JOURNAL OF DIFFERENTIAL EQUATIONS **124**, 108–131 (1996)
Article No. 0004

Solvability of the Neumann Problem in a Ball for $-\Delta u + u^{-\nu} = h(|x|)$, $\nu > 1$

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Received January 13, 1993; revised May 21, 1994

1. Introduction

This paper concerns mainly with the question of solvability of an elliptic problem of the form

$$\begin{aligned} -\Delta u + u^{-\nu} &= h \qquad \text{in } B \\ \frac{\partial u}{\partial n} &= 0 \qquad \text{on } \partial B, \end{aligned} \tag{1.1}$$

where B is the unit ball in $\mathbb{R}^N$, $N \geqslant 2$, $\nu > 1$ and $h \in C^0(\bar{B})$ is radially symmetric. Before specifying the exact meaning we will give to a solution of (1.1) we make the following observation: If $u \in C^0(\bar{B})$ is a nonnegative radial function satisfying distributionally

$$-\Delta u + u^{-\nu} = h \qquad \text{in } B$$

then $u > 0$ on $\bar{B} \backslash \{0\}$. In fact, if $u(\bar{r}) > 0$, we claim that $u(r) > 0$ for all $0 < r \leqslant \bar{r}$. To see this, we note that u satisfies the equation

$$(s^{N-1} u')' = (u^{-\nu} - h)\, s^{N-1}, \qquad s \in (r, \bar{r}).$$

* This work was completed while the first author was a member of The Institute for Advanced Study, supported by NSF Grant DMS-9100383.

0022-0396/96 $12.00

From here, it is easily seen that with no loss of generality we may assume that u is nondecreasing on $[r, \bar{r}]$ and $u > 0$ on $(r, \bar{r}]$. Multiplying by $s^{N-1}u'$ the above equation we find

$$\frac{d}{ds}\left(\frac{|s^{N-1}u'|^2}{2}\right) \geqslant -\frac{d}{ds}\left(\frac{u^{-\nu+1}(s)}{1-\nu}\, r^{2(N-1)} + mu\right)$$

where $m \geqslant \|h\|_\infty$. Then, integrating we find

$$r^{2(N-1)}\,\frac{u^{1-\nu}(s)}{\nu-1} \leqslant r^{2(N-1)}\,\frac{u^{1-\nu}(\bar{r})}{\nu-1} + mu(\bar{r}) + \frac{|\bar{r}^{N-1}u'(\bar{r})|^2}{2} \tag{1.2}$$

for $s \in (r, \bar{r})$. Since u is continuous, we conclude after letting $s \downarrow r$ that $u(r) > 0$. A similar argument shows that $u(r) > 0$ for $\bar{r} < r \leqslant 1$. By virtue of this observation, it seems natural to define a solution to (1.1) to be a $u \in C^2(\bar{B} - \{0\}) \cap C(\bar{B})$ satisfying $u > 0$ on $\bar{B} \backslash \{0\}$ and solving (1.1) in the distributional sense.

The question arises of whether a solution to (1.1) in the above sense may vanish at the origin. This is not the case in dimension $N = 1$, however for $N \geqslant 2$ one may have solutions to (1.1) such that $u(0) = 0$, as the following example shows:

Let η be a smooth function on $[0, 1]$ such that $0 \leqslant \eta \leqslant 1$ and satisfying $\eta(r) = 1$ for $r \in [0, \frac{1}{3}]$, $\eta(r) = 0$ for $r \in [\frac{2}{3}, 1]$. Then let

$$u(r) = c\eta(r)\, r^{2/(\nu+1)} + (1 - \eta(r)),$$

where $c = (2/(\nu+1) + N - 2)^{-1/(\nu+1)} > 0$. It is easily checked that u defined in this way solves weakly an equation of the form (1.1) for some smooth function h. This example marks an important difference between the one and higher dimensional cases. The behavior of vanishing weak solutions at the origin can be estimated, however. For any such solution one must have $u(r) \geqslant cr^{2/(\nu+1)}$ for some $c > 0$. This implies that $u^{-\nu} \in L^p(B)$ for some $p > N/2$, hence the solution u is actually in $W^{2,p}(B)$ and is Hölder continuous. This fact will be established later. Actually one can go further and prove that there are a priori estimates $\alpha, \beta > 0$ such that all solutions u to (1.1) satisfy

$$\alpha r^{2/(\nu+1)} \leqslant u(r) \leqslant \beta. \tag{1.3}$$

In order to attack the existence problem for equation (1.1), it seems natural to consider a family of approximating equation obtained truncating the nonlinearity $u^{-\nu}$ near $u=0$ and then extending nicely the truncation to the left. For example, one may consider the family of approximate problems

$$\begin{aligned} -\Delta u + (u \vee s)^{-\nu} &= h \qquad \text{in } B \\ \frac{\partial u}{\partial n} &= 0 \qquad \text{on } \partial B \end{aligned} \tag{1.4}$$

for small $s > 0$, where $u \vee s = \max\{u, s\}$. This approach works nicely in dimension $N=1$, see, e.g., [7] where a related periodic problem is treated. The reason is basically the existence of a uniform positive lower estimate for the approximating solutions. This is not the case if $N > 1$, thus introducing a technical difficulty. It seems therefore natural to consider an approximating scheme that somehow "pushes the solutions up" near the origin. To this purpose, we introduce the singular problems

$$\begin{aligned} -\Delta u + u^{-\nu} &= h + \varepsilon \delta_0 \qquad \text{in } B \\ \frac{\partial u}{\partial n} &= 0 \qquad \text{on } \partial B \\ u &> 0 \qquad \text{in } B \end{aligned} \tag{1.5}$$

where $\varepsilon > 0$ is small, and δ_0 denotes the Dirac measure supported at the origin. This problem, in appearance more delicate than the original one, has the nice feature of possessing a uniform positive lower estimate for its solutions. To construct a solution to (1.5) a second approximating scheme like (1.4) will work, as we shall see, yielding the following result.

THEOREM 1.1. *Problem* (1.5) *possesses at least one radial solution if* $0 < \varepsilon < \int_B h$ *or if* $\varepsilon > \int_B h$.

We do not know whether one can still show existence if $\varepsilon = \int_B h$, however uniformicity of the a priori estimates in ε is indeed lost at this level. Convenient a priori estimates for the solutions of (1.5) will permit us to take the limit as $\varepsilon \downarrow 0$ to obtain the following result:

THEOREM 1.2. *Problem* (1.1) *possesses at least one radial solution provided that* $\int_B h > 0$.

It is immediately checked that the condition $\int_B h > 0$ is actually necessary for existence. To provide a uniqueness result we consider, more generally than (1.1), the problem

$$\begin{aligned} -d\,\Delta u + u^{-\nu} &= h \qquad \text{in } B \\ \frac{\partial u}{\partial n} &= 0 \qquad \text{on } \partial B \end{aligned} \tag{1.6}$$

where $d > 0$. We have the following result.

