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Local mountain passes for semilinear elliptic problems in unbounded domains

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0 Introduction

This paper has been motivated by some works appeared in recent years concerning the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(z)\psi - \gamma \mid \psi \mid^{p-1} \psi. \tag{0.1}$$

We are interested in solutions of the form $\psi(z,t) = \exp(-iEt/\hbar)v(z)$ which are called *standing waves*. We observe that this ψ satisfies (0.1) if and only if the function $v(z)$ solves the elliptic equation

$$\frac{\hbar^2}{2m}\Delta v - (V(z) - E)v + \gamma \mid v \mid^{p-1} v = 0. \tag{0.2}$$

In [3], Floer and Weinstein consider the case $N = 1$ and $p = 3$. For a given nondegenerate critical point of the potential V, assumed globally bounded, and for $0 < E < \inf V$, they construct a standing wave provided that $\hbar$ is sufficiently small. This solution concentrates around the critical point as $\hbar \to 0$.

Their method, based on an interesting Lyapunov-Schmidt reduction, was extended by Oh in [6], [7] to conclude a similar result in higher dimensions, provided that $1 < p < \frac{N+2}{N-2}$. He restricts himself to potentials with "mild oscillation" at infinity, namely belonging to a Kato class. In case that V is bounded this restriction is not necessary as observed by Wang in [10].

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In [8], Oh uses this approach to construct solutions with multiple peaks which concentrate around any prescribed finite set of nondegenerate critical points of V.

The method in the works mentioned above, local in nature, seems to use in a essential way the nondegeneracy of the critical points, although this assumption can be somewhat relaxed, □□. In [9], Rabinowitz uses a global variational technique to find a solution with "minimal energy" for all small $\hbar$, when $1 < p < (N+2)/(N-2)$ and

$$\liminf_{|z|\to\infty} V(z) > \inf_{z\in R^N} V(z), \tag{0.3}$$

see also [2] for related results. A precise concentration statement (around a global minimum, possibly degenerate) for this solution is established in [10].

An interesting question, originally raised by Changfeng Gui, which motivates the present work, is whether one can find solutions which concentrate around local minima not necessarily nondegenerate.

As we will see, the answer is affirmative, and moreover the same is true if one considers the elliptic problem in an arbitrary domain in $\mathbb{R}^N$ with zero boundary conditions.

More generally, we deal with a semilinear elliptic problem of the form

$$\begin{cases} \varepsilon^2 \Delta u - V(z)u + f(u) = 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \quad u > 0 \text{ in } \Omega \end{cases} \tag{0.4}$$

where Ω is a domain in $\mathbb{R}^N$, possibly unbounded, with empty or smooth boundary. V will be assumed throughout this paper locally Hölder continuous and bounded below away from zero, say

$$V(z) \geq \alpha > 0, \quad \text{for all} \quad z \in \mathbb{R}^N. \tag{0.5}$$

We will also assume that $f : \mathbb{R}^+ \to \mathbb{R}$ is of class C^1 and satisfies the following conditions.

(f1) $f(\xi) = o(\xi)$ near $\xi \geq 0$.
(f2) $\lim_{\xi\to\infty} \frac{f(\xi)}{\xi^s} = 0$ for some $1 < s < \frac{N+2}{N-2}$.
(f3) For some $2 < \theta \leq s+1$ we have

$$0 \leq \theta F(\xi) < f(\xi)\xi \quad \text{for all} \quad \xi > 0 \tag{0.6}$$

where $F(\xi) = \int_0^\xi f(\tau) a\tau$.

(f4) The function $\xi \to \frac{f(\xi)}{\xi}$ in nondecreasing.

Our main result for equation (0.4) is the following.

Theorem 0.1 *Assume there is a bounded domain Λ compactly contained in Ω such that*

$$\inf_{\Lambda} V < \min_{\partial\Lambda} V. \tag{0.7}$$

Then there is an $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ a positive solution $u_\varepsilon \in H_0^1(\Omega)$ to problem (0.4) exists. Moreover, u_ε possesses just one local (hence global) maximum z_ε, which is in Λ. We also have that $V(z_\varepsilon) \to \inf_\Lambda V$, and

$$u_\varepsilon(z) \leq \alpha \exp(-\frac{\beta}{\varepsilon}|z - z_\varepsilon|), \tag{0.8}$$

for certain constants α, β.

We observe that no restriction on the global behavior of V is required other than (0.5). In particular, V is not required to be bounded or to belong to a Kato class.

On the other hand, the hypotheses on the nonlinearity f are fairly mild, in the sense that no special nondegeneracy or uniqueness assumptions need to be made on the "limiting" equation associated to (0.4), namely an equation of the form

$$\Delta u - u + f(u) = 0.$$

The proof of this result is variational, and relies on an elementary idea which permits to identify "local mountain passes". The energy functional associated to equation (0.4) generally satisfies the assumptions of the Mountain Pass Theorem, except for the P.S. condition. Indeed, a global assumption like (0.3) permits to recover P.S. at the mountain pass level when ε is small.

What we do in our situation is to build a convenient modification of the energy, in such a way that the functional satisfies P.S., and then prove that for a sufficiently small ε the associated mountain pass is indeed a solution to the original equation with the stated properties. The modification of the functional corresponds to a "penalization outside Λ", and this is why no other global assumptions are required. Though elementary, we believe this idea is flexible enough to be adapted to "catch mountain passes" in a number of interesting situations.

