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Local minimizers for the Ginzburg-Landau energy

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0 Introduction

Let Ω be a bounded, smooth domain in $\mathbb{R}^2$. We consider the Ginzburg-Landau system,

$$\begin{aligned} -\Delta u &= \frac{1}{\varepsilon^2}(1-|u|^2)u \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega \end{aligned} \tag{0.1}$$

where $u : \overline{\Omega} \to \mathbb{R}^2$, $g : \partial\Omega \to S^1$ is smooth and $\varepsilon > 0$ is a parameter. Associated to problem (0.1) is the energy functional

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (1-|u|^2)^2, \tag{0.2}$$

whose critical points in $H^1_g = \{u \in H^1(\Omega, \mathbb{R}^2) \;/\; u|_{\partial\Omega} = g\}$ correspond precisely to solutions of problem (0.1). The asymptotic behavior of these solutions as $\varepsilon \to 0$ has attracted considerable interest in recent years. Bethuel, Brezis and Hélein in [1] have described in detail the limit as $\varepsilon \to 0$ of the global minimizers of (0.2) in the case Ω starshaped. We describe next their result.

Let $\{\varepsilon_n\}$ be any sequence approaching zero, and u_n be a global minimizer of (0.2) for $\varepsilon = \varepsilon_n$. Then $\{u_n\}$ has a subsequence which has a limit u^*, which in complex form becomes

$$u^*(x) = \frac{(x-a_1)}{|x-a_1|}\cdots\frac{(x-a_d)}{|x-a_d|}e^{i\psi(x)}, \tag{0.3}$$

where ψ is real valued, $\Delta\psi = 0$ in Ω and $u^* = g$ on $\partial\Omega$. Here $a_1, \ldots, a_d \in \Omega$ and $d = \deg(g, \partial\Omega)$ is assumed non negative, without loss of generality.

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The convergence is smooth in compact subsets of $\overline{\Omega} \setminus \{a_1, \dots, a_d\}$. Moreover, the singularities $a_1, \dots, a_d$ correspond to global minimizers of the renormalized energy functional, $W : \Omega^d \to \mathbb{R}$, defined as

$$W(a_1, \dots, a_d) = -\pi \sum_{i \neq j} \log |\, a_i - a_j \,| + \frac{1}{2} \int_{\partial\Omega} \Phi(g \times g_\tau) - \pi \sum_{i=1}^{d} R(a_i). \quad (0.4)$$

Here Φ solves the equation

$$\begin{aligned} \Delta\Phi &= 2\pi \sum_{i=1}^{d} \delta_{a_i} \quad \text{in} \quad \Omega, \\ \frac{\partial\Phi}{\partial n} &= g \times g_\tau \text{ on } \partial\Omega. \end{aligned} \quad (0.5)$$

The function Φ is unique up to a constant. We normalize it taking Φ with mean value zero. R is the harmonic function $R(x) = \Phi(x) - \sum_{i=1}^{d} \log |\, x - a_i \,|$.

The assumption of star-shapedness of Ω is used in an application of Pohozaev identity in order to show that

$$\frac{1}{4\varepsilon^2} \int_\Omega (1 - |\, u \,|^2)^2 \leq C < +\infty \quad (0.6)$$

at any critical point u of (0.2). This estimate is crucial in obtaining the convergence results, and does not necessarily hold at arbitrary critical points of E_ε if starshapedness is violated, as shown via an example in [1]. However this estimate is still true for the minimizers in the case of an arbitrary domain, as established by Struwe in [11] and [12]. More generally, an estimate of the form

$$E_\varepsilon(u_\varepsilon) \leq k \log \frac{1}{\varepsilon}, \quad (0.7)$$

for arbitrary critical points still suffices, as shown by Lin in [8].

In view of the results in [1], it is natural to ask whether one can find families of solutions to (0.1) with asymptotic singularities $\{a_1, \dots, a_d\}$ located at other critical points of W. This question has been studied by Lin in [8], showing that the answer is affirmative in the case of a nondegenerate local minimum of W. His approach also covers some degenerate cases, see Section 4 in [8]. For example, if the minimum set is a k-dimensional manifold $1 \leq k \leq d-1$ and W is nondegenerate on the normal space at each point on the manifold, then there is a family of solutions with asymptotic singularities located at some point of it.

Lin's method is based on the study of the heat flow associated to (0.1),

$$\frac{\partial u}{\partial t} = \Delta u + \frac{1}{\varepsilon^2}(1 - |\, u \,|^2)u \quad \text{in } (0,\infty) \times \Omega$$

$$u(0,x) = U_\varepsilon(x) \quad \text{in } \Omega, \qquad u(t,\cdot) = g \quad \text{on } \partial\Omega.$$

In fact, for a suitable choice of the initial data U_ε it is established in [8] that $u(t,x)$ approaches, as $t \to \infty$, a steady state with the desired characteristics as

$\varepsilon \to 0$. Here, a delicate issue is the interchange of the limits $\varepsilon \to 0$ and $t \to \infty$, something not possible in general.

