega}_n = \varepsilon_n^{-1} \Omega$. Hence there is a sequence B_n of balls of radius one such that

$$\int_{B_n} v_n^2 \geqslant \sigma > 0. \tag{1.33}$$

In fact, otherwise we would get, from the H^1-boundedness, and the concentration compactness principle (see Lemma I.1 in [16] or Lemma 2.18 in [4]), that $v_n \to 0$ in any L^q with $2 < q < 2N/(N-2)$, contradicting (1.32).

Now, let us select $t_n > 0$ such that $I_{c-\delta}(t_n v_n) = \sup_{t>0} I_{c-\delta}(tv_n)$. Since v_n is bounded in H^1-norm we obtain

$$Ct_n^2 - \int_{\tilde{\Omega}_n} F(t_n v_n) \geqslant I_{c-\delta}(t_n v_n) \geqslant b^{c-\delta}.$$

But from assumption (f3) we have $F(s) \geqslant \delta s^{\mu}$, with $2 < \mu < 2N/(N-2)$, then

$$t_n^{\mu-2} \int_{\tilde{\Omega}_n} v_n^{\mu} \leqslant C. \tag{1.34}$$

This and (1.33) imply that t_n is bounded. Then, from (1.19) we have

$$\int_{\mathbb{R}^N \setminus \tilde{\Lambda}_n^+} (t_n v_n)^2 \to 0, \tag{1.35}$$

where $\tilde{\Lambda}_n^+ = \varepsilon_n^{-1} \Lambda^+$. Now, from (1.18) we have

$$b^{c-\delta} + o(1) \geqslant \varepsilon_n^{-N} J_{\varepsilon_n}(t_n u_n) \geqslant I_{c-\delta}(t_n v_n) - \frac{t_n^2}{2} \int_{\mathbb{R}^N \setminus \tilde{\Lambda}_n^+} (c - \delta + o(1))\, v_n^2,$$

and then, from (1.35) we obtain (1.31).

Following with the proof of (1.29), we set $w_n \equiv t_n v_n$, with t_n as above. From the definition of t_n, w_n belongs to the solution manifold of $I_{c-\delta}$, then it follows from (1.31) that w_n is a minimizing sequence of $I_{c-\delta}$ there. A standard application of Ekeland's variational principle, thus yields the existence of a Palais-Smale sequence $\tilde{w}_n$ of $I_{c-\delta}$ such that $w_n - \tilde{w}_n \to 0$ in the $H^1(\mathbb{R}^N)$-sense.

Thus there exists a sequence of points z_n such that $w_n(\cdot - z_n)$ converges in the H^1-sense to a solution w of equation (1.13) with $a = c - \delta$, radially symmetric with respect to the origin.

Let $\bar{y}_n = \varepsilon_n z_n$. Since w_n tends to zero away from $\tilde{\Lambda}_n$, we may assume, passing to a subsequence, that $\bar{y}_n \to \bar{y}$ in $\bar{\Lambda}$. Since

$$b^{c-\delta} \geqslant \lim_{n \to \infty} \varepsilon_n^{-N} J_{\varepsilon_n}(t_n u_n) = I_{V(\bar{y})}(w),$$

we have $b^{c-\delta} \geqslant b^{V(\bar{y})}$, so that $V(\bar{y}) \leqslant c - \delta$. But $\beta(u_n) \to \bar{y} \in \bar{\Lambda}$, thus (1.30) implies $V(\bar{y}) > c - \delta/2$. We have obtained a contradiction which proves (1.29), thus finishing the proof of the lemma. ∎

The above lemma shows that

$$\varepsilon^{-N} S_\varepsilon \geqslant b^{c-\delta} + \rho,$$

for some $\rho > 0$ and all ε sufficiently small. On the other hand, for all $\phi \in \Gamma_\varepsilon$ we have $\phi(y) = w_\varepsilon^y$ for $y \in B_0^\varepsilon$. We immediately check that for all small ε,

$$\sup_{y \in B_0^\varepsilon} \varepsilon^{-N} J_\varepsilon(\phi(y)) \leqslant b^{c-\delta} + \frac{\rho}{2}, \qquad \forall \phi \in \Gamma_\varepsilon.$$

These two facts, the validity of the Palais-Smale condition and a standard deformation argument yields the main result of this section, namely

PROPOSITION 1.1. *The number S_ε defined by* (1.15) *is a critical value of J_ε, namely there is a solution $u_\varepsilon \in H$ to the equation* (1.9) *such that $J_\varepsilon(u_\varepsilon) = S_\varepsilon$ for all ε sufficiently small.*

In the next section we will show that actually u_ε turns out to be a critical point of the original functional E_ε provided that ε is chosen small enough.