To illustrate this point, we will see in the last section how local properties of an unbounded domain lead to existence and localization of solutions to elliptic problems. As an example of the situation covered by Theorem 3.1 in §3, let us consider the case of an axially symmetric domain $\Omega \subset \mathbb{R}^N, N \geq 2$ given by

$$\Omega = \{(t,x) \in \mathbb{R}^{N-1} \times \mathbb{R} \ / \ |x| < \rho(t)\} \tag{0.9}$$

where $\rho : \mathbb{R} \to \mathbb{R}^+$ is a smooth function. We consider the problem

$$\begin{cases} \varepsilon^2 u_{tt} + \Delta u - u + f(u) = 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \quad u > 0 \text{ in } \Omega \end{cases} \tag{0.10}$$

Here Δ denotes the Laplacian with respect to the $N-1$-dimensional variable x, and f satisfies (f1)-(f4). We have

Theorem 0.2 *Assume that for some bounded interval I we have that*

$$\sup_I \rho > \rho|_{\partial I}. \tag{0.11}$$

Then there exists an $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$ *there is a positive solution* $u_\varepsilon \in H_0^1(\Omega)$ *to problem (0.10) such that for some sequence of* $t_\varepsilon \in I$ *with* $\rho(t_\varepsilon) \to \sup_I \rho$ *one has*

$$u_\varepsilon(x,t) \leq \alpha \exp(-\frac{\beta}{\varepsilon}|t - t_\varepsilon|), \tag{0.12}$$

for certain positive constants α *and* β.

The organization of this paper is as follows: In §1 we define the modification of the functional needed for the proof of Theorem 0.1, and prove some preliminary results. §2 is devoted to the proof of Theorem 0.1, while in §3 we use a similar approach to study the influence of the domain, and prove a general result from which Theorem 0.2 follows as a particular case.

1 Preliminaries

In this section we establish some preliminary results needed for the proof of Theorem 0.1 and the results of §3.

In the framework of Theorem 0.1, let us consider a function $f : \mathbb{R} \to \mathbb{R}$ satisfying (f1)-(f4) on $\mathbb{R}^+$ and defined as zero for negative values.

Associated to equation (0.4) is the "energy" functional

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega \varepsilon^2|\nabla u|^2 + V(z)u^2 - \int_\Omega F(u) \quad , \tag{1.1}$$

which is well defined for $u \in H$ where

$$H = \{u \in H_0^1(\Omega) \ / \ \int_\Omega V(z)u^2 < \infty\}.$$

H becomes a Hilbert space, continuously embedded in $H_0^1(\Omega)$, when endowed with the inner product

$$< u, v >_V = \int_\Omega \nabla u \cdot \nabla v + V(z)uv \tag{1.2}$$

whose associated norm we denote by $\|\cdot\|_H$.

Under the regularity assumptions on V and f, it is standard to check that the nontrivial critical points of J_ε correspond exactly to the positive classical solutions in $H_0^1(\Omega)$ of equation (0.4).

We will define a modification of this functional which satisfies the P.S. condition and to which we can apply the Mountain Pass Theorem. Let θ be a number as given by (f3), and let us choose $k > 0$ such that $k > \frac{\theta}{\theta-2}$. Let $a > 0$ be the value at which $\frac{f(a)}{a} = \frac{\alpha}{k}$, where α is as in (0.5). Let us set

$$\tilde{f}(s) = \begin{cases} f(s) & \text{if } s \le a \\ \frac{\alpha}{k} s & \text{if } s > a, \end{cases} \tag{1.3}$$

and define

$$g(\cdot, s) = \chi_\Lambda f(s) + (1 - \chi_\Lambda)\tilde{f}(s), \tag{1.4}$$

where Λ is a bounded domain as in the assumptions of Theorem 0.1. and χ_Λ denotes its characteristic function. It is easy to check that (f1)-(f4) implies that g defined in this way is a Caratheodory function and it satisfies the following assumptions:

(g1) $g(z,\xi) = o(\xi)$ near $\xi = 0$ uniformly in $z \in \Omega$.
(g2) $\lim_{\xi\to\infty} \frac{g(z,\xi)}{\xi^s} = 0$ for some $1 < s < \frac{N+2}{N-2}$ if $N \ge 3$ and no restriction on s if $N = 1, 2$.
(g3) There exist a bounded subset K of Ω, $\text{int}(K) \neq \emptyset$, and $2 < \theta < s+1$ so that
(i) $0 < \theta G(z,\xi) \le g(z,\xi)\xi$ for all $z \in K, \xi > 0$.
and
(ii) $0 \le 2G(z,\xi) \le g(z,\xi)\xi \le \frac{1}{k} V(z)\xi^2$ for all $\xi \in \mathbb{R}^+, z \notin K$ with the number k satisfying $k > \theta/(\theta - 2)$.
(g4) The function $\xi \to \frac{g(z,\xi)}{\xi}$ in nondecreasing for all $z \in \Omega$ a.e. and $\xi > 0$.

Here we have denoted $G(z,\xi) = \int_0^\xi g(z,\tau)d\tau$.

Since a similar approach will be used for the results of §3, we consider the more general framework of an equation of the form

$$\Delta u - V(z)u + g(z,u) = 0 \quad \text{in} \quad \Omega, \tag{1.5}$$

with V as before, satisfying (0.5), and g satisfying (g1)-(g4). Here we have set $\varepsilon = 1$ for notational simplicity.

Then we consider the functional $J : H \to \mathbb{R}$ associated to this equation,

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + V(z)u^2 - \int_\Omega G(z,u), \quad u \in H. \tag{1.6}$$

Then J is of class C^1 in H. Let us notice that the function f considered in the introduction satisfies the properties given for g, except for (g3) (ii). This last assumption implies that J satisfies the Palais Smale condition as we show next.