In this paper we revisit this question via a direct variational approach, which permits to cover the case of local minima with arbitrary degeneracy. More precisely we prove the following

Theorem 0.1 *Let Λ be an open subset of Ω^d such that $\overline{\Lambda} \subset \Omega^d$ and*

$$\inf_{\Lambda} W < \inf_{\partial\Lambda} W, \tag{0.8}$$

where W is given by (0.4). Then, any sequence $\varepsilon_n \to 0$ possesses a subsequence which we still denote ε_n, such that there is a family of solutions u_{ε_n} to (0.1) and a point $(a_1, \dots, a_d) \in \Lambda$ with

$$W(a_1, \dots, a_d) = \inf_{\Lambda} W, \tag{0.9}$$

such that $u_{\varepsilon_n} \to u^*$ uniformly and in the H^1 sense on compact subsets of $\overline{\Omega} \setminus \{a_1, \dots, a_d\}$. Moreover, u_{ε_n} is a local minimizer of E_{ε_n}.

It should be noticed that in the above situation, as stated in Section 4 in [8], Lin's method still provides solutions with asymptotic singularities in Λ. However it does not seem to be known if they satisfy (0.9), nor that they are local minimizers of E_{ε_n}.

The approach we will present also applies to the weighted Ginzburg-Landau equation

$$\begin{aligned} -\Delta u &= \frac{w(x)}{\varepsilon^2}(1- \mid u \mid^2)u \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{aligned} \tag{0.10}$$

where $0 < \alpha \leq w(x) \leq \beta$; c.f. open problem 2 in [1].

Thus we will also consider the problem of finding local minimizers of the associated energy

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \mid \nabla u \mid^2 + \frac{1}{4\varepsilon^2}\int_\Omega w(x)(1- \mid u \mid^2)^2, \tag{0.11}$$

$u \in H^1_g$. In fact, an estimate like (0.6), which is, as mentioned, crucial in the convergence results, is obtained in [1] when $w \equiv 1$ by a direct application of Pohozaev identity if the domain is starshaped, or by reducing this application to a suitable family of balls in the general case in [12]. This is not possible if, for example, w is not differentiable. Concentration of global minimizers in the case of a smooth weight has been studied in [10] and [7].

Estimate (0.6) still holds true however, for global minimizers of (0.11) under the only condition $0 < \alpha \leq w(x) \leq \beta$, without any regularity requirement. This follows from a standard upper estimate for the minimal energy and the following fact, which is actually a consequence of the results in [12].

Proposition 0.1 *Let* $\{u_\varepsilon\}$ *be a family of functions in* H^1_g, *bounded in* $L^2(\Omega)$ *and satisfying*

$$\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{\alpha}{4\varepsilon^2}\int_\Omega (|u_\varepsilon|^2 - 1)^2 \le \pi d \log\frac{1}{\varepsilon} + C_0, \tag{0.12}$$

where $\alpha > 0$. *Then there exists a constant* $C = C(\Omega, g, \alpha, C_0)$ *such that*

$$\frac{1}{\varepsilon^2}\int_\Omega (|u_\varepsilon|^2 - 1)^2 \le C \qquad \text{for all } \varepsilon > 0. \tag{0.13}$$

In fact, it follows from the results in [12] and [1] the validity of an estimate of the form

$$\pi d \log\frac{1}{\varepsilon} - C \le \inf_{v\in H^1_g} \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{\alpha}{8\varepsilon^2}\int_\Omega (|v|^2 - 1)^2, \tag{0.14}$$

for a certain constant C independent of ε. The conclusion of the proposition then follows from combining (0.14) and assumption (0.12). A different proof of Proposition 0.1, directly based on estimate (0.14) for a disk, can be found in [6].

Observe that u_ε is not assumed to be a critical point of the associated energy. This is important in our method of proof of Theorem 0.1. Moreover, it helps to extend Theorem 0.1 to the weighted case without the introduction of extra technical difficulty as follows.

Theorem 0.2 *Assume that the weight* w *is continuous in* Ω *and* $0 < \alpha \le w(x) \le \beta$ *for certain constants* α, β.

Then the statement of Theorem 0.1 holds true for equation (0.10), but with W replaced by the function

$$\tilde{W}(a_1, \dots, a_d) = W(a_1, \dots, a_d) + \frac{\pi}{2}\sum_{i=1}^{d} \log w(a_i). \tag{0.15}$$

The basic idea in the proof of this result is very simple. It consists of defining a suitable modification of the energy functional which penalizes with high values the appearance of singularities in places away from Λ. Then one proves that, as $\varepsilon \to 0$, its global minimizer is indeed a local minimizer for the original energy. This idea is in a similar spirit to that used by the authors in [4], where it is shown that in the nonlinear Schrödinger equation in $\mathbb{R}^N$

$$\varepsilon^2 \Delta u - V(x)u + u^p = 0, \qquad 1 < p < \frac{N+2}{N-2}, \tag{0.16}$$

one can find a family of solutions u_ε with the property that u_ε concentrates around a given, possibly degenerate local minimum of V, provided that V is bounded below away from zero. In this situation what one finds is a "local Mountain Pass" of the associated energy functional in $H^1(\mathbb{R}^N)$.

This paper is organized as follows. In §1 we define the modified energy functional J_ε given the assumptions of Theorem 0.2, and prove some preliminary estimates, the main of them being Proposition 0.1, which implies the uniform estimate (0.6) for the minimizers of J_ε.

