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MULTIPLE SOLUTIONS FOR THE p-LAPLACIAN UNDER GLOBAL NONRESONANCE

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ABSTRACT. Via the study of a simple Dirichlet boundary value problem associated with the one-dimensional p-Laplacian, $p > 1$, we show that in globally nonresonant problems for this differential operator the number of solutions may be arbitrarily large when $p \in (1, \infty)\setminus\{2\}$. From this point of view $p = 2$ turns out to be a very special case.

1. INTRODUCTION

Let us consider the boundary value problem

$$(1.1)\qquad (|u'|^{p-2}u')' + f(|u|^{p-2}u) = h(x), \qquad x \in (0, T),$$

$$(1.2)\qquad u(0) = u(T) = 0,$$

where $f \in C(\mathbf{R}, \mathbf{R})$ and $h \in C[0, T]$. In (1.1) and henceforth $' = d/dx$ and $1 < p < \infty$. It follows from the results of [2], (see also [3]), that (1.1)-(1.2) possesses at least one solution if f satisfies the asymptotic nonresonance condition

$$(1.3)\qquad \lambda_k < \liminf_{|s|\to\infty} f(s)/s \le \limsup_{|s|\to\infty} f(s)/s < \lambda_{k+1}$$

for some $k \in \mathbf{N}$. In (1.3), $\{\lambda_n\}_{n=1}^{\infty}$ is the sequence of eigenvalues corresponding to the problem

$$(1.4)\qquad (|u'|^{p-2}u')' + \lambda|u|^{p-2}u = 0, \qquad x \in (0, T),$$

$$(1.5)\qquad u(0) = u(T) = 0.$$

These eigenvalues are given by

$$(1.6)\qquad \lambda_k = \left(\frac{k\pi_p}{T}\right)^p, \qquad k \in \mathbf{N},$$

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where

$$\pi_p = 2(p-1)^{1/p} \int_0^1 \frac{ds}{(1-s^p)^{1/p}}, \tag{1.7}$$

(see [5], [6] or [3]).

We note that condition (1.3) is satisfied if, e.g., f is of class C^1 and there exist constants α, β such that

$$\lambda_k < \alpha \le f'(s) \le \beta < \lambda_{k+1} \tag{1.8}$$

for some $k \in \mathbf{N}$ and all $s \in \mathbf{R}$.

It is well known that in the case $p = 2$, the global nonresonance condition (1.8) also ensures uniqueness of the solution of (1.1), (1.2). At this point a natural question arises. Given $p \in (1, \infty)$, does (1.8) suffice for the unique solvability of (1.1), (1.2)? Our aim in this paper is to show that the answer is negative if $p \neq 2$. To do this, the rest of this paper will consider the boundary value problem

$$(P_\lambda) \quad \begin{aligned} &(|u'|^{p-2}u')' + \lambda|u|^{p-2}u = 1, \qquad x \in (0, T), && (1.9)\\ &u(0) = u(T) = 0 && (1.10)\end{aligned}$$

which is obtained from (1.1), (1.2) by setting $f(s) = \lambda s$, $s \in \mathbf{R}$, and $h \equiv 1$.

We note that in problem (P_λ) the nonresonance condition (1.8) holds if and only if $\lambda_k < \lambda < \lambda_{k+1}$ for some $k \in \mathbf{N}$.

Next let us denote the number of solutions of (P_λ) by $N_p(\lambda)$. For $p = 2$, that is, for the linear case, elementary calculations show that

$$N_2(\lambda) = \begin{cases} 1 & \text{if } \lambda \neq (k\pi/T)^2 \text{ for all } k \in \mathbf{N}, \\ 0 & \text{if } \lambda = ((2k-1)\pi/T)^2 \text{ for some } k \in \mathbf{N}, \\ \infty & \text{if } \lambda = (2k\pi/T)^2 \text{ for some } k \in \mathbf{N}. \end{cases} \tag{1.11}$$

Thus, for $p = 2$ and λ between consecutive eigenvalues, $N_2(\lambda) = 1$.

We will show in the next section that for $p \neq 2$, the number of solutions of (P_λ) can be arbitrarily large. Indeed, and as a consequence of our Theorem 2.1 we will obtain

$$\lim_{\lambda \to \infty} N_p(\lambda) = \infty. \tag{1.12}$$

In particular, for $p \neq 2$ and λ between consecutive eigenvalues, not only may $N_p(\lambda)$ be different from one but also λ large implies $N_p(\lambda)$ large. Thus, concerning $N_p(\lambda)$ there is a sharp contrast between problem (P_λ) under the nonresonant condition (1.8) for $p \neq 2$ and the case $p = 2$.

