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A Homotopic Deformation along p of a Leray–Schauder Degree Result and Existence for $(|u'|^{p-2}u')' + f(t, u) = 0$, $u(0) = u(T) = 0$, $p > 1$*

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1. INTRODUCTION

Let f: $[0, 1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and let ϕ_p: $\mathbb{R} \to \mathbb{R}$ be defined by $\phi_p(s) = |s|^{p-2} s$ for any fixed p greater than one.

We will consider the following Dirichlet BVP,

$$\text{(D)} \qquad \begin{aligned} (\phi_p(u'))' + f(t, u) &= 0 \qquad (1.1) \\ u(0) = u(T) &= 0, \qquad (1.2) \end{aligned}$$

where $' = d/dt$ and T is a positive real number. The notation we will use herein is mostly standard, albeit we will abbreviate it. Thus, the Banach space $C^i[0, T]$, where i is a nonnegative integer, endowed with the norm given by $|u|_i = \sum_{j=0}^{i} \sup_{t \in [0,T]} |u^{(j)}(t)|$ will be simply denoted by C^i. $B(0, r)$ will denote the ball center 0 and radius r in C^0, $\overline{B(0, r)}$ being its

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closure. We will also shorten $L^p(0, T)$, $W_0^{1,p}(0, T)$, and $C_0^\infty(0, T)$ to L^p, $W_0^{1,p}$, and C_0^∞, respectively. The norm in L^p is defined in the usual way and is denoted by $\| \;\|_p$. The norm in $W_0^{1,p}$ is denoted by $\| \;\|_{1,p}$ and is defined by $\|u\|_{1,p} = \|u'\|_p$. $C_0^\infty(0, T)$ is the set of functions of class C^∞ with compact support in $(0, T)$.

By a solution of problem (D) we will understand a function $u \in C^1$ such that $\phi_p(u') \in C^1$ and which satisfies (1.1) and (1.2).

Let $h \in C^0$ and consider the Dirichlet BVP

$$-u'' = h \tag{1.3}$$

$$u(0) = u(T) = 0. \tag{1.4}$$

It is an elementary fact that for every given $h \in C^0$ there is a unique solution $v \in C^2$ to the above problem. Using the fact that C^2 embeds compactly in C^0, we can define a compact linear mapping $G_2\colon C^0 \to C^0$ by $G_2(h) = v$. Let $\{\lambda_k\}_{k=1}^\infty$, where $\lambda_k = (k\pi/T)^2$, denote the sequence of eigenvalues of $-u''$ and let $\lambda \in (\lambda_n, \lambda_{n+1})$ for some positive integer n. Let I denote the identity in C^0 and let $T_2\colon C^0 \to C^0$ be defined by $T_2 = I - \lambda G_2$. It is well known (see for instance [9, Chap. V, p. 24]) that the Leray–Schauder degree of T_2 with respect to $B(0, r)$ and 0 is given by

$$d(T_2, B(0, r), 0) = (-1)^n \tag{1.5}$$

for any $r > 0$.

In Section 2 we reformulate problem (D). We begin by studying the equation

$$-(\phi_p(u'))' = h \tag{1.6}$$

under homogeneous Dirichlet conditions. We find a mapping $G_p\colon C^0 \to C^0$ which is completely continuous, i.e., continuous and compact such that $G_p(h)$ is the unique solution to (1.6). By means of this mapping G_p we are able to set down an equivalent abstract formulation for problem (D).

In Section 3 we briefly review the eigenvalue problem corresponding to the operator $-(\phi_p(u'))'$ under homogeneous Dirichlet conditions.

In Section 4 we establish our main result. We define $T_p\colon C^0 \to C^0$ by $T_p(u) = u - G_p(\lambda\phi_p(u))$ and prove, via a suitable homotopic deformation, that formula (1.5) still holds for any $p > 1$ if we substitute T_p for T_2.

In Section 5 we provide some sufficient conditions for the existence of solutions to problem (D). An application of the main result above, first, together with a direct consequence of Theorem 1.1 of [4] enables us to put forth sufficient conditions for this problem to have a solution when f is under nonuniform nonresonant conditions. The results we obtain are conceptually related with those in [7, 8] for the pde case and $p = 2$. We also prove sufficient conditions for existence of nontrivial solutions when $f(t, 0) = 0$ for all $t \in [0, T]$. Finally, we prove by means of a simple example that regarding the existence of solutions for problem (D) the case $p \neq 2$, $p > 1$, may be quite different from the case $p = 2$.

2. An Equivalent Abstract Equation

In this section we show that solving problem (D) is equivalent to finding fixed points of a certain completely continuous mapping G_p: $C^0 \to C^0$.