Given these estimates, we follow the method in [1] and [12] to show in §2 the convergence, up to a subsequence, of a family of minimizers u_ε to a map of the form (0.3). In this section we also prove two lemmas: Lemma 2.1 which provides a lower estimate for the energy and Lemma 2.2, which essentially tells us that the asymptotic singularities a_j cannot occur outside the given set Λ. We finally combine the results of the previous sections in §3 to complete the proof of Theorem 0.2.

In the rest of this paper we will assume that the boundary of Ω consists of N smooth, closed Jordan curves $\Gamma_0, \Gamma_1, \ldots, \Gamma_N$ with Γ_0 being the exterior component of $\partial\Omega$. We will also assume, without loss of generality, that $d = \deg(g, \partial\Omega) > 0$. We also set $d_0 = \deg(g_0, \Gamma_0)$ and $d_i = -\deg(g_i, \Gamma_i)$, with $g_i = g|_{\Gamma_i}$, $i = 1, \ldots, N$. Thus $d = \sum_{i=0}^N d_i$. We will denote by Ω_i the domain enclosed by Γ_i, $i = 1, \ldots, N$.

The results presented in this article were announced in [5].

1 The penalized energy. Preliminary estimates

Assume the hypothesis of Theorem 0.2, so that there is an open set $\Lambda \subset \Omega^d$ with

$$\inf_{\Lambda} \tilde{W} < \inf_{\partial\Lambda} \tilde{W}, \tag{1.1}$$

where $\tilde{W}$ is defined in (0.15). We look for local minimizers of

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega w(x)(1-|u|^2)^2, \tag{1.2}$$

with asymptotic singularities $(a_1, \ldots, a_d) \in \Lambda$, which minimize $\tilde{W}$ over Λ.

In order to find local minimizers of E_ε, we will modify E_ε. To do this, we first observe that the continuity of $\tilde{W}$ permits us to assume, without loss of generality, that $\Omega^d \setminus \overline{\Lambda}$ consists of the union of a finite number of open rectangles, say

$$\Omega^d \setminus \overline{\Lambda} = \bigcup_{i=1}^{J} (R_{i1} \times R_{i2} \times \ldots \times R_{id})$$

where R_{ij} is an open subset of Ω. We add to E_ε a term which penalizes with high energies asymptotic singularities on $\Omega^d \setminus \overline{\Lambda}$. Thus, we consider the functional J_ε defined on $H^1_g(\Omega)$ as

$$\begin{aligned} J_\varepsilon(u) \quad = \quad & \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega w(x)(1-|u|^2)^2 + \\ & + M \sum_{i=1}^{J} \left\{ \left(\prod_{j=1}^{d} \int_{R_{ij}} \frac{(1-|u|^2)^2}{\varepsilon^2} \right)^{\frac{1}{2d}} - \frac{1}{M} \right\}_+^2 . \end{aligned} \tag{1.3}$$

Here M is a large constant, independent of $\varepsilon > 0$, wich will be chosen in the course of the proof of Theorem 0.2. Here $\{a\}_+ = a$ if $a \geq 0$, and 0 otherwise. As we will show, the added term will be in fact zero at a global minimizer of J_ε, provided that the constant M was appropriately chosen and ε is small enough.

We begin with an upper estimate for the minimum value of J_ε. Thus we choose $u_\varepsilon \in H^1_g$ so that

$$J_\varepsilon(u_\varepsilon) = \min_{u \in H^1_g} J_\varepsilon(u).$$

As in [1], we use the notation

$$I(\varepsilon, \rho) = \inf\{\frac{1}{2}\int_{B_\rho} |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_{B_\rho}(1- |u|^2)^2 \mid u \in H_\rho\}, \tag{1.4}$$

where $u \in H_\rho$ means $u \in H^1(B_\rho)$ and $u(x) = x/|x|$ on ∂B_ρ. Here $B_\rho = B(0, \rho)$.

Lemma 1.1 *Let $a \in \Lambda$. Then there is a $\rho_0 > 0$ such that for all $0 < \rho < \rho_0$*

$$J_\varepsilon(u_\varepsilon) \leq dI(\varepsilon, \rho) + \pi d \log\frac{1}{\rho} + \tilde{W}(a) + o(1) \tag{1.5}$$

where $o(1) \to 0$ as $\rho \to 0$, uniformly on the parameter M.

Proof. We follow the proof of Lemma VII.1 in [1] to construct a suitable test function for J_ε which makes the penalization term zero and whose energy can be estimated as in (1.5).
Write $a = (a_1, \ldots, a_d)$ and choose ρ_0 so small that for $\rho < \rho_0$

$$B(a_1, \rho_1) \times \cdots \times B(a_d, \rho_d) \subset \Lambda,$$

where $\rho_i = \rho/w(a_i)$. Let us set $\Omega_\rho = \Omega \setminus \bigcup_{i=1}^d B(a_i, \rho_i)$. Then, following the proof of Theorem I.9 in [1], one can find a map $\hat{u}_\rho : \Omega_\rho \to S^1$ with the property that $\hat{u}_\rho = g$ on $\partial\Omega$ and

$$\int_{\Omega_\rho} |\nabla \hat{u}_\rho|^2 = \frac{\pi}{2}\sum_{i=1}^d \log\frac{w(a_i)}{\rho^2} + W(a) + O(\rho). \tag{1.6}$$

and $\hat{u}_\rho = \alpha_i \frac{(x-a_i)}{|x-a_i|}$ on $\partial B(a_i, \rho_i)$, with $\alpha_i \in \mathbb{C}$ such that $|\alpha_i| = 1$.

