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On the Number of 2π Periodic Solutions for $u'' + g(u) = s(1 + h(t))$ Using the Poincaré–Birkhoff Theorem*

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1. INTRODUCTION

In this paper we consider the differential equation

$$u'' + g(u) = s(1 + h(t)), \tag{1.1}$$

where $g: \mathbb{R} \to \mathbb{R}$ is of class C^1, $h: \mathbb{R} \to \mathbb{R}$ is continuous and 2π-periodic and s is a parameter.

We are interested in lower bounds for the number of 2π-periodic solutions of (1.1) under two distinct sets of conditions on g, namely

(i) For some nonnegative integer n

$$\lim_{t \to -\infty} g(t) = +\infty \qquad \text{and} \qquad 0 \leqslant n^2 < \lim_{t \to +\infty} g'(t) < (n+1)^2. \tag{1.2}$$

(ii) There exist positive integers k and n such that

$$(k-1)^2 < \alpha \equiv \lim_{t \to -\infty} g'(t) < k^2 \leqslant n^2 < \beta \equiv \lim_{t \to +\infty} g'(t) < (n+1)^2, \tag{1.3}$$

where α and β satisfy the condition $(2\sqrt{\alpha\beta}/(\sqrt{\alpha} + \sqrt{\beta}))$ is not an integer.

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In the rest of this paper we will refer to (P_1) (respectively (P_2)) as the problem of finding a lower bound for the number of 2π-periodic solutions of (1.1) under condition (i) (respectively (ii)).

We will use the following notation, $C[0, 2\pi](C(2\pi))$ will denote the usual Banach space of continuous (2π-periodic) functions $h\colon [0, 2\pi](\mathbb{R}) \to \mathbb{R}$ endowed with the sup norm, $\|h\|_0$. $C^1[0, 2\pi]$ will denote the Banach space of C^1 functions $r\colon [0, 2\pi] \to \mathbb{R}$ endowed with the norm

$$\|r\|_1 = \sup_{t \in [0, 2\pi]} |r(t)| + \sup_{t \in [0, 2\pi]} |r'(t)|.$$

Also we define l as

$$l = \operatorname{int}\left(\frac{2\sqrt{\alpha\beta}}{\sqrt{\alpha} + \sqrt{\beta}}\right), \tag{1.4}$$

where $\operatorname{int}(\gamma)$ denotes the greatest integer less than or equal to γ.

Our main result regarding problem (P_1) is

THEOREM 1.1. *Assume* (i) *is satisfied. Then there is an* h_0, $0 < h_0 < 1$, *and an* $s_0 = s_0(h_0)$, $s_0 > 0$ *such that for all* $s \geqslant s_0$ *and for all* $h \in C(2\pi)$ *with*

$$\|h\|_0 \leqslant h_0 < 1,$$

problem (P_1) *possesses at least* $2n + 2$ 2π*-periodic solutions.*

Analogously, for problem (P_2) we have

THEOREM 1.2. *Assume* (ii) *is satisfied. Then there is an* $h_0, 0 < h_0 < 1$, *and an* $s_0 = s_0(h_0)$, $s_0 > 0$ *such that for all* $h \in C(2\pi)$ *with*

$$\|h\|_0 \leqslant h_0 < 1$$

we have

(a) *For all* $s \geqslant s_0$, *problem* (P_2) *possesses at least* $2(n - l) + 1$ 2π*-periodic solutions.*

(b) *For all* s *negative with* $|s| \geqslant s_0$, *problem* (P_2) *possesses at least* $2(l - k + 1) + 1$ 2π*-periodic solutions.*

The main technique used to prove the above theorems is the Poincaré–Birkhoff Theorem as stated in [1].

In [7], the problem of a lower bound for the number of 2π-periodic solutions for the equation

$$u'' + g(u) = s(1 + \varepsilon h(t)) \tag{1.5}$$

under conditions (i) and (ii) was considerd. In (1.5), $h \in C(2\pi)$ and both s and ε are parameters. Due mainly to the use of the Implicit Function Theorem the results obtained in [7] hold for $s \geqslant s_0 > 0$ ($|s| \geqslant s_0$, s negative) and $|\varepsilon| \leqslant \varepsilon_0(s)$. Thus the important problem of the existence of a uniform lower bound for $\varepsilon_0(s)$, $s \geqslant s_0$ ($|s| \leqslant s_0$, s negative) arises as an open question. In particular, the related problem of the existence of a lower bound for the number of 2π-periodic solutions for the equation

$$u'' + g(u) = s + h(t) \tag{1.6}$$

when g satisfies either (i) or (ii) and $h \in C(2\pi)$ does not follow from the results of [7]. We note that (1.6) together with (i) or (ii) is known in the literature as a Jumping Nonlinearity problem. Equation (1.6) corresponds to the periodic case. The corresponding Dirichlet and Neumann cases have been dealt with in [2, 3, 5, 6, 8].

As an application of our results, in Section 7 of this paper, we generalize those of [7] in the sense that now s and ε are independent parameters. Furthermore, in that section we provide a lower bound for the number of 2π-periodic solutions for (1.6).

In Section 2 of this paper, we examine some preliminary results for problem (P_1). In Section 3 we show that 2π-periodic solutions of (P_1) are a-priori bounded and use this fact to formulate problem (P_1) in a form suitable for the use of the Poincaré–Birkhoff Theorem. In Section 4 we prove the first of our main theorems, i.e., Theorem 1.1.

In Section 5 we deal with some preliminary results for problem (P_2). Section 6 is dedicated to proving Theorem 1.2.

2. Preliminary Results for (P_1)

We begin this section by showing that for positive s and any $h \in C(2\pi)$, $\|h\|_0 \leqslant h_0 < 1$, h_0 defined below, (P_1) has two periodic solutions, one of them being strictly negative and the other strictly positive.

