Layers With Nonsmooth Interface in a Semilinear Elliptic Problem

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LAYERS WITH NONSMOOTH INTERFACE
IN A SEMILINEAR ELLIPTIC PROBLEM

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1. Introduction

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^m$, $m \geq 1$. In this paper we consider the semilinear Neumann problem

$$\varepsilon^2 \Delta u = f(u, x) \quad \text{in } \Omega$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega$$

where $f$ is of class $C^1$ and $f(\cdot, x)$ has precisely three zeros $h_-(x) < h_0(x) < h_+(x)$ for each $x \in \Omega$. We also assume that $h_\pm$ are nondegenerate and stable, namely

$$f_u(h_\pm(x), x) > 0 \quad \text{for all } x \in \Omega.$$  \hfill (1.2)

We are interested in solutions to (1.1) exhibiting a transition layer from $h_-$ to $h_+$ as $\varepsilon$ approaches zero. More precisely, we look for two open subsets $\Omega_+$ and $\Omega_-$ of $\Omega$ and a family of solutions $u_\varepsilon$ to problem (1.1) such that $u_\varepsilon$ approaches $h_\pm$ on compact subsets of $\Omega \pm$.

In a pioneering work, Fife and Greenlee [8] gave sufficient conditions for this phenomenon to take place. As pointed out by Caginalp and Fife [5], the following result for problem (1.1) follows from the methods in [8], where the Dirichlet case was treated. We denote

$$J(x) = \int_{h_-(x)}^{h_+(x)} f(s, x) ds.$$  \hfill (1.3)

1695
Theorem A. Let \( m = 2 \). Assume the existence of a closed smooth curve \( \Gamma \subset \Omega \) which divides \( \Omega \) into two smooth subdomains \( \Omega_+ \) and \( \Omega_- \) and such that \( J = 0 \) and \( \frac{\partial J}{\partial n} > 0 \) on \( \Gamma \). Here \( n \) denotes the normal direction to \( \Gamma \) towards \( \Omega_- \). Then there exist a positive number \( \varepsilon_0 \) and a family of solutions \( \{ u_{\varepsilon} \}_{\varepsilon \in (0, \varepsilon_0)} \) to problem (1.1) such that

\[
\lim_{\varepsilon \to 0} u_{\varepsilon}(x) = h_{\varepsilon}(x)
\]

uniformly on compacts of \( \Omega_\pm \).

It is observed in [8] that dimension is not an important restriction in this result. In fact, their method works in any dimension \( m \), with \( \Gamma \) replaced by a finite number of closed hypersurfaces.

What still seems to be an important restriction is that of the smoothness of the interface \( \Gamma \). Indeed, the method of [8] is based upon a careful formal approximation of the solution near \( \Gamma \) using a series expansion in powers of \( \varepsilon \). They then apply implicit function techniques based on this approximation. This method does not seem to apply once the smoothness hypothesis on \( \Gamma \) is removed.

It is noticed in [5] that a simpler, but still delicate, super-subsolutions approach could be devised along the lines of that paper. This is done in [7] for Dirichlet boundary conditions. That proof, however, also depends on the smoothness of the interface.

Here we present a completely different proof of Theorem A which permits to overcome this difficulty. Moreover, it allows \( \Gamma \) to be an arbitrary closed subset of \( \Omega \). In particular, \( \Gamma \) may intersect \( \partial \Omega \) and have an arbitrary number of components.

Our proof, based on standard elliptic theory and degree theoretical arguments, is rather simple and does not require the construction of first approximations or super-sub solutions based on the formal knowledge of the behavior of the solution near the interface. This flexibility may be useful in the study of more complex problems, like systems, in which the properties of the interface may not be entirely a priori known.

The following is the main result of this paper.

Theorem B. Let \( m \geq 1 \) and assume the existence of a closed set \( \Gamma \subset \bar{\Omega} \) and of open disjoint subsets of \( \Omega, \Omega_\pm \) and \( \Omega_- \) such that

\[
\Omega = \Omega_+ \cup \Gamma \cup \Omega_-
\]

Assume also the existence of an open neighborhood \( N \) of \( \Gamma \) such that

\[
J(x) > 0 \quad \text{for } x \in N \cap \Omega_- \setminus \Gamma
\]

and

\[
J(x) < 0 \quad \text{for } x \in N \cap \Omega_+ \setminus \Gamma
\]

Then there exist a positive number \( \varepsilon_0 \) and a family of solutions \( \{ u_{\varepsilon} \}_{\varepsilon < \varepsilon_0} \) to problem (1.1) such that

\[
\lim_{\varepsilon \to 0} u_{\varepsilon}(x) = h_{\varepsilon}(x)
\]

uniformly on compact subsets of \( \Omega_\pm \setminus \Gamma \), in particular of \( \partial \Omega_\pm \setminus \Gamma \).

Theorem A clearly follows from this result. Observe that the condition \( \frac{\partial J}{\partial n} > 0 \) has been replaced by a "change of sign" assumption for \( J \) on \( \Gamma \).
If \( f(\cdot, x) \) has more than one zero between \( h_-(x) \) and \( h_+(x) \), conditions (1.4) and (1.5) should respectively be replaced by

\[
\int_{h_-(x)}^{h_+(x)} f(s,x)\,ds > 0 \quad \text{for } u \in (h_-(x), h_+(x)) \text{ and } x \in \mathcal{N} \cap \Omega_\pm \setminus \Gamma \quad (1.4)'
\]

and

\[
\int_{h_+(x)}^{h_+(x)} f(s,x)\,ds < 0 \quad \text{for } u \in [h_+(x), h_+(x)) \text{ and } x \in \mathcal{N} \cap \Omega_+ \setminus \Gamma. \quad (1.5)'
\]

These conditions are equivalent to (1.4) and (1.5) in case that only one zero between \( h_- \) and \( h_+ \) exists, and what we will actually use in the proof.