2. PROOF OF THEOREM 0.1

We will show that the solution u_ε to equation (1.9) constructed in Proposition 1.1 is a solution of (0.6). The key step for that is the following.

PROPOSITION 2.1. *If m_ε is given by*

$$m_\varepsilon = \max_{x \in \partial \Lambda} u_\varepsilon(x), \tag{2.1}$$

then

$$\lim_{\varepsilon \to 0} m_\varepsilon = 0. \tag{2.2}$$

Moreover, for all ε sufficiently small, u_ε possesses at most one local maximum $x_\varepsilon \in \Lambda$ and we must have

$$c - \delta < \liminf_{\varepsilon \to o} V(x_\varepsilon) \leqslant \limsup_{\varepsilon \to o} V(x_\varepsilon) \leqslant c. \tag{2.3}$$

Before proving the proposition, let us see how Theorem 0.1 follows from it.

Proof of Theorem 0.1. The fact that u_ε solves (0.6) for small ε follows from Proposition 2.1 in the same way as in [5], where the case of a minimum of V was treated. We recall here the argument for completeness. There exists ε_0 such that for all $0 < \varepsilon < \varepsilon_0$,

$$u_\varepsilon(x) < a \qquad \text{for all} \quad x \in \partial \Lambda. \tag{2.4}$$

The function $u_\varepsilon \in H$ solves the equation

$$\varepsilon^2 \Delta u - V(x)\, u + g(x, u) = 0 \qquad \text{in } \Omega. \tag{2.5}$$

Since $u_\varepsilon \in H_0^1(\Omega)$ we can choose $(u_\varepsilon - a)_+$ as a test function in (2.5) so that, after integration by parts one gets

$$\int_{\Omega \setminus \Lambda} \varepsilon^2 \, |\nabla (u_\varepsilon - a)_+|^2 + c(x)(u_\varepsilon - a)_+^2 + c(x)\, a(u_\varepsilon - a)_+ = 0, \tag{2.6}$$

where

$$c(x) = V(x) - \frac{g(x, u_\varepsilon(x))}{u_\varepsilon(x)}. \tag{2.7}$$

The definition of g yields that $c(x) > 0$ in $\Omega \backslash \Lambda$, hence all terms in (2.6) are zero. We conclude in particular

$$u_\varepsilon(x) \leqslant a \qquad \text{for all} \quad x \in \Omega \backslash \Lambda.$$

Consequently u_ε is a solution to equation (0.6) for all small ε. The construction of the family of solutions u_ε depends on the particular $\delta > 0$ chosen at the beginning of Section 1. To emphasize this fact we denote this family as u_ε^δ. Let δ_j be any sequence of positive numbers such that $\delta_j \to 0$. Then there is a decreasing sequence of positive numbers $\varepsilon_j \to 0$ such that for all $0 < \varepsilon < \varepsilon_j$ one has that $u_\varepsilon^{\delta_j}$ solves (0.6), it has just one local maximum x_j^ε located in Λ and

$$c - \delta_j < V(x_j^\varepsilon) < c + \frac{1}{j}.$$

We define $u_\varepsilon = u_\varepsilon^{\delta_j}$ and $x_\varepsilon = x_j^\varepsilon$, if $\varepsilon_j \geqslant \varepsilon > \varepsilon_{j+1}$. Then we clearly have $V(x_\varepsilon) \to c$. Moreover $\nabla V(x_\varepsilon) \to 0$, as can be seen by using a result of [26]. This result applies since u_ε is a critical point of E_ε, $E_\varepsilon(u_\varepsilon) \leqslant C\varepsilon^N$ and $x_{\varepsilon_n} \to \bar{x} \in \Omega$. Finally, the exponential decay assertion, (0.9) follows from exactly the same argument provided in [5], thus concluding the proof of the theorem. ∎

It remains to prove Proposition 2.1. We do this next.

Proof of Proposition 2.1. We begin by establishing the following auxiliary fact: If $\varepsilon_n \downarrow 0$ and $x_n \in \bar{\Lambda}$ are such that $u_{\varepsilon_n}(x_n) \geqslant \gamma > 0$, then

$$\limsup_{n \to \infty} V(x_n) \leqslant c. \tag{2.8}$$

We assume, for contradiction, that passing to a subsequence, $x_n \to \bar{x} \in \bar{\Lambda}$ and

$$V(\bar{x}) > c. \tag{2.9}$$

We consider the sequence

$$v_n(z) = u_{\varepsilon_n}(x_n + \varepsilon_n z) \tag{2.10}$$

and study its behavior as n goes to infinity. The function v_n satisfies the equation

