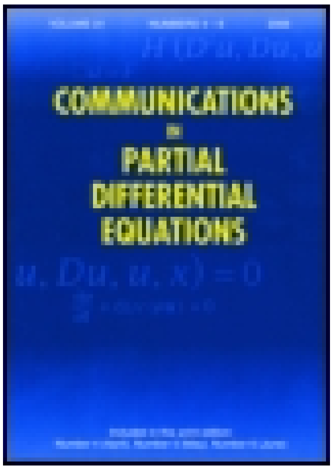

Communications in Partial Differential Equations
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lpde20

Layers With Nonsmooth Interface in a Semilinear Elliptic Problem

Manuel A. del Pino [a]
[a] University of Minnesota Minneapolis , Minnesota, 55455, U.S.A.
[b] Departamento de Matematicas , Fac. de Ciencias Universidad de Chile , Santiago 1, Chile
Published online: 08 May 2007.

COMMUN. IN PARTIAL DIFFERENTIAL EQUATIONS, 17(9&10), 1695-1708 (1992)

LAYERS WITH NONSMOOTH INTERFACE IN A SEMILINEAR ELLIPTIC PROBLEM

Manuel A. del Pino

School of Mathematics University of Minnesota
Minneapolis, Minnesota 55455 U.S.A.
and
Departamento de Matematicas, Fac. de Ciencias
Universidad de Chile, Casilla 653 Santiago 1, Chile.

1. Introduction

Let Ω be a smooth bounded domain in $\mathbf{R}^m$, $m \geq 1$. In this paper we consider the semilinear Neumann problem

$$\varepsilon^2 \Delta u = f(u,x) \quad \text{in } \Omega \tag{1.1}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega$$

where f is of class C^1 and $f(\cdot, x)$ has precisely three zeros $h_-(x) < h_0(x) < h_+(x)$ for each $x \in \bar{\Omega}$. We also assume that $h_\pm$ are nondegenerate and stable, namely

$$f_u(h_\pm(x), x) > 0 \quad \text{for all } x \in \bar{\Omega}. \tag{1.2}$$

We are interested in solutions to (1.1) exhibiting a transition layer from h_- to h_+ as ε approaches zero. More precisely, we look for two open subsets Ω_+ and Ω_- of Ω and a family of solutions $\{u^\varepsilon\}_{\varepsilon>0}$ to problem (1.1) such that u_ε approaches $h_\pm$ on compact subsets of $\Omega_\pm$.

In a pioneering work, Fife and Greenlee [8] gave sufficient conditions for this phenomenon to take place. As pointed out by Caginalp and Fife [5], the following result for problem (1.1) follows from the methods in [8], where the Dirichlet case was treated. We denote

$$J(x) = \int_{h_-(x)}^{h_+(x)} f(s,x)ds. \tag{1.3}$$

Theorem A. *Let $m = 2$. Assume the existence of a closed smooth curve $\Gamma \subset \Omega$ which divides Ω into two smooth subdomains Ω_+ and Ω_- and such that $J = 0$ and $\frac{\partial J}{\partial \nu} > 0$ on Γ. Here ν denotes the normal direction to Γ towards Ω_-. Then there exist a positive number ε_0 and a family of solutions $\{u^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ to problem (1.1) such that*

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = h_\pm(x)$$

uniformly on compacts of $\Omega_\pm$.

It is observed in [8] that dimension is not an important restriction in this result. In fact, their method works in any dimension m, with Γ replaced by a finite number of closed hypersurfaces.

What still seems to be an important restriction is that of the smoothness of the interface Γ. Indeed, the method of [8] is based upon a careful formal approximation of the solution near Γ using a series expansion in powers of ε. They then apply implicit function techniques based on this approximation. This method does not seem to apply once the smoothness hypothesis on Γ is removed.

It is noticed in [5] that a simpler, but still delicate, super-subsolutions approach could be devised along the lines of that paper. This is done in [7] for Dirichlet boundary conditions. That proof, however, also depends on the smoothness of the interface.

Here we present a completely different proof of Theorem A which permits to overcome this difficulty. Moreover, it allows Γ to be an arbitrary closed subset of $\bar{\Omega}$. In particular, Γ may intersect $\partial\Omega$ and have an arbitrary number of components.

Our proof, based on standard elliptic theory and degree theoretical arguments, is rather simple and does not require the construction of first approximations or super-sub solutions based on the formal knowledge of the behavior of the solution near the interface. This flexibility may be useful in the study of more complex problems, like systems, in which the properties of the interface may not be entirely *a priori* known.

The following is the main result of this paper.

Theorem B. *Let $m \geq 1$ and assume the existence of a closed set $\Gamma \subset \bar{\Omega}$ and of open disjoint subsets of Ω, Ω_+ and Ω_- such that*

$$\bar{\Omega} = \bar{\Omega}_+ \cup \Gamma \cup \bar{\Omega}_-.$$

Assume also the existence of an open neighborhood $\mathcal{N}$ of Γ such that

$$J(x) > 0 \quad \text{for } x \in \mathcal{N} \cap \bar{\Omega}_- \setminus \Gamma \tag{1.4}$$

and

$$J(x) < 0 \quad \text{for } x \in \mathcal{N} \cap \bar{\Omega}_+ \setminus \Gamma \tag{1.5}$$

Then there exist a positive number ε_0 and a family of solutions $\{u^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ to problem (1.1) such that

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = h_\pm(x)$$

uniformly on compact subsets of $\bar{\Omega}_\pm \setminus \Gamma$, in particular of $\partial\Omega_\pm \setminus \Gamma$.

Theorem A clearly follows from this result. Observe that the condition $\frac{\partial J}{\partial \nu} > 0$ has been replaced by a "change of sign" assumption for J on Γ.