Let us choose a minimizer v_ε associated to $I(\varepsilon, \rho)$ and define

$$v^i_\varepsilon(x) = \alpha_i v_\varepsilon(w(a_i)(x - a_i)) \qquad \text{for} \quad x \in B(a_i, \rho_i).$$

We consider the function $\tilde{u}_\varepsilon : \Omega \to \mathbb{C}$ as $\tilde{u}_\varepsilon = v^i_\varepsilon$ on $B(a_i, \rho_i)$, and $\tilde{u}_\varepsilon = \hat{u}_\rho$ on Ω_ρ. Then

$$\begin{aligned} \frac{1}{2}\int_{\Omega\setminus\Omega_\rho} |\nabla\tilde{u}_\varepsilon|^2 \quad &+ \quad \frac{1}{4\varepsilon^2}\int_{\Omega\setminus\Omega_\rho}\frac{w(x)}{4\varepsilon^2}(1- |\tilde{u}_\varepsilon|^2)^2 \\ &= \quad dI(\varepsilon, d) + \sum_{i=1}^d \int_{B(a_i,\rho_i)} \frac{(w(x) - w(a_i))}{4\varepsilon^2}(1- |v_\varepsilon|^2)^2 \\ &= \quad dI(\varepsilon, d) + o(1). \end{aligned} \tag{1.7}$$

Here we have used the continuity of w, together with the known fact that $\frac{1}{4\varepsilon^2}\int_{B_\rho}(1-|v_\varepsilon|^2)^2$ is uniformly bounded. See [1].

Using (1.6), (1.7), and the fact that $|\tilde{u}_\varepsilon|=1$ on Ω_ρ, we conclude that $J_\varepsilon(u_\varepsilon)$ is bounded above by a quantity of the form (1.5), and the lemma follows. □

Fixing a small $\rho > 0$, and using the fact that $I(\varepsilon, \rho) \leq \pi \log \frac{1}{\varepsilon} + C$ (see [1], Ch. III), one obtains the following

Corollary 1.1 *There is a constant $C_0 > 0$, independent of M and ε such that*

$$\inf_{H^1_g(\Omega)} J_\varepsilon(u) \leq \pi d \log \frac{1}{\varepsilon} + C_0. \tag{1.8}$$

The above corollary makes Proposition 0.1 applicable to the minimizer of u_ε of J_ε, so that the following key estimate is obtained.

Lemma 1.2 *There is a constant $C > 0$ such that*

$$\frac{1}{\varepsilon^2}\int_\Omega (|u_\varepsilon|^2 - 1)^2 \leq C, \tag{1.9}$$

where C is independent of $\varepsilon > 0$ and $M > 0$.

2 Minimizers for the penalized energy

In this section we start the proof of Theorem 0.2 by considering the minimizers of the penalized energy. In the rest of the paper we will denote by u_ε a minimizer of J_ε, the penalized energy, as defined in (1.3). What we will prove first is that, up to subsequences, u_ε converges in a suitable sense to a harmonic map u^*, with singularities at points $a_1, a_2, \ldots, a_d$ in Ω. Then we will obtain a precise estimate from below for the value of the minimizers and finally we study carefully the sequence of minimizers to discard the development of singularities outside Λ for a conveniently chosen M. For that purpose we prove Lemma 2.2 which gives an upper estimate for M provided that the singularities are away from the boundary and from each other. The proof of Theorem 0.2 is completed in the next section where we will see that an appropriate choice of the parameter M in the definition of J_ε makes it impossible that $(a_1, a_2, \ldots, a_d) \in \Omega^d \setminus \bar{\Lambda}$. Finally, combining a precise lower estimate for the energy with the upper estimate found in Lemma 1.1 we will see that actually this point is in Λ and it minimizes $\tilde{W}$ in that set. This will also imply that the penalization term is zero for small $\varepsilon > 0$ therefore the full statement of the theorem.

To carry out this program, we begin by observing that since u_ε is a critical point of J_ε, it satisfies the equation

$$\begin{aligned} \Delta u &= \frac{w_\varepsilon(x)}{\varepsilon^2}(|u|^2 - 1)u \quad \text{in} \quad \Omega \\ u &= g \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.1}$$

where

$$w_\varepsilon(x) = w(x) + \frac{2M}{d} \sum_{i=1}^{J} \mu_{i\varepsilon}\lambda_{i\varepsilon} \sum_{j=1}^{d} \lambda_{ij\varepsilon}\chi_{ij}(x), \tag{2.2}$$

with

$$\lambda_{ij\varepsilon} = \prod_{l\neq j} \int_{R_{il}} \varepsilon^{-2}(1-|u_\varepsilon|^2)^2, \tag{2.3}$$

$$\lambda_{i\varepsilon} = \left(\prod_{l=1}^{d} \int_{R_{il}} \varepsilon^{-2}(1-|u_\varepsilon|^2)^2 \right)^{\frac{1}{2d}-1}, \tag{2.4}$$

$$\mu_{i\varepsilon} = \left\{ \left(\prod_{l=1}^{d} \int_{R_{il}} \varepsilon^{-2}(1-|u_\varepsilon|^2)^2 \right)^{\frac{1}{2d}} - \frac{1}{M} \right\}_+ \tag{2.5}$$

and where χ_{ij} denotes the characteristic function of the open sets R_{ij}.

