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POSITIVE SOLUTIONS OF A SEMILINEAR ELLIPTIC EQUATION ON A COMPACT MANIFOLD

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LET (M, g) be a compact Riemannian manifold of dimension $N \geq 1$. We consider the problem of finding positive solutions to the following semilinear elliptic equation

$$\Delta u + \lambda u - h(x)u^p = 0 \qquad \text{in } M, \tag{1}$$

where Δ represents the Laplace–Beltrami operator associated to the metric g. $p > 1, \lambda$ are constants and $h(x)$ a given function on M.

Equation (1) was considered by Kazdan and Warner [1] in the context of the classical problem of conformally deforming a given metric on M to another with prescribed scalar curvature. Among other results, they established that if h is smooth and $h > 0$ on M, then problem (1) possesses a unique positive solution for any $\lambda > 0$. They also conjectured that $h \geq 0$, $h \not\equiv 0$ should indeed suffice for the validity of this result. The situation, however, turns out to be more subtle in such a case, as has been recently established by Ouyang in [2]. To state his result, we let M_0 be the interior of the set where h vanishes. Denote by $\lambda_1(M_0)$ the first eigenvalue of the problem

$$\Delta u + \lambda u = 0 \qquad \text{in } M_0$$

$$u = 0 \qquad \text{on } \partial M_0,$$

where we understand $\lambda_1(M_0) = +\infty$ in the case where M_0 is empty. Under certain additional regularity assumptions that we discuss below, the following result holds.

THEOREM 1. Assume $h \in C^\infty(M)$ is nonnegative and not identically zero. Then problem (1) has a unique positive solution u_λ for all $0 < \lambda < \lambda_1(M_0)$. If M_0 is nonempty, then no positive solution exists if $\lambda \geq \lambda_1(M_0)$. Moreover,

$$\lim_{\lambda \to \lambda_1(M_0)} \|u_\lambda\|_{L^2(M)} = +\infty.$$

It should be remarked that some steps in the proof of this result in [2] require regularity on the boundary of the set $M_+ = \{x \in M \mid h(x) > 0\}$. This is the case of the argument on pp. 522–524 of [2], where, also, the additional fact that ∂M_+ and ∂M_0 coincide is implicitly used. We note that regularity of ∂M_0 is also used in the argument on p. 521, where Hopf's lemma is applied.

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In this paper we will provide a short proof of the above result based on a direct variational approach and a tool introduced by Brezis and Oswald in [3]. As well as relaxing the above-mentioned regularity conditions, our proof also avoids a delicate a priori estimate contained in lemma 4 of [2], whose proof makes use of smoothness of the coefficient h. We point out that Kazdan and Warner's result only needs $h \in L^q(M)$ for some $q > N$ and $h > 0$ a.e., see [1, theorem 2.11]. We will only require that h satisfies this integrability condition. On the other hand, we will assume the existence of an open set M_0 with boundary of measure zero such that $h = 0$ a.e. on M_0 and $h > 0$ a.e. on $M \backslash M_0$. Note that if h is continuous, this assumption is equivalent to the fact that the boundary of the set where h is positive has a measure of zero.

Before stating our first result, we make precise the definition of $\lambda_1(M_0)$ when M_0 is an open subset of M with a boundary of measure of zero. For an open neighborhood Ω of M_0 with smooth boundary, we define classically its first Dirichlet eigenvalue $\lambda_1(\Omega)$ as

$$\lambda_1(\Omega) = \inf\left\{\int_\Omega |\nabla u|^2 \,\middle|\, u \in H_0^1(\Omega),\ \int_\Omega u^2 = 1\right\}.$$

Then we let

$$\lambda_1(M_0) = \sup\{\lambda_1(\Omega) \mid \Omega \text{ is a smooth neighborhood of } \bar{M}_0\}.$$

We also use the convention $\lambda_1(M_0) = +\infty$ in the case where M_0 is empty. We designate by $H_*^1(M_0)$ the space of all functions $u \in H^1(M)$ such that $u = 0$ a.e. on $M \backslash M_0$. (This space coincides with $H_0^1(M_0)$ if M_0 has a sufficiently regular boundary.) The definition of $\lambda_1(M_0)$ yields, after a simple approximation argument involving the fact that ∂M_0 has measure zero,

$$\lambda_1(M_0) = \inf\left\{\int_{M_0} |\nabla u|^2 \,\middle|\, u \in H_*^1(M_0),\ \int_{M_0} u^2 = 1\right\}. \tag{2}$$