If $f(\cdot, x)$ has more than one zero between $h_-(x)$ and $h_+(x)$, conditions (1.4) and (1.5) should respectively be replaced by

$$\int_{h_-(x)}^{u} f(s,x)ds > 0 \quad \text{for } u \in (h_-(x), h_+(x)] \text{ and } x \in \mathcal{N} \cap \bar{\Omega}_- \setminus \Gamma \qquad (1.4)'$$

and

$$\int_{u}^{h_+(x)} f(s,x)ds < 0 \quad \text{for } u \in [h_+(x), h_+(x)) \text{ and } x \in \mathcal{N} \cap \bar{\Omega}_+ \setminus \Gamma. \qquad (1.5)'$$

These conditions are equivalent to (1.4) and (1.5) in case that only one zero between h_- and h_+ exists, and what we will actually use in the proof.

Remark. A different method for the obtention of layered families of solutions is the direct variational approach, which has been used in e.g. [1], [2], [11] and [13]. Alikakos and Simpson [2] have studied several properties of the global minimizer of the associated energy for a special case of (1.1) under radial symmetry. In our case, these global minimizers constitute a family of layered solutions as in Theorem B when one takes $\Omega_+ = \{x | J(x) < 0\}$, $\Omega_- = \{x | J(x) > 0\}$ and the Lebesgue measure of $\Gamma = \{x | J(x) = 0\}$ is zero. This is not hard to establish, by conveniently adapting the argument in [2], but only L^p-convergence is clear. Uniform convergence away from the layer as that given by Theorem B is not obvious from this characterization. That property is useful in deriving other convergence features of the family. See [4].

Remark. Our method does not predict existence of a family of solutions with layer in the "opposite direction", that is, approaching h_- in Ω_+ and h_+ in Ω_-. In the one-dimensional case such a family is known to exist and be unstable. See [10].

The literature on layers in elliptic and parabolic semilinear equations with a bi-stable nonlinearity is today extense. We refer the reader for example to [1], [3], [4], [5], [10], [11], [13] and references therein for problems related to the one treated here.

2. Preliminar Results

Theorem B will be a consequence of some lemmas which we state and prove next.

Our first lemma gives the existence of families of unlayered solutions converging uniformly to each of the stable zeros of f.

Lemma 2.1. *There exist a number ε_0 and families of solutions $\{h_-^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ and $\{h_+^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ to problem (1.1) such that*

$$\lim_{\varepsilon \to 0} h_\pm^\varepsilon(x) = h_\pm(x)$$

uniformly on $\bar{\Omega}$.

Proof. We will prove the existence of h_ε^-. The proof for h_ε^+ is the same.

Consider for $t \in [0,1]$ the problem

$$\varepsilon^2 \Delta u = tf(u,x) + (1-t)(u - h_-(x)) \quad \text{in } \Omega \tag{2.1}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Fix $\rho_0 > 0$ so small that $f_u(s,x) > 0$ whenever $|s - h_-(x)| \le \rho_0$ and $x \in \bar{\Omega}$.

Claim. Given $0 < \rho \le \rho_0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \le \varepsilon_0$ and all $t \in [0,1]$, every solution of (2.1) in B_{ρ_0} is in B_ρ. Here B_ρ denotes the open ball center h_- radius ρ in $C(\bar{\Omega})$.

Proof of the claim. Assume the contrary. Then there exist sequences $\varepsilon_n \to 0$, $t_n \to \bar{t} \in [0,1]$ and u_n, solution of (2.1) for $t = t_n$, $\varepsilon = \varepsilon_n$, such that

$$\sup_{x \in \bar{\Omega}} |u_n(x) - h_-(x)| = \rho \quad \text{for all } n \in \mathbb{N}. \tag{2.2}$$

Let $\bar{x}_n \in \bar{\Omega}$ be a point where the supremum (2.2) is attained. Assume also that $x_n \to \bar{x} \in \bar{\Omega}$. We consider two cases.

Case 1. $\bar{x} \in \Omega$. In this case, for all sufficiently large n, the ball $\bar{B}(x_n, \varepsilon_n)$ is contained in Ω. For $y \in \bar{B}(0,1)$ we define

$$U_n(y) = u_n(x_n + \varepsilon_n y).$$

Then U_n satisfies the equation

$$\Delta U_n = t_n f(U_n, x_n + \varepsilon_n y) + (1 - t_n)(U_n - h_-(x_n + \varepsilon_n y)) \tag{2.3}$$

on $\bar{B}(0,1)$.

From (2.2), we see that U_n is uniformly bounded, as well as the right hand side of (2.3). Then, L^p and Schauder estimates give the existence of a subsequence of U_n converging in the $C^{2,\alpha}(\bar{B}(0,1))$-sense to a solution U of

$$\Delta U = \bar{t} f(U, \bar{x}) + (1 - \bar{t})(U - h_-(\bar{x}))$$

on $\bar{B}(0,1)$.

From the fact that $|U - h_-(\bar{x})| \le \rho$ and our choice of ρ, we find that $V \equiv U - h_-(\bar{x})$ satisfies on $\bar{B}(0,1)$ an equation of the form

$$\Delta V - c(y)V = 0$$

with $c > 0$. But, from the definition of x_n, either V or $-V$ attains a nonnegative maximum at $y = 0$. This contradicts the maximum principle and shows the impossibility of Case 1.

Case 2. $\bar{x} \in \partial\Omega$. Here we distinguish two subcases:

(a) There exists a number $\delta > 0$ and a subsequence of ε_n, relabeled again ε_n such that $B(x_n, \varepsilon_n \delta) \subset \Omega$ for all n.