Remark. A different method for the obtention of layered families of solutions is the direct variational approach, which has been used in e.g. [1], [2], [11] and [13]. Alikakos and Simpson [2] have studied several properties of the global minimal energy for a special case of (1.1) under radial symmetry. In our case, these global minimizers constitute a family of layered solutions as in Theorem B when one takes \( \Omega_+ = \{ x | J(x) < 0 \} \), \( \Omega_- = \{ x | J(x) > 0 \} \) and the Lebesgue measure of \( \Gamma = \{ x | J(x) = 0 \} \) is zero. This is not hard to establish, by conveniently adapting the argument in [2], but only \( L^p \)-convergence is clear. Uniform convergence away from the layer as that given by Theorem B is not obvious from this characterization. That property is useful in deriving other convergence features of the family. See [4].

Remark. Our method does not predict existence of a family of solutions with layer in the "opposite direction", that is, approaching \( h_- \) in \( \mathbb{R}^+ \) and \( h_+ \) in \( \mathbb{R}^- \). In the one-dimensional case such a family is known to exist and be unstable. See [10].

The literature on layers in elliptic and parabolic semilinear equations with a bistable nonlinearity is today extensive. We refer the reader for example to [1], [3], [4], [5], [10], [11], [13] and references therein for problems related to the one treated here.

2. Preliminary Results

Theorem B will be a consequence of some lemmas which we state and prove next.

Our first lemma gives the existence of families of unlayered solutions converging uniformly to each of the stable zeros of \( f \).

Lemma 2.1. There exists a number \( \varepsilon_0 \) and families of solutions \( \{ h_{-\epsilon} \}_{0 < \epsilon < \varepsilon_0} \) and \( \{ h_{+\epsilon} \}_{0 < \epsilon < \varepsilon_0} \) to problem (1.1) such that

\[
\lim_{\epsilon \to 0} h_{\pm\epsilon}(x) = h_{\pm}(x)
\]

uniformly on \( \hat{\Omega} \).

Proof. We will prove the existence of \( h_{-\epsilon} \). The proof for \( h_{+\epsilon} \) is the same.

Consider for \( t \in [0, 1] \) the problem
\[ \varepsilon^2 \Delta u = tf(u, x) + (1 - t)(u - h_-(x)) \quad \text{in } \Omega \tag{2.1} \]

\[ \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega. \]

Fix \( \rho_0 > 0 \) so small that \( f_\varepsilon(s, x) > 0 \) whenever \( |s - h_-(x)| \leq \rho_0 \) and \( x \in \bar{\Omega} \).

**Claim.** Given \( 0 < \rho \leq \rho_0 \), there exists \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \leq \varepsilon_0 \) and all \( t \in [0, 1] \), every solution of (2.1) in \( B_{\rho_0} \) is in \( B_{\rho} \). Here \( B_\rho \) denotes the open ball center \( h_- \) radius \( \rho \) in \( C(\bar{\Omega}) \).

**Proof of the claim.** Assume the contrary. Then there exist sequences \( \varepsilon_n \to 0 \), \( t_n \to t \in [0, 1] \) and \( u_n \), solution of (2.1) for \( t = t_n, \varepsilon = \varepsilon_n \), such that

\[ \sup_{x \in \partial \Omega} |u_n(x) - h_-(x)| = \rho \quad \text{for all } n \in \mathbb{N}. \tag{2.2} \]

Let \( \bar{x}_n \in \bar{\Omega} \) be a point where the supremum (2.2) is attained. Assume also that \( x_n \to \bar{x} \in \bar{\Omega} \). We consider two cases.

**Case 1.** \( \bar{x} \in \Omega \). In this case, for all sufficiently large \( n \), the ball \( B(x_n, \varepsilon_n) \) is contained in \( \Omega \). For \( y \in B(0, 1) \) we define

\[ U_n(y) = u_n(x_n + \varepsilon_n y). \]

Then \( U_n \) satisfies the equation

\[ \Delta U_n = t_n f(U_n, x_n + \varepsilon_n y) + (1 - t_n)(u_n - h_-(x_n + \varepsilon_n y)) \quad \text{on } B(0, 1). \tag{2.3} \]

From (2.2), we see that \( U_n \) is uniformly bounded, as well as the right hand side of (2.3). Then, \( L^p \) and Schauder estimates give the existence of a subsequence of \( U_n \) converging in the \( C^{1,1}(B(0,1)) \)-sense to a solution \( U \) of

\[ \Delta U = tf(U, \bar{x}) + (1 - t)(U - h_-(\bar{x})) \]

on \( B(0, 1) \).

From the fact that \( |U - h_-(\bar{x})| \leq \rho \) and our choice of \( \rho \), we find that \( V \equiv U - h_-(\bar{x}) \) satisfies on \( B(0, 1) \) an equation of the form

\[ \Delta V - c(y)V = 0 \]

with \( c > 0 \). But, from the definition of \( \varepsilon_n \), either \( V \) or \( -V \) attains a nonnegative maximum at \( y = 0 \). This contradicts the maximum principle and shows the impossibility of Case 1.