These findings reveal that the uniqueness problem for (1.1)–(1.2) under global nonresonance can be highly nontrivial.

Remark. If we let $f(0) = 0$, $h \equiv 0$ in (1.1), (1.2) then (1.8) ensures that $u \equiv 0$ is the unique solution of (1.1), (1.2). This result is easily shown from Sturm's theorem for equations of the form

$$(|u'|^{p-2}u')' + a(x)|u|^{p-2}u = 0,$$

(see for example [4]), and the fact that an eigenfunction associated with $\lambda = \lambda_k$ in (1.4), (1.5) possesses exactly $k-1$ zeros in $(0, T)$.

2. Main result and consequences

In this section we state our main result, Theorem 2.1, and derive some consequences from it. Theorem 2.1 will be proved in the next section.

Together with the sequence $\{\lambda_k\}_{k=1}^{\infty}$ of eigenvalues of (1.4), (1.5) we will consider the sequence $\{\mu_k\}_{k=1}^{\infty}$ defined by

$$\mu_k = (kp'\pi_p/T)^p, \qquad k \in \mathbf{N}. \tag{2.1}$$

In (2.1) and henceforth, $p' = p/(p-1)$. We observe that $\mu_k < (=)(>)\lambda_{2k}$ if $p < (=)(>)2$. These numbers μ_k, $k \in \mathbf{N}$, will play an important role in our results. Indeed, it is their nonuniform distribution with respect to the λ_k's for $p \neq 2$ which produces the existence of a large number of solutions to (P_λ) for large λ.

We will say that a function $u \in C^1[0, T]$ belongs to E_k^+ (E_k^0) (E_k^-) if u possesses exactly $k-1$ zeros in $(0, T)$ and $u'(0) > (=)(<)0$.

Theorem 2.1. (a) *If $\lambda \in (0, \lambda_1)$, then (P_λ) possesses exactly one solution u, and $u \in E_1^-$.*

(b) *If $\lambda = \lambda_1$, then (P_λ) has no solution.*

(c) *If λ is strictly between λ_{2k-1} and μ_k, then (P_λ) possesses at least one solution $u \in E_{2k-1}^+$.*

(d) *If λ is strictly between μ_k and λ_{2k+1}, then (P_λ) possesses at least one solution $u \in E_{2k+1}^+$.*

(e) *If $\lambda = \mu_k$ then (P_λ) possesses a solution $u \in E_k^0$.*

(f) *If λ is strictly between μ_k and λ_{2k}, then (P_λ) possesses a solution u in E_{2k}^+ and a solution v in E_{2k}^-.*

Remark. In contrast with (a) of Theorem 2.1, it is shown in [3] that for $p > 2$ and $\lambda \in (0, \lambda_1)$ one can always find an $h \in C[0, I]$ such that the problem

$$(|u'|^{p-2}u')' + \lambda|u|^{p-2}u = h(x), \qquad x \in (0, T),$$
$$u(0) = u(T) = 0$$

admits at least two solutions. We also remark that Anane and Gossez [1] have studied the resonance-nonresonance problem for the p-Laplacian in the case where the nonlinearity "lies" to the left of the first eigenvalue.

By combining (a)–(f) of Theorem 2.1 we can easily obtain the following.

Corollary 2.2. *Let $p \in (1, \infty)\backslash\{2\}$. Then (P_λ) is solvable for all $\lambda > 0$ except $\lambda = \lambda_1$ and, eventually, those numbers λ of the form $\lambda = \lambda_{2k-1}$ for $k < 1/|p' - 2|$.*

From this corollary we obtain, in particular, that (P_λ) is solvable for all large positive λ. Furthermore, as λ goes to infinity the number of solutions $N_p(\lambda)$ of (P_λ) goes to infinity, as the following estimate shows.

Proposition 2.3. *Let* $p \in (1, \infty)$, $p \neq 2$. *Then the number of solutions* $N_p(\lambda)$ *of* (P_λ) *satisfies*

$$N_p(\lambda) \geq \frac{3T\lambda^{1/p}}{\pi_p}\left|\frac{1}{p'} - \frac{1}{2}\right| - 3 \tag{2.2}$$

for all $\lambda > 0$. *In particular,* $\lim_{\lambda\to\infty} N_p(\lambda) = \infty$.