(b)
$$\lim_{n \to \infty} \frac{dist(x_n, \partial\Omega)}{\varepsilon_n} = 0.$$

If (a) holds we will obtain the same situation of Case 1, hence (a) is not possible. If (b) holds we argue as follows: consider a local chart $\phi : \mathcal{U} \to \mathbf{R}^m$ where $\mathcal{U}$ is some open neighborhood of $\bar{x}$ such that

$$\phi(\mathcal{U} \cap \Omega) = \mathbf{R}^m_+ = \{z = (z', z_m) \mid z' \in \mathbf{R}^{m-1}, z_m > 0\}$$

and $\phi(\bar{x}) = 0$. Then $\tilde{u}_n(z) \equiv u_n(\phi^{-1}(z))$ satisfies

$$\varepsilon_n^2 L\tilde{u}_n = t_n f(\tilde{u}_n, \phi^{-1}(z)) + (1 - t_n)(\tilde{u}_n - h_-(\phi^{-1}(z)) \quad \text{in } \mathbf{R}^m_+ \tag{2.4}$$

$$\frac{\partial \tilde{u}_n}{\partial z_n} = 0 \quad \text{on } \partial \mathbf{R}^m_+$$

where L is a strongly elliptic operator of the form

$$L = \sum_{i,j} a_{ij} \frac{\partial^2}{\partial z_i \partial z_j} + \sum_i b_i \frac{\partial}{\partial z_i}.$$

Moreover, after a convenient choice of the change of coordinates, we may also assume $a_{ij}(0) = \delta_{ij}$.

Let π be the orthogonal projection onto $\partial\Omega$ which is well defined and smooth in some neighborhood of $\partial\Omega$. Set

$$\tilde{U}_n(y) = \tilde{u}_n(\phi^{-1}(\phi(\pi(x_n)) + \varepsilon_n y))$$

for $y \in B(0,1) \cap \mathbf{R}^m_+$. After writing (2.4) in the y-coordinates, elliptic estimates imply convergence of a subsequence of U_n to some $U \in C^2(B(0,1) \cap \mathbf{R}^m_+)$ satisfying

$$\Delta U = \bar{t} f(U, \bar{x}) + (1 - \bar{t})(U - h_-(\bar{x})) \tag{2.5}$$

$$\frac{\partial \tilde{U}}{\partial y_m} = 0 \quad \text{on } \partial \mathbf{R}^m_+ \cap B(0,1).$$

But the Neumann boundary condition permits us to extend U evenly to the whole $B(0,1)$, so that the extension still satisfies (2.5). A similar straightening-reflection argument appears in [12].

On the other hand, since (b) holds, we find that

$$\frac{\phi(\pi(x_n)) - \phi(x_n)}{\varepsilon_n} \to 0 \quad \text{as } n \to \infty.$$

But $U(\frac{\phi(\pi(x_n)) - \phi(x_n)}{\varepsilon_n}) = u_n(x_n)$, and hence $|U(0) - h_-(\bar{x})| = \rho$. At this point we are in the same situation of Case 1 and a contradiction comes from the maximum principle. This concludes the proof of the claim.

The conclusion of the lemma follows from the claim and a degree-theoretical argument. Let R_ε denote the inverse of $(\varepsilon^2\Delta - I)$ under Neumann boundary conditions. Then R_ε applies compactly $C(\bar{\Omega})$ into itself. Observe that equation (1.1) is equivalent to the fixed point problem in $C(\bar{\Omega})$:

$$u = R_\varepsilon(f(u, x) - u) \equiv T^\varepsilon(u). \tag{2.6}$$

Consider the compact homotopy

$$Q_t^\varepsilon(u) = R_\varepsilon(t(f(u,x) - u) - (1-t)h_-). \tag{2.7}$$

Fix ρ sufficiently small. From the claim, we know that Q_t^ε has no fixed points on ∂B_ρ for all $t \in [0,1]$, provided that ε is sufficiently small. Since $Q_1^\varepsilon = T^\varepsilon$ and $Q_0^\varepsilon = -R_\varepsilon(h_-)$, the invariance of the degree under compact homotopies implies

$$\deg\,(I - T^\varepsilon, B_\rho, 0) = \deg\,(I + R_\varepsilon(h_-), B_\rho, 0).$$

If $-R_\varepsilon(h_-)$ is included in B_ρ, the latter degree equals one and we would conclude existence of a solution of (2.6) and hence of (1.1) in B_ρ. Thus, it only remains to verify that the unique solution of

$$-\varepsilon^2\Delta u + u = h_- \quad \text{in } \Omega \tag{2.8}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega$$

lies on B_ρ for all sufficiently small ε. But (2.8) corresponds to (2.1) for $t = 0$. Observe that for $t = 0$ the claim holds true without smallness restriction on ρ_0. Hence, it suffices to show that the solution of (2.8) has an L^∞-estimate independent of ε. This is true. Indeed, it is easily seen that

$$\inf_\Omega h_- \le u \le \sup_\Omega h_-$$

for a solution u of (2.8) and for all $\varepsilon > 0$: just use the H^1-functions $(u - \inf_\Omega h_-)^-$ and $(u - \sup_\Omega h_-)^+$ as test functions for (2.8).

We conclude the existence of a solution h_-^ε to (1.1) in B_ρ. The uniform convergence of h_-^ε to h_- is a consequence of the claim for $t = 1$. ■

Our proof of Theorem B will require the construction of "nice" approximations of the sets Ω_- and Ω_+. We will do this using the following lemma.

Lemma 2.2. *Let $K \subset \mathbf{R}^m$ be a compact set and M a smooth boundaryless hypersurface. Then, given $\delta > 0$, there exists an open neighborhood N_δ of K such that*
(a)

$$\{x \mid dist(x,K) \le \frac{\delta}{2}\} \subset N_\delta \subset \{x \mid dist(x,K) \le \delta\}$$

(b) N_δ is a finite union of smooth domains.
(c) Either ∂N_δ does not intersect M or does it orthogonally.