THEOREM 1.3. *Assume* $\int_B h > 0$. *Then* (1.6) *possesses at least one radial solution for any* $d > 0$. *Moreover, there is a* $d_0 > 0$ *such that this solution is unique for all* $d \geqslant d_0$.

The rest of this paper will be mostly devoted to the proof of the above results. As we mentioned before, the existence result for (1.1) is already known in dimension $N = 1$, as established in [6]. In that reference the authors consider the equation

$$-u'' + u^{-\nu} = h(t),$$

under T-periodic boundary conditions, and prove existence of a positive solution provided that $\int_0^T h > 0$. Their proof carries over with only minor variations to the Neumann case. This result was extended in [9] to a second order singular potential system. T-periodic problems, including equations of the form

$$-u'' + u^{-\nu} - f(u) = h(t),$$

where f is continuous on $[0, \infty)$ and unbounded above, have been treated in [2] and [3], respectively for the cases of an asymptotically linear and a superlinear f.

It should be remarked that in the higher dimensional case, the related problem

$$\Delta u + k(x)\, u^{-\nu} = 0$$

under zero Dirichlet boundary conditions on a bounded smooth domain, where the coefficient k is positive, has been considered by several authors, see for example [1], [6], [5], [4] and their references. This problem is actually of a very different nature than (1.1). From the existence point of view, this problem is in some sense simpler than (1.1), since the sign of the nonlinearity makes the Maximum Principle applicable to obtain estimates which allow the use of e.g. a super-subsolutions scheme. It should be emphasized that the above works mostly focuse on the behavior of the solution (which is unique) near the boundary, where it vanishes.

It is worth mentioning that the problem

$$\Delta u + u^{-\nu} = h(x)$$

under Neumann boundary conditions on a bounded, smooth domain possesses a (unique) positive solution in case that h is, for example, strictly positive. In fact, in such case a small and a large positive constant represent respectively a sub and a supersolution to the above problem, thus providing existence. Uniqueness follows easily from the Maximum Principle.

This paper is organized as follows. In Section 2 we prove some preliminary lemmas needed in Section 3 to find a priori estimates for the singular problem (1.5) and for a family of truncated approximations to it. In Section 4 we prove Theorem 1.1. Finally, in Section 5 we prove Theorems 1.2 and 1.3.

2. Preliminary Lemmas

In this section we prove some preliminary results useful for deriving estimates for solutions of an equation of the form

$$\begin{aligned} u'' + \frac{N-1}{r} u' = u^{-\nu} - h(r), \qquad r \in (0, 1) \\ u'(1) = 0 \end{aligned} \tag{2.1}$$

where $h \in C^0[0, 1]$. These results, of elementary nature, will play a key role in the derivation of a priori estimates for the singular problem (1.5) in the next section.

Lemma 2.1. *Assume that u is of class C^2 on $[\delta, 1]$ for some $0 \leqslant \delta < 1$ and satisfies the differential inequality*

$$\begin{aligned} -\left(u'' + \frac{N-1}{r} u'\right) \leqslant m, \qquad r \in [\delta, 1] \\ u'(\delta),\ u'(1) = 0 \end{aligned} \tag{2.2}$$

for some $m > 0$. Then, for any $\delta < \rho \leqslant 1$ one has

$$\sup_{\delta \leqslant r \leqslant 1} u \leqslant \inf_{\delta \leqslant r \leqslant 1} u + mc(f(\rho) + 1)$$

$$\text{where} \quad c = c(N) \quad \text{and} \quad f(\rho) = \begin{cases} \rho^{2-N}, & \text{if} \quad N > 2 \\ -\log \rho, & \text{if} \quad N = 2. \end{cases} \tag{2.3}$$

Proof. Since u satisfies

$$-(r^{N-1}u')' \leqslant mr^{N-1} \tag{2.4}$$

we have that

$$-r^{N-1}u'(r) \leqslant \frac{m}{N}(r^N - \delta^N), \qquad r \in [\delta, 1].$$

Hence, integrating again, we find

$$u(r) \leqslant u(1) + \frac{m}{2N}(1 - r^2), \qquad r \in [\delta, 1]$$

so that

$$\sup_{\delta \leqslant r \leqslant 1} u \leqslant u(1) + mc. \tag{2.5}$$

On the other hand, we also have from (2.4),

$$u(1) - u(r) \leqslant m \int_r^1 \frac{1}{Ns^{N-1}}(1 - s^N)\, ds, \qquad r \in (\delta, 1].$$

It follows that

$$u(1) \leqslant \inf_{\rho \leqslant r \leqslant 1} u + mc(f(\rho) + 1) \tag{2.6}$$

From (2.5) and (2.6) we obtain the validity of (2.3). ∎

LEMMA 2.2. *Assume that u is of class C^2, positive on some interval $[a, b]$ with $0 \leqslant a \leqslant b \leqslant 1$, satisfying the differential inequality*

$$u'' + \frac{N-1}{r}u' - u^{-\nu} + m \geqslant 0 \tag{2.7}$$

where $m > 0$. Moreover, assume $u(a) \geqslant \mu > 0$, $u'(a) = 0$ and $u'(r) \leqslant 0$ on $[a, b]$.

Then there exists a number $\theta = \theta(\nu, m, \mu) > 0$ such that $u(b) > \theta$.

Proof. Let us denote $u_0 = u(a)$. Multiplying both sides of (2.7) by u' we see that

$$\frac{d}{dr}\left\{\frac{u'^2}{2} + \frac{1}{\nu - 1}\frac{1}{u^{\nu-1}} + mu\right\} \leqslant -\frac{N-1}{r}u'^2 \leqslant 0$$

on $[a, b]$. Hence, since $b \leqslant 1$ and $u'(a) = 0$ we obtain for $r \in [a, b]$,

$$\frac{u'^2(r)}{2} + \frac{1}{v-1}\frac{1}{u^{v-1}(r)} + mu(r) \leqslant \frac{1}{v-1}\frac{1}{u_0^{v-1}} + mu_0. \tag{2.8}$$

Assume that $u(b)$ is less than a positive number $\theta < \mu$. Then (2.8) and the fact that $u_0 \geqslant \mu$ yield that

$$u_0 \geqslant \frac{c}{\theta^{v-1}} \tag{2.9}$$

where $c = c(v, m, \mu) > 0$. On the other hand, (2.8) implies that

$$\int_{u(b)}^{u_0} \frac{dt}{\sqrt{2\int_t^{u_0} (m - s^{-v})\, ds}} \leqslant b - a \leqslant 1,$$

hence

$$c \int_{\mu}^{u_0} \frac{dt}{\sqrt{u_0 - t}} \leqslant 1 \tag{2.10}$$

for some $c = c(v, m, \mu)$. But the left hand side of (2.10) becomes arbitrarily large if u_0 does. Therefore u_0 is bounded by some number depending only on v, m, μ. Thus, we conclude from (2.9) the existence of a lower estimate for θ of the desired form. This concludes the proof. ∎

COROLLARY 2.1. *Assume* $\|h\|_\infty \leqslant m$. *Then there exists a number* $\theta = \theta(m, v) > 0$ *such that if* u *is a solution of* (2.1) *for which there are numbers* $a \leqslant \rho < 1$ *with* $u(\rho) < \theta$, $u'(a) = 0$ *and* $u'(\rho) \geqslant 0$, *then* u *is nondecreasing on* $[a, \rho]$.