Lemma 1.1 *Let $\{u_n\}$ be a sequence in H such that $J(u_n)$ is bounded and $J'(u_n) \to 0$. Then u_n has a convergent subsequence.*

Proof. We have that $\{u_n\}$ is bounded in H. In fact, using (g3) we easily see that

$$\int_\Omega |\nabla u_n|^2 + V(z)u_n^2 \ge \int_K g(z,u_n)u_n + o(\|u_n\|_H). \tag{1.7}$$

It also follows from the assumptions that

$$\frac{1}{2}\int_{\Omega} |\nabla u_n|^2 + V(z)u_n^2 = \int_{\Omega} G(z,u_n) + O(1)$$
$$\leq \int_K G(z,u_n) + \frac{1}{2k}\int_{\Omega\setminus K} V(z)u_n^2 + O(1). \quad (1.8)$$

Thus, from (1.7), (1.8) and ($g3$) we find

$$(\frac{\theta}{2} - 1)\int_{\Omega} |\nabla u_n|^2 + V(z)u_n^2 \leq \frac{\theta}{2k}\int_{\Omega\setminus K} V(z)u_n^2 + o(\|u_n\|_H) + O(1). \quad (1.9)$$

Then, it follows from the choice of k in (g3) that $\{u_n\}$ is bounded as desired.

Let us choose a subsequence, still denoted by $\{u_n\}$, weakly convergent to u in H. This convergence is actually strong. Indeed, it suffices to show that, given $\delta > 0$, there is an $R > 0$ such that

$$\limsup_{n\to\infty} \int_{\Omega\setminus B_R} \{|\nabla u_n|^2 + V(z)u_n^2\} < \delta, \quad (1.10)$$

where B_R denotes the ball with center 0 and radius R. We may assume that R is chosen so that $K \subset B_{R/2}$.

Let η_R be a cut-off function so that $\eta_R = 0$ on $B_{R/2}, \eta_R = 1$ on $\Omega \setminus B_R, 0 \leq \eta_R \leq 1$ and $|\nabla\eta_R| \leq c/R$. Since $\{u_n\}$ is a bounded P.S. sequence, we have that

$$< J'(u_n), \eta_R u_n >= o(1), \quad (1.11)$$

so that

$$\int_{\Omega}\{|\nabla u_n|^2 + V(z)u_n^2\}\eta_R + \int_{\Omega} u_n\nabla u_n \cdot \nabla\eta_R$$
$$= \int_{\Omega} g(z,u_n)u_n\eta_R + o(1) \leq \frac{1}{k}\int_{\Omega} V(z)u_n^2\eta_R + o(1). \quad (1.12)$$

We conclude that

$$\int_{\Omega\setminus B_R} |\nabla u_n|^2 + V(z)u_n^2 \leq \frac{C}{R}\|u_n\|_{L^2(\Omega)}\|\nabla u_n\|_{L_2(\Omega)} + o(1), \quad (1.13)$$

from where (1.10) follows. □

The previous lemma makes the Mountain Pass Theorem applicable to the functional J. In fact, on the one hand, it is easily checked using (g1) that

$$J(u) \geq c\|u\|_H^2 \quad (1.14)$$

for $\|u\|_H$ small enough. Next, choosing $\phi \in H \setminus \{0\}$ non negative, with its support contained in the set K given in (g3), we see that

$$J(t\phi) \to -\infty \quad \text{as } t \to +\infty. \tag{1.15}$$

The Mountain Pass Theorem thus implies that the following is a critical value of J

$$c = \inf_{\gamma \in \mathscr{P}} \sup_{t \in [0,1]} J(\gamma(t)), \tag{1.16}$$

where $\mathscr{P} = \{\gamma \in C([0,1], H) \ / \ \gamma(0) = 0 \text{ and } J(\gamma(1)) < 0\}$.

Condition (g4) implies that the critical value c can be characterized in a simpler way, as has been essentially established in [9] and [2]. We provide a proof for completeness.

Lemma 1.2 *If (g1)-(g4) hold then*

$$c = \inf_{\substack{u \in H \\ u \neq 0}} \sup_{\tau \geq 0} J(\tau u). \tag{1.17}$$

Proof. Let $u \in H, u \neq 0$, then the function $\tau \to J(\tau u)$ has at most one nontrivial critical point for $\tau \geq 0$ and it satisfies

$$\int_\Omega |\nabla u|^2 + V(z)u^2 = \int_\Omega \frac{g(z, \tau u)u}{\tau}, \tag{1.18}$$

this is because (g2), (g4) and (g5). We define $\varphi(u)$ as the unique solution of (1.18) if a solution exists, and we put $\varphi(u) = \infty$ when no solution exists. If we put C^* the right hand side in (1.17) we see that $C \leq C^*$. In order to prove the other inequality we note that

$$C^* = \inf_{v \in M} J(v) \tag{1.19}$$

where $M = \{v = \varphi(u)u \ / \ u \in H, u \neq 0, \varphi(u) < \infty\}$. Thus we only have to show that given $\gamma \in \mathscr{P}$ there exist $\bar{\tau} \in [0,1]$ such that $\gamma(\bar{\tau}) \in M$. Assuming the contrary we have $\gamma(\tau) \notin M$ for all $\tau \in [0,1]$. In view of (g1) and (1.18)

$$\frac{1}{2}\int_\Omega |\nabla\gamma(\tau)|^2 + \gamma(\tau)^2 > \frac{1}{2}\int_\Omega g(z, \gamma(\tau))\gamma(\tau). \tag{1.20}$$