Proof. Let $\psi_\varepsilon(x)$ be the usual approximation of the identity, i.e.,

$$\psi_\varepsilon(x) = \frac{1}{\varepsilon^m}\psi(\frac{x}{\varepsilon})$$

where $\psi \in C_c^\infty(|x| < 1)$ and $\int \psi = 1$.

Set $K_\delta = \{x \mid dist(x,K) \le \frac{\delta}{3}\}$ and define

$$g_\varepsilon = \psi_\varepsilon * \chi_{K_\delta}$$

where * denotes convolution product and χ_{K_δ} the characteristic function of the set K_δ. g_ε is smooth. Observe also that for sufficiently small ε,

$$g_\varepsilon(x) = 1 \quad \text{if } dist(x,K) \le \frac{\delta}{2} \tag{2.9}$$

and

$$g_\varepsilon(x) = 0 \quad \text{if } dist(x,K) \ge \delta. \tag{2.10}$$

Fix such a small $\varepsilon > 0$.

Let π denote the orthogonal projection onto M, which is well defined and smooth in some neighborhood V of M. Consider a smooth function $d(x)$ which vanishes outside V and equals one on some smaller neighborhood of M. Define

$$\tilde{g}(x) = d(x)g_\varepsilon(\pi(x)) + (1-d(x))g_\varepsilon(x).$$

Observe that, reducing ε and V if necessary, $\tilde{g}$ still satisfies (2.9) and (2.10). Finally, set

$$N_\delta = \{x \mid \tilde{g}(x) > 1-\rho\}$$

where, using Sard's Theorem, we choose $\rho > 0$ small and so that $1-\rho$ is a regular value of $\tilde{g}$. We easily see that N_δ satisfies (a) and (b). Now, if ∂N_δ intersects M at some point $\bar{x}$, we see from the definition of $\tilde{g}$ that

$$\nabla\tilde{g}(\bar{x}) = P_{\bar{x}}\nabla g_\varepsilon(\bar{x})$$

where, $P_{\bar{x}}$ denotes the orthogonal projection onto the tangent space to M at $\bar{x}$. It follows that ∂N_δ and M intersect orthogonally, at $\bar{x}$, hence (c) holds. ∎

Next fix $\delta > 0$ and let Ω_+, Ω_-, Γ be as in the hypotheses of Theorem B. Let N_δ be a neighborhood of Γ as given by Lemma 2.2. Define the open sets Ω_+^δ, Ω_-^δ by

$$\Omega_\pm^\delta = \Omega_\pm \setminus N_\delta.$$

Observe that $\Omega_\pm^\delta$ is a finite union of bounded domains. Moreover, $\partial\Omega_\pm^\delta$ is smooth, except near $\partial\Omega \cap \partial N_\delta$, which consists of a finite union of orthogonal corners.

Also, for sufficiently small δ we have

$$\pm J(x) > 0 \quad \text{for } x \in \partial N_\delta \cap \bar{\Omega}_\pm \tag{2.11}$$

where J is defined by (1.3). We will assume this henceforth.

Let $h_\pm^\varepsilon$ be the families of unlayered solutions predicted by Lemma 2.1. Let ψ be a C^∞ function such that $\psi \equiv 1$ on $\Omega_{\frac{\delta}{3}}^-$ whose support is compact and does not intersect $\Omega_{\frac{\delta}{3}}^+$. Define

$$h_\varepsilon \equiv h_-^\varepsilon\psi + h_+^\varepsilon(1-\psi) \tag{2.12}$$

Denote by χ_δ^+ (resp. χ_δ^-) the characteristic function of $\Omega_{\frac{\delta}{2}}^+$ (resp. $\Omega_{\frac{\delta}{2}}^-$).

Our proof of Theorem B is based on the study of the following family of problems.

$$\varepsilon^2\Delta u = tf(u,x) + (1-t)\{\chi_\delta^-(u-h_-^\varepsilon) + \chi_\delta^+(u-h_+^\varepsilon) + \varepsilon^2\Delta h^\varepsilon\} \tag{2.13}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

We intend to apply degree theory to (2.13), as we did in the proof of Lemma 2.1, in some appropriate open subset of $C(\bar{\Omega})$ to conclude the existence of solutions of (1.1) with the desired characteristics. First we require a lemma.

Lemma 2.3. *There exists an L^∞-bound $M > 0$, independent of all small ε and all $t \in [0,1]$, for the solutions of (2.13).*

Proof. Set $v \equiv u - h^\varepsilon$. Then (2.13) takes the form

$$\varepsilon^2 \Delta v = t(f(h^\varepsilon + v, x) - \varepsilon^2 \Delta h^\varepsilon) + (1-t)(\chi_\delta^+ + \chi_\delta^-)v \tag{2.14}$$

$$\frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Assume first $t > 0$. Fix a number $R > 0$. Using $(v-R)^+ \in H^1(\Omega)$ as a test function for (2.14) we obtain

$$-\varepsilon^2 \int_{(v>R)} |\nabla v|^2 - (1-t)\int_{(v>R)} (\chi_\delta^+ + \chi_\delta^-)v(v-R) = \int_{(v>R)} t(f(h^\varepsilon + v, x) - \varepsilon^2 \Delta h^\varepsilon)(v-R). \tag{2.15}$$

If the set $(v > R)$ were nonempty, (2.15) would imply the existence of $x_0 \in \Omega$ and a number $s > R$ such that

$$f(h^\varepsilon(x_0) + s, x_0) \leq \varepsilon^2 \Delta h_\varepsilon. \tag{2.16}$$

On the other hand, it is easily seen that any solution to (1.1) is between $\inf h_-$ and $\sup h_+$. Hence, redefining f if necessary, we may assume that $f(t,x) \geq ct$ for all large t and all $x \in \bar{\Omega}$, some $c > 0$. But $\varepsilon^2 \Delta h^\varepsilon$ is uniformly bounded. Indeed, $\varepsilon^2 \Delta h_\pm^\varepsilon = f(h_\pm^\varepsilon, x) \to 0$ as $\varepsilon \to 0$, uniformly. Since $h_\pm^\varepsilon$ is bounded, elliptic estimates imply that $\varepsilon^2 \nabla h_\pm^\varepsilon$ also is. From this and the definition of h^ε in (2.11) the conclusion is immediate.