Proof. If u were not nondecreasing on $[a, \rho]$, it is easy to see that there would be a point $a < b_1 < \rho$ such that $u'(b_1) < 0$, $u(b_1) < \theta$. We let $a_1 \geqslant a$ be the first point to the left of b_1 such that $u'(a_1) = 0$.

Note that

$$(r^{N-1}u')' = \left(\frac{1}{u^v} - h\right) r^{N-1} > \left(\frac{1}{u(a_1)^v} - m\right) r^{N-1}$$

on $[a_1, b_1]$, hence

$$0 > b_1^{N-1}u'(b_1) > \frac{1}{N}(b_1^N - a_1^N)\left(\frac{1}{u(a_1)^v} - m\right),$$

therefore

$$u(a_1) > \frac{1}{m^{1/\nu}}.$$

Then we apply the last lemma with this m and $\mu = m^{-1/\nu}$. To conclude the existence of a number $\theta(m, \nu) > 0$ such that if $\theta < \theta(m, \nu)$, the above situation is impossible. This concludes the proof. ∎

3. A Priori Estimates for the Singular Problem

We consider in this section the problem of finding a priori estimates for the radial solutions of

$$\begin{aligned} -\Delta u + (u \vee s)^{-\nu} &= h + \varepsilon \delta_0 \qquad \text{in } B \\ \frac{\partial u}{\partial n} &= 0 \qquad \text{on } \partial B, \end{aligned} \tag{3.1}$$

where δ_0 denotes the Dirac measure supported at 0, $\varepsilon > 0$, $s > 0$, $\nu > 1$. We will designate by $g_N(r)$ the fundamental solution for the Laplacian, namely

$$g_N(r) = \begin{cases} \dfrac{1}{\omega_N (N-2)} r^{2-N}, & \text{if} \quad N > 2 \\ -\dfrac{1}{2\pi} \log r, & \text{if} \quad N = 2, \end{cases} \tag{3.2}$$

where ω_N denotes the surface measure of the unit sphere in $\mathbb{R}^N$. We will prove two propositions, respectively yielding an upper and a lower estimate for the solutions of (3.1).

Proposition 3.1. *Assume $h \in C(\bar{B})$ is radially symmetric, and such that $\int_B h > 0$. Then, given $0 < \varepsilon_0 < \int_B h$ there exist numbers $s_0 > 0$, $\beta > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, $0 < s \leqslant s_0$ and any radial solution $u(r)$ to (3.1) for such s, ε one has*

$$u(r) - \varepsilon g_N(r) \leqslant \beta \qquad \forall r \in (0, 1].$$

Proof. Let us assume that $N > 2$. Since u satisfies (3.1), it follows that

$$v(r) = u(r) - \frac{\varepsilon}{\omega_N (N-2)} r^{2-N}$$

is of class C^2 on B and satisfies

$$-\Delta v \leqslant h \qquad \text{in } B.$$

Then, the same integration that leaded us to inequality (2.5) yields

$$\sup_B v \leqslant v(1) + c\,\|h\|_\infty,$$

so that to prove the proposition it suffices to show that $v(1)$ is uniformly bounded or equivalently that $u(1)$ is. To prove this, we argue by contradiction, namely we assume the existence of sequences $0 < \varepsilon_n < \varepsilon_0$, $s_n \downarrow 0$ and solutions u_n to (3.1) for $\varepsilon = \varepsilon_n$, $s = s_n$ so that

$$u_n(1) \to \infty. \tag{3.3}$$

It follows from Lemma 2.1 that we actually have

$$\inf_{\rho \leqslant r \leqslant 1} u_n \to \infty \tag{3.4}$$

for all $\rho > 0$. Now, the fact that

$$\int_B \frac{1}{(u_n \vee s_n)^\nu} = \int_B h + \varepsilon_n$$

and (3.4) imply that for any small number $\theta > 0$,

$$\inf_B u_n < \theta$$

for all sufficently large n. Since $u_n(0^+) = \infty$, this infimum must be attained at some number $\delta_n \in (0, 1]$. Note then that, from Lemma 2.2, we must have that u_n is nonincreasing on $(0, \delta_n]$ for any large n.

Next, for a small number $\theta > 0$ we denote

$$\delta'_n = \inf\{0 < \rho < 1 \mid u_n(r) \geqslant \theta \text{ for all } r \geqslant \rho\}. \tag{3.5}$$

From (3.4) we must necessarily have

$$\delta'_n \to 0 \qquad \text{as} \quad n \to \infty.$$

Moreover, Corollary 2.1 guarantees that if θ is chosen small enough then u_n is non-decreasing on $[\delta_n, \delta'_n]$ for large n since $u'_n(\delta'_n) \geqslant 0$.

At this point we introduce the change of variable $r = t^{-1/(N-2)}$. With the notation $\tilde{f}(t) = f(t^{-1/(N-2)})$, it is easily checked that $\tilde{u}_n$ satisfies

$$\tilde{u}''_n = \frac{1}{(N-2)^2\, t^\lambda}\,(f_n(\tilde{u}_n) - \tilde{h}(t)), \qquad t \in [1, \infty) \tag{3.6}$$

where $f_n(\tau) = (\tau \vee s_n)^{-\nu}$, $\lambda = 2(N-1)/(N-2)$. We use the notations $R_n = \delta_n^{2-N}$, $R'_n = \delta_n'^{2-N}$. We also denote by α the positive number

$$\alpha = \frac{1}{(N-2)^2} \int_1^\infty \frac{\tilde{h}(t)}{t^\lambda}\, dt = \frac{1}{(N-2)\,\omega_N} \int_B h.$$

We will prove the following facts.

(a) $\lim_{n\to\infty} \tilde{u}_n(R'_n + \tau) = -\alpha$ uniformly on compacts subsets of $[0, \theta/\alpha)$.

(b) $\lim_{n\to\infty} (R_n - R'_n) = \theta/\alpha$.