$$\begin{cases} \Delta v_n - V(x_n + \varepsilon_n z)\, v_n + g(x_n + \varepsilon_n z, v_n) = 0 & \text{in } \Omega_n \\ v_n = 0 & \text{on } \partial\Omega_n, \end{cases} \tag{2.11}$$

where $\Omega_n = \varepsilon_n^{-1}\{\Omega - x_n\}$. We see that v_n is bounded in $H^1(\mathbb{R}^N)$, and from elliptic estimates it can be assumed to converge uniformly on compacts subsets of $\mathbb{R}^N$ to a nontrivial function $v \in H^1(\mathbb{R}^N)$. The sequence of functions $\chi_n(z) \equiv \chi_\Lambda(x_n + \varepsilon_n z)$ can also be assumed to converge weakly in any L^p over compacts subsets of $\mathbb{R}^N$ to a function χ with $0 \leqslant \chi \leqslant 1$. Therefore, v satisfies the limiting equation

$$\Delta v - V(\bar{x})\, v + \bar{g}(z, v) = 0 \qquad \text{in } \mathbb{R}^N, \tag{2.12}$$

where

$$\bar{g}(z, s) = \chi(z)\, f(s) + (1 - \chi(z))\, \tilde{f}(s). \tag{2.13}$$

Thus v is a nontrivial critical point of the functional $\bar{I}$: $H^1(\mathbb{R}^N) \to \mathbb{R}$ defined as

$$\bar{I}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(\bar{z})\, w^2 - \int_{\mathbb{R}^N} \bar{G}(z, u),\ u \in H^1(\mathbb{R}^N), \tag{2.14}$$

where $\bar{G}(z, s) = \int_0^s \bar{g}(z, \tau)\, d\tau$. Then we are exactly in the same setting as in Lemma 2.2 of [5], so that we have the estimate

$$\liminf_{n \to \infty} \varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}) \geqslant \bar{I}(v).$$

This and (1.16) imply then that $\bar{I}(v) \leqslant b^c$. But, $\bar{I}(u) \geqslant I_{V(\bar{x})}(u)$ for all $u \in H^1(\mathbb{R}^N)$. Then, using that v is a critical point of $\bar{I}$, we have that

$$b^c \geqslant \bar{I}(v) = \max_{\tau \geqslant 0} \bar{I}(\tau v) \geqslant \max_{\tau \geqslant 0} I_{V(\bar{x})}(\tau v) \geqslant b^{V(\bar{x})}. \tag{2.15}$$

It follows that $V(\bar{z}) \leqslant c$, which certainly contradicts (2.9) and the proof of (2.8), is thus complete.

Now we will establish the validity of assertion (2.2). We assume, by contradiction, that there is a sequence of points $x_n \in \partial\Lambda$, with $x_n \to \bar{x} \in \partial\Lambda$ such that

$$u_{\varepsilon_n}(x_n) \geqslant \gamma > 0.$$

We claim that $V(\bar{x}) > c - \delta$. To prove this we first recall that from (1.16) Lemma 1.1,

$$b^c \geqslant \liminf_{n \to \infty} \varepsilon_n^{-N} J_{\varepsilon_n}(u_n) > b^{c-\delta}. \tag{2.16}$$

Defining v_n and v as above, we see that v_n cannot converge strongly in the H^1-sense to v. In fact, otherwise $\liminf_{n \to \infty} \varepsilon_n^{-N} J_{\varepsilon_n}(u_n) = \bar{I}(v)$, where $\bar{I}$ was defined in (2.14). After a rotation, we see that since $x_n \in \partial \Lambda$, v will be a nontrivial solution of the equation

$$\Delta v - (c - \delta)\, v + \chi_{\{z_1 < 0\}} f(v) + \chi_{\{z_1 > 0\}} \tilde{f}(v) = 0, \tag{2.17}$$

where $b = V(\bar{x}) = c - \delta > 0$. It was proven in [8], Lemma 2.3 that v actually solves

$$\Delta v - (c - \delta)\, v + f(v) = 0. \tag{2.18}$$

Then $\bar{I}(v) = b^{c-\delta}$ and (2.16) would be impossible. Thus a concentration-compactness argument yields that v_n must concentrate on balls $B_1(z_n)$, where $|z_n| \to \infty$. Similarly as in the proof of Proposition 2.1 in [5], we would end up with at least two concentration regions, so that $\liminf_{n \to \infty} \varepsilon_n^{-N} J_{\varepsilon_n}(u_n) \geqslant 2b^{c-\delta}$. Thus $2b^{c-\delta} \leqslant b^c$ which is impossible if δ was taken sufficiently small. Thus the claim is proved. Observe that we have also established that $v_n \to v$ strongly in $H^1(\mathbb{R}^N)$.