Lemma 2.1. *Suppose that $\|h\|_0 < 1$ in (P_1). Then there exist an $s_1 > 0$ such that for all $s \geqslant s_1$ (1.1) has a strictly negative 2π-periodic solution.*

Proof. Let us rewrite (1.1) as

$$u'' + g(u) - s(1 + h(t)) = 0. \tag{2.1}$$

Since $g(u) \to +\infty$ as $|u| \to \infty$ we have that there exist an $s_1 > 0$ and constants $\underline{u}_s < \bar{u}_s < 0$ such that

$$g(\bar{u}_s) - s(1 + h(t)) < 0 < g(\underline{u}_s) - s(1 + h(t)) \tag{2.2}$$

for all $s \geqslant s_1$ and for all $t \in \mathbb{R}$. Hence, $\bar{u}_s$ and $\underline{u}_s$ are respectively upper and lower solutions of (1.1). It is well known that this implies the existence of a 2π-periodic solution $\tilde{u}_s$ of (P_1) such that

$$\underline{u}_s \leqslant \tilde{u}_s(t) \leqslant \bar{u}_s < 0 \tag{2.3}$$

for all $t \in \mathbb{R}$. This shows the lemma. ▮

Next, let $w \in C(2\pi)$ and $R(w)$ be the unique 2π-periodic solution of

$$u'' + \tilde{\beta} u = -w, \tag{2.4}$$

where $\tilde{\beta} = \lim_{t \to +\infty} g'(t)$. We recall that $R\colon C^0(2\pi) \to C^0(2\pi)$ is a bounded linear operator such that $\tilde{\beta}\, \|R\| \geqslant 1$.

Let z denote the unique 2π-periodic solution of

$$u'' + \tilde{\beta} u = 1 + h(t), \tag{2.5}$$

i.e., $z = R(-(1 + h))$ and let h_0 be a real number such that $0 < h_0 < 1/\tilde{\beta}\, \|R\|$. From (2.5) we obtain immediately.

PROPOSITION 2.2. *If h in (2.5) satisfies $\|h\|_0 \leqslant h_0$, then*

$$z(t) \geqslant \frac{1}{\tilde{\beta}} - \|R\|\ \|h\|_0 \geqslant \frac{1}{\tilde{\beta}} - \|R\|\ h_0 > 0. \tag{2.6}$$

Note. Since $\tilde{\beta}\, \|R\| \geqslant 1$, h_0 satisfies $h_0 < 1$. In particular any $h \in C(2\pi)$ such that $\|h\|_0 \leqslant h_0$ meets the conditions of Lemma 2.1.

LEMMA 2.2. *Suppose that h in (1.1) satisfies $\|h\|_0 \leqslant h_0$ and let $\delta_0 = 1/\tilde{\beta} - h_0\, \|R\|$. Then for any δ, $0 < \delta < \delta_0$, there is an $s_0 = s_0(h_0)$ such that for any $s \geqslant s_0$, (1.1) has a unique strictly positive 2π-periodic solution, say $u_s(t)$, such that*

$$\|u_s - sz\|_0 \leqslant s\delta. \tag{2.7}$$

Proof. It is clear that finding 2π-periodic solutions of (1.1) is equivalent to solving the fixed point problem

$$u = R(f(u) - s(1 + h)), \tag{2.8}$$

where

$$f(u) = g(u) - \tilde{\beta} u. \tag{2.9}$$

Setting $v = u/s$ and using the linearity of R and the fact that $R(-(1+h)) = z$, we conclude that (2.8) is equivalent to $v = \Phi_s(v)$, where

$$\Phi_s(v) := R\left(\frac{f(sv)}{s}\right) + z. \tag{2.10}$$

Next, let δ be a fixed real number such that $0 < \delta < \delta_0$. Thus if $v \in \overline{B(z, \delta)}$ then $v(t) > 0$ for all $t \in \mathbb{R}$. Also let $z_0 > 0$ be such that for all $t \geqslant z_0$

$$|f'(t)| \leqslant \frac{1}{3\,\|R\|}. \tag{2.11}$$

Let us define $s_2 \geqslant s_1$ by $s_2 = z_0/(\delta_0 - \delta)$. Then from (2.10) and (2.11) it is easy to see that for all $s \geqslant s_2$ and all $v, w \in \overline{B(z, \delta)}$ we have that

$$\|\Phi_s(v) - \Phi_s(w)\|_0 \leqslant \tfrac{1}{3}\,\|v - w\|_0 \tag{2.12}$$

and hence Φ_s is a contractive mapping. To show that for large s, Φ_s maps $\overline{B(z, \delta)}$ into itself, we note that from (2.10) it follows that

$$\|\Phi_s - z\|_0 \leqslant \|R\| \left\|\frac{f(sv)}{s}\right\|_0. \tag{2.13}$$

Now, from

$$f(sv(t)) = f(sz(t)) + \int_0^1 f'(s(\tau v(t) + (1-\tau)\,z(t))\,s(v(t) - z(t))\,d\tau, \tag{2.14}$$

(2.11), and a choice of $s_0 \geqslant s_2$ such that $|f(s(z(t)))/s| \leqslant 2\delta/3\,\|R\|$, for all $s \geqslant s_0$ and $t \in \mathbb{R}$, we obtain that

$$\left\|\frac{f(sv)}{s}\right\|_0 \leqslant \frac{2\delta}{3\,\|R\|} + \frac{\|v - z\|_0}{3\,\|R\|} \leqslant \frac{\delta}{\|R\|}. \tag{2.15}$$

From (2.13) and (2.15) we find that

$$\|\Phi_s(v) - z\|_0 \leqslant \delta. \tag{2.16}$$

The Banach Fixed-Point Theorem, (2.12) and (2.16) imply the existence of a unique $v_s \in \overline{B(z, \delta)}$, $s \geqslant s_0$, such that

$$v_s = \Phi_s(v_s). \tag{2.17}$$

Next, setting

$$u_s = sv_s, \tag{2.18}$$

we obtain that u_s is a positive 2π-periodic solution of (1.1).

Finally, on multiplying (2.16) by s and calling on (2.17) and (2.18) we obtain (2.7) and hence the lemma. ▮

Remark. We observe in Lemma 2.2 that the only restriction on h is $\|h\|_0 \leqslant h_0$. Also we note that for fixed δ, with $0 < \delta < \delta_0$, s_0 depends only on h_0.