Before proving (a) and (b), let us see how a contradiction is derived from them. Consider a small, fixed number $\eta > 0$ to be chosen later. Without loss of generality we assume that θ was chosen so small that

$$(1-\eta)\, f_n(\tau) < f_n(\tau) - \tilde{h}(t) < (1+\eta)\, f_n(\tau) \tag{3.7}$$

whenever $\tau < \theta$, for all sufficiently large n. Let ω_n denote the function

$$\omega_n(s) = \tilde{u}_n(R_n + s).$$

Then

$$\omega_n''(s) \geqslant \frac{c}{R_n^\lambda}(1-\eta)\, f_n(\omega_n(s)) \qquad \text{if} \quad s<0 \tag{3.8}$$

$$\omega_n''(s) \leqslant \frac{c}{R_n^\lambda}(1+\eta)\, f_n(\omega_n(s)) \qquad \text{if} \quad s>0 \tag{3.9}$$

provided that $\omega_n(s) < \theta$. Here, $c = (N-2)^{-2}$. Thus (3.8) holds on the interval $[-(R_n - R'_n), 0]$.

Multiplying (3.8) by ω'_n and integrating we obtain that

$$\frac{1}{2}\, \omega'_n(s)^2 \geqslant \frac{c}{R^{\lambda_n}}(1-\eta) \left[\int_{\omega_n(0)}^{\omega_n(s)} f_n(\tau)\, d\tau \right] \tag{3.10}$$

for $s \in [-(R_n - R'_n), 0]$. Hence (3.10) implies

$$(1-\eta)^{1/2}\,(-s) \leqslant \int_{\omega_n(0)}^{\omega_n(s)} \left(\frac{2c}{R_n^\lambda} \int_{\omega_n(0)}^{z} f_n(\tau)\, d\tau \right)^{-1/2} dz. \tag{3.11}$$

Similarly, we see from (3.9) that if we fix $s \in (-(R_n - R'_n), 0)$ and $\bar{s} > 0$ is such that $\omega_n(\bar{s}) = \omega_n(s)$, then

$$(1+\eta)^{1/2}\, \bar{s} \geqslant \int_{\omega_n(0)}^{\omega_n(s)} \left(\frac{2c}{R_n^\lambda} \int_{\omega_n(0)}^{z} f_n(\tau)\, d\tau \right)^{-1/2} dz. \tag{3.12}$$

We conclude from (3.11) and (3.12) that

$$\omega_n(s) \geqslant \omega_n(-\mu s) \tag{3.13}$$

whenever $s \in (-(R_n - R'_n), 0)$, where

$$\mu = \left(\frac{1-\eta}{1+\eta}\right)^{1/2}.$$

Next we let $a = \mu\theta/2\alpha$. Then using (a) and (b) we obtain $-a/\mu \in (-(R_n - R'_n), 0)$ for large n, and

$$\alpha = -\lim_{n\to\infty} \omega'_n\left(-\frac{a}{\mu}\right) = \lim_{n\to\infty} c\int_{-a/\mu}^{0} \frac{f_n(\omega_n(\tau))\,d\tau}{(R_n+\tau)^{\lambda}}. \tag{3.15}$$

But, from (3.13), and the fact that f_n is nonincreasing we obtain

$$\int_0^a \frac{f_n(\omega_n(\tau))}{(R_n+\tau)^{\lambda}}\,d\tau \geqslant \int_{-a}^{0} \frac{f_n(\omega_n(\tau/\mu))}{(R_n-\tau)^{\lambda}}\,d\tau$$

$$\geqslant \left(1+\frac{(1-\mu)\,a}{\mu R_n}\right)^{-1} \int_{-a/\mu}^{0} \frac{f_n(\omega_n(\tau))}{(R_n+\tau)^{\lambda}}\,d\tau. \tag{3.16}$$

But

$$\omega'_n(a) = c\int_0^a \frac{f_n(\omega_n(\tau))}{(R_n+\tau)^{\lambda}}\,d\tau + o(1).$$

Therefore, from (3.14) we obtain

$$\omega'_n(a) \geqslant \mu\alpha + o(1).$$

Now let b_n denote the (unique) positive value where $\omega_n(b_n) = \theta$. Then $a < b_n$, from (3.13). Since ω_n may be assumed to be convex on $(0, b_n)$, we have that

$$\omega'_n(+\infty) \geqslant \omega'_n(a) + c\int_{b_n}^{\infty} \frac{d\tau}{(R_n+\tau)^{\lambda}}\,(f_n(\omega_n) - \tilde{h}). \tag{3.18}$$

But $\omega_n \geqslant \theta$ on $[b_n, \infty)$, hence the second term on the right hand side of (3.18) is $o(1)$. Using this, (3.17) and the fact that

$$\omega'_n(+\infty) = \frac{1}{(N-2)\,\omega_N}\,\varepsilon_n \leqslant \frac{1}{(N-2)\,\omega_N}\,\varepsilon_0,$$

we arrive to the inequality

$$\frac{\mu}{(N-2)\,\omega_N}\int_B h=\mu\alpha\leqslant\frac{1}{(N-2)\,\omega_N}\,\varepsilon_0. \tag{3.19}$$

Since, recall, μ given by (3.14) can be chosen arbitrarily close to 1 and $\varepsilon_0<\int_B h$, we have arrived to a contradiction which proves the proposition, modulo the proof of assertions (a) and (b), which we carry out next. To see (a), we first observe that

$$\lim_{n\to\infty}\tilde{u}_n'(R_n')=\lim_{n\to\infty}c\int_1^{R_n}\left(\frac{1}{\tilde{u}_n^{\nu}}-\tilde{h}\right)\frac{dt}{t^{\lambda}}=-c\int_1^{\infty}\frac{\tilde{h}\,dt}{t^{\lambda}}=-\alpha \tag{3.20}$$

where we have used the fact that $\tilde{u}_n^{-\nu}$ is uniformly bounded on $(1, R_n)$, (3.4) and dominated convergence. Now, $\tilde{u}_n$ is decreasing and convex on (R_n', R_n), hence for any given $\varepsilon>0$ we have that

$$v_n(t)\equiv u_n(R_n'+t)\geqslant\theta-(\alpha+\varepsilon)\,t$$

for all $t\in(0, R_n-R_n')$. It follows that $f_n(v_n)$ is uniformly bounded on $(0, \theta(\alpha+2\varepsilon)^{-1})$, thus

$$v_n''(t)=(f_n(v_n)-h)\,\frac{1}{(R_n'+t)^{\lambda}}\to 0$$

as $n\to\infty$, uniformly on t in this interval. From here and (3.20), assertion (a) follows.