Then, using (g3) (i) and (ii) we find for all $\tau \in [0,1]$

$$J(\gamma(\tau)) > (\frac{\theta}{2} - 1)\int_K G(z, \gamma(\tau)) \geq 0, \tag{1.21}$$

and this contradicts the definition of $\mathscr{P}$. □

We end this section with a comparison result between critical values of an "autonomous" functional which will be used in §3. Let f satisfy assumptions (f1)-(f4) and let us consider two open subsets of $\mathbb{R}^N$, Ω_1 and Ω_2. Then we write

$$I(u) = \frac{1}{2}\int_{R^N} |\nabla u|^2 + u^2 - \int_{R^N} F(u). \tag{1.22}$$

Let us consider the mountain pass values c_i, $i = 1, 2$ of I when restricted to $H_0^1(\Omega_i)$, namely

$$c_i = \inf_{\substack{u \in H_0^1(\Omega_i) \\ u \neq 0}} \sup_{\tau \geq 0} I(\tau u). \tag{1.23}$$

Then we have,

Lemma 1.3 *If $\Omega_1 \subset \Omega_2$, $\Omega_1 \neq \Omega_2$ and c_1 is a critical value of the restriction of I to $H_0^1(\Omega_1)$, then*

$$c_2 < c_1. \tag{1.24}$$

Proof. A proof of this fact was given in [1], so that we only provide a sketch. Let $u_1 \in H_0^1(\Omega_1)$ be a critical point associated to c_1. Then u_1 is a positive solution of an elliptic equation in Ω_1, but it does not satisfies the same equation in Ω_2, by unique continuation. Thus u_1 is not a critical point of I in $H_0^1(\Omega_2)$. Then we can deform the compact set $\mathscr{K} = \{tu_1 / \tau \in [0, \tau_1]\}$, where τ_1 is such that $I(\tau u_1) < 0$, using the negative gradient flow. This will provide a path with values strictly less that c_1. □

2 Local mountain pass: the potential

We devote this section to the proof of Theorem 0.1. As we already mentioned, in order to localize the mountain pass, we consider the modification of the function f given by g in (1.4), and the associated functional J_ε defined as

$$J_\varepsilon(u) = \frac{\varepsilon^2}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2} \int_\Omega V(z) u^2 - \int_\Omega G(z, u), \quad u \in H.$$

Since the modified function g satisfies assumptions (g1)-(g4), then the results of §1 yield the following lemma.

Lemma 2.1 *The functional J_ε possesses a positive critical point $u_\varepsilon \in H$ so that*

$$J_\varepsilon(u_\varepsilon) = \inf_{\substack{u \in H \\ u \neq 0}} \sup_{\tau \geq 0} J_\varepsilon(\tau u). \tag{2.1}$$

Next we use a test function to derive a useful estimate of the number in (2.1). Set $V_0 = \min_\Lambda V$ and let $w \in H^1(\mathbb{R}^N)$ be a least energy solution to

$$\Delta w - V_0 w + f(w) = 0,$$

that is, w satisfies

$$\underline{c} = I_0(w) = \inf_{\substack{v \in H^1(R^N) \\ v \neq 0}} \sup_{\tau \geq 0} I_0(\tau v), \tag{2.2}$$

where I_0 is defined as

$$I_0(v) = \frac{1}{2} \int_{R^N} |\nabla v|^2 + V_0 v^2 - \int_{R^N} F(v). \tag{2.3}$$

We may assume that w maximizes at zero. Also set Then, let $z_0 \in \Lambda$ be such that $V(z_0) = V_0$ and set $u(z) = \eta(z)w((z - z_0)/\varepsilon)$ where η is a smooth function compactly supported in Ω and with $\eta \equiv 1$ in a neighborhood of z_0. A direct computation then shows that

$$J_\varepsilon(u_\varepsilon) \leq \sup_{t>0} J_\varepsilon(tu) = \varepsilon^N\{\underline{c} + o(1)\}, \tag{2.4}$$

with $o(1) \to 0$ as $\varepsilon \to 0$.

On the other hand, combining this, the fact that $J_\varepsilon'(u_\varepsilon)u_\varepsilon = 0$ and using the definitions of g and the number θ we easily obtain that

$$\gamma \int_\Omega \varepsilon^2 \mid \nabla u_\varepsilon \mid^2 + V(z)u_\varepsilon^2 \leq C\varepsilon^N, \tag{2.5}$$

where $C > 0$ and

$$\gamma = \frac{\theta}{2}(\frac{\theta - 2}{\theta} - \frac{1}{k}) > 0$$

thanks to our choice of k.

We devote what remains of this section to prove that when ε is sufficiently small u_ε actually solves (0.4) and satisfies (0.8). Let $a > 0$ be the number chosen in the definition of g. Then the desired result will follow if we show that for all small ε one has $u_\varepsilon < a$ on $\Omega \setminus \Lambda$. Crucial step in the proof of this fact is the following

Proposition 2.1 *If m_ε is given by*

$$m_\varepsilon = \max_{z \in \partial\Lambda} u_\varepsilon(z), \tag{2.6}$$

then

$$\lim_{\varepsilon \to o} m_\varepsilon = 0. \tag{2.7}$$

Moreover, for all ε sufficiently small u_ε possesses at most one local maximum $z_\varepsilon \in \Lambda$ and we must have

$$\lim_{\varepsilon \to o} V(z_\varepsilon) = V_0 = \min_{z \in \Lambda} V(z), \tag{2.8}$$

Remark. This proposition can be regarded as a local version of the global concentration result of Wang. See Theorem 2.1 in [10].

Before proving this proposition, let us see how Theorem 0.1 follows from it.