We conclude that (2.16) is impossible if R is chosen sufficiently large. Hence, for some large R, $v \leq R$ in $\bar{\Omega}$. A similar procedure gives a lower bound for v, from which a uniform bound for u follows in case that $t > 0$.

Now, if $t = 0$, (2.14) becomes

$$\varepsilon^2 \Delta v = (\chi_\delta^+ + \chi_\delta^-)v \tag{2.17}$$

$$\frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

It easily follows that $v \equiv 0$, hence $u = h_\varepsilon$ is uniformly bounded. This completes the proof. ■

For $\rho > 0$ we consider the open subsets of $C(\bar{\Omega})$ Λ_ρ^+ and Λ_ρ^- defined by

$$\Lambda_\rho^\pm = \{u \in C(\bar{\Omega}) \mid \sup_{x \in \bar{\Omega}_\delta^\pm} |u(x) - h_\pm(x)| \leq \rho\}. \tag{2.18}$$

The following is a key step in the proof of Theorem B, where the fact (2.11) will play a main role.

Lemma 2.4 *Given any $\rho > 0$ sufficiently small there exists a number $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ and all $t \in [0,1]$ there is no solution of (2.14) on $\partial\Lambda_\rho^+ \cup \partial\Lambda_\rho^-$.*

Proof. Assume the contrary. then there exist sequences $\varepsilon_n \to 0$, $t_n \to \bar{t} \in [0,1]$ and a sequence of solutions u_n to (2.14) for $t = t_n$, $\varepsilon = \varepsilon_n$ such that either

$$\sup_{x \in \bar{\Omega}_\delta^-} |u_n(x) - h_-(x)| = \rho. \quad \text{for all } n \in \mathbb{N} \tag{2.19}$$

or

$$\sup_{x \in \bar{\Omega}_\delta^+} |u_n(x) - h_+(x)| = \rho. \quad \text{for all } n \in \mathbb{N}.$$

Assume that (2.19) occurs. The other case is similar. Let x_n be a point where the supremum (2.19) is attained. Without loss of generality we may assume that $x_n \to \bar{x}$ in $\bar{\Omega}_-^\delta$. As in the proof or Lemma 2.1, we consider different cases for the position of $\bar{x}$.

Case 1. $\bar{x} \in \Omega_-^\delta$. In this case the ball $\bar{B}(x_n, \varepsilon_n)$ lies on Ω_-^δ for all sufficiently large n. Define

$$U_n(y) = u_n(x_n + \varepsilon_n y)$$

for $y \in \bar{B}(0,1)$. Thus U_n satisfies in this ball

$$\Delta U_n = t_n f(U_n, x_n + \varepsilon_n y) + (1 - t_n)((U_n - h_-^\varepsilon(x_n + \varepsilon_n y)) + (1 - t_n)\varepsilon_n^2 \Delta h_-^{\varepsilon_n}.$$

Since $h_-^{\varepsilon_n} \to h_-$ and $\varepsilon_n^2 \Delta h_-^{\varepsilon_n} \to 0$ uniformly, we obtain as in Lemma 2.1 the existence of a subsequence of U_n convergent in the C^2-sense to some U satisfying in $\bar{B}(0,1)$ the equation

$$\Delta U = \bar{t} f(U, \bar{x}) + (1 - \bar{t})(U - h_-(\bar{x}))$$

and we obtain, as in Case 1 of Lemma 2.1, a contradiction to the Maximum Principle.

Case 2. $\bar{x}$ is in $\partial\Omega \setminus \bar{N}_\delta$. As in Case 1, we can reach a contradiction by slightly modifying the proof in Case 2 of Lemma 2.1.

Case 3. $\bar{x} \in \partial N_\delta \cap \Omega$. Here we distinguish two subcases

(a) For some subsequence of x_n, again labeled x_n, and some $\lambda > 0$ we have

$$\bar{B}(x_n, \lambda\varepsilon_n) \subset \Omega_\delta^- \quad \text{for all } n \in \mathbb{N}$$

or

(b)

$$\lim_{n \to \infty} \frac{dist(x_n, \partial\Omega_-^\delta)}{\varepsilon_n} = 0.$$

If (a) holds, we may proceed exactly as in Case 1, just changing $B(0,1)$ by $B(0,\lambda)$. In case (b) further considerations are needed. Here the fact that $J(\bar{x}) > 0$ will permit us to reach a contradiction.