Let us now prove (b). We assume that, on the contrary, there is a number $\eta>0$ such that $R_n\geqslant R_n'+\theta/\alpha+2\eta$. Note that the above proof also shows that $v_n(t)\to\theta-\alpha t$ uniformly on compacts subsets of $[0, \theta/\alpha)$. Since v_n is decreasing and convex on $[0, \theta/\alpha+\eta]$, it follows that

$$v_n'\left(\frac{\theta}{\alpha}+\eta\right)\to 0.$$

Hence, we have

$$v_n'\left(\frac{\theta}{\alpha}+\eta\right)-v_n'\left(\frac{\theta}{\alpha}-\eta\right)\to\alpha \tag{3.21}$$

and also,

$$v_n'\left(\frac{\theta}{\alpha}+3\eta\right)-v_n'\left(\frac{\theta}{\alpha}+\eta\right)\to 0. \tag{3.22}$$

But, on the other hand,

$$
\begin{aligned}
& v_n'\left(\frac{\theta}{\alpha}+\eta\right)-v_n'\left(\frac{\theta}{\alpha}-\eta\right) \\
& \qquad = c\int_{\theta/\alpha-\eta}^{\theta/\alpha+\eta} \frac{f_n(v_n(t))}{(R_n'+t)^\lambda}\,dt+o(1) \\
& \qquad \leqslant c\left(1-\frac{2\eta}{R_n}\right)^{-\lambda}\int_{\theta/\alpha+\eta}^{\theta/\alpha+3\eta}\frac{f_n(v_n(t-2\eta))}{(R_n'+t)^\lambda}\,dt+o(1).
\end{aligned}
\tag{3.23}
$$

Since f_n is nonincreasing and $v_n(t)\leqslant v_n(t-2\eta)$, we get, using (3.22), (3.23) that

$$
v_n'\left(\frac{\theta}{\alpha}+\eta\right)-v_n'\left(\frac{\theta}{\alpha}-\eta\right)=o(1),
$$

which contradicts (3.21). Thus (b) is stablished, and the proof of the proposition when $N>2$ is complete. For $N=2$ the same proof applies, except that the change of variable $r=t^{-1/(N-2)}$ in the above argument should be replaced by $r=e^{-t}$, $t\in[0,\infty)$. ∎

PROPOSITION 3.2. *Given $0<\varepsilon<\int_B h$, there exist numbers $s_0>0$, $\alpha_\varepsilon>0$ such that for any solution u to* (3.1) *for $0<s<s_0$, one has*

$$
\alpha_\varepsilon\leqslant u(r)\qquad \forall r\in[0,1].
$$

Proof. We assume $N>2$ and argue again by contradiction, assuming the existence of a sequence $s_n\downarrow 0$ and solutions u_n to (3.1) for $s=s_n$ such that

$$
\inf_B u_n\leqslant o(1).
$$

We consider again the change of variable $t=r^{2-N}$ and employ the notation of the proof of Proposition 3.1, so that $\tilde{u}_n$ satisfies equation (3.6). We denote by R_n the point where $\tilde{u}$ minimizes and for a small $\theta>0$ we define

$$
R_n'(\theta)=\sup\{R>1 \mid \tilde{u}(t)\geqslant\theta\ \forall 1\leqslant t\leqslant R\}.
$$

As we will see next, the above numbers are well defined for any sufficiently small θ. Observe first that

$$
|\tilde{u}_n(t)|=c\int_1^t (f_n(\tilde{u}_n)-h)\,\frac{dt}{t^\lambda}\leqslant 2c\int_1^\infty |h|\,\frac{dt}{t^\lambda}+\frac{\varepsilon}{(N-2)\,\omega_N},
$$

so that $\tilde{u}_n'$ is uniformly bounded. Note that this implies the following:

If $t_n \geqslant 1$ is a sequence such that

$$\tilde{u}_n(t_n) \leqslant o(1) \tag{3.26}$$

then $t_n \to +\infty$. Indeed, let us fix a number $\delta > 0$. Then,

$$\begin{aligned}\frac{\varepsilon}{(N-2)\,\omega_N} + c\int_1^\infty \tilde{h}\,\frac{dt}{t^\lambda} &\geqslant c\int_{t_n}^{t_n+\delta} f_n(\tilde{u}_n)\,\frac{dt}{t^\lambda}\\
&\geqslant \frac{1}{(\delta+t_n)^\lambda}\,c\int_{t_n}^{t_n+\delta} f_n(\tilde{u}_n)\,dt\\
&\geqslant \frac{1}{(\delta+t_n)^\lambda}\,c\int_{t_n}^{t_n+\delta} f_n(\tilde{u}_n(t_n)+k(t-t_n))\,dt\\
&\geqslant \frac{c}{(\delta+t_n)^\lambda}\,\frac{\delta}{\max\{s_n,\tilde{u}_n(t_n)+k\delta\}^\nu}\end{aligned}$$

for some $k > 0$. Hence, from (3.26),

$$\liminf_{n\to\infty}\,(t_n+\delta)^\lambda \geqslant \frac{K}{\delta^{\nu-1}}$$

for some $K > 0$. Since δ is arbitrary, $t_n \to +\infty$ follows. Thus, in particular, we have that $u_n(1)$ remains bounded below away from zero, so that the numbers (3.25) are well defined for any small $\theta > 0$. Moreover, this also shows that $R_n \to +\infty$. A similar argument also provides us with an estimate for the number $R'_n(\theta)$. In fact, for a large n,

$$c \geqslant \int_1^{R'_n(\theta)} f_n(\tilde{u}_n)\,\frac{dt}{t^\lambda} = \int_1^{R'_n(\theta)} \frac{1}{\tilde{u}_n^\nu}\,\frac{dt}{t^\lambda} \geqslant \frac{1}{R'_n(\theta)^\lambda}\int_1^{R'_n(\theta)} \frac{dt}{(\theta+k(R'_n(\theta)-t))^\nu},$$

from where it follows

$$R'_n(\theta) \geqslant \frac{c}{\theta^{(\nu-1)/\lambda}} \tag{3.27}$$

for any sufficiently smal $\theta > 0$.

Next we claim the following: Given $\rho > 0$ one has

$$\limsup_{n\to\infty}\,|\tilde{u}'_n(R'_n(\theta))| < \rho$$

for all sufficiently small $\theta > 0$. Let us fix a small number θ_0. Then $\tilde{u}_n$ will be decreasing and convex on $[R'_n(\theta_0), R_n]$ for all large n. We distinguish two subcases

(a) $\lim_{n\to\infty} R_n'(\theta_0) = +\infty$ and

(b) $R_n'(\theta_0) \leqslant R_0 < \infty$.