Proof. We will assume $p > 2$. The case $p < 2$ can be treated similarly.

Let us fix $\lambda > 0$ and denote by M_j the number of positive integers such that (f) of Theorem 2.1 holds, i.e.,

$$M_f = \operatorname{card}\left\{k \in \mathbf{N} \,\middle|\, \frac{p'\pi_p k}{T} < \lambda^{1/p} < \frac{2\pi_p k}{T}\right\}. \tag{2.3}$$

Clearly

$$M_f \geq \max\left\{k \in \mathbf{N} \,\middle|\, k < \frac{T\lambda^{1/p}}{p'\pi_p}\right\} - \min\left\{k \in \mathbf{N} \,\middle|\, \frac{T\lambda^{1/p}}{2\pi_p} < k\right\} + 1 \tag{2.4}$$

and hence

$$M_f \geq \left(\frac{T\lambda^{1/p}}{p'\pi_p} - 1\right) - \left(\frac{T\lambda^{1/p}}{2\pi_p} + 1\right) = \frac{T\lambda^{1/p}}{\pi_p}\left(\frac{1}{p'} - \frac{1}{2}\right) - 1. \tag{2.5}$$

Next, let us denote by M_c (M_d) the number of positive integers such that (c) ((d)) of Theorem 2.1 holds. Estimates similar to those for M_f yield

$$M_c \geq \frac{T\lambda^{1/p}}{\pi_p}\left(\frac{1}{p'} - \frac{1}{2}\right) - \frac{3}{2}, \tag{2.6}$$

$$M_d \geq \frac{T\lambda^{1/p}}{\pi_p}\left(\frac{1}{p'} - \frac{1}{2}\right) - \frac{1}{2}. \tag{2.7}$$

From (2.5)–(2.7) we obtain

$$N_p(\lambda) \geq M_c + M_d + m_f \geq \frac{3T\lambda^{1/p}}{\pi_p}\left(\frac{1}{p'} - \frac{1}{2}\right) - 3, \tag{2.8}$$

and hence the proposition. □

3. Proof of Theorem 2.1

In this section we will prove Theorem 2.1. To this end, we will first study some properties of the solution to the initial value problem

$$(|u'|^{p-2}u')' + \lambda|u|^{p-2}u = 1, \tag{3.1}$$

$$u(0) = 0 \quad u'(0) = \alpha. \tag{3.2}$$

We will construct a global solution to this problem. Multiplying both sides of (3.1) by u' and integrating from 0 to t we find that a solution u to (3.1), (3.2) must satisfy the energy relation

$$\frac{|u'(t)|^p}{p'} + \lambda\frac{|u(t)|^p}{p} = \frac{|\alpha|^p}{p'} + u(t). \tag{3.3}$$

Let $t_\lambda(\alpha)$ be the first positive zero of u'. Thus for $t \in (0, t_\lambda(\alpha))$

$$t = \int_0^{u(t)} dw/(\alpha^p + p'w - \lambda\tfrac{w^p}{p-1})^{1/p} \tag{3.4}$$

if $\alpha \geq 0$, and

$$t = \int_0^{-u(t)} dw/(|\alpha|^p - p'w - \lambda\tfrac{w^p}{p-1})^{1/p} \tag{3.5}$$

if $\alpha < 0$. Thus, considering the function

$$F(s) = \int_0^{s} dw/(|\alpha|^p + p'\,\mathrm{sgn}(\alpha)w - \lambda\tfrac{w^p}{p-1})^{1/p}, \tag{3.6}$$

where

$$\mathrm{sgn}(\alpha) = \begin{cases} 1 & \text{if } \alpha \geq 0, \\ -1 & \text{if } \alpha < 0 \end{cases}$$

it follows that

$$t_\lambda(\alpha) = F(q(\alpha)), \tag{3.7}$$

where $q(\alpha)$ is the unique positive root of the equation

$$\lambda x^p/(p-1) - p'\,\mathrm{sgn}(\alpha)x = |\alpha|^p. \tag{3.8}$$

Also, from (3.4)–(3.6) and for $t \in (0, t_\lambda(\alpha)]$, we have

$$u(t) = \begin{cases} F^{-1}(t) & \text{if } \alpha \geq 0, \\ -F^{-1}(t) & \text{if } \alpha < 0. \end{cases} \tag{3.9}$$

Conversely, if we have a function u of the form (3.9) it can be directly verified that u satisfies (3.1), (3.2) and hence is the unique solution of this initial value problem on the interval $(0, t_\lambda(\alpha)]$ with $t_\lambda(\alpha)$ defined by (3.7).