Moreover, this infimum is attained at some nonnegative function $\phi_1 \in H_*^1(M_0) \cap C^\infty(M_0)$ which satisfies

$$\Delta\phi_1 + \lambda_1(M_0)\phi_1 = 0 \qquad \text{in } M_0.$$

We call such a ϕ_1 a positive eigenfunction associated with $\lambda_1(M_0)$. Note that the Strong Maximum Principle implies $\phi_1 > 0$ on any component of M_0 where it does not vanish identically.

We assume in the next result that $h \in L^q(M)$ for some $q > N$. By a solution to (1) in such a case we understand a function $u \in W^{2,q}(M)$ satisfying (1) in the strong sense. Observe that such a u is actually of class $C^1(M)$. In the case that h is Hölder continuous, this concept reduces to the classical one.

Theorem 2. Assume $h \in L^q(M)$ for some $q > N$ and that there exists an open subset M_0 of M with boundary of measure zero, such that $h = 0$ a.e. on M_0 and $h > 0$ a.e. in $M \backslash M_0$. Then problem (1) has a unique positive solution u_λ for all $0 < \lambda < \lambda_1(M_0)$. If M_0 is nonempty, then no positive solution exists if $\lambda \geq \lambda_1(M_0)$ and

$$\lim_{\lambda \to \lambda_1(M_0)} \|u_\lambda\|_{L^2(M)} = +\infty.$$

The proof of theorem 1 in [2] actually provides an interesting by-product concerning the behavior of the solution u_λ: it remains uniformly bounded on compact subsets of the set where h is positive as $\lambda \to \lambda_1(M_0)$. We will provide an alternative proof of this fact. Moreover, our next result additionally establishes that the blow-up of u_λ as $\lambda \to \lambda_1(M_0)$ is uniform on compact subsets of M_0 provided that M_0 is connected.

In the sequel we denote by M_+ the set

$$M_+ = \{x \in M \mid h(x) > 0\}. \tag{3}$$

THEOREM 3. Under the assumptions of theorem 2:

(i) assume that h is continuous. Then for every compact set $K \subset M \subset M_+$ we have

$$\sup_{\lambda \in I} \|u_\lambda\|_{L^\infty(K)} < +\infty,$$

where I is any bounded subinterval of $(0, \lambda_1(M_0))$;

(ii) if M_0 is connected and nonempty, then for any compact set $K \subset M_0$ we have

$$\inf_K u_\lambda \to +\infty \qquad \text{as } \lambda \to \lambda_1(M_0).$$

Before going into the proofs of these results, we remark that Ouyang has continued his study of problem (1) in [4], where he considers the case in which h changes sign in M and $\int_M h > 0$.

The proof of theorem 2 is based on direct variational arguments applied to the functional J on $H^1(M)$ with values on $(-\infty, \infty]$ defined in the following manner.

$$J(u) = \frac{1}{2}\left\{\int_M |\nabla u|^2 - \lambda u^2\right\} + \frac{1}{p+1}\int_M h|u|^{p+1} \tag{4}$$

if $\int_M h|u|^{p+1} < +\infty$ and $J(u) = +\infty$ otherwise.

By a critical point of J we understand a $u \in H^1(M)$ with $J(u) < +\infty$ such that

$$\frac{\partial}{\partial t} J(u + t\varphi)|_{t=0} = 0 \qquad \text{for all } H^1(M) \cap L^\infty(M). \tag{5}$$

Hence, if $u \geq 0$ is a critical point of J, then u solves (1) in the following weak sense

$$\int_M \nabla u \, \nabla \varphi + \int_M h u^p \varphi = \lambda \int_M u\varphi \qquad \text{for all } \varphi \in H^1(M) \cap L^\infty(M). \tag{6}$$