The existence of u_s, as follows from the above lemma will allow us to modify Eq. (1.1) in such a way that finding 2π-periodic solutions of that equation will be equivalent to finding nontrivial 2π-periodic solutions of an equivalent equation. Thus let $\varepsilon^* > 0$ be such that

$$n^2 < \tilde{\beta} - \varepsilon^* < (n+1)^2 \tag{2.19}$$

and let us increase s_0, if necessary, so that

$$\| g'(u_s) - \tilde{\beta} \|_0 = \| f'(u_s) \|_0 \leqslant \varepsilon^*. \tag{2.20}$$

Let u be any 2π-periodic solution of (1.1) and define

$$v(t) = u(t) - u_s(t) \tag{2.21}$$

for all $t \in \mathbb{R}$. Then v is a 2π-periodic solution of

$$x'' + F(t, x) = 0, \tag{2.22}$$

where

$$F(t, x) = g(u_s(t) + x) - g(u_s(t)). \tag{2.23}$$

Thus $F: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of class C^1 and 2π-periodic in t. Furthermore, it satisfies

$$\lim_{x \to 0} \frac{F(t, x)}{x} = g'(u_s(t)) \geqslant \tilde{\beta} - \varepsilon^* > n^2 \tag{2.24}$$

$$\lim_{x \to +\infty} \frac{F(t, x)}{x} = \tilde{\beta} \tag{2.25}$$

and

$$\lim_{x \to -\infty} F(t, x) = +\infty, \tag{2.26}$$

all these three limits being uniform in t. We observe that $x = 0$ is a trivial 2π-periodic solution of (2.22). Also we note that (2.25) and (2.26) imply the existence of an $x_0 > 0$ and an $M > 0$ such that

$$F(t, x) > 0 \tag{2.27}$$

for all $t \in \mathbb{R}$ and $x \in (-\infty, -x_0) \cup (x_0, +\infty)$ and

$$F(t, x) > -M \tag{2.28}$$

for all $(t, x) \in \mathbb{R}_2$. In Sections 3 and 4, s will be a fixed number, $s \geqslant s_0$, so that Lemmas 2.1 and 2.2 hold true.

3. 2π-Periodic Solutions for (P_1) Are A-Priori Bounded

In this section we will prove

Lemma 3.1. *2π-periodic solutions of* (2.22) *are a-priori bounded.*

Proof. Suppose that v is a 2π-periodic solution of (2.22) which attains a minimum at $t = t_m$. Then from (2.22), we have

$$F(t_m, v(t_m)) \leqslant 0 \tag{3.1}$$

and from (2.27), $v(t_m) \in [-x_0, x_0]$. On integrating (2.22), with v in the place of x, from t_m to $t \in [t_m, t_m + 2\pi]$ and calling on (2.28) we obtain

$$v'(t) = -\int_{t_m}^{t} F(r, v(r))\, dr \leqslant M(t - t_m). \tag{3.2}$$

On integrating (3.2) again from t_m to $t \in [t_m, t_m + 2\pi]$ we find that

$$v(t) \leqslant v(t_m) + \frac{M}{2}(t - t_m)^2 \leqslant x_0 + 2\pi^2 M. \tag{3.3}$$

Hence,

$$-x_0 \leqslant v(t) \leqslant x_0 + 2\pi^2 M. \tag{3.4}$$

Since v was any 2π-periodic solution of (2.22) the lemma follows. ∎

We use Lemma 3.1 to modify (2.22) into a form suitable for the application of the Poincaré–Birkhoff theorem. Thus suppose that $x_1 \in \mathbb{R}$ satisfies $x_1 \geqslant x_0 + 2\pi^2 M$, and define $G\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$G(t, x) = \begin{cases} F(t, x) & \text{for } (t, x) \in \mathbb{R} \times (-x_1, x_1) \\ F(t, -x_1) & \text{for } (t, x) \in \mathbb{R} \times (-\infty, -x_1] \\ F(t, x_1) & \text{for } (t, x) \in \mathbb{R} \times [x_1, +\infty). \end{cases} \tag{3.5}$$

Then G is a continuous function which is 2π-periodic in t and locally lipschitzian in x.

By repeating the argument in this proof of Lemma 3.1 with G in the place of F, it is clear that (2.22) and

$$v'' + G(t, v) = 0 \tag{3.6}$$

have the same 2π-periodic solutions. We note now that G is a bounded continuous function and hence in particular the unique solution to the initial value problem for (3.6) can be extended to the whole real line. Also, if v is a nontrivial solution of (3.6) then necessarily $v^2(t) + v'^2(t) \neq 0$ for all $t \in \mathbb{R}$.

4. At Least $2n+2$ Solutions for (P_1)

From previous Sections we know that searching for additional 2π-periodic solutions of (1.1) different from $\tilde{u}_s$ and u_s is equivalent to searching for nontrivial 2π-periodic solutions of (3.6) other than $\tilde{u}_s - u_s$. We note that $\tilde{u}_s - u_s < 0$ for all $t \in \mathbb{R}$. To obtain these nontrivial solutions we will use a generalization of the Poincaré–Birkhoff theorem due to W. Y. Ding, see [1, Th. 1].

Let us rewrite (3.6) as

$$v' = z \tag{4.1}$$

$$z' = -G(t, v). \tag{4.2}$$

For any $(a, b) \in \mathbb{R}^2$, let $(v(t, a, b), z(t, a, b))$ denote the solutions of (4.1)–(4.2) such that $(v(0, a, b), z(0, a, b)) = (a, b)$. Define $P: \mathbb{R}^2 \to \mathbb{R}^2$, the Poincaré map induced by (4.1) and (4.2), by

$$P(a, b) = (v(2\pi, a, b, z(2\pi, a, b)). \tag{4.3}$$

We recall that P is an area preserving homeomorphism which in our case satisfies $P(0, 0) = (0, 0)$. By defining $v(t) = R(t) \cos \Theta(t)$, $z(t) = R(t) \sin \Theta(t)$ we obtain the equivalent polar system

$$R' = -G(t, R \cos \Theta) \sin \Theta + R \sin \Theta \cos \Theta. \tag{4.4}$$

$$\Theta' = -\frac{G(t, R \cos \Theta)}{R} \cos \Theta - \sin^2 \Theta. \tag{4.5}$$