**Case 2.** \( \bar{x} \in \partial \Omega \). Here we distinguish two subcases:

(a) There exists a number \( \delta > 0 \) and a subsequence of \( \varepsilon_n \), relabeled again \( \varepsilon_n \) such that \( B(x_n, \varepsilon_n \delta) \subset \Omega \) for all \( n \).

(b) \[ \lim_{n \to \infty} \frac{\text{dist}(x_n, \partial \Omega)}{\varepsilon_n} = 0. \]
If (a) holds we will obtain the same situation of Case 1, hence (a) is not possible. If (b) holds we argue as follows: consider a local chart $\phi : U \to \mathbb{R}^m$ where $U$ is some open neighborhood of $z$ such that

$$\phi(U \cap \Omega) = \mathbb{R}^m = \{z = (x', x_m) | x' \in \mathbb{R}^{m-1}, x_m > 0\}$$

and $\phi(z) = 0$. Then $\bar{u}_n(z) \equiv u_n(\phi^{-1}(z))$ satisfies

$$\varepsilon_n^2 L \bar{u}_n = t_n f(\bar{u}_n, \phi^{-1}(z)) + (1 - t_n) \bar{u}_n - h_-(\phi^{-1}(z)) \quad \text{in } \mathbb{R}^m_+$$

where $L$ is a strongly elliptic operator of the form

$$L = \sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i}$$

Moreover, after a convenient choice of the change of coordinates, we may also assume $a_{ij}(0) = b_{ij}$. Let $\pi$ be the orthogonal projection onto $\partial \Omega$ which is well defined and smooth in some neighborhood of $\partial \Omega$. Set

$$\bar{U}_n(y) = \bar{u}_n(\phi^{-1}(\phi(\pi(y)) + \varepsilon_n y))$$

for $y \in B(0,1) \cap \mathbb{R}^m_+$. After writing (2.4) in the $y$-coordinates, elliptic estimates imply convergence of a subsequence of $U_n$ to some $U \in C^2(B(0,1) \cap \mathbb{R}^m_+)$ satisfying

$$\Delta U = \mathcal{I} f(U, \pi) + (1 - \mathcal{I})(U - h_-(\pi))$$

But the Neumann boundary condition permits us to extend $U$ evenly to the whole $B(0,1)$, so that the extension still satisfies (2.5). A similar straightening-reflection argument appears in [12].

On the other hand, since (b) holds, we find that

$$\phi(\pi(x_n)) - \phi(x_n) \to 0 \quad \text{as } n \to \infty$$

But $U(\phi(\pi(x_n)) - \phi(x_n)) = u_n(x_n)$, and hence $|U(0) - h_-(\pi)| = \rho$. At this point we are in the same situation of Case 1 and a contradiction comes from the maximum principle. This concludes the proof of the claim.

The conclusion of the lemma follows from the claim and a degree-theoretical argument. Let $R_\mathcal{I}$ denote the inverse of $(\varepsilon^2 \Delta - I)$ under Neumann boundary conditions. Then $R_\mathcal{I}$ applies compactly $C(\bar{\Omega})$ into itself. Observe that equation (1.1) is equivalent to the fixed point problem in $C(\bar{\Omega})$:

$$u = R_\mathcal{I}(f(u, x) - u) \equiv T^*(u).$$
Consider the compact homotopy
\[ Q_t(u) = R_t(f(u,x) - u) - (1-t)h_. \] (2.7)

Fix \( \rho \) sufficiently small. From the claim, we know that \( Q_t \) has no fixed points on \( \partial B_{\rho} \) for all \( t \in [0,1] \), provided that \( \varepsilon \) is sufficiently small. Since \( Q'_t = T^* \) and \( Q'_0 = -R_t(h_) \), the invariance of the degree under compact homotopies implies
\[ \deg (I - T^*, B_{\rho}, 0) = \deg (I + R_t(h_), B_{\rho}, 0) . \]

If \( -R_t(h_) \) is included in \( B_{\rho} \), the latter degree equals one and we would conclude existence of a solution of (2.6) and hence of (1.1) in \( B_{\rho} \). Thus, it only remains to verify that the unique solution of
\[ -\varepsilon^2 \Delta u + u = h_\quad \text{in } \Omega \]
\[ \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega \] (2.8)
lies on \( B_{\rho} \) for all sufficiently small \( \varepsilon \). But (2.8) corresponds to (2.1) for \( t = 0 \). Observe that for \( t = 0 \) the claim holds true without smallness restriction on \( \rho_0 \). Hence, it suffices to show that the solution of (2.8) has an \( L^\infty \)-estimate independent of \( \varepsilon \). This is true. Indeed, it is easily seen that
\[ \inf_{\Omega} h_\leq u \leq \sup_{\Omega} h_ \]
for a solution \( u \) of (2.8) and for all \( \varepsilon > 0 \); just use the \( H^1 \)-functions \( (u - \inf_{\Omega} h_)^- \) and \( (u - \sup_{\Omega} h_)^+ \) as test functions for (2.8).

We conclude the existence of a solution \( h^* \) to (1.1) in \( B_{\rho} \). The uniform convergence of \( h^* \) to \( h_ \) is a consequence of the claim for \( t = 1 \). \[ \square \]

Our proof of Theorem B will require the construction of "nice" approximations of the sets \( R^- \) and \( R^+ \). We will do this using the following lemma.