For $t \geq 2$ and $R > 0$ we choose $\varphi = (\min\{u, R\})^{t-1}$ as a test function in (6). Applying the Sobolev embedding and letting $R \to \infty$, we arrive at an inequality of the form

$$\|u\|_{L^{t+(2t/(N-2))}(M)} \leq C\|u\|_{L^t(M)} \tag{7}$$

in the case where $N \geq 3$. It follows from (7) that $u \in L^t(M)$ for all $t \geq 2$. Obviously the same conclusion remains if $N \leq 2$ since $u \in H^1(M)$. Standard elliptic regularity applied to (6) then shows that $u \in W^{2,q}(M)$, so that u solves (1) in the strong sense. Moreover, the Strong Maximum Principle for $W^{2,N}$-solutions (see [5]) implies $u > 0$ in the case where u is not identically zero. Thus, the problem of finding positive solutions to (1) is equivalent to the one of finding nonnegative, not identically zero critical points of J.

Proof of theorem 2. Standard arguments show that the functional J defined by (4) is weakly lower semicontinuous. Assume first $0 < \lambda < \lambda_1(M_0)$. We will show that J possesses a minimizer. To do this, it suffices to verify that J is coercive, that is

$$J(u) \to +\infty \qquad \text{as } \|u\|_{H^1(M)} \to +\infty. \tag{8}$$

We assume the contrary, namely the existence of a sequence $\{u_n\}$ such that $\|u_n\|_{H^1(M)} \to +\infty$ and $J(u_n)$ remains bounded above. Observe that this implies $\|u_n\|_{L^2(M)} \to +\infty$. Define $\hat{u}_n = u_n/\|u_n\|_{L^2(M)}$. Then we find that

$$\limsup_{n\to\infty}\left\{\frac{1}{2}\left(\int_M |\nabla \hat{u}_n|^2 - \lambda\right) + \frac{1}{p+1}\int_M h|\hat{u}_n|^{p+1}\|u_n\|_{L^2(M)}^{p-1}\right\} \leq 0. \tag{9}$$

In particular, $\|\hat{u}_n\|_{H^1(M)}$ is bounded. Thus, we may assume $\hat{u}_n \to \hat{u}$ weakly in $H^1(M)$ and strongly in $L^2(M)$. From (9), the fact that $\|u_n\|_{L^2(M)} \to \infty$ and Fatou's lemma, we obtain that $\int_M h|\hat{u}|^{p+1} = 0$. Since $h > 0$ a.e. on $M\backslash M_0$, this immediately yields a contradiction in the case where $M_0 = \emptyset$. Assume the contrary. Then $\hat{u} \in H_*^1(M_0)$. Again from (9) we obtain

$$\int_{M_0} |\nabla \hat{u}|^2 \leq \lambda.$$

This contradicts the characterization $\lambda_1(M_0)$ in (2) since $\|\hat{u}\|_{L^2(M_0)} = 1$ and, therefore, (8) holds true. We conclude that J possesses a minimizer $u_0 \in H^1(M)$. u_0 is not identically zero since evaluating J at the constant function $t > 0$ we get

$$J(t) = -\frac{\lambda}{2}t^2|M| + \frac{t^{p+1}}{p+1}\int_M h < 0 = J(0),$$

in the case where t is chosen sufficiently small.

Finally, since $|u_0|$ also minimizes J, we conclude the existence of a nonnegative, nonzero critical point of J in the sense of (6) and, hence, of a positive solution to (1). Existence is thus established in the case where $0 < \lambda < \lambda_1(M_0)$. For uniqueness, as well as for the proof of the second part of the theorem, we will make use of the following fact.

CLAIM. For any $\lambda > 0$, there is at most one critical point $u_0 > 0$ of J and it must be a minimizer.

We prove this claim by making use of a tool introduced by Brezis and Oswald [3]. We consider the functional I defined on the convex cone of nonnegative functions v such that $v^{1/2} \in H^1(M)$ as

$$I(v) = \frac{1}{2}\int_M |\nabla v^{1/2}|^2 - \frac{\lambda}{2}\int_M v + \frac{1}{p+1}\int_M hv^{(p+1)/2}$$

if $\int_M hv^{(p+1)/2}$ is finite, and $I(v) = +\infty$ otherwise.

Then, $J(u) = I(u^2)$ for all $u \in H^1(M)$. Let $u_0 > 0$ be a critical point of J. The claim clearly follows if we prove:

(a) $v_0 = u_0^2$ minimizes I;

(b) I has at most one positive minimizer.