Assume that case (a) holds. We claim that

$$\tilde{u}_n'(R_n'(\theta_0)) \to 0 \qquad \text{as} \quad n \to \infty,$$

from where (3.28) follows immediately. In fact assume that, otherwise, we have

$$-\tilde{u}_n'(R_n'(\theta_0)) \geqslant \alpha_0 > 0 \qquad \text{for all large } n.$$

Observe then that the sequence

$$v_n(t) = \tilde{u}_n(R_n'(\theta_0) - t)$$

satisfies

$$v_n''(t) = \frac{1}{(R_n'(\theta_0) - t)^\lambda} \left(\frac{1}{v_n^\nu(t)} - \tilde{h}(R_n'(\theta_0) - t) \right), \qquad 0 < t < R_n'(\theta_0) - 1.$$

Hence $v_n''(t) \to 0$ uniformly on compact subsets of $[0, \infty)$. It follows that for any $s > 0$ one has

$$v_n(s) \geqslant \theta_0 + \alpha_0 s + o(1)\, s^2$$

and $v_n'(s) \to \alpha_0$. It follows that v_n and hence $\tilde{u}_n$ may take arbitrarily large values at points where $\tilde{u}_n'$ is negative. But this is impossible, from the result of Proposition 3.1. Hence (3.29) holds and the proof of (3.28) in case (a) is complete. Next we assume that (b) holds. Since $\tilde{u}_n$ is decreasing and convex on $(R_n'(\theta_0), R_n'(\theta))$, where $\theta < \theta_0$, we have that, using (3.27),

$$-\tilde{u}_n'(R_n'(\theta)) \leqslant \frac{\theta_0 - \theta}{R_n'(\theta) - R_n'(\theta_0)} \leqslant \frac{\theta_0}{c\theta^{(1-\nu)/\lambda} + R_0} < \rho,$$

in case that θ is chosen small enough, thus (3.28) holds and the proof of the claim is now complete.

Next we fix small numbers $\rho, \eta > 0$ and choose a $\theta > 0$ such that (3.28) holds for this ρ and so that

$$(1-\eta)\, f_n(\tau) \leqslant f_n(\tau) + \tilde{h}(t) \leqslant (1+\eta)\, f_n(\tau) \qquad \forall t,$$

for all large n, whenever $\tau < \theta$. Then an integration similar to that leading to (3.11) and (3.12) yields that for some $c > 0$

$$|\tilde{u}_n'(R_n'(\theta))| \geqslant \left((1-\eta)\, 2c \int_{\tilde{u}_n(R_n)}^{\theta} f_n(\tau)\, d\tau \right)^{1/2}$$

$$\geqslant \left(\frac{1-\eta}{1+\eta} \right)^{1/2} |\tilde{u}_n'(R_n''(\theta))| \tag{3.30}$$

where $R_n''(\theta)$ is the first value $R > R_n$ at which $\tilde{u}_n(R) = \theta$. Note also that u_n is increasing on $[R_n''(\theta), \infty)$ and that $R_n''(\theta) \to \infty$.

Now, we have

$$\tilde{u}_n'(+\infty) = \tilde{u}_n'(R_n''(\theta)) + c \int_{R_n''(\theta)}^{\infty} \left(\frac{1}{\tilde{u}_n^{\nu}} - \tilde{h} \right) \frac{dt}{t^{\lambda}}. \tag{3.31}$$

The second summand in the right hand side of (3.31) is $o(1)$ since $R_n''(\theta) \to \infty$. Then we conclude, using (3.30) and (3.31) that

$$\frac{\varepsilon}{(N-2)\,\omega_N} \leqslant \left(\frac{1+\eta}{1-\eta} \right)^{1/2} \rho, \tag{3.32}$$

which leads to a contradiction, since ρ can be chosen arbitrarily small. This concludes the proof in case that $N > 2$. This proof carries over similarly when $N = 2$, using the change of variable $r = e^{-t}$. ∎

Remark 3.1. A priori estimates for (3.1) in the sense of the above two propositions can also be obtained in the case $\varepsilon > \int_B h$. Indeed, a lower positive estimate as in Proposition 3.2 follows by observing that

$$\limsup_{n \to \infty} |\tilde{u}_n'(R_n'(\theta))| \leqslant \limsup_{n \to \infty} \int_1^{R_n(\theta)} \frac{\tilde{h}}{t^{\lambda}}\, dt \leqslant \frac{1}{(N-2)\,\omega_N} \left(\int_B h + \rho \right),$$

where ρ is any fixed small positive number, provided that θ is chosen small enough. Then, following the argument in the above proof, one sees that inequality (3.32) can be replaced by

$$\varepsilon \leqslant \left(\frac{1+\eta}{1-\eta} \right)^{1/2} \left(\int_B h + \rho \right)$$

which also leads to a contradiction if η, ρ are chosen sufficiently small. Thus a lower estimate $\alpha_\varepsilon > 0$ as in Proposition 3.2, does exist.

Now, an upper estimate for $u_n - \varepsilon g_N$ also exists. Indeed, it is enough to see that $u_n(1)$ remains bounded above. Otherwise we may assume, as before, $u_n \to \infty$ a.e. on B. But $u_n \geqslant \alpha_\varepsilon > 0$ and

$$\int_B \frac{1}{u_n^\nu} = \int_B h + \varepsilon$$

for large n, hence $\int_B h + \varepsilon = 0$ which is a contradiction. This proves the analogue of Proposition 3.1 for this case.

4. Existence for the Singular Problem

Our main purpose in this section is to prove Theorem 1.1. For the proof of this result we begin by observing that the a priori estimates of Proposition 3.2 for $\varepsilon < \int_B h$ (or those of Remark 3.1 for $\varepsilon > \int_B h$) reduce the problem to showing that for any $s > 0$ sufficiently small the problem

$$\begin{aligned} -\Delta u + (u \vee s)^{-\nu} &= h + \varepsilon\delta_0 && \text{in } B \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \partial B \end{aligned} \tag{4.1}$$

has a radial solution. We will do this by means of a variational argument. Let us observe first that (4.1) is equivalent to the problem

$$\begin{aligned} -\Delta v + f_s(\varepsilon\varphi(r) + v) &= h_\varepsilon && \text{in } B \\ \frac{\partial v}{\partial n} &= 0 && \text{on } \partial B \end{aligned} \tag{4.2}$$

where $f_s(\tau) = (\tau \vee s)^{-\nu}$ and $\varphi(r)$ is a $C^2(\bar{B} \setminus \{0\})$ radial function such that $\varphi'(1) = 0$ and $\psi(r) \equiv \varphi(r) - g_N(r)$ is of class $C^2(\bar{B})$, where g_N is defined in (3.2), and such that $\Delta\psi \geqslant 0$ in $\bar{B}$. It is easily seen that such a function φ indeed exists. h_ε is given by $h_\varepsilon = h + \varepsilon\, \Delta\psi$, so that $h_\varepsilon \in C(\bar{B})$ and $\int_B h > 0$.