From Lemma 1.2 we have that

$$\frac{1}{\varepsilon^2} \int_\Omega (1-|u_\varepsilon|^2)^2 \le C, \tag{2.6}$$

with a constant C independent of ε and M. It follows then, in particular, that w_ε is uniformly bounded by a constant possibly depending on M. Then, an application of the Maximum Principle and elliptic regularity as in [3], or in [12], Lemma 2.2, yields the estimates

$$|u_\varepsilon| \le 1, \quad |\nabla u_\varepsilon| \le \frac{C}{\varepsilon} \tag{2.7}$$

with the constant C possibly dependent on M. Let us set $S_\varepsilon = \{x \in \Omega \mid u_\varepsilon(x) < 1/2\}$. Then, if $x \in S_\varepsilon$, the estimate for the gradient tells us that

$$\frac{1}{\varepsilon^2} \int_{\Omega\cap B_{\varepsilon/5}(x)} (1-|u_\varepsilon|^2)^2 \ge \gamma > 0$$

with γ independent of ϵ and x. This and estimate (2.6) tell us that S_ε can be covered by a finite number of balls I' of radius ε. Here we choose a sequence $\varepsilon_n \to 0$ so that I' remains constant.

Let $x_i^n = x_i^{\varepsilon_n}$, $i = 1, \ldots, I'$ be the centers of these balls. Then we can also assume that these points converge to distinct points $x_1, \ldots, x_I \in \overline{\Omega}$, with $I \le I'$. Exactly as in Proposition 3.3. in [12] we then get the estimate

$$\int_{\Omega_\sigma} |\nabla u_n|^2 \le 2\pi d \log \frac{1}{\sigma} + C$$

where $\Omega_\sigma = \Omega \setminus \cup_{i=1}^{I} B_\sigma(x_i)$ and $u_n = u_{\varepsilon_n}$. Following [12], we obtain that u_n converges to a function u, weakly locally in $H^1(\Omega)$, away from the singularities, and weak in $W^{1,p}(\Omega)$ for all $p < 2$. Now, the arguments in [1], Chapter VI, reproduce without changes to prove that the asymptotic singularities of u_n are away from the boundary, they all have local degree +1 and there are exactly d of them. These points will be denoted by $a_1, \dots, a_d$. The limiting function u satisfies $|u| = 1$ a.e. and satisfies distributionally the equation $\operatorname{div}(u \times \nabla u) = 0$, since u_n does. Then, applying the arguments in [12] we see that u is smooth away from the singularities $a_1, \dots, a_d$. Since they have degree 1 and $u \in W^{1,1}(\Omega)$, it follows from section I.3 in [1] that u has the form (0.3).

On the other hand, the convergence of u_n is in the uniform and H^1 senses, away from the singularities. This can be obtained following the arguments in [1] and [2], observing that the presence of the possibly discontinuous weight does not introduce any extra technical difficulty. We also obtain that $\varepsilon_n^{-2} \int_K (1- \mid u_n \mid^2)^2 \to 0$ on any compact K away from the singularities.

We do not know whether it is still possible to obtain stronger convergence results for u_n, but they will not be necessary for our purposes.

We gather the information obtained so far on u_ε and u in the following proposition.

Proposition 2.1 *There is a sequence $\varepsilon_n \to 0$ and a subsequence $u_n = u_{\varepsilon_n}$ of u_ε and d points $a_1, \dots, a_d$ in Ω such that u_n converges to the harmonic map*

$$u^*(x) = \left(\frac{x - a_1}{\mid x - a_1 \mid} \right) \dots \left(\frac{x - a_d}{\mid x - a_d \mid} \right) e^{i\psi(x)}, \tag{2.8}$$

where $\Delta\psi = 0$ in Ω and $u^* = g$ on $\partial\Omega$. The convergence is in the H^1 and uniform senses over any compact set K of $\overline{\Omega} \setminus \{a_1, \dots, a_d\}$.

Moreover, the sequence u_n also satisfies

$$\lim_{n\to\infty} \frac{1}{\varepsilon_n^2} \int_K (1- \mid u_n \mid^2)^2 = 0. \tag{2.9}$$

A precise estimate from below for the penalized energy $J_{\varepsilon_n}(u_n)$ will be needed in the proof of Theorem 0.2. In view of the convergence properties of the sequence u_n given in Proposition 2.1 we have

Lemma 2.1 *For any sequence $\varepsilon_n \to 0$ and $u_n = u_{\varepsilon_n}$ of minimizers of J_{ε_n} satifying the conclusions of Proposition 2.1, there is $\rho_0 > 0$ such that, for every $0 < \rho < \rho_0$ there is $N \in \mathbb{N}$ such that the following holds*

$$J_{\varepsilon_n}(u_n) \geq dI(\varepsilon_n, \rho) + \frac{\pi}{2} \sum_{i=1}^{d} \log w(a_i) + W(a_1, \dots, a_d) + \pi d \log \frac{1}{\rho} + o(1) \tag{2.10}$$

for all $n \geq N(\rho)$. Here $o(1)$ means that $\lim_{\rho\to 0} o(1) = 0$.