Let us prove these facts. For $v_1 \geq 0$ such that $v_1^{1/2} \in H^1(M) \cap L^\infty(M)$, we consider the function

$$\varphi(t) = I(v_0 + tk),$$

where $t \in [0, 1]$ and $k = v_1 - v_0$. It is easy to see that φ is finite, twice differentiable on $[0, 1)$ and

$$\varphi'(0) = \frac{\partial}{\partial t} J(u_0 + t\omega)|_{t=0}$$

where $\omega = 1/2((v_1/u_0) - u_0)$.

Note that $\omega \in H^1(M) \cap L^\infty(M)$ since $u_0 > 0$ on M and $u_0 \in W^{2,q}(M) \subset C^1(M)$. Hence, by definition of a critical point of J, $\varphi'(0) = 0$.

Next, we compute $\varphi''(t)$. We find,

$$\varphi''(t) = \int_M \left|\nabla\left(\frac{k}{v_0 + tk}\right)\right|^2 (v_0 + tk) + \frac{p-1}{4}\int_M h(v_0 + tk)^{(p-3)/2} k^2$$

for $t \in [0, 1)$. Note that this number is well defined since $v_0 + tk \geq (1 - t)v_0$ and v_0 is away from zero in M. It is easily checked that $\varphi''(t) > 0$ for all $t \in [0, 1)$, and, hence, φ is strictly convex on $[0, 1]$. Since $\varphi'(0) = 0$, we obtain that $I(v_0) < I(v_1)$ and (a) follows. The same argument shows (b) and the validity of the claim is proved.

In particular, the claim implies the uniqueness assertion of the first part of the theorem. Let us prove the second part.

Let us assume $\lambda \geq \lambda_1(M_0)$. Let $\phi_1 \in H^1_*(M_0)$ be a positive eigenfunction associated to $\lambda_1(M_0)$. Note that, from the strong maximum principle, the function ϕ_1 does not solve the equation

$$\Delta u + \lambda u = 0 \qquad \text{in } M.$$

Therefore, we can find a function $u_0 \in H^1(M)$ such that the number

$$a = \int_{M_0} (\nabla u_0 \nabla \phi_1 - \lambda u_0 \phi_1)$$

is strictly positive. Let us write $u_t = t\phi_1 + u_0$. Then

$$J(u_t) = \frac{t^2}{2}\int_{M_0} (|\nabla\phi_1|^2 - \lambda\phi_1^2) + \frac{1}{2}\int_M (|\nabla u_0|^2 - \lambda u_0^2) + t\int_{M_0} (\nabla u_0 \nabla\phi_1 - \lambda u_0\phi_1)$$
$$+ \frac{1}{p+1}\int_M h(t\phi_1 + u_0)^{p+1}.$$

Hence,

$$J(u_t) = \frac{t^2}{2}(\lambda_1(M_0) - \lambda) + ta + b,$$

where

$$b = \frac{1}{p+1}\int_M h u_0^{p+1} + \frac{1}{2}\int_M (|\nabla u_0|^2 - \lambda u_0^2).$$

Since $a > 0$ and $\lambda_1(M_0) - \lambda \leq 0$, it follows that $J(u_t) \to -\infty$ as $t \to -\infty$. Hence, J is not bounded below. Since any positive solution to (1) must be a minimizer of J, we conclude that no such solution exists.

Finally, we prove that if M_0 is nonempty, then

$$\lim_{\lambda \to \lambda_1(M_0)} \|u_\lambda\|_{L^2(M)} = +\infty. \tag{10}$$

Assume, by contradiction, that there is a sequence $\lambda_n \uparrow \lambda_1(M_0)$ such that $\|u_{\lambda_n}\|_{L^2(M)}$ is bounded. Since u_{λ_n} minimizes J for $\lambda = \lambda_n$, we conclude that $\|u_{\lambda_n}\|_{H^1(M)}$ is also bounded. Passing to a subsequence, we may assume that there is a $u \in H^1(M)$ such that $u_{\lambda_n} \to u$ weakly in $H^1(M)$ and strongly in $L^2(M)$. It easily follows that this u must be a minimizer of J for $\lambda = \lambda_1(M_0)$. Since no such minimizer exists, the validity of (10) follows concluding the proof of the theorem. ■