Proof of Theorem 0.1. By Proposition 2.1 there exists ε_0 such that for all $0 < \varepsilon < \varepsilon_0$,

$$u_\varepsilon(z) < a \quad \text{for all} \quad z \in \partial\Lambda. \tag{2.9}$$

The function $u_\varepsilon \in H$ solves the equation

$$\varepsilon^2 \Delta u - V(x)u + g(x, u) = 0 \qquad \text{in } \Omega. \tag{2.10}$$

Since $u_\varepsilon \in H_0^1(\Omega)$ we can choose $(u_\varepsilon - a)_+$ as a test function in (2.10) so that, after integration by parts one gets

$$\int_{\Omega\setminus\Lambda} \varepsilon^2 \mid \nabla(u_\varepsilon - a)_+ \mid^2 + c(z)(u_\varepsilon - a)_+^2 + c(z)a(u_\varepsilon - a)_+ = 0, \tag{2.11}$$

where

$$c(z) = V(z) - \frac{g(z, u_\varepsilon(z))}{u_\varepsilon(z)}. \tag{2.12}$$

The definition of g yields that $c(z) > 0$ in $\Omega \setminus \Lambda$, hence all terms in (2.11) are zero. We conclude in particular

$$u_\varepsilon(z) \leq a \quad \text{for all} \quad z \in \Omega \setminus \Lambda.$$

Consequently u_ε is a solution to equation (0.4). To obtain (0.8), we note that the maximum value of u_ε is achieved at a point $z_\varepsilon \in \Lambda$ and it is away from zero. Then we set

$$v_\varepsilon(z) = u_\varepsilon(z_\varepsilon + \varepsilon z).$$

From (2.5) we see that

$$\|v_\varepsilon\|_{H^1(R^N)} \leq C \tag{2.13}$$

for some $C > 0$. Note also that v_ε satisfies

$$\Delta v_\varepsilon - V(z_\varepsilon + \varepsilon z)v_\varepsilon + f(v_\varepsilon) = 0.$$

Now, each sequence $\varepsilon_j \to 0$ possesses a subsequence which we label in the same way such that $z_{\varepsilon_j} \to \bar{z}$, with $V(\bar{z}) = V_0$. From (2.13), and elliptic estimates, we have that this subsequence can be chosen in such a way that $v_{\varepsilon_j} \to v$ uniformly over compacts, where $v \in H^1(\mathbb{R}^N)$ maximizes at zero and solves

$$\Delta v - V_0 v + f(v) = 0 \qquad \text{in } \mathbb{R}^N. \tag{2.14}$$

It is well known that positive solutions to (2.14) decay exponentially and they are radially symmetric. See the work by Gidas, Ni and Nirenberg [4]. It follows from (2.13), (2.14) and the growth assumptions on f that

$$I_0(v) \leq C \tag{2.15}$$

where the constant C does not depend on v, but on the family $\{v_\varepsilon\}$. Then we obtain the existence of $R > 0$ so that $v(z) < a$ for all $|z| = R$.

From these facts, one can easily conclude the existence of a number $R > 0$ such that $v_\varepsilon(z) < a$ if $|z| = R$, for all $\varepsilon > 0$ sufficiently small. Since no other local maxima of v_ε exists, as Proposition 2.1 states, we conclude that actually $v_\varepsilon(z) < a$ for all $|z| > R$. Then we can use the maximum principle to conclude that $v_\varepsilon \leq w_0$ for $\mid z \mid > R$, where w is an appropriate multiple of the fundamental solution of $\Delta w - bw = 0$ with $b = \alpha - f(a)/a > 0$. Since w_0 decays exponentially, relation (0.8) follows, and the proof of the theorem is complete. □

To prove the proposition we will establish the following fact: If $\varepsilon_n \downarrow 0$ and $z_n \in \overline{\Lambda}$ are such that $u_{\varepsilon_n}(z_n) \geq b > 0$, then

$$\lim_{n\to\infty} V(z_n) = V_0. \tag{2.16}$$

We argue by contradiction. Thus we assume, passing to a subsequence, that $z_n \to \bar{z} \in \overline{\Lambda}$ and

$$V(\bar{z}) > V_0. \tag{2.17}$$

Similarly to the proof of the theorem given above, we consider the sequence

$$v_n(z) = u_{\varepsilon_n}(z_n + \varepsilon_n z) \tag{2.18}$$

and study its behavior as n goes to infinity. The function v_n satisfies the equation

$$\begin{cases} \Delta v_n - V(x_n + \varepsilon_n z)v_n + g(x_n + \varepsilon_n z, v_n) = 0 \text{ in } \Omega_n, \\ \qquad\qquad\qquad\qquad\qquad\qquad v_n = 0 \text{ on } \partial\Omega_n \end{cases} \tag{2.19}$$

where $\Omega_n = \varepsilon_n^{-1}\{\Omega - x_n\}$. Again from (2.5), we see that v_n is bounded in $H^1(\mathbb{R}^N)$, and from elliptic estimates it can be assumed to converge uniformly on compacts subsets of $\mathbb{R}^N$ to a function $v \in H^1(\mathbb{R}^N)$. Now, the sequence of functions $\chi_n(z) \equiv \chi_\Lambda(z_n + \varepsilon_n z)$ can also be assumed to converge weakly in any L^p over compacts to a function χ with $0 \leq \chi \leq 1$. Therefore, v satisfies the limiting equation

$$\Delta v - V(\bar{z})v + \bar{g}(z, s) = 0 \qquad \text{in } \mathbb{R}^N. \tag{2.20}$$

where

$$\bar{g}(z, s) = \chi(z)f(s) + (1 - \chi(z))\tilde{f}(s). \tag{2.21}$$

Associated to the equation (2.20) we have the functional $\bar{J} : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined as

$$\bar{J}(u) = \frac{1}{2}\int_{R^N} |\nabla u|^2 + V(\bar{z})w^2 - \int_{R^N} \bar{G}(z, u)\,, \quad u \in H^1(\mathbb{R}^N) \tag{2.22}$$

where $\bar{G}(z, s) = \int_0^s \bar{g}(z, \tau)d\tau$. Then v is a critical point of $\bar{J}$. We also set

$$J_n(u) = \frac{1}{2}\int_{\Omega_n} |\nabla u|^2 + V(z_n + \varepsilon_n z)w^2 - \int_{\Omega_n} \bar{G}(z_n + \varepsilon_n z, u)\,, \quad u \in H_0^1(\Omega_n).$$