Let $H = \{(r, \theta) \mid r > 0,\ \theta \in \mathbb{R}\}$ and let T be the mapping from H into itself defined by

$$T(r, \theta) = (R(1\pi, r, \theta), \Theta(2\pi, r, \theta)), \tag{4.6}$$

where $(R(t, r, \theta), \Theta(t, r, \theta))$ denotes the unique solution to (4.4)–(4.5) such that

$$R((0, r, \theta), \Theta(0, r, \theta)) = (r, \theta).$$

We have that the mapping T is an area preserving homeomorphism form H into itself which satisfies

$$T(r, \theta + 2\pi) = T(r, \theta) + (0, 2\pi). \tag{4.7}$$

Next let j be any integer and define $T_j\colon H \to H$ by

$$T_j(r, \theta) = T(r, \vartheta) + (0, 2\pi j). \tag{4.8}$$

Clearly each mapping T_j, $j \in \mathbb{Z}$, is an area preserving homeomorphism from H onto its image which, because of (4.7), satisfies

$$T_j(r, \theta + 2\pi) = T_j(r, \theta) + (0, 2\pi). \tag{4.9}$$

We have now

PROPOSITION 4.1. *There is a $\mu_0 > 0$ such that for any μ, $0 < \mu \leqslant \mu_0$ the solution $(R(t, \mu, \theta), \Theta(t, \mu, \theta))$ of (4.4)–(4.5) at $t = 2\pi$ satisfies*

$$\theta - \Theta(2\pi, \mu, \theta) > 2\pi n \tag{4.10}$$

for any $\theta \in \mathbb{R}$.

Proof. Let $(\mu, \theta) \in H$ and let $(v(t, a, b), z(t, a, b))$ be the solutions of (4.1)–(4.2) such that $a = \mu \cos\theta$ and $b = \mu \sin\theta$. Recalling that $(v(t, 0, 0), z(t, 0, 0)) = (0, 0)$ for all $t \in \mathbb{R}$, then from the continuity with respect to initial conditions we have that given $\varepsilon_1 > 0$ there is a $\mu_0 > 0$ such that $0 < \mu \leqslant \mu_0$ implies that $\max_{t \in [0, 2\pi]} |v(t, a, b)| < \varepsilon_1$. Define $\tilde{\alpha}\colon \mathbb{R} \to \mathbb{R}$ by

$$\tilde{\alpha}(t) = \begin{cases} -G(t, v(t, a, b))/v(t, a, b) & \text{if } \ v(t, a, b) \neq 0 \\ g'(u_s(t)) & \text{if } \ v(t, a, b) = 0. \end{cases} \tag{4.11}$$

Then $\tilde{\alpha}$ is is continuous and $v(t, a, b)$ is a solution of the linear equation

$$v'' + \tilde{\alpha}(t)v = 0 \tag{4.12}$$

for all $t \in [0, 2\pi]$. We note that from (2.24) and having chosen ε_1 sufficiently small, we obtain that

$$\tilde{\alpha}(t) > n^2 \tag{4.13}$$

for all $t \in [0, 2\pi]$. Since (4.12) is a Sturm majorant for

$$\bar{v}'' + n^2 \bar{v} = 0,$$

(4.10) follows from [4, Proof of Th. 3.1] (i.e., Sturm's First Comparison Theorem). Hence, the proposition. ∎

PROPOSITION 4.2. *There is a* $\Delta_0 > \mu_0$ *such that for any* $\Delta \geqslant \Delta_0$ *the solution of* (4.4)–(4.5) *at* $t = 2\pi$, $(R(2\pi, \Delta, \theta), \Theta(2\pi, \Delta, \theta))$ *satisfies*

$$\theta - \Theta(2\pi, \Delta, \theta) < 2\pi \tag{4.14}$$

for any $\theta \in \mathbb{R}$.

Proof. Suppose this Δ_0 does not exist. Then there is a sequence $(\Delta_k, \varphi_k)_{k=1}^{\infty}$ with $\Delta_k \to +\infty$ as $k \to +\infty$, such that $v_k(t) := v(t, a_k, b_k)$ possesses at least two zeros in $[0, 2\pi]$. Here $a_k = \Delta_k \cos \varphi_k$, $b_k = \Delta_k \sin \varphi_k$, $k \in \mathbb{N}$. Let us define the sequence of functions $\{\hat{v}_k\}_{k=1}^{\infty}$ by $\hat{v}_k(t) = v_k(t)/\|v_k\|_1$, $t \in [0, 2\pi]$. Then $\|\hat{v}_k\|_1 = 1$, $k \in \mathbb{N}$. From the fact that for $t \in [0, 2\pi]$, $v_k(t)$, $k \in \mathbb{N}$, satisfies (3.6) we obtain that

$$\hat{v}_k(t) = \hat{v}_k(0) + \hat{v}'_k(0)t - \int_0^t \int_0^\tau \frac{G(\tau, v_k(\tau))}{\|v_k\|_1} d\tau\, dt. \tag{4.15}$$

This, together with the boundedness of G and the Ascoli–Arzela's theorem implies that $\{\hat{v}_k\}_{k=1}^{\infty}$ possesses a uniformly convergent subsequence in $C^1[0, 2\pi]$. Denoting this subsequence again by $\{\hat{v}_k\}_{k=1}^{\infty}$, its limit by $\hat{v}$, and letting $k \to +\infty$ in (4.15) it follows that

$$\hat{v}(t) = \hat{v}(0) + \hat{v}'(0)t. \tag{4.16}$$

But this implies that for sufficiently large k, $v_k(t)$ can have at most one zero in $[0, 2\pi]$. This is a contradiction and hence the proposition. ∎

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We only need to prove that (3.6) possesses at least $2n$ nontrivial 2π-periodic solutions different from $\tilde{u}_s - u_s$. To do this let us consider the area preserving homeomorphism T_j, $j \in \mathbb{Z}$ defined in (4.8) and set

$$T_j(r, \theta) = (R_j(r, \theta), \Theta_j(r, \theta)), \tag{4.17}$$

$j \in \mathbb{Z}$. From (4.6) and (4.8) we obtain that $R_j(r, \theta) = R(r, \theta)$ and that

$$\Theta_j(r, \theta) = \Theta(2\pi, r, \theta) + 2\pi j, \tag{4.18}$$

$j \in \mathbb{Z}$. Then from Propositions 4.1 and 4.2 we have that

$$\Theta_j(\mu, \theta) - \theta < 2\pi(j-n) \tag{4.19}$$

and

$$\Theta_j(\Delta, \theta) - \theta > 2\pi(j-1), \tag{4.20}$$

any $j \in \mathbb{Z}$, any $\mu \in (0, \mu_0]$ and any $\Delta \geqslant \Delta_0$.