Proof of theorem 3. To prove part (i) it clearly suffices to show that u_λ remains bounded on compacts of Ω, where Ω is any open set with smooth boundary contained in M_+. Thus, fix such a neighborhood Ω. As is well known, the function $x \mapsto \operatorname{dist}(x, \partial\Omega)$ is smooth on $\Omega \cup V$, where V is a sufficiently small neighborhood of $\partial\Omega$, and we assume it smoothly extended to a positive function $d(x)$ defined on $\bar{\Omega}$. Next set

$$v(x) = Cd(x)^{-\alpha},$$

where C and α are positive constants yet to be determined.

Note that we have

$$\Delta d^{-\alpha} = \alpha(\alpha + 1)|\nabla d|^2 d^{-(\alpha+2)} - \alpha d^{-(\alpha+1)} \Delta d$$

and, therefore,

$$\int_\Omega \nabla v \, \nabla \varphi = C \int_\Omega \{\alpha d^{-(\alpha+1)} \Delta d - \alpha(\alpha + 1)|\nabla d|^2 d^{-(\alpha+2)}\}\varphi \tag{11}$$

for all $\varphi \in H^1(M)$ with compact support contained in Ω. Also, setting $u = u_\lambda$, we have

$$\int_\Omega \nabla u \, \nabla \varphi + \int_\Omega (hu^p - \lambda u)\varphi = 0 \tag{12}$$

for all these φs. In particular, choosing $\varphi = (u - v)_+$ we obtain, after subtraction of (12) and (11)

$$\int_\Omega |\nabla \varphi|^2 + \int_\Omega (hu^p - \lambda u - \underline{h} v^p)\varphi$$
$$= \int_\Omega (C\{\alpha(\alpha + 1)|\nabla d|^2 d^{-(\alpha+2)} - \alpha d^{-(\alpha+1)} \Delta d\} - C^p \underline{h} d^{-p\alpha})\varphi, \tag{13}$$

where $\underline{h}$ is any positive constant. We choose $\underline{h} = \frac{1}{2} \inf_\Omega h$ which is positive, by continuity of h.

Let us assume $\lambda \in I$, with I a bounded interval. Suppose that $u(x) \geq Cd(x)^{-\alpha}$. Then, if we choose C large enough, independently of x we can arrange that for any $\lambda \in I$

$$h(x)u^p(x) - \lambda u(x) \geq \underline{h} u^p(x)$$

and also that

$$C\{\alpha(\alpha - 1)|\nabla d|^2 d^{-(\alpha+2)} - \alpha d^{-(\alpha+1)} \Delta d\} - C^p \underline{h} d^{-\alpha} \leq 0 \qquad \text{on } \Omega,$$

provided that α is such that $\alpha > 2/(p-1)$. Choosing such numbers C and α in the definition of v, we obtain from (13)

$$\int_\Omega \underline{h}(u^p - v^p)(u-v)_+ \leq 0.$$

Hence, $(u-v)_+ \equiv 0$, which means $u(x) \leq Cd(x)^{-\alpha}$ for every $x \in \Omega$. This implies u is locally uniformly bounded on $M \backslash \bar{\Omega}$ as desired and the result of (i) follows.

Let us next prove part (ii). Choose any sequence $\lambda_n \uparrow \lambda_1(M_0)$ and denote $u_n = u_{\lambda_n}$. u_n clearly satisfies

$$\int_M |\nabla u_n|^2 + \int_M h u_n^{p+1} = \lambda_n \int_M u_n^2. \tag{14}$$

Let us set $\hat{u}_n = u_n / \|u_n\|_{L^2(M)}$. Then we obtain that for a subsequence of $\hat{u}_n$ which we relabel in the same way, $\hat{u}_n \rightharpoonup \hat{u}$ in $H^1(M)$, $\hat{u}_n \to \hat{u}$ in $L^2(M)$ where $\hat{u}$ satisfies