Then $J_n(v_n) = \varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n})$, so that the key step in the proof of the proposition is the following

Lemma 2.2

$$\liminf_{n\to\infty} J_n(v_n) \geq \bar{J}(v).$$

In particular, $\bar{J}(v) \leq \underline{c}$ where $\underline{c}$ is given by (2.2).

Proof. Write

$$h_n = \frac{1}{2}(|\nabla v_n|^2 + V(z_n + \varepsilon_n z)v_n^2) - G(z_n + \varepsilon_n z, v_n).$$

Then, choose $R > 0$. Since v_n converges in the C^1-sense over compacts to v we get

$$\lim_{n\to\infty} \int_{B_R} h_n = \frac{1}{2} \int_{B_R} |\nabla v|^2 + V(\bar{z})v^2 - \int_{B_R} \bar{G}(z, v).$$

Since $v \in H^1(\mathbb{R}^N)$, we have that for each given $\delta > 0$,

$$\lim_{n\to\infty} \int_{B_R} h_n \geq \bar{J}(v) - \delta,$$

provided that R was chosen sufficiently large. Thus it only suffices to check that

$$\liminf_{n\to\infty} \int_{\Omega_n \setminus B_R} h_n \geq -\delta, \tag{2.23}$$

for large enough R. Let us consider a smooth cut-off function η_R such that $\eta_R \equiv 0$ on B_{R-1}, $\eta_R \equiv 1$ on $\mathbb{R}^N \setminus B_R$, $0 \leq \eta_R \leq 1$ and $|\nabla \eta_N| \leq C$, C independent of R.

We use $w_n = \eta_R v_n \in H^1(\Omega_n)$ as a test function for $J_n'(v_n) = 0$ to obtain

$$0 = J_n'(v_n)w_n = E_n + \int_{\Omega_n \setminus B_R} 2h_n + g_n \tag{2.24}$$

where $g_n = 2G(z_n + \varepsilon z) - g(z_n + \varepsilon_n z, v_n)v_n$ and E_n is given by

$$E_n = \int_{B_R \setminus B_{R-1}} \nabla v_n \cdot \nabla(\eta_R v_n) + V(z_n + \varepsilon_n z)\eta_R v_n^2 - \int_{B_R \setminus B_{R-1}} g(z_n + \varepsilon z, v_n)\eta_R v_n. \tag{2.25}$$

Again, the convergence of v_n in the C^1-sense over compacts to v and the fact that $v \in H^1(\mathbb{R}^N)$ implies that for $R > 0$ sufficiently large we will have $\lim_{n\to\infty} |E_n| \leq \delta$. On the other hand, the definition of g together with the properties of f give that $g_n \leq 0$. Using this in (2.24), (2.23) follows, and the proof of the lemma is complete. □

Now we are ready to obtain a contradiction with (2.16). Since v is a critical point of $\bar{J}$, and $\bar{g}$ satisfies hypothesis (g4) in §1, we have that

$$\bar{J}(v) = \max_{\tau \geq 0} \bar{J}(\tau v) \tag{2.26}$$

Then, since $f(s) \geq \tilde{f}(s)$ for all s we have that

$$\bar{J}(v) \geq \inf_{\substack{u \in H^1(\mathbb{R}^N) \\ u \neq 0}} \sup_{\tau > 0} I_{\bar{z}}(\tau u) \equiv \bar{c} \tag{2.27}$$

where $I_{\bar{z}}$ is given by

$$I_{\bar{z}}(u) = \frac{1}{2}\int\limits_{R^N} |\nabla u|^2 + V(\bar{z})u^2 - \int\limits_{R^N} F(u)\,, \quad u \in H^1(\mathbb{R}^N). \tag{2.28}$$

But, since $V(\bar{z}) > V_0$, we clearly have that $\bar{c} > \underline{c}$ where $\underline{c}$ is the number defined in (2.3). Hence $\bar{J}(v) > \underline{c}$. But this contradicts the previous lemma, and the proof of the claim, i.e. (2.16), is thus complete. □

To conclude the proof of Proposition 2.1, we need to show that u_ε possesses at most one local maximum in Λ. Assume the contrary, namely the existence of a sequence $\varepsilon_n \to 0$ such that u_{ε_n} possesses two local maxima z_n^1, $z_n^2 \in \Lambda$. Then $u_{\varepsilon_n}(z_n^i) \geq b > 0$, $i = 1, 2$. From (2.16) we have that these sequences stay away from the boundary of Λ. Set $v_n(z) = u_{\varepsilon_n}(z_n^1 + \varepsilon_n z)$. Then, it turns out, that after passing to a subsequence v_n converges in the C^2 sense over compacts to a solution v in $H^1(\mathbb{R}^N)$ of equation (2.14), where V_0 is replaced by $V(\bar{z}^1)$. Here $\bar{z}^1 = \lim z_n^1$. The function v has a local maximum at zero, it is radially symmetric and radially decreasing, as the arguments in [4] show. Then zero is a nondegenerate global maximum. This and the local C^2 convergence of v_n to v clearly implies that the second local maximum of v_n must go away, namely $z_n = \varepsilon_n^{-1}(z_n^2 - z_n^1)$ satisfies $|z_n| \to +\infty$. Using this fact, a slight variation of the argument in the proof of Lemma 2.2 yields

$$\liminf_{n\to\infty} \varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}) \geq 2I_0(w) = 2\underline{c}.$$

This is in obvious contradiction with (2.4), and the proof of the proposition is complete. □

3 Local mountain pass: the domain

In this section we will prove Theorem 0.2. The hypotheses on the behavior of the domain, given through the function ρ, has certain analogy with the hypotheses on V in Theorem 0.1. We exploit this analogy, and we obtain results on existence of positive solution to elliptic equations in unbounded domains.