We look for a classical solution to (4.2), which is equivalent to searching for a critical point of the functional defined in H_r^1 by

$$J_s(v) = \tfrac{1}{2} \int_B |\nabla v|^2 + \int_B F_s(\varepsilon\varphi + v) - \int_B h_\varepsilon v \tag{4.3}$$

where $F_s(z) = \int_0^z f_s(\tau)\, d\tau$ and H_r^1 denotes the subspace of radial elements of $H^1(B)$. We need a lemma.

LEMMA 4.1. *The functional J_s given by* (4.3) *satisfies the Palais–Smale condition for any sufficiently small $s>0$.*

Proof. Let $\{v_n\}$ be a Palais–Smale sequence for J_s, namely $\{J_s(v_n)\}$ is bounded and $J_s'(v_n) \to 0$.

We need to show that $\{v_n\}$ is precompact in H_r^1. A standard argument yields that it actually suffices to show that $\{v_n\}$ is bounded in the H^1-norm. To do this, we decompose

$$v_n = w_n + \lambda_n$$

where $\lambda_n \in \mathbb{R}$ and $\int_B w_n = 0$. We will first show that w_n is bounded in H^1. Note that since $J_s'(v_n) \to 0$ we have that

$$\int_B |\nabla w_n|^2 = \int_B (h_\varepsilon - f_s(u_n))\, w_n + o(1)\, \|w_n\|_{H^1}. \tag{4.4}$$

But $\|w_n\|_{H^1} \leqslant c\, \|\nabla w_n\|_{L^2}$. Since $\int_B w_n = 0$. But f_s is uniformly bounded, hence (4.4) yields

$$\|w_n\|_{H^1}^2 \leqslant (cs_1\, \|h_\varepsilon\|_\infty + o(1))\, \|w_n\|_{H^1},$$

from where the boundedness of w_n follows. Now, again from $J'(v_n) \to 0$ we have that

$$\int_B f_s(w_n + \lambda_n) = \int_B h_\varepsilon + o(1) \tag{4.5}$$

If $\lambda_n \to +\infty$, the fact that w_n is H^1-bounded would imply $f_s(w_n + \lambda_n) \to 0$ a.e. Then dominated convergence would yield $\int_B h_\varepsilon = 0$, a contradiction. Instead, if $\lambda_n \to \infty$ then $f_s(w_n + \lambda_n) \to s^{-\mu}$, hence (4.5) would imply

$$s^{-\mu}\, |B| = \int_B h_\varepsilon$$

which is again impossible if s is sufficiently small. Then λ_n is uniformly bounded in such case. We conclude that the Palais–Smale condition holds true for J_s provided that

$$s < \left(\frac{1}{|B|} \int_B h_\varepsilon \right)^{-1/\nu}.$$

Next we show that J_s has a critical point for any small $s>0$. For this, we consider again the decomposition

$$H_r^1 = \Lambda \oplus W$$

where Λ is the space of constant functions and $W = \{w \in H^1_r \mid \int_B w = 0\}$. We note that J_s is bounded below on W. Indeed,

$$J_s(w) \geqslant \frac{1}{2} \int_B |\nabla w|^2 - c(s, \|h_\varepsilon\|_\infty) \left(\int_B w^2 \right)^{1/2}$$

and $\int_B |\nabla w|^2 \geqslant c \int_B w^2$ from where the assertion follows.

On the other hand, for $\lambda \in \Lambda$ we have

$$\frac{1}{\lambda} J_s(\lambda) = \int_B \frac{F_s(\varepsilon\varphi + \lambda)}{\lambda} - \int_B h$$

hence

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} J_s(\lambda) = \frac{|B|}{s^\nu} - \int_B h > 0,$$

for all sufficiently small s. We conclude that

$$\lim_{|\lambda| \to \infty} J_s(\lambda) = -\infty.$$

Since J_s also satisfies Palais–Smale for small s, we see that the assumptions of Rabinowitz's Saddle Point Theorem, see, e.g., [8], are satisfied on the decomposition $\Lambda \oplus W$. We conclude the existence of at least one critical point of J_s for all small s, and the proof of the theorem is complete. ∎

5. Proof of Theorems 1.2 and 1.3

In this section we prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. From Theorem 1.1, we know that there is a solution u_ε to the problem

$$\begin{aligned} -\Delta u + u^{-\nu} &= h + \varepsilon\delta_0 && \text{in } B \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \partial B \\ u &> 0 && \text{in } B \end{aligned}$$

for any $0 < \varepsilon < \int_B h$. Let us set

$$v_\varepsilon = u_\varepsilon - \varepsilon g_N \tag{5.1}$$

where g_N is defined in (3.2), so that v_ε satisfies

$$\Delta v_\varepsilon = u_\varepsilon^{-\nu} - h \qquad \text{in } B \tag{5.2}$$

and is uniformly bounded above for small ε, thanks to Proposition 3.1. Let us assume first that

$$\inf_B u_\varepsilon \geqslant c > 0$$

for all small ε. Then $u_\varepsilon^{-\nu}$ is uniformly bounded, hence elliptic estimates imply that we may assume, passing to a subsequence if necessary, that $v_\varepsilon \to v$ in the $C^1(\bar{B})$ sense as $\varepsilon \downarrow 0$. This v will clearly be positive and satisfy weakly

$$-\Delta v + v^{-\nu} = h \qquad \text{in } B$$

$$\frac{\partial v}{\partial n} = 0 \qquad \text{on } \partial B.$$

Hence v solves (1.1) and the theorem follows in case that (5.3) holds. Thus, we now assume that, passing to a subsequence we have

$$\inf_B u_\varepsilon \to 0.$$

Let $\delta_\varepsilon > 0$ be a point where this minimum is attained. Then $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$, as e.g. the argument right after (3.26) shows. Moreover, this argument also shows that for any $\theta > 0$ sufficiently small the number

$$\delta'_\varepsilon = \inf\{0 < \rho < 1 \mid u_\varepsilon(r) \geqslant \theta, \forall \rho \leqslant r \leqslant 1\}$$

is well defined. Let us fix such a small θ. Then, as usual, u_ε is increasing on $[\delta_\varepsilon, \delta'_\varepsilon]$. We claim that the following estimate hold,

$$u_\varepsilon(r) \geqslant c(r - \delta_\varepsilon)^{2/(\nu+1)}, \qquad \forall r \in [\delta_\varepsilon, 1] \tag{5.4}$$

where c is independent of ε for ε small. In fact fact, note that

$$r^{N-1} u'_\varepsilon(r) = \int_{\delta_\varepsilon}^{r} \left(\frac{1}{u_\varepsilon^\nu(s)} - h(s) \right) s^{N-1}\, ds. \tag{5.5}$$

We assume that θ was chosen so small that

$$\frac{1}{\tau^\nu} - h(s) \geqslant \frac{1}{2\tau^\nu}, \qquad \forall 0 < \tau < \theta, \quad \forall s.$$