Let us choose a $\mu \in (0, \mu_0]$ and a $\Delta \geqslant \Delta_0$. Then for $j = 1, ..., n$ we have that

$$\Theta_j(\mu, \theta) - \theta < 0 \qquad \text{and} \qquad \Theta_j(\Delta, \theta) - \theta > 0. \tag{4.21}$$

Let us define $H_{\Delta\mu}$ by $H_{\Delta\mu} = \{(r, \theta) \mid \mu r \leqslant \Delta, \theta \in \mathbb{R}\}$. Then from [1, Th. 1] it follows that for each $j = 1, ..., n$ the mapping T_j from $H_{\Delta\mu}$ onto its image possesses two different fixed points. Let us denote these fixed points by $r_{ij}, \theta_{ij}, i = 1, 2$. Also let $a_{ij} = r_{ij} \cos\, \theta_{ij}$, $b_{ij} = r_{ij} \sin \theta_{ij}$, $i = 1, 2$. Then for each $j = 1, ..., n$ the points (a_{ij}, b_{ij}), $i = 1, 2$, are fixed points of the Poincaré mapping P. Each one of these points is a pair of initial conditions for a 2π-periodic solution of (3.6), say $v(t, a_{ij}, b_{ij})$, possessing exactly $2j$ zeros in $[0, 2\pi)$, $j = 1, ..., n$, $i = 1, 2$. Thus (3.6) has at least $2n$ nontrivial 2π-periodic solutions other han $\tilde{u}_s = u_s$ and the theorem is proved. ∎

5. Preliminary Results for (P_2)

Let $w \in C(2\pi)$ and respectively denote by $R_\alpha(w)$ and $R_\beta(w)$ the unique 2π-periodic solutions of the equations

$$u'' + \alpha u = -w \tag{5.1}$$

$$u'' + \beta u = -w, \tag{5.2}$$

where α and β are as in (1.3).

We have that $R_\alpha, R_\beta \colon C(2\pi) \to C(2\pi)$ are bounded linear operators such that $\alpha \,\|R_\alpha\| \geqslant 1$, $\beta \,\|R_\beta\| \geqslant 1$. Let us set $z_\alpha = < R_\alpha(-(1+h))$, $z_\beta = R_\beta(-(1+h))$, where $h \in C(2\pi)$. The validity of the following proposition is easily checked.

Proposition 5.1. *Let h_0 be a real number such that $0 < h_0 < \min\{1/\beta \,\|R_\beta\|, 1/\alpha \,\|R_\alpha\|\}$. Then if h satisfies $\|h\|_0 \leqslant h_0$ we have that*

$$z_\alpha(t) \geqslant \frac{1}{\alpha} - \|R_\alpha\| \, \|h\|_0 \geqslant \frac{1}{\alpha} - \|R_\alpha\| \, h_0 > 0 \tag{5.3}$$

$$z_\beta(t) \geqslant \frac{1}{\beta} - \|R_\beta\| \, \|h\|_0 \geqslant \frac{1}{\beta} - \|R_\beta\| \, h_0 > 0. \tag{5.4}$$

In a similar form as we proved Lemma 2.2 we can now prove

LEMMA 5.2. *Suppose h in (1.1) satisfies $\|h\|_0 \leqslant h_0$ and let $\delta_0 = \min\{1/\alpha - \|R_\alpha\|\, h_0,\ 1/\beta - \|R_\beta\|\, h_0\}$. Then for fixed δ, with $0 < \delta < \delta_0$, there is an $s_0 = s_0(h_0) > 0$ such that for all $s \geqslant s_0$ (1.1) possesses a unique 2π-periodic solution u_s^+ which is positive and satisfies*

$$\|u_s^+ - sz_\beta\|_0 \leqslant s\delta. \tag{5.5}$$

Also if $s \leqslant -s_0$ then (1.1) possesses a unique 2π-periodic solution u_s^- which is negative and satisfies

$$\|u_s^- - sz_\alpha\|_0 \leqslant |s|\ \delta. \tag{5.6}$$

As in Section 2, it is clear from the definition of h_0 that $h_0 < 1$.

In Sections 6 and 7, s will be a fixed number, $|s| \geqslant s_0$, so that Lemma 5.2 holds true.

6. MULTIPLE 2π-PERIODIC SOLUTIONS FOR THE SECOND CASE

In this section we will prove Theorem 1.2. As we did before, we will reduce problem (P_2) to a search for nontrivial 2π-periodic solutions of an equivalent problem.

Let us first consider the case $s \geqslant s_0$, with s_0 as in Section 5. We increase s_0, if necessary, so that

$$\|g'(u_s^+) - z_\beta\|_0 \leqslant \varepsilon^*, \tag{6.1}$$

where ε^* satisfies

$$n^2 < \beta - \varepsilon^* < (n+1)^2. \tag{6.2}$$

Setting $u = u_s^+ + v$ we have that u is a solution of (1.1) if and only if v is a solution of

$$v'' + F(t, v) = 0, \tag{6.3}$$

where $F\colon \mathbb{R} + \mathbb{R} \to \mathbb{R}$ is defined by

$$F(t, x) = g(u_s^+(t) + x) - g(u_s^+(t)). \tag{6.4}$$

F is of class C^1 and satisfies

$$\lim_{x \to 0} \frac{F(t, x)}{x} = g'(u_s^+(t)) \geqslant \beta - \varepsilon^* > n^2 \tag{6.5}$$

$$\lim_{x \to +\infty} \frac{F(t, x)}{x} = \beta, \qquad \lim_{x \to -\infty} \frac{F(t, x)}{x} = \alpha, \tag{6.6}$$

with all these limits being uniform in t.

We note that $v \equiv 0$ is a solution of (6.3). Also we observe that local existence and uniqueness of solutions to the initial value problem associated with (6.3) are ensured. The extendibility of these solutions to the whole real line follows from the sublinearity of F.