**Lemma 2.2.** Let \( K \subset \mathbb{R}^m \) be a compact set and \( M \) a smooth boundaryless hypersurface. Then, given \( \delta > 0 \), there exists an open neighborhood \( N_\delta \) of \( K \) such that
(a) \( \{ x \mid \text{dist}(x,K) \leq \delta \} \subset N_\delta \subset \{ x \mid \text{dist}(x,K) \leq \frac{\delta}{2} \} \)
(b) \( N_\delta \) is a finite union of smooth domains.
(c) Either \( \partial N_\delta \) does not intersect \( M \) or does it orthogonally.

**Proof.** Let \( \psi(x) \) be the usual approximation of the identity, i.e.,
\[ \psi_\varepsilon(x) = \frac{1}{\varepsilon^m} \psi \left( \frac{x}{\varepsilon} \right) \]
where \( \psi \in C^\infty_c(\mathbb{R}^m) \) and \( \int \psi = 1 \).

Set \( K_\varepsilon = \{ x \mid \text{dist}(x,K) \leq \frac{\varepsilon}{2} \} \) and define
\[ g_\varepsilon = \psi_\varepsilon \ast \chi_{K_\varepsilon} \]
where \( \ast \) denotes convolution product and \( \chi_{K_\varepsilon} \) the characteristic function of the set \( K_\varepsilon \). \( g_\varepsilon \) is smooth. Observe also that for sufficiently small \( \varepsilon \),
LAYERS WITH NONSMOOTH INTERFACE

\begin{equation}
g_\varepsilon(x) = \begin{cases} 1 & \text{if } \text{dist}(x, K) \leq \frac{\delta}{2} \\ 0 & \text{if } \text{dist}(x, K) \geq \delta. \end{cases}
\end{equation}

Fix such a small \(\varepsilon > 0\).

Let \(\pi\) denote the orthogonal projection onto \(M\), which is well defined and smooth in some neighborhood \(V\) of \(M\). Consider a smooth function \(d(x)\) which vanishes outside \(V\) and equals one on some smaller neighborhood of \(M\). Define

\[ \hat{g}(x) = d(x)g_\varepsilon(\pi(x)) + (1 - d(x))g_\varepsilon(x). \]

Observe that, reducing \(\varepsilon\) and \(V\) if necessary, \(\hat{g}\) still satisfies (2.9) and (2.10). Finally, set

\[ N_\varepsilon = \{ x \mid \hat{g}(x) > 1 - \rho \} \]

where, using Sard’s Theorem, we choose \(\rho > 0\) small and so that \(1 - \rho\) is a regular value of \(\hat{g}\). We easily see that \(N_\varepsilon\) satisfies (a) and (b). Now, if \(\partial N_\varepsilon\) intersects \(M\) at some point \(\hat{x}\), we see from the definition of \(\hat{g}\) that

\[ \nabla \hat{g}(x) = P_\varepsilon \nabla g_\varepsilon(\hat{x}) \]

where, \(P_\varepsilon\) denotes the orthogonal projection onto the tangent space to \(M\) at \(\hat{x}\). It follows that \(\partial N_\varepsilon\) and \(M\) intersect orthogonally, at \(\hat{x}\), hence (c) holds.

Next fix \(\delta > 0\) and let \(\Omega^+\), \(\Omega^-\), \(\Gamma\) be as in the hypotheses of Theorem B. Let \(\Omega_\varepsilon\) be a neighborhood of \(\Gamma\) as given by Lemma 2.2. Define the open sets \(\Omega^+_\varepsilon\), \(\Omega^-\varepsilon\) by

\[ \Omega^+_\varepsilon = \Omega_\varepsilon \setminus N_\varepsilon. \]

Observe that \(\Omega^+_\varepsilon\) is a finite union of bounded domains. Moreover, \(\partial \Omega^+_\varepsilon\) is smooth, except near \(\partial \Omega_\varepsilon \cap \partial N_\varepsilon\), which consists of a finite union of orthogonal corners.

Also, for sufficiently small \(\delta\) we have

\[ \pm J(x) > 0 \quad \text{for } x \in \partial N_\varepsilon \cap \Omega_\varepsilon \]

where \(J\) is defined by (1.3). We will assume this henceforth.

Let \(h^\pm\) be the families of unlayered solutions predicted by Lemma 2.1. Let \(\psi\) be a \(C^\infty\) function such that \(\psi \equiv 1\) on \(\Omega^+_\varepsilon\) whose support is compact and does not intersect \(\Omega^-\varepsilon\). Define

\[ h_\varepsilon \equiv h^+\varepsilon \psi + h^-\varepsilon(1 - \psi) \]

Denote by \(x^\pm_\varepsilon\) (resp. \(x^-_\varepsilon\)) the characteristic function of \(\Omega^+_\varepsilon\) (resp. \(\Omega^-\varepsilon\)).

Our proof of Theorem B is based on the study of the following family of problems.

\[ \varepsilon^2 \Delta u = tf(u, x) + (1 - t)(x^+_\varepsilon(u - h^+_\varepsilon) + x^-_\varepsilon(u - h^-_\varepsilon) + \varepsilon^2 \Delta h^+) \]

\[ \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega. \]
We intend to apply degree theory to (2.13), as we did in the proof of Lemma 2.1, in some appropriate open subset of \( C(\Omega) \) to conclude the existence of solutions of (1.1) with the desired characteristics. First we require a lemma.

**Lemma 2.3.** There exists an \( L^\infty \)-bound \( M > 0 \), independent of all small \( \varepsilon \) and all \( t \in [0,1] \), for the solutions of (2.13).