Theorem 0.2 is a particular case of a more general result, Theorem 3.2 below. Before stating it, let us see another special case: a domain defined as the region 'below' the graph of a positive function. Let g be a smooth function $g : \mathbb{R}^{N-1} \to \mathbb{R}$ with $0 < g(x) \quad \forall x \in \mathbb{R}^{N-1}$. We consider

$$\Omega = \{(t, x) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid \quad 0 < x < g(t)\} \tag{3.1}$$

and the problem

$$\begin{cases} \varepsilon^2 \Delta_t u + u_{xx} - u + f(u) = 0 \text{ in } \Omega \\ \\ u = 0 \text{ on } \partial\Omega, \quad u > 0 \text{ in } \Omega. \end{cases} \tag{3.2}$$

Here $\varepsilon > 0$ is a parameter and Δ_t denotes the Laplacian with respect to the $N-1$-dimensional variable t. We have

Theorem 3.1 *Assume there is a bounded neighborhood $\Lambda \subset \mathbb{R}^{N-1}$ containing* 0 *such that*

$$\max_{\Lambda} g > \max_{\partial\Lambda} g. \tag{3.3}$$

Then there is an $\varepsilon_0 > 0$ so that for all $0 < \varepsilon < \varepsilon_0$ there is a positive solution $u_\varepsilon \in H^1(\Omega)$ to problem (3.2). Moreover a property like (0.12) holds.

Theorems 0.2 and 0.3 are consequences of a much more general existence result which we state next. Assume $N \geq 2$ and let $1 \leq \ell \leq N-1$. We identify $\mathbb{R}^N = \mathbb{R}^\ell \times \mathbb{R}^{N-\ell}$ and write $z \in \mathbb{R}^N$ as $z = (t,x)$ with $t \in \mathbb{R}^\ell, x \in \mathbb{R}^{N-\ell}$.

We consider a smooth domain Ω in $\mathbb{R}^N$ satisfying the following set of properties:

(Ω1) There is a bounded domain Λ in $\mathbb{R}^\ell$ containing 0_{R^ℓ} with $D \equiv (\Lambda \times \mathbb{R}^{N-\ell}) \cap \Omega$ bounded. Additionally, for each $z \in \partial D \cap \partial\Omega$ the normal vector to $\partial\Omega$ at z has a non-zero x-component.

In what follows we denote by Ω^t the t-section of Ω, that is

$$\Omega^t = \{x \in \mathbb{R}^{N-\ell} \ / \ (t,x) \in \Omega\},$$

where $t \in \overline{\Lambda}$.

(Ω2) For all $t \in \partial\Lambda$ we have $\overline{\Omega^t} \subseteq \Omega^0$ and $\Omega^t \subset \Omega^0$ for all $t \in \Lambda$.

For a domain Ω satisfying the properties given above we consider the semilinear elliptic problem

$$\begin{cases} \varepsilon^2 \Delta_t u + \Delta_x u - u + f(u) = 0 \text{ in } \Omega \\ \\ u = 0 \text{ on } \partial\Omega, \quad u > 0 \text{ in } \Omega, \end{cases} \tag{3.4}$$

where Δ_t and Δ_x represent the Laplacian operators with respect to the t and x variables respectively. Note that the situation in theorems 0.2 and 3.1 falls within this framework with $\ell = 1$ and $\ell = N-1$ respectively.

On problem (3.4) we have the following result, which includes Theorems 0.2 and 3.1 as special cases.

Theorem 3.2 *Assume (Ω1), (Ω2) and (f1) – (f4) hold. Then there is an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, problem (3.4) has a positive solution $u_\varepsilon \in H^1(\Omega)$. Moreover the function u_ε concentrates on Λ in the sense that there is a sequence $t_\varepsilon \in \Lambda$ with the property that*

$$u_\varepsilon(t,x) \leq \alpha \exp\{-\frac{\beta}{\varepsilon}|t - t_\varepsilon|\} \tag{3.5}$$

for all $(t,x) \in \Omega$ and some positive constants α, β.

Existence of "least energy" positive solutions of semilinear equations in unbounded domains has been considered by several authors. See for example [1], [5] and references therein. In these works global assumptions on the domain are made.

The proof of Theorem 3.2 is similar to that of Theorem 0.1, except that now we rely on a different limiting problem.