Proof. Let $\Omega_\rho = \Omega \setminus \cup_{i=1}^d B_\rho(a_i)$, where $1/2 \geq \rho > 0$ is small enough so that $B_\rho(a_i) \subset \Omega$, for all $i = 1, \dots, d$. From Proposition 2.1 we know that u_n converges to u^* in the H^1 sense over Ω_ρ so that we can find $N_1 \in \mathbb{N}$ such that

$$\frac{1}{2} \int_{\Omega_\rho} \mid \nabla u_n \mid^2 + \frac{1}{4\varepsilon_n^2} \int_{\Omega_\rho} w(x)(\mid u_n \mid^2 -1)^2 \geq \frac{1}{2} \int_{\Omega_\rho} \mid \nabla u^* \mid^2 + O(\rho), \tag{2.11}$$

for $n \geq N_1$. Then using Theorem I.8 in [1] we find

$$\frac{1}{2}\int_{\Omega_\rho} |\nabla u_n|^2 + \frac{1}{4\varepsilon_n^2}\int_{\Omega_\rho} w(x)(|u_n|^2 - 1)^2$$
$$\geq \quad W(a_1, \ldots, a_d) + \pi d \log\frac{1}{\rho} + O(\rho). \tag{2.12}$$

In what follows we study the energy over each ball $B_\rho(a_i)$; for notational convenience we write $a_i = a$ and we put $A_\rho = \overline{B_\rho}(a) \setminus B_{\rho/2}(a)$.

First we define a function v_n as

$$v_n(x) = (2\frac{|x-a|}{\rho} - 1)(\frac{u_n}{|u_n|} - u_n) + u_n \tag{2.13}$$

for $x \in A_\rho$, and $v_n(x) = u_n(x)$, for $x \in B_{\rho/2}(a)$. From Proposition 2.1 and an easy computation we see that

$$v_n \to u^* \quad \text{in} \quad L^\infty(A_\rho) \qquad \text{and} \quad \nabla v_n \to \nabla u^* \quad \text{in} \quad H^1(A_\rho). \tag{2.14}$$

Next we define an auxilliary multiplier as

$$Q(x) = e^{i(2|x-a|/\rho - 1)(H(a) - H(x))} \tag{2.15}$$

for $x \in A_\rho$ and $Q(x) = 1$ if $x \in B_{\rho/2}(a)$. Here H is such that $u^*(x) = e^{i(\theta + H(x))}$ near $x = a$. Again, after a simple computation using that H is smooth, we see that $|Q(x)| \leq c$ for all $x \in A_\rho$ and for an appropriate constant c.

Finally we define the function $\tilde{v}_n \in H^1(B_\rho(a))$ as $\tilde{v}_n(x) = Q(x)v_n(x)$ for all $x \in B_\rho(a)$. We observe that $\tilde{v}_n(x) = u_n(x)$ for $x \in B_{\rho/2}(a)$ and that $\tilde{v}_n(x) = e^{i(\theta + H(a))}$ for all $x \in \partial B_\rho(a)$.

Now we can do our estimates.

$$\begin{aligned}
&\frac{1}{2}\int_{B_\rho(a)} |\nabla u_n|^2 + \frac{1}{4\varepsilon_n^2}\int_{B_\rho(a)} w(x)(|u_n|^2 - 1)^2 \\
= \quad &\frac{1}{2}\int_{B_\rho(a)} |\nabla \tilde{v}_n|^2 + \frac{w(a)}{4\varepsilon_n^2}\int_{B_\rho(a)} (|\tilde{v}_n|^2 - 1)^2 \\
&+ \frac{1}{2}\int_{B_\rho(a)} |\nabla u_n|^2 - |\nabla \tilde{v}_n|^2 \\
&+ \frac{1}{4\varepsilon_n^2}\int_{B_\rho(a)} (w(x) - w(a))(|u_n|^2 - 1)^2 \\
&+ \frac{w(a)}{4\varepsilon_n^2}\int_{B_\rho(a)} (|u_n|^2 - 1)^2 - (|\tilde{v}_n|^2 - 1)^2 \\
\geq \quad &I(\varepsilon_n, \rho) + \frac{\pi}{2}\log w(a) + I_1 + I_2 + I_3 + o(1).
\end{aligned} \tag{2.16}$$

The inequality in (2.16) follows from the definition of $I(\varepsilon, \rho)$ and Lemma IX.1 in [1] . Next we prove that $I_1 + I_2 + I_3 = o(1)$, so that from (2.12) and (2.16) the result follows.

From Proposition 2.1 we see that, over A_ρ, we have

$$\lim_{n\to\infty} \nabla\tilde{v}_n = \lim_{n\to\infty} v_n \nabla Q + Q\nabla v_n = u^*\nabla Q + Q\nabla u^* \tag{2.17}$$

in $L^2(A_\rho)$. Then

$$\lim_{n\to\infty} \int_{A_\rho} |\nabla u_n|^2 - |\nabla\tilde{v}_n|^2 = -\int_{A_\rho} |u^*\nabla Q|^2 + \int_{A_\rho} u^* Q\nabla Q\cdot\nabla u^*. \tag{2.18}$$

But, since ∇Q is bounded, and $|\nabla u^*| \leq c/\rho$ over A_ρ, from (2.18) we find that $|I_1| \leq c\rho$ if $n \geq N_2(\rho)$, where $N_2 \geq N_1$.