**Proof.** Set \( v \equiv u - h^\varepsilon \). Then (2.13) takes the form

\[
\varepsilon^2 \Delta v = t(f(h^\varepsilon + v, x) - \varepsilon^2 \Delta h^\varepsilon) + (1 - t)(\chi^+_\varepsilon + \chi^-_\varepsilon)v
\]

\[
\frac{\partial v}{\partial n} = 0 \quad \text{on } \partial \Omega.
\]

Assume first \( t > 0 \). Fix a number \( R > 0 \). Using \((v - R)^+ \in H^1(\Omega)\) as a test function for (2.14) we obtain

\[
-\varepsilon^2 \int_{\{v > R\}} |\nabla v|^2 - (1 - t) \int_{\{v > R\}} (\chi^+_\varepsilon + \chi^-_\varepsilon)v(v - R) = \int_{\{v > R\}} t(f(h^\varepsilon + v, x) - \varepsilon^2 \Delta h^\varepsilon)(v - R).
\]

If the set \( \{v > R\} \) were nonempty, (2.15) would imply the existence of \( x_0 \in \Omega \) and a number \( s > R \) such that

\[
f(h^\varepsilon(x_0) + s, x_0) \leq \varepsilon^2 \Delta h.
\]

On the other hand, it is easily seen that any solution to (1.1) is between \( \inf h_- \) and \( \sup h_+ \). Hence, redefining \( f \) if necessary, we may assume that \( f(t, x) \geq ct \) for all large \( t \) and all \( x \in \Omega \), some \( c > 0 \). But \( \varepsilon^2 \Delta h^\varepsilon \) is uniformly bounded. Indeed, \( \varepsilon^2 \Delta h^\varepsilon = f(h^\varepsilon, x) \to 0 \) as \( \varepsilon \to 0 \), uniformly. Since \( h^\varepsilon \) is bounded, elliptic estimates imply that \( \varepsilon^2 \Delta h^\varepsilon \) also is. From this and the definition of \( h^\varepsilon \) in (2.11) the conclusion is immediate.

We conclude that (2.16) is impossible if \( R \) is chosen sufficiently large. Hence, for some large \( R, v \leq R \) in \( \Omega \). A similar procedure gives a lower bound for \( v \), from which a uniform bound for \( u \) follows in case that \( t > 0 \).

Now, if \( t = 0 \), (2.14) becomes

\[
\varepsilon^2 \Delta v = (\chi^+_\varepsilon + \chi^-_\varepsilon)v
\]

\[
\frac{\partial v}{\partial n} = 0 \quad \text{on } \partial \Omega.
\]

It easily follows that \( v \equiv 0 \), hence \( u = h_0 \) is uniformly bounded. This completes the proof. \( \blacksquare \)

For \( \rho > 0 \) we consider the open subsets of \( C(\Omega) \) \( \Lambda^+_\varrho \) and \( \Lambda^-_\varrho \) defined by

\[
\Lambda^+_\varrho = \{ u \in C(\Omega) \mid \sup_{x \in \partial \Omega^+} |u(x) - h_+(x)| \leq \rho \}.
\]

The following is a key step in the proof of Theorem B, where the fact (2.11) will play a main role.

**Lemma 2.4.** Given any \( \rho > 0 \) sufficiently small there exists a number \( \varepsilon_0 > 0 \) such that for all \( 0 < \varepsilon \leq \varepsilon_0 \) and all \( t \in [0,1] \) there is no solution of (2.14) on \( \partial \Lambda^+_\varrho \cup \partial \Lambda^-_\varrho \).

**Proof.** Assume the contrary, then there exist sequences \( \varepsilon_n \to 0 \), \( t_n \to t \in [0,1] \) and a sequence of solutions \( u_n \) to (2.14) for \( t = t_n, \varepsilon = \varepsilon_n \) such that either
LAYERS WITH NONSMOOTH INTERFACE

\[ \sup_{x \in \Gamma_1^+} |u_n(x) - h_+(x)| = \rho \quad \text{for all } n \in N \tag{2.19} \]

or

\[ \sup_{x \in \Gamma_1^+} |u_n(x) - h_-(x)| = \rho \quad \text{for all } n \in N. \]

Assume that (2.19) occurs. The other case is similar. Let \( x_n \) be a point where the supremum (2.19) is attained. Without loss of generality we may assume that \( x_n \to \bar{x} \) in \( \Omega_\lambda^+ \). As in the proof or Lemma 2.1, we consider different cases for the position of \( \bar{x} \).

Case 1. \( \bar{x} \in \Omega_\lambda^+ \). In this case the ball \( B(x_n, \varepsilon_n) \) lies on \( \Omega_\lambda^+ \) for all sufficiently large \( n \). Define

\[ U_n(y) = u_n(x_n + \varepsilon_n y) \]

for \( y \in B(0,1) \). Thus \( U_n \) satisfies in this ball

\[ \Delta U_n = \tau_n f(U_n, x_n + \varepsilon_n y) + (1 - \tau_n)(h_+(x_n + \varepsilon_n y) - h_-(x_n + \varepsilon_n y)) + (1 - \tau_n)\varepsilon_n^2 \Delta h_+^\varepsilon_n. \]

Since \( h_+^\varepsilon_n \to h_+ \) and \( \varepsilon_n^2 \Delta h_+^\varepsilon_n \to 0 \) uniformly, we obtain as in Lemma 2.1 the existence of a subsequence of \( U_n \) convergent in the \( C^2 \)-sense to some \( U \) satisfying in \( B(0,1) \) the equation

\[ \Delta U = f(U, \bar{x}) + (1 - \bar{f})(U - h_-(\bar{x})) \]

and we obtain, as in Case 1 of Lemma 2.1, a contradiction to the Maximum Principle.