Rewriting (6.3) as

$$v' = z \tag{6.7}$$

$$z' = -F(t, v) \tag{6.8}$$

and defining $v(t) = R(t) \cos \Theta(t)$, $z(t) = R(t) \sin \Theta(t)$ we obtain the equivalent polar system

$$R' = -F(t, R \cos \Theta) \sin \Theta + R \sin \Theta \cos \Theta \tag{6.9}$$

$$\Theta' = -\frac{F(t, R \cos \Theta) \cos \Theta}{R} - \sin^2 \Theta. \tag{6.10}$$

We denote by $(R(t, r, \Theta), \Theta(t, r, \theta))$ the unique solution of (6.9)–(6.10), such that $(R(0, r, \theta), \Theta(0, r, \theta)) = (r, \theta) \in H$, with H is as in Section 4.

The next proposition can be proved in the same form as we proved Proposition 4.1.

PROPOSITION 6.1. *There is a $\mu_0 > 0$ such that for any $0 < \mu \leqslant \mu_0$ the solution $(R(t, \mu, \theta), \Theta(t, \mu, \theta)$ of (6.9)–(6.10) at $t = 2\pi$ satisfies*

$$\theta - \Theta(2\pi, \mu, \theta) > 2\pi n \tag{6.11}$$

for any $\theta \in \mathbb{R}$.

Next, we have

PROPOSITION 6.2. *There is a Δ_0, with $\Delta_0 > \mu_0 > 0$ such that for any $\Delta \geqslant \Delta_0$ the solution of (6.9)–(6.10) at $t = 2\pi$ satisfies*

$$\theta - \Theta(2\pi, \Delta, \theta) < 2(l+1)\pi \tag{6.12}$$

for any $\theta \in \mathbb{R}$.

The proof of this proposition is based on the following two simple propositions which we state without proof.

PROPOSITION 6.3. *Let* $\{u_m\}_{m=1}^{\infty}$ *be a sequence in* $C^0[0, 2\pi]$ *such that*

$$\lim_{m\to\infty} \|u_m\|_0 = +\infty.$$

Suppose that $u_m/\|u_m\|_0$ *converges to* $\hat{u}$ *in* $C^0[0, 2\pi]$ *as* $m \to +\infty$. *Then the sequence*

$$\left\{\frac{F(\cdot, u_m(\cdot))}{\|u_m\|_0}\right\}_{m=1}^{\infty},$$

where F is as in (6.4), *possesses a subsequence which is weakly convergent in* $L^q(0, 2\pi)$, $q > 1$, *to* $\beta\hat{u}^+ - \alpha\hat{u}^-$.

PROPOSITION 6.4. *Let* $\{u_m\}_{m=1}^{\infty}$ *be a sequence of solutions of* (6.3) *such that* $\|u_m\|_1 \to +\infty$ *as* $m \to +\infty$, *then* $\|u_m\|_0 \to +\infty$ *as* $m \to +\infty$.

We are now ready to prove Proposition 6.2.

Proof of Proposition 6.2. Assume that such a Δ_0 does not exist. Then there is a sequence of solutions $\{R(t, \Delta_k, \theta_k), \Theta(t, \Delta_k, \theta_k)\}_{k=1}^{\infty}$, $t \in \mathbb{R}$, of (6.9)–(6.10) with $\Delta_k \to +\infty$ as $k \to \infty$ and such that for any $k \in \mathbb{N}$

$$\theta_k - \Theta(2\pi, \Delta_k, \theta_k) \geqslant 2(l+1)\pi. \tag{6.13}$$

Let us set

$$(v_k(t), z_k(t)) = (R(t, \Delta_k, \theta_k) \cos \Theta(t, \Delta_k, \theta_k), R(t, \Delta_k, \theta_k) \sin \Theta(t, \Delta_k, \theta_k)), \tag{6.14}$$

$k \in \mathbb{N}$. Since $\Delta_k \to \infty$ we have that $\|v_k\|_1 \to +\infty$ as $k \to +\infty$. From Proposition (6.4), $\|v_k\|_0 \to +\infty$ as $k \to \infty$. Define the sequences $\{\hat{v}_k\}_{k=1}^{\infty}$ and $\{\hat{z}_k\}_{k=1}^{\infty}$ by $\hat{v}_k(t) = v_k(t)/\|v_k\|_0$, $\hat{z}_k(t) = z_k(t)/\|v_k\|_0$, $k \in \mathbb{N}$. From (6.7), (6.8), (6.13), and (6.14) we obtain that for each $k \in \mathbb{N}$, there are $t_k, t_k^* \in [0, 2\pi]$ such that $\hat{v}_k(t_k) = 0 = \hat{v}_k'(t_k^*)$. From this fact, the sublinearity of F, the Ascoli–Arzela theorem, and (6.3), we find that $\{\hat{v}_k\}_{k=1}^{\infty}$ contains a subsequence $\{\hat{v}_{k_j}\}_{k=1}^{\infty}$ such that $\hat{v}_{k_j} \to \hat{v}$ in $C^1[0, 2\pi]$, as $j \to +\infty$. Setting $w_j(t) = F(t, v_{k_j}(t))/\|v_{k_j}\|_0$, $j \in \mathbb{N}$, it follows from Proposition (6.3) that the sequence $\{w_j\}_{j=1}^{\infty}$ possesses a subsequence which we denote again by $\{w_j\}_{j=1}^{\infty}$, converging weakly in $L_q(0, 2\pi)$, $q > 1$, to $\beta\hat{v}^+ - \alpha\hat{v}^-$.