Next, let us extend u to obtain a global solution $u_\lambda(\alpha, t)$ to (3.1), (3.2). Thus define $u_\lambda(\alpha, t) = u(t)$ for $t \in (0, t_\lambda(\alpha)]$, $u_\lambda(\alpha, t) = u(2t_\lambda(\alpha) - t)$ for $t \in [t_\lambda(\alpha), 2t_\lambda(\alpha)]$, and $u_\lambda(\alpha, t) = u_\lambda(-\alpha, 2(t_\lambda(\alpha)+t_\lambda(-\alpha))-t)$ for $t \in [2t_\lambda(\alpha), 2(t_\lambda(\alpha)+t_\lambda(-\alpha))]$. Finally we periodically extend this function to the whole real line in a $2(t_\lambda(\alpha) + t_\lambda(-\alpha))$-periodic manner. It is easily verified that $u_\lambda(\alpha, t)$ is of class C^1 and solves (3.1), (3.2). It can be shown that this is actually the unique solution of (3.1), (3.2).

We note that the zeros of $u_\lambda(\alpha, t)$ are the numbers $2kt_\lambda(\alpha)+2(k-\varepsilon)t_\lambda(-\alpha)$, $k \in \mathbf{N}$, $\varepsilon = 0$ or 1. Clearly, whenever one of these numbers equals T wc obtain a solution of (P_λ).

Next, let us show some properties of the function $t_\lambda(\alpha)$ which will be needed in the proof of Theorem 2.1.

Lemma 3.1. *The function* $t_\lambda(\alpha)$ *satisfies*

(a) t_λ *is strictly decreasing and continuous on the intervals* $(-\infty, 0)$ *and* $[0, \infty)$.

(b) t_λ *presents a jump discontinuity at* $\alpha = 0$; *more precisely,*

$$0 = \lim_{\alpha\to 0^-} t_\lambda(\alpha) < t_\lambda(0) = p'\pi_p/2\lambda^{1/p}. \tag{3.10}$$

(c)

$$\lim_{\alpha\to-\infty} t_\lambda(\alpha) = \pi_p/2\lambda^{1/p} = \lim_{\alpha\to+\infty} t_\lambda(\alpha). \tag{3.11}$$

Proof. From (3.6) and (3.7) follows

$$t_\lambda(\alpha) = \int_0^{q(\alpha)} dw/(|\alpha|^p + p'\operatorname{sgn}(\alpha)w - \lambda\tfrac{w^p}{p-1})^{1/p}. \tag{3.12}$$

Substituting $w = sq(\alpha)$ in (3.12) and calling on (3.8) with $x = q(\alpha)$ we obtain

$$t_\lambda(\alpha) = \int_0^1 ds/(-\tfrac{1}{q(\alpha)^{p-1}}p'\operatorname{sgn}(\alpha)(1-s) + \tfrac{\lambda}{p-1}(1-s^p))^{1/p}. \tag{3.13}$$

To examine the behavior of $q(\alpha)$ with respect to α we note that the Implicit Function Theorem and (3.8) imply that $q(\alpha)$ is of class C^1 on $\mathbf{R}\backslash\{0\}$ and

$$\frac{dq}{d\alpha}(\alpha) = \frac{p|\alpha|^{p-2}\alpha}{p'(\lambda q(\alpha)^{p-1} - \operatorname{sgn}(\alpha))}. \tag{3.14}$$

From the definition of $q(\alpha)$, we easily see that the denominator on the right-hand side of (3.14) is positive and hence that $q(\alpha)$ is strictly increasing and continuous for $\alpha \in [0, \infty)$ and strictly decreasing and continuous for $\alpha \in (-\infty, 0)$. Thus (a) follows immediately from (3.13).