Case 2. \( \bar{x} \) is in \( \partial \Omega_\lambda \setminus \Pi_\lambda \). As in Case 1, we can reach a contradiction by slightly modifying the proof in Case 2 of Lemma 2.1.

Case 3. \( \bar{x} \in \partial \Omega_\lambda \cap \Omega \). Here we distinguish two subcases

(a) For some subsequence of \( x_n \), again labeled \( x_n \), and some \( \lambda > 0 \) we have

\[ B(x_n, \lambda \varepsilon_n) \subset \Omega_\lambda^- \quad \text{for all } n \in N \]

or

(b) \[ \lim_{n \to \infty} \frac{\text{dist}(x_n, \partial \Omega^+_\lambda)}{\varepsilon_n} = 0. \]

If (a) holds, we may proceed exactly as in Case 1, just changing \( B(0,1) \) by \( B(0,\lambda) \). In case (b) further considerations are needed. Here the fact that \( J(\bar{x}) > 0 \) will permit us to reach a contradiction.

We straighten \( \partial \Omega^+_\lambda \) near \( \bar{x} \), as we did in Case 2 of Lemma 2.1. After a standard diagonal procedure to obtain the \( C^2 \) convergence on compacts of some subsequence of the rescaling \( U_n \) (recall from Lemma 2.3 that \( u_n \) is uniformly bounded) we obtain the existence of \( U \in C^2(\mathbb{R}^m) \) bounded and satisfying in \( \mathbb{R}^m \)

\[ \Delta U = f(U, \bar{x}) + (1 - \bar{f})(U - h_-(\bar{x})). \tag{2.20} \]

Moreover,

\[ |U(y) - h_-(\bar{x})| \leq \rho \quad \text{on } \mathbb{R}^m \]

with equality at the origin. We will assume

\[ U(0) - h_-(\bar{x}) = \rho. \]
The other case is similar. Denote by $g$ the function defined by
\[
g(s) = \tilde{f}(s, \tilde{x}) + (1 - \tilde{r})(s - h_-(\tilde{x})).
\] (2.21)
Let $w_0(t)$ be the unique solution of the differential equation
\[
w''(t) = g(w(t))
\] (2.22)
and consider $\tilde{U} \in C^2(\mathbb{R}^m)$ defined by
\[
\tilde{U}(y', y_m) = w_0(y_m).
\]
Recall that $\inf h_- \leq u \leq \sup h_+$ for every solution $u$ to (1.1). Hence, we do not lose
generality in assuming
\[
\lim_{s \to \pm \infty} f(s, x) = \pm \infty
\]
nuniformly on $x \in \Omega$. Since this holds at $-\infty$ and $\inf_{x \in \Omega} f_s(h_-(x), x) > 0$, we see that if $\rho$ is chosen sufficiently small, then
\[
\frac{g(s) - g(r)}{s - r} > 0
\] (2.23)
whenever $|s - h_-(x)| \leq \rho$ and $r - h_-(x) \leq \rho$.

Let us assume that $w_0$ satisfying (2.22) is increasing on $(-\infty, 0]$. We will prove this
fact later. We shall next show that this implies that $U = \tilde{U}$ on $\mathbb{R}^m$. Indeed, observe that
\[
\Delta(\tilde{U} - U) = \frac{g(U) - g(\tilde{U})}{U - \tilde{U}}(\tilde{U} - U) = 0
\] (2.24)
and that (2.23) holds for $s = U$ and $r = \tilde{U}$. Since
\[
(\tilde{U} - U) = \rho - (U - h_-(x)) \geq 0
\]
on $\partial \mathbb{R}^m$,
(2.24) and the maximum principle imply $\tilde{U} - U > 0$ in $\mathbb{R}^m$ unless $(\tilde{U} - U) \equiv 0$. But
since $(\tilde{U} - U)(0) = 0$, Hopf's Boundary Point Lemma (e.g. [9], p. 34) implies that the
former case can only occur if $\frac{\partial \tilde{U}}{\partial y_m}(\tilde{U} - U)(0) < 0$ which is false by definition of $\tilde{U}$. Hence,
necessarily $U = \tilde{U}$ on $\mathbb{R}^m$ if $w_0$ is increasing. We will next show that this is indeed the case

Observe first that
\[
w'_0(0) = \frac{\partial U}{\partial y_m}(0) \geq 0
\]
since $U$ maximizes on $\mathbb{R}^m$ at $0$. We must actually have $w'_0(0) > 0$. Indeed, otherwise
$w_0(t) > h(\tilde{x}) + \rho$ for all sufficiently small $t < 0$, since $w'_0(0) > 0$. Hence $(\tilde{U} - U) > 0$
on some ball $B \subset \mathbb{R}^m$ such that $0 \in \partial B$. From this we immediately find a contradiction
with Hopf's Lemma. Hence $w'_0(0) > 0$.

Assume that $w_0$ is not increasing on $(-\infty, 0]$ and let $-\infty < -\tilde{r} < 0$ be the first
negative point where $w'_0(-\tilde{r}) = 0$. Since $w_0$ satisfies an autonomous O.D.E., the reflection
LAYERS WITH NONSMOOTH INTERFACE

of $w_0$ through $-r$ coincides with $w_0$. Hence $w_0(t) \leq h_-(z) + \rho$ on $[-2r, 0]$ with equality at the endpoints and $w_0'(-2r) = -w_0'(0) < 0$.