We start by studying the following auxiliary problem. For a given $h \in L^q$, with $q > 1$, we look for a function $u \in C^1$ satisfying

$$\text{(AP)} \qquad \begin{aligned} -(\phi_p(u'))' &= h \qquad \text{a.e.} \quad \text{on } [0, T] & (2.1) \\ u(0) = u(T) &= 0, & (2.2) \end{aligned}$$

with $\phi_p(u')$ an absolutely continuous function on $[0, T]$. Clearly, if u is such a solution then it satisfies

$$\int_0^T \phi_p(u')v' = \int_0^T hv, \tag{2.3}$$

for all $v \in W_0^{1,p}$. Conversely, if $u \in W_0^{1,p}$ satisfies (2.3) for all $v \in W_0^{1,p}$ and we let, henceforth, $p' = p/(p-1)$ and take $r = \min\{p', q\}$ then $\phi_p(u')$ and h belong to L^r and satisfy (2.3) for all $v \in C_0^\infty$. Hence $\phi_p(u') \in W_0^{1,r}$. From this and Theorem VIII.2 of [2] we can see that $\phi_p(u')$ is an absolutely continuous function which satisfies (2.1). Since $\phi_p(u')$ is absolutely continuous, $\phi_{p'}$ is the inverse function of ϕ_p, and $u \in W_0^{1,p}$, using Remark 6 of [2] we find that $u \in C^1$, and satisfies (2.2).

Next we observe that searching for $u \in W_0^{1,p}$ satisfying (2.3) is equivalent to finding critical points of the functional ψ_h: $W_0^{1,p} \to \mathbb{R}$ defined by

$$\psi_h(u) = \frac{1}{p} \int_0^T |u'|^p - \int_0^T hu. \tag{2.4}$$

We find that ψ_h is a continuous strictly convex functional such that $\psi_h(u) \to +\infty$ as $\|u\|_{1,p} \to \infty$. Hence (see for instance [6]) it possesses a unique critical point at which it reaches its global minimum.

From the previous arguments we conclude that (AP) has a unique solution $w \in C^1$. Thus we can define a mapping G_p: $L^q \to C^1$ by $G_p(h) = w$.

Our next step is to prove that G_p seen as a mapping from L^q into C^0 is completely continuous. Instead of doing this directly, we will establish and prove, anticipatively, two propositions which will actually be needed later and, for the present case, make the result immediate.

PROPOSITION 2.1. *Let p_0 be a real number greater than one and let us define ϕ: $[p_0, +\infty) \times C^0 \to C^0$ by $\phi(p, u)(t) = \phi_p(u(t))$, for all $t \in [0, T]$. Then the mapping ϕ is continuous and sends bounded sets of $[p_0, +\infty) \times C^0$ into bounded sets of C^0.*

Proof. It follows directly from the definition of ϕ. ∎

In the next proposition, and henceforth, weak convergence will be indicated by the symbol $\rightharpoonup$.

Let p_0 be a real number greater than one and let G: $[p_0, +\infty) \times L^q \to C^0$ be the mapping defined by $G(p, h) = G_p(h)$.

PROPOSITION 2.2. *If $\{p_n\}_{n=1}^{\infty}$ is a sequence in $[p_0, +\infty)$ such that $\lim_{n \to \infty} p_n = p$ and $\{h_n\}_{n=1}^{\infty}$ is a sequence in L^q such that $h_n \rightharpoonup h \in L^q$ as $n \to \infty$ then $\lim_{n \to \infty} G(p_n, h_n) = G(p, h)$.*

Proof. Let $\{p_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ be sequences in $[p_0, +\infty)$ and L^q, respectively, such that $\lim_{n \to \infty} p_n = p$ and $h_n \rightharpoonup h$ as $n \to \infty$. Suppose that $G(p_n, h_n)$ does not converge to $G(p, h)$ as $n \to \infty$. Hence there exists an $\varepsilon > 0$ and a subsequence of $\{(p_n, h_n)\}_{n=1}^{\infty}$, which we will call again $\{(p_n, h_n)\}_{n=1}^{\infty}$, such that

$$|G(p_n, h_n) - G(p, h)|_0 \geqslant \varepsilon \tag{2.5}$$

for all $n \in \mathbb{N}$.