We straighten $\partial\Omega_-^\delta$ near $\bar{x}$, as we did in Case 2 of Lemma 2.1. After a standard diagonal procedure to obtain the C^2 convergence on compacts of some subsequence of the rescaling U_n, (recall from Lemma 2.3 that u_n is uniformly bounded) we obtain the existence of $U \in C^2(\mathbf{R}^m)$ bounded and satisfying in $\mathbf{R}^m$

$$\Delta U = \bar{t} f(U, \bar{x}) + (1 - \bar{t})(U - h_-(\bar{x})). \tag{2.20}$$

Moreover,

$$|U(y) - h_-(\bar{x})| \leq \rho \quad \text{on } \bar{\mathbf{R}}_-^m$$

with equality at the origin. We will assume

$$U(0) - h_-(\bar{x}) = \rho.$$

The other case is similar. Denote by g the function defined by

$$g(s) = \bar{t}f(s, \bar{x}) + (1 - \bar{t})(s - h_-(\bar{x})). \tag{2.21}$$

Let $w_0(t)$ be the unique solution of the differential equation

$$w''(t) = g(w(t)) \tag{2.22}$$

$$w(0) = h(\bar{x}) + \rho, \quad w'(0) = \frac{\partial U}{\partial y_m}(0)$$

and consider $\tilde{U} \in C^2(\mathbf{R}^m)$ defined by

$$\tilde{U}(y', y_m) = w_0(y_m).$$

Recall that $\inf h_- \leq u \leq \sup h_+$ for every solution u to (1.1). Hence, we do not lose generality in assuming

$$\lim_{s \to \pm\infty} f(s, x) = \pm\infty$$

uniformly on $x \in \bar{\Omega}$. Since this holds at $-\infty$ and $\inf_{x \in \bar{\Omega}} f_u(h_-(x), x) > 0$, we see that if ρ is chosen sufficiently small, then

$$\frac{g(s) - g(r)}{s - r} > 0 \tag{2.23}$$

whenever $|s - h_-(\bar{x})| \leq \rho$ and $r - h_-(\bar{x}) \leq \rho$.

Let us assume that w_0 satisfying (2.22) is increasing on $(-\infty, 0]$. We will prove this fact later. We shall next show that this implies that $U = \tilde{U}$ on $\bar{\mathbf{R}}^m_-$. Indeed, observe that

$$\Delta(\tilde{U} - U) - \frac{g(U) - g(\tilde{U})}{(U - \tilde{U})}(\tilde{U} - U) = 0 \tag{2.24}$$

and that (2.23) holds for $s = U$ and $r = \tilde{U}$. Since

$$(\tilde{U} - U) = \rho - (U - h_-(\bar{x})) \geq 0 \text{ on } \partial\mathbf{R}^m_-,$$

(2.24) and the maximum principle imply $\tilde{U} - U > 0$ in $\mathbf{R}^m_-$ unless $(\tilde{U} - U) \equiv 0$. But since $(\tilde{U} - U)(0) = 0$, Hopf's Boundary Point Lemma (e.g. [9], p. 34) implies that the former case can only occur if $\frac{\partial}{\partial y_m}(\tilde{U} - U)(0) < 0$ which is false by definition of $\tilde{U}$. Hence, necessarily $U = \tilde{U}$ on $\bar{\mathbf{R}}^m_-$ if w_0 is increasing. We will next show that this is indeed the case

Observe first that

$$w_0'(0) = \frac{\partial U}{\partial y_m}(0) \geq 0$$

since U maximizes on $\bar{\mathbf{R}}^m_-$ at 0. We must actually have $w_0'(0) > 0$. Indeed, otherwise $w_0(t) > h(\bar{x}) + \rho$ for all sufficiently small $t < 0$, since $w_0''(0) > 0$. Hence $(\tilde{U} - U) > 0$ on some ball $B \subset \bar{\mathbf{R}}^m_-$ such that $0 \in \partial B$. From this we immediately find a contradiction with Hopf's Lemma. Hence $w_0'(0) > 0$.

Assume that w_0 is not increasing on $(-\infty, 0]$ and let $-\infty < -\bar{r} < 0$ be the first negative point where $w_0'(-\bar{r}) = 0$. Since w_0 satisfies an autonomous O.D.E., the reflection of w_0 through $-\bar{r}$ coincides with w_0. Hence $w_0(t) \le h_-(\bar{x}) + \rho$ on $[-2\bar{r}, 0]$ with equality at the endpoints and $w_0'(-2\bar{r}) = -w_0'(0) < 0$.

Again from (2.23) with $s = U$, $r = \tilde{U}$ and (2.24), we conclude that $U(y) = \tilde{U}(y)$ for all $y = (y', y_m) \in \bar{\mathbf{R}}^m_-$ such that $-2\bar{r} \le y_m \le 0$. In particular, U maximizes at $y = (0_{R^{m-1}}, -2\bar{r})$ and hence $\nabla U = 0$ at that point. But

$$\frac{\partial U}{\partial y_m}(0_{R^{m-1}}, -2\bar{r}) = w_0'(-2\bar{r}) < 0.$$

We have obtained a contradiction which shows that $w_0' > 0$ on $(-\infty, 0]$, and hence $U = \tilde{U}$ on $\bar{\mathbf{R}}^m_-$. We will next see what happens on $\mathbf{R}^m_+$. First, a property of w_0 which follows from the key fact $J(\bar{x}) > 0$.

Claim. $w_0(t) \to +\infty$ as $t \to +\infty$. Indeed, we know that w_0 is increasing on $(-\infty, 0]$. Since w_0 is also bounded there (from $\tilde{U} = U$), it easily follows that $w_0(-\infty) = h_-(\bar{x})$ and $w_0'(-\infty) = 0$. We thus obtain from (2.22)

$$\frac{w_0'(t)^2}{2} = \int_{h_-(\bar{x})}^{w_0(t)} g(s)ds. \tag{2.25}$$

Recall that $w_0'(0) > 0$, $w_0(0) = h_-(\bar{x}) + \rho$. If w_0' vanished at some point $\bar{s} > 0$, we would have

$$0 = \int_{h_-(\bar{x})}^{w_0(\bar{s})} g(s)ds = \bar{t}\int_{h_-(\bar{x})}^{w_0(\bar{s})} f(s, \bar{x})ds + (1 - \bar{t})\frac{(w_0(\bar{s}) - h_-(\bar{x}))^2}{2}. \tag{2.26}$$

But, since $w_0(\bar{s}) > h_-(\bar{x}) + \rho$ and $J(\bar{x}) > 0$, it follows that the right hand side of (2.26) is bounded below by a strictly positive constant, and we get a contradiction which shows that w_0 is strictly increasing. Observe that actually (1.4)$'$ at $x = \bar{x}$ has been used here. For the same reason, (2.25) implies that w_0 is unbounded and the claim follows.