Again from (2.23) with $s = U, r = \hat{U}$ and (2.24), we conclude that $U(y) = \hat{U}(y)$ for all $y = (y', y_m) \in \mathbb{R}^n$ such that $-2r \leq y_m \leq 0$. In particular, $U$ maximizes at $y = (0, y_{m-1}, -2r)$ and hence $\nabla U = 0$ at that point. But

$$\frac{\partial U}{\partial y_m}(0, y_{m-1}, -2r) = w_0'(-2r) < 0.$$  

We have obtained a contradiction which shows that $w_0 > 0$ on $(-\infty, 0]$, and hence $U = \hat{U}$ on $\mathbb{R}^+$. We will next see what happens on $\mathbb{R}^+$. First, a property of $w_0$ which follows from the key fact $J(z) > 0$.

Claim. $w_0(t) \to +\infty$ as $t \to +\infty$. Indeed, we know that $w_0$ is increasing on $(-\infty, 0]$. Since $w_0$ is also bounded there (from $U = \hat{U}$), it easily follows that $w_0(-\infty) = h_-(z)$ and $w_0'(-\infty) = 0$. We thus obtain from (2.22)

$$\frac{w_0'(t)^2}{2} = \int_{h_-(z)}^{w_0(t)} g(s)ds. \tag{2.25}$$

Recall that $w_0'(0) > 0$, $w_0(0) = h_-(z) + \rho$. If $w_0$ vanished at some point $z > 0$, we would have

$$0 = \int_{h_-(z)}^{w_0(t)} g(s)ds = \int_{h_-(z)}^{w_0(0)} f(s, \hat{U})ds + (1 - \delta_0) \int_{h_-(z)}^{w_0(t)} (w_0(s) - h_-(z))^2 \frac{ds}{2}. \tag{2.26}$$

But, since $w_0(z) > h_-(z) + \rho$ and $J(z) > 0$, it follows that the right hand side of (2.26) is bounded below by a strictly positive constant, and we get a contradiction which shows that $w_0$ is strictly increasing. Observe that actually (1.4)' at $x = \hat{x}$ has been used here. For the same reason, (2.25) implies that $w_0$ is unbounded and the claim follows.

It follows from the claim that $\hat{U}$ is unbounded. We will reach a contradiction from this fact by means of the following argument.

Denote by $w_4(t)$ the unique solution of the O.D.E.

$$w_4(t) = g(w(t))$$

$$w(0) = h_-(z) + \rho, \; w'(0) = \frac{\partial U}{\partial y_m}(0) + \delta$$

and set $\hat{U}_4(y', y_m) \equiv w_4(y_m)$. Then $\hat{U}_4$ solves (2.20). Also, $\hat{U}_4 = \hat{U}$.

Let $R > 0$ and define $H_R = \{(y', y_m) \mid 0 < y_m < R\}$. Denote by $\Gamma_R$ and $\Gamma_R$ respectively the left and right boundaries of $H_R$.

Since $w_4$ is increasing and unbounded and $U$ is bounded, we find that for all $R > 0$ sufficiently large $\hat{U}_4 > U$ on $\Gamma_R$ for all $\delta \geq 0$. Fix such an $R$. We claim that there exists a number $\delta^* > 0$ so large that $\hat{U}_4 > U$ on $H_R$.

Assume the contrary. Then there exist sequences $\delta_n \to \infty$ and points $y_n \in H_R, y^n = (y^n, y^n_m)$ such that

$$U(y^n) \geq \hat{U}_{\delta_n} \tag{2.27}$$
Define

\[ U^n(z) = U(y^n + z). \]  

(2.28)

Then \( \Delta U^n = g(U^n) \). Since \( U^n \) is bounded, elliptic estimates imply that we do not lose any generality in assuming that \( U^n \) converges in the \( C^1 \) sense on \( \bar{B}(0, R) \). In particular, \( \frac{\partial U^n}{\partial r} \) remains bounded there.

But from (2.27), the fact that \( \frac{\partial U^n}{\partial r} > w_0'(0) + \delta_n \) in \( H_R \), and since \( U = \tilde{U}_0 \), \( \frac{\partial U^n}{\partial r} < \frac{\partial U}{\partial r} \) on \( \Gamma_0 \), the mean value theorem implies the existence of a point \( z^n \in B(0, R) \) such that \( \frac{\partial U^n}{\partial r}(z^n) \to \infty \) as \( n \to \infty \). We have reached a contradiction which proves the claim.

Fix a number \( \delta^* > 0 \) such that \( \tilde{U}_0 > U \) on \( H_R \) and set

\[ E = \{ \delta \in [0, \delta^*] \mid \tilde{U}_0 > U \text{ in } H_R \} \]

\( E \) is nonempty. It is also closed, for let \( \delta_n \in E \) such that \( \delta_n \to \delta \in [0, \delta^*] \). Then \( \tilde{U}_0 > U \) in \( H_R \). Since \( \tilde{U}_0 > U \) on \( \Gamma_R \), it follows from the Maximum Principle that \( \tilde{U}_0 > U \) in \( H_R \). Hence \( \delta \in E \) and \( E \) is closed.