Let us consider $D = (\Lambda \times \mathbb{R}^{N-\ell}) \cap \Omega$, as given in ($\Omega$1). Then we modify f in the following way outside D: we set

$$g(z,s) = \chi_D(z)f(s) + (1-\chi_D(z))\tilde{f}(s). \tag{3.6}$$

Thus we search for critical points of the functional

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla_x u|^2 + \varepsilon^2|\nabla_t u|^2 + u^2 - \int_\Omega G(z,u), \qquad u \in H_0^1(\Omega). \tag{3.7}$$

Then it follows from the results of Section 1, the existence of a critical point u_ε of J_ε with a variational characterization like (2.1). To estimate $J_\varepsilon(u_\varepsilon)$ we set

$$I(u) = \frac{1}{2}\int |\nabla u|^2 + u^2 - \int F(u). \tag{3.8}$$

Then I possesses a critical point w in $H^0 = H_0^1(\mathbb{R}^l \times \Omega^0)$ so that

$$I(w) = \bar{c} = \inf_{\substack{u\in H^0\\ u\neq 0}} \sup_{\tau\geq 0} I(\tau u). \tag{3.9}$$

Similarly to §2, we can use this w to construct a test function in the variational characterization of u_ε to obtain

$$J_\varepsilon(u_\varepsilon) \leq \varepsilon^l(\bar{c} + o(1)). \tag{3.10}$$

The result of the theorem will follow if we prove that

$$\max_{\partial D} u_\varepsilon \to 0 \quad \text{ as } \varepsilon \to 0, \tag{3.11}$$

since the same argument given to prove Theorem 0.1 from Proposition 2.1 applies. To prove (3.11) we argue again by contradiction, assuming the existence of sequences $\varepsilon_n \to 0$ and $z_n = (t_n, x_n) \in (\partial\Lambda \times \mathbb{R}^{N-l}) \cap \Omega$ with $t_n \to \bar{t} \in \partial\Lambda$ and $u_{\varepsilon_n}(z_n) \geq b > 0$. Then we write $z = (t,x) \in \mathbb{R}^l \times \mathbb{R}^{N-l}$ and set

$$v_n(z) = u_{\varepsilon_n}(t_n + \varepsilon_n t, x).$$

where, we assume that v_n is extended as zero outside its domain of definition $\Omega_n \equiv \{(t,x) \ / \ (t_n + \varepsilon_n t, x) \in \Omega\}$, so that v_n is understood as an element of $H^1(\mathbb{R}^N)$.

Then v_n satisfies the equation

$$\Delta v_n - v_n + g(t_n + \varepsilon_n t, x, v_n) = 0 \quad \text{for all} \quad (t,x) \in \Omega_n, \tag{3.12}$$

and (3.10) implies that the sequence v_n is bounded in $H^1(\mathbb{R}^N)$. This, interior and boundary elliptic estimates, using the regularity of Ω and (Ω1), imply that v_n converges uniformly and in $H^1(\mathbb{R}^N)$ over compact sets to a nonzero function $v \in H_0^1(\mathbb{R}^\ell \times \Omega^{\bar{t}})$ satisfying

$$\begin{cases} \Delta v - v + \bar{g}(z,v) = 0 & \text{in } \mathbb{R}^\ell \times \Omega^{\bar{t}} \\ u = 0 \text{ on } \mathbb{R}^\ell \times \partial\Omega^{\bar{t}} \end{cases} \tag{3.13}$$

where $\bar{g}(z,s) = \chi(z)f(s) + (1-\chi(z))\tilde{f}(s)$ and χ is some measurable function such that $0 \leq \chi \leq 1$.

Now, let J_n be the energy functional associated to (3.12) and $\bar{J}$ that of (3.13). Then, exactly the same argument used in the proof of Proposition 2.1 yields

$$\liminf_{n\to\infty} J_n(v_n) \geq \bar{J}(v).$$

Therefore, from (3.10) one gets $\bar{J}(v) \leq \bar{c}$. But this is impossible, since clearly $\bar{J}(v)$ is greater than or equal to the mountain pass value of the functional I given by (3.8) over $H_0^1(\mathbb{R}^l \times \Omega^{\bar{t}})$, which is strictly less than $\bar{c}$.

This proves that $m_\varepsilon \to 0$. From here it follows, as in the previous section, that u_ε actually solves equation (3.4) for all small ε and that all local maxima of u_ε must lie in $\Lambda \times \mathbb{R}^{N-\ell}$. Let $z_\varepsilon = (t_\varepsilon, x_\varepsilon) \in \Omega$ with $t_\varepsilon \in \Lambda$ such that $u_\varepsilon(z_\varepsilon) > a$ is a local maximum of u_ε. We define $v_\varepsilon(t,x) = u_\varepsilon(t_\varepsilon + \varepsilon t, x)$ for $z = (t,x) \in \mathbb{R}^\ell \times \mathbb{R}^{N-\ell}$. Next with essentially the same arguments given above, and recalling that $\Omega^t \subset \Omega^0$ for all $t \in \Lambda \setminus \{0\}$, we find that $t_\varepsilon \to \bar{t}$, where $\bar{t}$ is such that $\Omega^{\bar{t}} = \Omega^0$, and that there is a sequence $\varepsilon_n \to 0$ and $v_n = v_{\varepsilon_n} \to v$ uniformly and in $H^1(\mathbb{R}^N)$ over compacts. The function v satisfies the equation

$$\begin{cases} \Delta v - v + f(v) = 0 & \text{in } \mathbb{R}^\ell \times \Omega^0 \\ v = 0 \text{ on } \mathbb{R}^\ell \times \partial\Omega^0 \end{cases} \tag{3.14}$$

and $v(z) > 0$ for $z \in \mathbb{R}^\ell \times \Omega^0$. Using the moving planes argument as in [4], only in the t variables, one can get that the positive solutions of (3.14) are radially symmetric in the variable t and increasing along the t-rays. These facts allow us to show that for a number $R > 0$ $v(z) < a$ if $|z| = R$. Here we use that for the solutions v of (3.14) we are interested in, the functional I given in (3.8) is uniformly bounded. Then, using the uniform convergence of the sequence v_n over compact sets, and the fact that the local maxima of v_n must be at a finite distance, we find that $v_n(z) < a$ for all $|z| > R$. From here the constants α and β can be found, depending only on the family $\{v_\varepsilon\}$, so that (3.5) holds.

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