Next we use that w is a continuous function, together with the fact that $1/4\varepsilon_n^2 \int_\Omega (|u_n|^2 - 1)^2$ is bounded, to obtain that $I_2 = o(1)$.

Finally we estimate the integral I_3. By definition of $\tilde{v}_n$ we observe that $|u_n| \leq |v_n| = |\tilde{v}_n|$. Then we have

$$\begin{aligned} |I_3| &= \frac{|w(a)|}{4\varepsilon_n^2}\int_{A_\rho} (|u_n|^2-1)^2 + (|\tilde{v}_n|^2-1)^2 \\ &\leq \frac{2|w(a)|}{4\varepsilon_n^2}\int_{A_\rho} (|u_n|^2-1)^2 \end{aligned} \tag{2.19}$$

but this last integral converges to zero as $n\to\infty$, proving in this way that $I_3 = o(1)$. This completes the proof of the proposition. □

Remark 2.1 If w is smooth we can obtain $O(\rho)$ instead of $o(1)$ in (2.10). Moreover, if $\nabla H(a) = 0$, then it was shown in [1] that it is possible to obtain $O(\rho^2)$ in (2.10). We notice, however that in the presence of a non trivial weight we do not have $\nabla H(a) = 0$ in general.

The next lemma is the first step towards discarding the possibility that $(a_1, \ldots, a_d) \in \Omega\setminus\overline{\Lambda}$.

Lemma 2.2 *Assume that* $(a_1, \ldots, a_d) \in \Omega\setminus\overline{\Lambda}$ *and that for a certain* $\delta > 0$ *we have*

$$dist(a_i, \partial\Omega) \geq 2\delta, \quad |a_i - a_j| \geq 2\delta$$

for $i, j = 1, \ldots, d$. *Then there exists a constant* $C(\delta) > 0$, *depending on* δ *but not on* M, *such that necessarily* $M \leq C$.

Proof. Since u_ε satisfies equation (2.1), then it is also a critical point of the functional

$$\tilde{E}_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega w_\varepsilon(x)(1-|u|^2)^2,$$

where w_ε is given by (2.2). Note also that

$$\tilde{E}_\varepsilon(u_\varepsilon) = E_\varepsilon(u_\varepsilon) + M\sum_{i=1}^{J} \lambda_{i\varepsilon}^{1/(1-2d)}\mu_{i\varepsilon}$$

where $\mu_{i\varepsilon}, \lambda_{i\varepsilon}$ are given by (2.4) and (2.5), respectively. It then follows from the definition of $\mu_{i\varepsilon}$ and J_ε that

$$\tilde{E}_\varepsilon(u_\varepsilon) \le E_\varepsilon(u_\varepsilon) + \sum_{i=1}^{J} M\,\mu_{i\varepsilon}^2 + \mu_{i\varepsilon} \le J_\varepsilon(u_\varepsilon) + C\,, \tag{2.20}$$

with C independent of M. Here we used that $\mu_{i\varepsilon}$ is uniformly bounded independently of M, thanks to (2.6).

Now we consider a sequence $\varepsilon_n \to 0$ so that $u_{\varepsilon_n} = u_n$ satisfies the convergence properties given in Proposition 2.1. Let $u^* = \lim_{n\to\infty} u_n$ having singularities $(a_1, \ldots, a_d)$. By hypothesis, we have that for some i the singularities satisfy $(a_1, \ldots, a_d) \in R_{i1} \times R_{i2} \times \ldots \times R_{id}$. Note that the balls $B(a_j, \delta)$, $j = 1, \ldots, d$ are disjoint and they are inside Ω. We make the following observation: since ${\varepsilon_n}^{-2} \int_K (1 - |u_n|^2)^2 \to 0$ on any compact set K not containing the singularities, we have that for the i above

$$\max_{j=1,\ldots,d} \lambda_{ijn} \ge \lambda_{kln} + \theta(\varepsilon_n)$$

for all k, l, where $\theta(\varepsilon_n) \to 0$ as $\varepsilon_n \to 0$ and $\lambda_{ijn} = \lambda_{ij\varepsilon_n}$.

Next we estimate $\tilde{E}_{\varepsilon_n}(u_n)$ from below. We have that

$$\begin{aligned}\tilde{E}_{\varepsilon_n}(u_n) \quad \ge \quad & \sum_{j=1}^{d} \frac{1}{2} \int_{B(a_j,\delta)} |\nabla u_n|^2 + \frac{\alpha_n^j}{4\varepsilon_n^2} \int_{B(a_j,\delta)} (1 - |u_n|^2)^2 \\ & - \frac{\alpha_n^j}{4\varepsilon_n^2} \int_{B(a_j,\delta)\setminus B(a_j,\rho)} (1 - |u_n|^2)^2,\end{aligned} \tag{2.21}$$

with

$$\alpha_n^j = \alpha + \frac{2M}{d}\,\mu_{in}\lambda_{in}\lambda_{ijn}\,,$$

where α is a positive lower bound for w, the numbers $\lambda_{ijn} = \lambda_{ij\varepsilon_n}$, $\lambda_{in} = \lambda_{i\varepsilon_n}$ and $\mu_{in} = \mu_{i\varepsilon_n}$ are given by (2.3)-(2.5), and ρ is chosen so small that $B(a_j, \rho) \subset R_{ij}$. We note that the third summand in the right hand side of (2.21) tends to zero with ε_n.