We will next show that \( E \) is also open. Otherwise, there exists \( \delta \in [0, \delta^*] \) such that

\[ \tilde{U}_0 > U \text{ in } H_R \]

(2.29)

and sequences \( \delta_n \in [0, \delta^*] \), \( y^n = (y_1^n, y_2^n) \) such that \( \delta_n \to \delta \), \( y^n_m \to y_m \in [0, R] \) as \( n \to \infty \), and

\[ \tilde{U}_0(y^n_m) = w_0(y^n_m) \leq U(y^n). \]

(2.30)

As before, define \( U^n(z) = U(y^n + z) \). We may assume that \( U^n \to U^\infty \) in the \( C^2 \)-sense over compacts, where \( U^\infty \) satisfies

\[ \Delta U^\infty = g(U^\infty). \]

From (2.29) and the Maximum Principle, we obtain

\[ w_0(y_0 + z_m) > U^\infty(z) \quad \text{for } z_m \in (-y_m, R - y_m). \]

(2.31)

It follows from (2.30) that \( y_m = 0 \), hence \( U^\infty(0) = w_0(0) \), but (2.30) also implies

\[ \frac{\partial}{\partial z_m}(w_0(z_m) - U^\infty(z))|_{z=0} = 0. \]

This and (2.31) easily yield a contradiction with Hopf’s Lemma. Hence \( E \) is open, so that \( E = [0, \delta^*] \). In particular,

\[ \tilde{U} > U \text{ in } H_R. \]

But \( \frac{\partial U}{\partial r} = \frac{\partial \tilde{U}}{\partial r} \), \( U = \tilde{U} \) on \( \Gamma_0 \), and we obtain again a contradiction with Hopf’s Lemma. We have proved that Case 3 is not possible.

In the second part of the above proof, we have essentially used a variation of the so-called “Sweeping Principle”. See [6] for a statement and applications of this method to a problem related to ours.

**Case 4.** \( \alpha \in \partial N_1 \cap \partial \Omega \). After arguments similar to those given in Case 3, and recalling that \( \partial N_1 \) and \( \partial \Omega \) meet orthogonally, we reduce Case 4 to the following situation.
LAYERS WITH NONSムフSH INTERFACE

Denote $y \in \mathbb{R}^m$ as $y = (y''', y_{m-1}, y_m)$, where $y'' \in \mathbb{R}^{m-2}$, and $H^- = \{ y | y_{m-1} < 0 \}$. Then there exists $U \in C^2(\bar{H}^-)$ such that

$$\Delta U = g(U) \quad \text{on} \quad H^-$$

(2.32)

where $g$ is as in (2.21),

$$\frac{\partial U}{\partial y_{m-1}} = 0 \quad \text{on} \quad \partial H^-$$

(2.33)

and

$$\sup_{y'' \in \partial H^-} |U - h_-(x)| = |U(0) - h_-(x)| = \rho.$$  

(2.34)

Using (2.32) and (2.33), we can extend $U$ through $\partial H^-$ evenly, so that the extension still satisfies (2.32), now on the whole $\mathbb{R}^m$. At this point we are in the same situation we found in Case 3, hence Case 4 is not possible. This concludes the proof of the lemma.

3. Proof of the main result

We are now in a position to prove Theorem B.

Proof of Theorem B. Let $M > 0$ be as in Lemma 2.3. Choose a small $\rho > 0$ and $\varepsilon_0$ as in Lemma 2.4. Set

$$\Lambda_{\rho, \varepsilon} = \{ u \in C(\bar{\Omega}) \mid |u - h_+| < \rho \text{ on } \Omega^+_0 \text{ and } |u| < M \text{ on } \partial \Omega \}.$$  

From the same arguments used in Lemma 2.1, we obtain from Lemma 2.4

$$\deg (I - T^\varepsilon, h_+, 0) = \deg (I - h^\varepsilon, \Lambda_{\rho, \varepsilon}, 0)$$

for all sufficiently small $\varepsilon > 0$. Here $T^\varepsilon$ is the operator defined in (2.6) and $h^\varepsilon$ the function given by (2.12). But $h^\varepsilon$ is in $\Lambda_{\rho, \varepsilon}$ for small $\varepsilon$, hence the latter degree equals one. As in Lemma 2.1, we conclude the existence of a solution to (1.1) in $\Lambda_{\rho, \varepsilon}$ for all small $\varepsilon$.

Next set $\rho = \delta$, and let $u^\varepsilon_\delta$ be the predicted solution in $\Lambda_{\delta, \delta}$. Define

$$\delta = \inf \{ \delta > 0 \mid \exists \varepsilon_0 > 0 \forall \varepsilon \leq \varepsilon_0 \ |u^\varepsilon_\delta - h_+| < \delta \text{ on } \Omega^+_0 \}.$$  

A simple indirect argument yields that $\delta = 0$. It follows the existence of a decreasing sequence $\delta_n \rightarrow 0$ such that

$$3\varepsilon_n > 0 \forall \varepsilon \leq \varepsilon_n \ |u_n^\varepsilon - h_+| < \delta_n \text{ on } \Omega^+_0.$$  

Without loss of generality we may assume that $\varepsilon_n$ is decreasing. Finally, define

$$u_\varepsilon = u^\varepsilon_{\varepsilon_n} \quad \text{if} \quad \varepsilon \in (\varepsilon_{n+1}, \varepsilon_n).$$

Clearly $\{ u_\varepsilon \}_{0 < \varepsilon < \varepsilon_0}$ defined in this manner satisfies the requirements of Theorem B. This concludes the proof.

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