$$\int_M |\nabla \hat{u}|^2 \leq \lambda_1(M_0). \tag{15}$$

But $\hat{u} \equiv 0$ a.e. on $M \backslash M_0$, since clearly we get from (14) $\int_M h\hat{u}^{p-1} = 0$. Hence, $\hat{u} \in H^1_*(M_0)$ and from (15) we get $\hat{u} = \phi_1$ on M, where $\phi_1 \in C^\infty(M_0) \cap H^1_*(M_0)$ is a positive eigenfunction associated to $\lambda_1(M_0)$ such that $\|\phi_1\|_{L^2(M_0)} = 1$ and $\phi_1 > 0$ on M_0 (here is where connectedness is used). Moreover, since $\hat{u}_n$ satisfies

$$\Delta \hat{u}_n + \lambda_n \hat{u}_n = 0 \qquad \text{in } M_0,$$

interior elliptic estimates imply that the convergence of $\hat{u}_n$ to ϕ_1 is uniform over compacts of M_0. Since ϕ_1 is strictly positive on such sets and $\|u_n\|_{L^2(M)} \to \infty$, the result of part (ii) follows. This finishes the proof. ■

We conclude with some remarks concerning the above proofs.

Remark 1. The method in the proof of part (i) of theorem 3 can also be used to obtain estimates for the growth rate of u_λ near ∂M_+. For example, if ∂M_+ is a smooth $(N-1)$-dimensional submanifold of M and we assume

$$h(x) \geq A \operatorname{dist}(x, \partial M_+)^\eta \qquad \text{on } M_+$$

for some constants A, $\eta > 0$, then for a given bounded interval I we have that for each $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that

$$u_\lambda(x) \leq C_\varepsilon \operatorname{dist}(x, \partial M_+)^{-(2+\eta+\varepsilon)/(p-1)} \qquad \text{for all } x \in M_+,\ \lambda \in I.$$

On the other hand, if we do not assume regularity on ∂M_+ but h is smooth, we obtain

$$u_\lambda(x) \leq C_\varepsilon h(x)^{-(2+\varepsilon)/(p-1)} \qquad \text{for all } x \in M_+.$$

The problem of finding optimal growth rates of u_λ near ∂M_+, as well as an estimate for $\|u_\lambda\|_{L^\infty(M)}$ as $\lambda \to \lambda_1(M_0)$ arises as an open question.

Remark 2. If in part (ii) of theorem 3 M_0 is not connected but has a component M_1 such that

$$\lambda_1(M_1) < \lambda_1(M')$$

for any other component M' of M_0, then the result of (ii) of theorem 3 holds true for M_1 in place of M_0. Indeed, it is not hard to check from the variational characterization (2) that in such a case we have $\lambda_1(M_0) = \lambda_1(M_1)$ and any positive eigenfunction ϕ_1 associated to $\lambda_1(M_0)$ will have the form $\phi_1(x) = \tilde{\phi}_1(x)$ if $x \in M_1$ and $\phi_1(x) = 0$ otherwise, where $\tilde{\phi}_1$ is a positive eigenfunction associated with $\lambda_1(M_1)$.

Remark 3. Let Ω be a bounded, smooth domain in $\mathbf{R}^N$. The proofs of theorems 2 and 3 work equally well for the case of the problem

$$\Delta u + \lambda u - h(x)u^p = 0 \qquad \text{in } \Omega$$

$$u > 0 \qquad \text{in } \Omega$$

under homogeneous Neumann or Dirichlet boundary conditions on $\partial\Omega$. In the latter case, the condition $\lambda > 0$ in the first part of theorem 2 should be replaced by $\lambda > \lambda_1(\Omega)$.

REFERENCES

1. Kazdan J. L. & Warner F. W., Scalar curvature and conformal deformation of Riemannian structure, *J. diff. Geometry* **10**, 113–134 (1975).
2. Ouyang T., On the positive solutions of semilinear equations $\Delta u + \lambda u - hu^p = 0$ on compact manifolds, *Trans. Am. math. Soc.* **331**, 503–527 (1992).
3. Brezis H. & Oswald L., Remarks on sublinear elliptic equations, *Nonlinear Analysis* **10**, 55–64 (1986).
4. Ouyang T., On the positive solutions of semilinear equations $\Delta u + \lambda u - hu^p = 0$ on compact manifolds. Part II, *Indiana Univ. math. J.* **40**, 1083–1141 (1991).
5. Gilbarg D. & Trudinger N., *Elliptic Partial Differential Equations of Second Order*, 2nd edition. Springer, New York (1983).