Now, since $u_n \to u^*$ uniformly on $\partial B(a_j, \delta)$, u^* given by (2.8), we conclude that

$$\frac{1}{2} \int_{B(a_j,\delta)} |\nabla u_n|^2 + \frac{\alpha_n^j}{4\varepsilon_n^2} \int_{B(a_j,\delta)} (1 - |u_n|^2)^2 \ge \pi \log \frac{1}{\varepsilon} + \frac{1}{2}\pi \log \alpha_n^j - C$$

where C depends on δ and u^*. This inequality is obtained using Lemma VIII.2 in [1], after an "interpolation" argument as that given in the proof of Lemma 2.1. Moreover, since we are assuming that the singularities are away from each other and from the boundary at a distance at least 2δ, the constant C can actually be chosen to depend only on δ and not on the particular location of the singularities which determines the u^*.

Hence, we obtain

$$\tilde{E}_{\varepsilon_n}(u_n) \geq d\pi \log \frac{1}{\varepsilon_n} + \frac{1}{2} \sum_{j=1}^{d} \log \alpha_n^j - C(\delta) \tag{2.22}$$

for ε_n sufficiently small, possibly depending on M. Combining this estimate with (2.20) and Corollary 1.1 we may then conclude that

$$\limsup_{\varepsilon_n \to 0} \frac{2M}{d} \mu_{in} \lambda_{in} \lambda_{ijn} \leq C \quad \text{for all } j = 1, \ldots, d. \tag{2.23}$$

Here C depends on δ but not on M. It follows from (2.23), and the definition of w_n that $w_n \leq C(\delta)$, independently of M. We conclude, as in [12], that there is a constant $C(\delta) > 0$, independent of M, such that $|\nabla u_n| \leq C/\varepsilon_n$ for sufficiently small ε_n. This in turn implies, as in [12], that

$$\frac{1}{\varepsilon_n^2} \int_{R_{ij}} (1 - |\, u_n\, |^2)^2 \geq \gamma$$

for some positive constant γ depending on δ but not on M, for all sufficiently small ε_n. Therefore

$$M \mu_{in}^2 \geq M(\gamma^{1/2} - \frac{1}{M})_+^2.$$

On the other hand, from [1] we know that $\inf E_{\varepsilon_n} \geq d \log \varepsilon_n^{-1} - C$. Hence

$$\pi d \log \frac{1}{\varepsilon_n} - C + M(\gamma^{1/2} - \frac{1}{M})_+^2 \leq J_{\varepsilon_n}(u_n) \leq \pi d \log \frac{1}{\varepsilon_n} + C,$$

for some C independent of M. Here we have used Corollary 1.1. From this, an upper estimate for M depending only on δ readily follows, concluding the proof of the lemma. □

3 Proof of theorem 0.2

Now we are in a position to finish the proof of Theorem 0.2. Combining the upper estimate in Lemma 1.1 and the lower estimate in Lemma 2.1, we obtain

$$W(a) + \frac{1}{2} \sum_{j=1}^{d} \log w(a_j) \leq W(\tilde{a}) + \frac{1}{2} \sum_{j=1}^{d} \log w(\tilde{a}_j),$$

namely $\tilde{W}(a) \leq \tilde{W}(\tilde{a})$, where $\tilde{a}$ minimizes $\tilde{W}$ on Λ. This estimate immediatelly tells us that the a_i's must remain at a uniform distance from $\partial\Omega$, independent of M as the mutual distances between the a_j's do, for otherwise $W(a)$ would become too large. Therefore Lemma 2.2 is applicable for some fixed $\delta > 0$, and we choose an $M > C(\delta)$ there. The Lemma then tells us that a cannot be in $\Omega^d \setminus \overline{\Lambda}$. Moreover, thanks to our assumption (0.8), we must necessarily have that $a \in \Lambda$ and it minimizes $\tilde{W}$ there.

Finally, since $\int_K \varepsilon_n^{-2}(1-|u_n|^2)^2 \to 0$ over compacts away from the singularities, we have that the penalization term $M\sum_{i=1}^J \mu_{in}^2 = 0$ for all sufficiently small ε_n. Moreover, the same is true if one evaluates this term at any u in a sufficiently small ε_n-dependent H^1 neighbourhood of u_n. Since u_n is minimizing J_{ε_n}, it follows that u_n is a local minimizer of E_{ε_n}. This concludes the proof of the theorem.

Remark 3.1 In [9], Lin has used the heat flow method to detect the presence of other type of solutions to the Ginzburg-Landau system. In fact, he finds boundary conditions g of degree d so that solutions with d asymptotic singularities of degree 1 plus an arbitrary number of pairs of asymptotic singularities with degrees ± 1 exist. The idea is to adjust the boundary condition so that local minimizers for the corresponding extension of the renormalized energy associated to these singularities appear. It would be interesting to study whether the direct method we have developed in this paper can provide further insight into these phenomena.

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