It follows from the claim that $\tilde{U}$ is unbounded. We will reach a contradiction from this fact by means of the following argument.

Denote by $w_\delta(t)$ the unique solution of the O.D.E.

$$w''(t) = g(w(t))$$

$$w(0) = h_-(\bar{x}) + \rho, \quad w'(0) = \frac{\partial U}{\partial y_m}(0) + \delta$$

and set $\tilde{U}_\delta(y', y_m) \equiv w_\delta(y_m)$. Then $\tilde{U}_\delta$ solves (2.20). Also, $\tilde{U}_0 = \tilde{U}$.

Let $R > 0$ and define $H_R = \{(y', y_m) \mid 0 < y_m < R\}$. Denote by Γ_0 and Γ_R respectively the left and right boundaries of H_R.

Since w_δ is increasing and unbounded and U is bounded, we find that for all $R > 0$ sufficiently large $\tilde{U}_\delta > U$ on Γ_R for all $\delta \ge 0$. Fix such an R. We claim that there exists a number $\delta^* > 0$ so large that

$$\tilde{U}_{\delta^*} > U \quad \text{on } H_R.$$

Assume the contrary. Then there exist sequences $\delta_n \to \infty$ and points $y_n \in H_R$, $y^n = ({y'}^n, y_m^n)$ such that

$$U(y^n) \ge \tilde{U}_{\delta_n} \tag{2.27}$$

Define

$$U^n(z) = U(y^n + z). \tag{2.28}$$

Then $\Delta U^n = g(U^n)$. Since U^n is bounded, elliptic estimates imply that we do not lose any generality in assuming that U^n converges in the C^1 sense on $\bar{B}(0,R)$. In particular, $\frac{\partial U^n}{\partial z_m}$ remains bounded there.

But from (2.27), the fact that $\frac{\partial \tilde{U}_{\delta_n}}{\partial y_m} > w_0'(0) + \delta_n$ in H_R, and since $U = \tilde{U}_{\delta_n}$, $\frac{\partial U}{\partial y_m} < \frac{\partial \tilde{U}_{\delta_n}}{\partial y_m}$ on Γ_0, the mean value theorem implies the existence of a point $z^n \in B(0,R)$ such that $\frac{\partial U^n}{\partial z_m}(z^n) \to \infty$ as $n \to \infty$. We have reached a contradiction which proves the claim.

Fix a number $\delta^* > 0$ such that $\tilde{U}_{\delta^*} > U$ on H_R and set

$$E = \{\delta \in [0,\delta^*] \mid \tilde{U}_\delta > U \text{ in } H_R\}$$

E is nonempty. It is also closed, for let $\delta_n \in E$ such that $\delta_n \to \bar{\delta} \in [0,\delta^*]$. Then $\tilde{U}_{\bar{\delta}} \geq U$ in H_R. Since $\tilde{U}_{\bar{\delta}} > U$ on Γ_R, it follows from the Maximum Principle that $\tilde{U}_{\bar{\delta}} > U$ in H_R. Hence $\bar{\delta} \in E$ and E is closed.

We will next show that E is also open. Otherwise, there exists $\bar{\delta} \in [0,\delta^*]$ such that

$$\tilde{U}_{\bar{\delta}} > U \quad \text{in } H_R \tag{2.29}$$

and sequences $\delta_n \in [0,\delta^*]$, $y^n = (y'^n, y_m^n)$ such that $\delta_n \to \bar{\delta}$, $y_m^n \to \bar{y}_m \in [0,R]$ as $n \to \infty$. and

$$\tilde{U}_{\delta_n}(y^n) = w_{\delta_n}(y_m^n) \leq U(y^n). \tag{2.30}$$

As before, define $U^n(z) = U(y^n + z)$. We may assume that $U^n \to U^\infty$ in the C^2-sense over compacts, where U^∞ satisfies

$$\Delta U^\infty = g(U^\infty).$$

From (2.29) and the Maximum Principle, we obtain

$$w_{\bar{\delta}}(\bar{y}_m + z_m) > U^\infty(z) \quad \text{for } z_m \in (-\bar{y}_m, R - \bar{y}_m]. \tag{2.31}$$

It follows from (2.30) that $\bar{y}_m = 0$, hence $U^\infty(0) = w_{\bar{\delta}}(0)$. but (2.30) also implies

$$\frac{\partial}{\partial z_m}(w_{\bar{\delta}}(z_m) - U^\infty(z))|_{z=0} = 0.$$

This and (2.31) easily yield a contradiction with Hopf's Lemma. hence E is open, so that $E = [0,\delta_*]$. In particular,

$$\tilde{U} > U \quad \text{in } H_R.$$

But $\frac{\partial \tilde{U}}{\partial y_m} = \frac{\partial U}{\partial y_m}$, $U = \tilde{U}$ on Γ_0, and we obtain again a contradiction with Hopf's Lemma. We have proved that Case 3 is not possible.

In the second part of the above proof, we have essentially used a variation of the so-called "Sweeping Principle". See [6] for a statement and applications of this method to a problem related to ours.

Case 4. $\bar{x} \in \partial N_\delta \cap \partial \Omega$. After arguments similar to those given in Case 3, and recalling that ∂N_δ and $\partial\Omega$ meet orthogonally, we reduce Case 4 to the following situation.