Then, from (5.5) we see that for $r \in (\delta_\varepsilon, \delta'_\varepsilon)$ we have

$$r^{N-1} u'_\varepsilon(r) \geqslant \frac{1}{2N} \frac{1}{u^\nu(r)} (r^N - \delta_\varepsilon^N).$$

Hence

$$u_\varepsilon^{\nu+1}(r) - u_\varepsilon^{\nu+1}(\delta_\varepsilon) \geqslant \frac{\nu+1}{2N} \int_{\delta_\varepsilon}^{r} (r - \delta_\varepsilon)\, dr$$

$\forall r \in (\delta_\varepsilon, \delta'_\varepsilon)$, from where we obtain

$$u_\varepsilon(r) \geqslant c_1 (r - \delta_\varepsilon)^{2/(\nu+1)}, \qquad r \in (\delta_\varepsilon, \delta'_\varepsilon),$$

some $c_1 > 0$. But $u_\varepsilon \geqslant \theta$ on $(\delta'_\varepsilon, 1)$, hence letting $c = \min\{\theta, c_1\}$ we obtain the validity of (5.4). Note that, in particular $u_\varepsilon^{-\nu}$ remains uniformly bounded on each annulus $A_\rho = \{x \mid \rho < |x| < 1\}$. Since

$$\Delta u_\varepsilon = u_\varepsilon^{-\nu} - h \qquad \text{in } A_\rho,$$

and u_ε is uniformly bounded in A_ρ, we conclude the existence of a subsequence of $\{u_\varepsilon\}$ converging uniformly on compact subsets of $\bar{B} \backslash \{0\}$ to a function $u \in C^2(\bar{B} - \{0\})$ satisfying

$$\begin{aligned} \Delta u &= u^{-\nu} - h && \text{in } \bar{B} - \{0\} \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \partial B \\ u &> 0 && \text{in } \bar{B} \backslash \{0\}. \end{aligned} \tag{5.6}$$

Note that $u \in L^\infty(B)$, thanks to Proposition 3.1. But we also have, from (5.4) that

$$u(r) \geqslant c r^{2/(\nu+1)}, \qquad r \in (0, 1]. \tag{5.7}$$

It follows that $u^{-\nu} - h \in L^p(B)$ for some $p > N/2$. Let v be the unique solution of

$$\begin{aligned} \Delta v &= u^{-\nu} - h && \text{in } B \\ v &= u && \text{on } \partial B. \end{aligned}$$

Then $v \in W^{2,p}(B) \subset C^0(\bar{B})$. Then $\varphi = v - u$ is bounded and satisfies

$$\Delta \varphi = 0 \quad \text{on } \bar{B} \backslash \{0\}, \qquad \varphi = 0 \quad \text{on } \partial B.$$

It follows from a standard argument that $\varphi \equiv 0$, so that u extends to a full solution of (1.1). This concludes the proof of the theorem. ▮

Proof of Theorem 1.3. The proof of existence of a solution to (1.6) follows in exactly the same way as that of (1.1), $d=1$ having been chosen only for notational simplicity. In order to prove uniqueness for a large d, we begin by claiming that for any sequence $d_n \to \infty$ and solutions u_n to (1.6) for $d=d_n$ one has

$$u_n \to \left(\frac{1}{|B|}\int_B h\right)^{-1/\nu} \qquad \text{uniformly on } \bar{B}. \tag{5.8}$$

Let us observe that, since a solution u to (1.6) satisfies this equation distributionally on B, exactly the same argument that leaded us to estimate (5.7) also shows that this u must satisfy

$$u(r) \geqslant c r^{2/(\nu+1)} d^{-1/(\nu+1)} \tag{5.9}$$

for a certain universal constant c. Now, we have

$$\Delta u_n = f_n \qquad \text{in } B \tag{5.10}$$

where $f_n = d_n^{-1}(u^{-\nu} - h)$. Then, from (5.9) we see that

$$|f_n(r)| \leqslant c\left(\frac{r^{-2\nu/(\nu+1)}}{d_n^{1/(\nu+1)}} + \frac{1}{d_n}\|h\|_\infty\right).$$

It follows that for some $p > N/2$ we have

$$\int_B |f_n|^p \to 0 \qquad \text{as} \quad n \to \infty. \tag{5.11}$$

Now, from (5.10) we have

$$r^{N-1}|u_n'(r)| \leqslant c\left(\int_B |f_n|^p\right) r^{N/q} \tag{5.12}$$

where $q = p/(p-1) < N/(N-2)$. Then, it follows from (5.12) and (5.11) that

$$\int_0^1 |u_n'(r)|\, dr \to 0 \qquad \text{as} \quad n \to \infty \tag{5.13}$$

But, on the other hand, $\int_B (u_n^{-\nu} - h) = 0$, hence there is a $r_n \in [0, 1]$ such that

$$u_n(r_n) = \left(\frac{1}{|B|}\int_B h\right)^{-1/\nu}.$$

This and (5.13), clearly imply the validity of (5.8).

Next we assume that the uniqueness assertion of the theorem does not hold, so that there exists a sequence $\{d_n\} \to \infty$ and solutions $u_n \neq v_n$ to (1.6) for $d = d_n$. We will show that this is not possible for large n. Let us set $w_n = u_n - v_n$. Then w_n satisfies

$$\begin{aligned} \Delta w_n + a_n(r)\, w_n = 0 \qquad & \text{in } B \\ \frac{\partial w}{\partial n} = 0 \qquad & \text{on } \partial B \end{aligned} \tag{5.14}$$

where

$$a_n(r) = \frac{1}{d_n}\int_0^1 \frac{\nu\, dt}{(u_n(r) + t(u_n(r) - v_n(r)))^{\nu+1}}.$$

Then from (5.8) we have that

$$d_n a_n(r) \to \nu \left(\frac{1}{|B|}\int_B h\right)^{(\nu+1)/\nu} > 0, \tag{5.15}$$

uniformly, as $n \to \infty$. Let λ_1 denote the first nonzero radial eigenvalue of the problem

$$\begin{aligned} \Delta\phi + \lambda\phi = 0 \qquad & \text{in } B \\ \frac{\partial \phi}{\partial n} = 0 \qquad & \text{on } \partial B. \end{aligned}$$

Then (5.15) implies that for n large enough one has

$$0 < a_n(r) < \lambda_1,$$

so that (5.14) implies $w_n \equiv 0$, a contradiction which finishes the proof of the theorem. ∎

Remark 5.1. With a method similar to the above used, one can prove that if, say, $d = 1$ the following holds: There exist positive constants α, β such that for any radial solution of (1.1) one has

$$\alpha r^{2/(\nu+1)} \leqslant u(r) \leqslant \beta.$$

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