We may assume that δ given at the beginning of Section 1 is so small that our $\bar{x}$ lies in a region where $\partial \Lambda$ is smooth and $\nabla_T V(\bar{x}) \neq 0$. However, we will show next that this is impossible since our indirect assumption of concentration on the boundary actually implies $\nabla_T V(\bar{x}) = 0$.

Let us prove this. With no loss of generality, we may assume that $\bar{x} = 0$ and that in a small neighborhood $B(0, 2\rho)$ the domain Λ can be described as

$$\Lambda \cap B(0, 2\rho) = \{(x', x_N) \in B(0, 2\rho) \mid x' \in \mathbb{R}^{N-1}, x_N < \psi(x')\},$$

where ψ is a smooth function such that, $\psi(0) = 0$, $\nabla \psi(0) = 0$. In this setting the function $v_n(z)$ satisfies

$$\Delta v_n - V(x_n + \varepsilon_n z)\, v_n + \chi_{\{z_N > \varepsilon^{-1} \psi(\varepsilon z')\}} f(v_n) + \chi_{\{z_N < \varepsilon^{-1} \psi(\varepsilon z')\}} \tilde{f}(v_n) = 0 \tag{2.19}$$

in $B(0, \rho/\varepsilon_n)$. Now, the fact that $v_n \to v$ in H^1, implies that v_n will be uniformly small outside of a ball $B(0, R)$, hence it satisfies an equation of the form

$$\Delta v_n - a_n(z)\, v_n = 0 \qquad \text{in} \quad \Omega_n \setminus B(0, R),$$

where $0 < \gamma_1 \leqslant a_n(z) \leqslant \gamma_2$, and $v_n = 0$ on $\partial\Omega_n$. Thus a comparison argument yields that v_n and its derivatives decay exponentially, say

$$v_n(z) + |\nabla v_n(z)| \leqslant Ce^{-\beta |z|}, \tag{2.20}$$

for some positive numbers C, β. Let $1 \leqslant i \leqslant N-1$. Multiplying (2.19) by $\partial v_n / \partial z_i$ and integrating by parts we obtain

$$\int_{|z| = \rho/\varepsilon_n} \frac{\partial v_n}{\partial r} \frac{\partial v_n}{\partial z_i} \, d\sigma - \int_{|z| = \rho/\varepsilon_n} |\nabla v_n|^2 \, v_i \, d\sigma - \int_{|z| < \rho/\varepsilon_n} V(x_n + \varepsilon_n z) \frac{\partial}{\partial z_i} \frac{v_n^2}{2}$$
$$+ \int_{|z| < \rho/\varepsilon_n} \chi_{\{z_N > \varepsilon^{-1}\psi(\varepsilon z')\}} \frac{\partial}{\partial z_i} F(v_n) + \chi_{\{z_N < \varepsilon^{-1}\psi(\varepsilon z')\}} \frac{\partial}{\partial z_i} \tilde{F}(v_n) = 0. \tag{2.21}$$

Here and in what follows

$$v_i = \frac{1}{\sqrt{1 + |\nabla\psi|^2 \, (\varepsilon z')}} \frac{\partial \psi}{\partial x_i} (\varepsilon z').$$

Integrating by parts once again and using (2.20) we see that

$$\int_{D_n} (F(v_n) - \tilde{F}(v_n)) \, v_i \, d\sigma = \varepsilon_n \int_{|z| < \rho/\varepsilon} V_{x_i}(x_n + \varepsilon_n z) \frac{v_n^2}{2} + O(e^{-\beta/\varepsilon_n}), \tag{2.22}$$

where $D_n = \{z_N = \varepsilon_n^{-1} \psi(\varepsilon_n z'), |z| < \rho/\varepsilon_n\}$. Then, dividing (2.22) through ε_n and letting $n \to \infty$ we obtain

$$\int_{\mathbb{R}^{N-1}} (F(v) - \tilde{F}(v)) \, z' \cdot \mathbf{b} \, dz' = V_{x_i}(\bar{x}) \int_{\mathbb{R}^N} \frac{v^2}{2}, \tag{2.23}$$

where $\mathbf{b} = \partial \nabla\psi / \partial x_i(0)$. But from Lemma 2.2 in [7] we actually have $F(v) = \tilde{F}(v)$ so we get $V_{x_i}(\bar{x}) = 0$. This can be done for $i = 1, ..., N-1$, so that $\nabla_T V(\bar{x}) = 0$, as desired. This finishes the proof of $m_\varepsilon \to 0$.

Finally, to prove that u_ε possesses at most one local maximum in Λ, we can follow the arguments given in Proposition 2.1 of [5]. The proof is thus complete. ∎

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