Now, it is easy to see that for each $j \in \mathbb{N}$ we have

$$\theta_{k_j} - \Theta_{k_j}(2\pi, \Delta_{k_j}, \theta_{k_j}) = \int_0^{2\pi} \frac{w_j(t)\, \hat{v}_{k_j}(y) + \hat{z}_{k_j}^2(t)}{\hat{v}_{k_j}^2(t) + \hat{z}_{k_j}^2(t)}\, dt. \tag{6.15}$$

Letting $j \to +\infty$ in (6.15) and setting $\theta^* = \lim_{k \to \infty} \arctan(\hat{z}_{k_j}(0)/\hat{v}_{k_j}(0))$, we obtain that

$$\Theta \equiv \lim_{j \to +\infty} \Theta_{k_j}(2\pi, \Delta_{k_j}, \theta_{k_j}) = \theta^* - \int_0^{2\pi} \frac{(\beta\hat{v}^+ - \alpha\hat{v}^-)\hat{v} + \hat{v}'^2}{\hat{v}^2 + \hat{v}'^2}\,dt. \quad (6.16)$$

Thus, from (6.13) it follows that

$$\theta^* - \Theta = \int_0^{2\pi} \frac{(\beta\hat{v}^+ - \alpha\hat{v}^-)\hat{v} - \hat{v}'^2}{\hat{v}^2 + \hat{v}'^2}\,dt \geqslant 2(l+1)\pi. \quad (6.17)$$

On the other hand, from (6.3) we have that

$$\hat{v}'_{k_j}(t) = \hat{v}'_{k_j}(0) + \int_0^t w_j(\tau)\,ds \quad (6.18)$$

for $j \in \mathbb{N}$ and $t \in [0, 2\pi]$. Letting $j \to +\infty$ in (6.18) we obtain

$$\hat{v}'(t) = \hat{v}'(0) + \int_0^t (\beta v^+(\tau) - \alpha v^-(\tau))\,d\tau \quad (6.19)$$

and hence $\hat{v}$ is a solution of the equation

$$x'' + \beta x^+ - \alpha x^- = 0. \quad (6.20)$$

Let us denote by $(R_x(t, r, \theta), \Theta_x(t, r, \theta))$, $t \in \mathbb{R}$, the polar representation of the solution of (6.20) with initial data $(r, \theta) \in H$. In particular the initial polar data for the solution $\hat{v}$ is $(r^*, \theta^*) \in H$, where $r^* = \sqrt{\hat{v}^2(0) + \hat{v}'^2(0)}$. Then it is obvious that

$$\Theta_{\hat{v}}(2\pi, r^*, \theta^*) = \Theta \quad (6.21)$$

and hence from (6.17)

$$\theta^* - \Theta_{\hat{v}}(2\pi, r^*, \theta^*) \geqslant 2(l+1)\pi. \quad (6.22)$$

Next, let us consider the equation

$$w'' + cw^+ - dw^- = 0, \quad (6.23)$$

where $c > \beta$ and $d > \alpha$ are given by

$$c = \frac{1}{4}(j+1)^2\left[1 + \sqrt{\frac{\beta}{\alpha}}\right] \quad (6.24)$$

$$d = \frac{1}{4}(l+1)^2\left[1 + \sqrt{\frac{\alpha}{\beta}}\right]. \quad (6.25)$$

Let $(\tilde{R}_w(t, r, \theta), \tilde{\Theta}_w(t, r, \theta))$, $t \in \mathbb{R}$, be the polar representation of the solution of (6.23) with initial data $(r, \theta) \in H$. From the choice of the coefficients c and d it follows that any solution w of (6.23) is 2π-periodic and furthermore satisfies

$$\theta^* - \Theta_w(2\pi, r^*, \theta^*) = 2(l+1)\pi. \tag{6.26}$$

Now it is not difficult to prove that

$$\Theta_v'(t, r^{**}, \theta^*) > \tilde{\Theta}_w'(t, r^*\theta^*) \tag{6.27}$$

for all $t \in [0, 2\pi]$. Integrating (6.27) from 0 to 2π we obtain

$$\theta^* - \Theta_{\bar{v}}(2\pi, r^*, \theta^*) < \theta^* - \tilde{\Theta}_w(2\pi, r^*, \theta^*). \tag{6.28}$$

We conclude the proof of the proposition by noting that (6.22), (6.28) lead to a contradiction. ∎

Using Propositions (6.1) and (6.2) in the same manner as we did Propositions (4.1), (4.2) in order to prove Theorem 4.3 and proceeding in an entirely similar form as we did with Theorem 4.3, we can establish the first half of our main result for this section.

THEOREM 6.5. *For $s \geqslant s_0$, Eq. (6.3) possesses at least $2(n-l)$ nontrivial 2π-periodic solutions, $v_{ij}(t)$, $i = 1, 2$, $j = l+1, ..., n$. For $i = 1, 2$, $v_{ij}(t)$ has exactly $2j$ zeros in $[0, 2\pi)$, $j = l+1, ..., n$. Correspondingly, (P_2) has at least $2(n-l)$ 2π-periodic solutions of the form $u_s^+(t) + v_{ij}(t)$, $t \in \mathbb{R}$, $i = 1, 2, j = l+1, ..., n$. These solutions together with $u_s^+(t)$ give us a total of at least $2(n-l)+1$ 2π-periodic solutions for (P_2).*

We continue the search for 2π-periodic solutions of (1.1) by considering the case $|s| \geqslant s_0$, s negative. Again, we first reduce the problem to a problem of finding nontrivial 2π-periodic solutions of an equivalent equation. Thus by increasing s_0, if necessary, we assume that for all s negative with $|s| \geqslant s_0$ we have

$$\| g'(u_s^-) - \alpha \|_0 \leqslant \varepsilon^{**}, \tag{6.29}$$

where ε^{**} is such that

$$(k-1)^2 < \alpha + \varepsilon^{**} < k^2. \tag{6.30}$$

Setting $u = u_s^- + v$ we have that v is a solution of (1.1) if and only if v is a solution of

$$v'' + \tilde{F}(t, v) = 0, \tag{6.31}$$

where

$$\tilde{F}(t, x) = g(x + u_s^-(t)) - g(u_s^-(t)). \tag{6.32}$$

It follows that $\tilde{F}$: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^1, $F(t, 0) = 0$ for all $t \in \mathbb{R}$, and from (1.13), (6.30), and (6.32) that $\tilde{F}$ satisfies

$$\lim_{x \to 0} \frac{\tilde{F}(t, x)}{x} = g'(u_s^-(t)) \leqslant \alpha + \varepsilon^{**} < k^2, \tag{6.33}$$

$$\lim_{x \to -\infty} \frac{\tilde{F}(t, x)}{x} = \alpha, \qquad \lim_{x \to +\infty} \frac{\tilde{F}(t, x)}{x} = \beta, \tag{6.34}$$

with all these limits being uniform in t.