Denote $y \in \mathbf{R}^m$ as $y = (y'', y_{m-1}, y_m)$, where $y'' \in \mathbf{R}^{m-2}$, and $H^- = \{y \mid y_{m-1} < 0\}$. Then there exists $U \in C^2(\bar{H}^-)$ such that

$$\Delta U = g(U) \quad \text{on } H^- \tag{2.32}$$

where g is as in (2.21),

$$\frac{\partial U}{\partial y_{m-1}} = 0 \quad \text{on } \partial H^- \tag{2.33}$$

and

$$\sup_{\mathbf{R}^m_- \cap H^-} |U - h_-(\bar{x})| = |U(0) - h_-(\bar{x})| = \rho. \tag{2.34}$$

Using (2.32) and (2.33), we can extend U through ∂H^- evenly, so that the extension still satisfies (2.32), now on the whole $\mathbf{R}^m$. At this point we are in the same situation we found in Case 3, hence Case 4 is not possible. This concludes the proof of the lemma. ■

3. Proof of the main result

We are now in a position to prove Theorem B.

Proof of Theorem B. Let $M > 0$ be as in Lemma 2.3. Choose a small $\rho > 0$ and ε_0 as in Lemma 2.4. Set

$$\Lambda_{\rho,\delta} = \{u \in C(\bar{\Omega}) \mid |u - h_\pm| < \rho \text{ on } \bar{\Omega}^\delta_\pm \text{ and } |u| < M \text{ on } \bar{\Omega}\}.$$

From the same arguments used in Lemma 2.1, we obtain from Lemma 2.4

$$\deg (I - T^\varepsilon, \Lambda_{\rho,\delta}, 0) = \deg (I - h^\varepsilon, \Lambda_{\rho,\delta}, 0)$$

for all sufficiently small $\varepsilon > 0$. Here T^ε is the operator defined in (2.6) and h^ε the function given by (2.12). But h^ε is in $\Lambda_{\rho,\delta}$ for small ε, hence the latter degree equals one. As in Lemma 2.1, we conclude the existence of a solution to (1.1) in $\Lambda_{\rho,\delta}$ for all small ε.

Next set $\rho = \delta$, and let u^δ_ε be the predicted solution in $\Lambda_{\delta,\delta}$. Define

$$\bar{\delta} = \inf\{\delta > 0 \mid \exists \varepsilon_0 > 0 \ \forall \varepsilon \le \varepsilon_0 \ |u^\delta_\varepsilon - h_\pm| < \delta \text{ on } \bar{\Omega}^\delta_\pm\}$$

A simple indirect argument yields that $\bar{\delta} = 0$. It follows the existence of a decreasing sequence $\delta_n \to 0$ such that

$$\exists \varepsilon_n > 0 \ \forall \varepsilon \le \varepsilon_n \ |u^{\delta_n}_\varepsilon - h_\pm| < \delta_n \text{ on } \bar{\Omega}^{\delta_n}_\pm.$$

Without loss of generality we may assume that ε_n is decreasing. Finally, define

$$u_\varepsilon \equiv u^{\delta_n}_\varepsilon \quad \text{if } \varepsilon \in [\varepsilon_{n+1}, \varepsilon_n).$$

Clearly $\{u_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ defined in this manner satisfies the requirements of Theorem B. This concludes the proof. ■

ACKNOWLEDGEMENTS

This paper is part of the author's Ph.D. thesis at the University of Minnesota. He gratefully acknowledges the encouragement received from his adviser, Prof. Wei-Ming Ni, in the course of this work.

REFERENCES

[1] N.D. Alikakos and P.W. Bates. *On the singular limit in a phase field model of phase transitions.* Ann. Inst. Henri Poincaré 5, (1988) 141-178.

[2] N.D. Alikakos and H. Sympson. *A variational approach for a class of singular perturbation problems and applications* Proc. Roy. Soc. Edinburgh 107 A, (1987) 27-42.

[3] S.B. Angenent, J. Mallet-Paret and L.A. Peletier. *Stable transition layers in a semilinear boundary value problem* J. Diff. Equations 67, (1987) 212-242.

[4] M. Bardi and B. Parthame. *Exponential decay to stable states in phase transitions via a double log-transformation.* Comm. Part. Diff. Eqns. 15 (12), (1990) 1649-1669.

[5] G. Caginalp and P.C. Fife. *Elliptic problems involving phase boundary satisfying a curvature condition* IMA J. Appl. Math. 38, (1987) 195-217.

[6] P. Clement P and G. Sweers. *Existence and multiplicity results for a semilinear elliptic boundary value problem* Ann. Sc. Norm. Sup. Pisa 14, (1987) 97-121.

[7] A.J. De Santi. *Boundary and interior layer behavior of solutions of some singularly perturbed semilinear elliptic boundary value problems.* J. Math pures appl. 65, (1986) 227-262.

[8] P.C. Fife and W.M. Greenlee. *Interior transition layers for elliptic boundary value problems with a small parameter.* Russian Math. Surveys 29, (1974) 103-131.

[9] D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order* second edition. Springer-Verlag, Berlin-New York (1983).

[10] J.K. Hale and K. Sakamoto. *Existence and stability of transition layers.* Japan J. Appl. Math. 5, (1988) 367-405.

[11] R.V. Kohn and P. Sternberg. *Local minimisers and singular perturbations* Proc. Roy. Soc. Edinburgh 111 A, (1989) 69-84.

[12] C.-S. Lin, W.-M. Ni and I. Takagi. *Large amplitude stationary solutions to a chemotaxis system.* J. Diff. Equations 72, (1988) 1-27.

[13] L. Modica. *The gradient theory of phase transitions and the minimal interface criterion.* Arch. Rational Mech. Anal. 98, (1987) 123-142

Received March 1991
Revised May 1991