To show (b) we first observe that $q(0^-) = 0$ and hence from (3.12) we conclude that $\lim_{\alpha\to 0^-} t_\lambda(\alpha) = 0$. Next, setting $\alpha = 0$ in (3.13) and using the fact that $q(0) = (p/\lambda)^{1/(p-1)}$ from (3.8), we obtain

$$t_\lambda(0) = \left(\frac{p-1}{\lambda}\right)^{1/p} \int_0^1 \frac{ds}{(s-s^p)^{1/p}}. \tag{3.15}$$

The substitution $s = \tau^{p'}$ in (3.15) yields

$$t_\lambda(0) = \left(\frac{p-1}{\lambda}\right)^{1/p} \int_0^1 \frac{p'\,d\tau}{(1-\tau^p)^{1/p}} = \frac{p'\pi_p}{2\lambda^{1/p}}. \tag{3.16}$$

This shows (b). Finally, to show (c), we note that $q(\alpha) \to +\infty$ as $|\alpha| \to \infty$. Letting α go to $\pm\infty$ in (3.15), it follows from the Dominated Convergence Theorem that

$$\lim_{\alpha\to\pm\infty} t_\lambda(\alpha) = \left(\frac{p-1}{\lambda}\right)^{1/p} \int_0^1 \frac{ds}{(1-s^p)^{1/p}} = \frac{\pi_p}{2\lambda^{1/p}}. \tag{3.17}$$

This concludes the proof of (c) and hence the lemma. □

With these preliminaries we are now ready to prove Theorem 2.1. For notational simplicity, throughout the proof we will assume $T = \pi_p$. In this case $\lambda_k = k^p$ and $\mu_k = (p'k)^p$, $k \in \mathbf{N}$.

Proof of Theorem 2.1. (a) Assume that $0 < \lambda < 1$. Then from Lemma 2.1 we find $2t_\lambda(\alpha) > \pi_p$ for $\alpha \geq 0$. Hence there is no solution of (P_λ) with nonnegative derivative at $t = 0$. Again from Lemma 3.1, but for $\alpha < 0$, we see that there is a unique $\alpha^* < 0$ such that $2t_\lambda(\alpha^*) = \pi_p$. It follows that $u_\lambda(\alpha^*, t)$ is the unique solution of (P_λ) and it belongs to E_1^-.

(b) Suppose that $\lambda = 1$. The absence of solutions to (P_λ) in this case follows directly from the facts that $2t_\lambda(\alpha) > \pi_p$ for $\alpha \geq 0$, and $2t_\lambda(\alpha) < \pi_p$ for $\alpha < 0$.

(c) We assume λ strictly between λ_{2k-1} and μ_k, i.e., $\lambda^{1/p}$ is between $2k-1$ and $p'k$. Let us consider the function

$$f(\alpha) = 2(k-1)(t_\lambda(\alpha) + t_\lambda(-\alpha)) + 2t_\lambda(\alpha). \tag{3.18}$$

From Lemma 3.1 we see that f is continuous on $(0, \infty)$, $f(0^+) = p'k\pi_p/\lambda^{1/p}$, and $\lim_{\alpha\to+\infty} f(\alpha) = (2k-1)\pi_p/\lambda^{1/p}$. Now, the fact that $\lambda^{1/p}$ is strictly between $2k-1$ and $p'k$ implies the existence of $\overline{\alpha} > 0$ such that $f(\overline{\alpha}) = \pi_p$, and hence $u(t) \equiv u_\lambda(\overline{\alpha}, t)$ is a solution to (P_λ) with exactly $2k-2$ inner zeros. Clearly $u \in E_{2k-1}^+$.

(d) The proof is analogous to that of (c) except that this time we consider the function

$$f(\alpha) = 2k(t_\lambda(\alpha) + t_\lambda(-\alpha)) + 2t_\lambda(\alpha) \tag{3.19}$$

on the interval $(-\infty, 0)$.

(e) If $\lambda = \mu_k$, then $\lambda^{1/p} = kp'$. This implies that $2kt_\lambda(0) = \pi_p$ and, therefore, $u(t) \equiv u_\lambda(0, t)$ is a solution of (P_λ) with exactly $k-1$ inner zeros. Clearly $u \in E_k^0$. We note that in this case all the zeros of u are double.

(f) Suppose finally that λ is strictly in between kp' and $2k$. For this case we define

$$f(\alpha) = 2k(t_\lambda(\alpha) + t_\lambda(-\alpha)). \tag{3.20}$$

Clearly f is continuous on each of the intervals $(-\infty, 0)$ and $(0, \infty)$. Also $f(0^+) = p'k\pi_p/\lambda^{1/p}$ and $\lim_{\alpha\to\pm\infty} f(\alpha) = 2k\pi_p/\lambda^{1/p}$. Reasoning as before we obtain the existence of solutions u, v of (P_λ) with $u \in E_{2k}^+$ and $v \in E_{2k}^-$. This concludes the proof of the theorem. □

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