Rewriting (6.31) as a system, we obtain

$$v' = z \tag{6.35}$$

$$z' = -\tilde{F}(t, v). \tag{6.36}$$

Letting $v(t) = R(t) \cos \Theta(t)$, $z(t) \sin \Theta(t)$ we obtain the equivalent polar system

$$R' = -\tilde{F}(t, R \cos \Theta) \sin \Theta + R \sin \Theta \cos \Theta \tag{6.37}$$

$$\Theta' = -\frac{\tilde{F}(t, R \cos \Theta)}{R} \cos \Theta - \sin^2 \Theta. \tag{6.38}$$

We denote by $(\tilde{R}(t, r, \theta), \tilde{\Theta}(t, r, \theta))$ the unique solution of Eqs. (6.37) and (6.38) such that $(\tilde{R}(0, r, \theta), \tilde{\Theta}(0, r, \theta)) = (r, \theta) \in H$.

Using the same techniques we used for the case $s \geqslant s_0$ and some obvious modifications, we now obtain the following straightforward analogue of Propositions 6.1, 6.2, and Theorem 6.5.

PROPOSITION 6.6. *There is a $\mu_0 > 0$ such that for any $0 < \mu \leqslant \mu_0$ the solution $(\tilde{R}(t, \mu, \theta), \tilde{\Theta}(t, \mu, \theta)$ of (6.37)–(6.38) at $t = 2\pi$ satisfies*

$$\theta - \tilde{\Theta}(2\pi, \mu, \theta) < 2\pi k \tag{6.39}$$

for any $\theta \in \mathbb{R}$.

PROPOSITION 6.7. *There is a $\Delta_0 > \mu_0$ such that for any $\Delta \geqslant \Delta_0$ the solution $(\tilde{R}(t, \Delta, \theta), \tilde{\Theta}(t, \Delta, \theta))$ of (6.37)–(6.38) at $t = 2\pi$ satisfies*

$$\theta - \tilde{\Theta}(2\pi, \Delta, \theta) > 2l\pi \tag{6.40}$$

for any $\theta \in \mathbb{R}$.

Thus we can now establish the second half of our main result.

THEOREM 6.8. *For $|s| \geqslant s_0$, s negative, Eq.* (6.31) *possesses at least* $2(l-k+1)$ *nontrivial 2π-periodic solutions, $\tilde{v}_{ij}(t)$, $i = 1, 2$, $j = k, ..., l$. For $i = 1, 2$, $\tilde{v}_{ij}(t)$ has exactly $2j$ zeros in $[0, 2\pi)$, $j = k, ..., l$. Correspondingly, (P_2) has at least $2(l-k+1)$ 2π-periodic solutions of the form $u_s^-(t) + \tilde{v}_{ij}(t)$, $t \in \mathbb{R}$, $i = 1, 2$, $j = k, ..., l$. These solutions together with $u_s^-(t)$ give us a total of at least $2(l-k+1)+1$ 2π-periodic solution for (P_2).*

Finally, combining Theorems 6.5 and 6.8 we obtain Theorem 1.2.

7. Some Applications of Our Main Results

In this Section we apply our main results to the existence of a lower bound for the number of 2π-periodic solutions for (1.5) and (1.6) when g in these equations satisfies either (i) or (ii).

We consider first (1.6). We note that this equation can be written as

$$u'' + g(u) = s\left(1 + \frac{h(t)}{s}\right). \tag{7.1}$$

for s different of zero. Letting s_0 and h_0 to be as in Theorems 1.1 and 1.2, increasing s_0 if necessary, so that additionally we have

$$\frac{\|h\|_0}{|s|} \leqslant h_0 \tag{7.2}$$

for all $s \geqslant s_0$ or for all $|s| \geqslant s_0$, s negative, and recalling the definition of l given in (1.4) we obtain

THEOREM 7.1. (a) *If* (i) *of the Introduction holds then there is an $s_0 > 0$ such that for any $s \geqslant s_0$, Eq.* (1.6) *possesses at least $2n+2$ 2π-periodic solutions.*

(b) *If* (ii) *holds then there is an $s_0 > 0$ such that for any $s \geqslant s_0$, Eq.* (1.6) *possesses at least $2(n-l)+1$ 2π-periodic solutions. Also for any $|s| \geqslant s_0$, s negative,* (1.6) *has at least $2(l-k+1)+1$ 2π-periodic solutions.*

Thus the number of 2π-periodic solutions of (1.6) *for $s \geqslant s_0$ plus the number of solutions for $|s| \geqslant s_0$, s negative is equal to $2Q+2$, where Q is the number of squares of integers lying in the interval (α, β).*

Next, let us consider the problem of a lower bound for the number of 2π-periodic solutions of (1.5). Just by taking

$$|\varepsilon| \leqslant \varepsilon_0 = \frac{h_0}{\|h\|_0}, \tag{7.3}$$

in Theorems 1.1 and 1.2, s_0 and h_0 like in these theorems, we obtain

THEOREM 7.2. (a) *If* (i) *of the Introduction is satisfied, then there is an* $s_0 > 0$ *such that for any* $s \geqslant s_0$ *and any* $|\varepsilon| \leqslant \varepsilon_0$, *Eq.* (1.5) *possesses at least* $2n+2$ 2π*-periodic solutions.*

(b) *If* (ii) *of the Introduction is satisfied, then there is an* $s_0 > 0$ *such that for any* $s \geqslant s_0$ *and any* $|\varepsilon| \leqslant \varepsilon_0$, *Eq.* (1.5) *has at least* $2(n-l)+1$ 2π*-periodic solutions. Also for any* $|s| \geqslant s_0$, s *negative,* (1.5) *has at least* $2(l-k+1)+1$ 2π*-periodic solutions.*

Thus for $|\varepsilon| \leqslant \varepsilon_0$ *the number of* 2π*-periodic solutions of* (1.5) *for* $s \geqslant s_0$ *plus the number of solutions for* $|s| \geqslant s_0$, s *negative, is equal to* $2Q+2$, *where* Q *is defined as above.*

We note that in Theorem 7.2, s and ε are independent parameters. We have thus extended the main results of [7].

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