Based on the definition of the mappings ϕ and G, the fact that for fixed $p > 1$ and u a solution of (AP) $\phi(p, u')$ is an absolutely continuous function on $[0, T]$, and setting $u_n = G(p_n, h_n)$, we find that

$$-(\phi(p_n, u_n'))' = h_n \tag{2.6}$$

for each fixed $n \in \mathbb{N}$. Equation (2.6) and the boundedness of $\{h_n\}_{n=1}^{\infty}$ tell us that the sequence $\{\phi(p_n, u_n)\}_{n=1}^{\infty}$ meets the requirements of Ascoli–Arzela's theorem in C^0. Hence there exists a subsequence of $\{\phi(p_n, u_n)\}_{n=1}^{\infty}$ which is convergent in C^0. We label this subsequence again by $\{\phi(p_n, u_n)\}_{n=1}^{\infty}$. From Proposition 2.1 and since $u_n = \phi(p_n, \phi(p_n, u_n))$, $n \in \mathbb{N}$, we find that $\{u_n\}_{n=1}^{\infty}$ is convergent in C^0. This fact and Ascoli–Arzela's theorem imply indeed that $\{u_n\}_{n=1}^{\infty}$ contains a convergent subsequence in C^1. We label this subsequence again by $\{u_n\}_{n=1}^{\infty}\}$. Let $u = \lim_{n \to \infty} u_n$. We note that from (2.5)

$$|u_n - G(p, h)|_0 \geqslant \varepsilon. \tag{2.7}$$

Letting $u = u_n$ and $h = h_n$ in (2.3) we find

$$\int_0^T \phi(p_n, u_n')v = \int_0^T h_n v \tag{2.8}$$

for all $n \in \mathbb{N}$ and for all $v \in C_0^\infty(0, T)$. Recalling from Proposition (2.1) that ϕ is continuous we can let n go to infinity in (2.8) to obtain

$$\int_0^T \phi(p, u')v' = \int_0^T \phi_p(u')v' = \int_0^T hv \tag{2.9}$$

for all $v \in C_0^\infty$ and hence for all $v \in W_0^{1,p}$. From (2.9) we find that $G(p, h) = G_p(h) = u$ which is a contradiction in light of (2.7). Thus the proposition is proved. ∎

COROLLARY 2.3. (i) *The mapping G seen as a mapping from $[p_0, +\infty) \times L^q \to C^0$ is completely continuous.*

(ii) *The mapping G seen as a mapping from $[p_0, +\infty) \times C^0 \to C^0$ is completely continuous.*

Proof. (i) It follows directly from Proposition 2.2.

(ii) It follows from (i) and the fact that the canonical embedding of C^0 into L^q, $q > 1$, is continuous. ∎

From this corollary and for any fixed $p \in (1, +\infty)$ we find that the mapping $G(p, \cdot) = G_p$ seen as a mapping from L^q or C^0 into C^0 is a completely continuous mapping.

Let us now define F: $C^0 \to C^0$ by $F(u)(t) = f(t, u(t))$ for all $t \in [0, T]$. It is well known that F is a continuous operator which sends bounded sets of C^0 into bounded sets of C^0. Furthermore if we let I denote the identity in C^0 then $I - G_p \circ F$: $C^0 \to C^0$ is a compact perturbation of the identity. It is now easy to see that solving problem (D) is equivalent to solving the abstract equation

$$u - G_p(F(u)) = 0. \tag{2.10}$$

3. AN EIGENVALUE PROBLEM

In this section we will briefly review and refine some results of Drábek [5] concerning the eigenvalue problem

$$\text{(E)} \qquad \begin{aligned} -(\phi_p(u'))' &= \lambda \phi_p(u) && (3.1) \\ u(0) = u(T) &= 0. && (3.2) \end{aligned}$$

A real number λ such that (E) possesses a nontrivial solution will be called an eigenvalue of (E). The associated nontrivial solution will be refered to as an eigenfunction of (E) corresponding to λ.

It is easy to check that if λ is an eigenvalue of (E) and u is a corresponding eigenfunction then

$$\lambda = \frac{\|u'\|_p^p}{\|u\|_p^p} \tag{3.3}$$

and hence all the eigenvalues of (E) must be positive. Correspondingly, we take $\lambda > 0$ in (3.1).

In order to solve problem (E) in a similar form as it is done when $p = 2$ we start by studying the IVP

$$\text{(I)} \qquad \begin{aligned} &-(\phi_p(u'))' = \lambda \phi_p(u) && (3.4) \\ &u(0) = 0, \qquad u'(0) = \alpha \in \mathbb{R}. && (3.5) \end{aligned}$$

A change of the independent variable t to $\tau = \lambda^{1/p} t$ reduces (I) to

$$\text{(I}_1\text{)} \qquad \begin{aligned} &-(\phi_p(u'))' = \phi_p(u) && (3.6) \\ &u(0) = 0, \qquad u'(0) = \alpha \lambda^{-1/p}, && (3.7) \end{aligned}$$

where now $' = d/d\tau$. This IVP can be solved by direct integration. In fact, we let for any real $p > 1$

$$\pi_p = 2 \int_0^{(p-1)^{1/p}} \frac{ds}{(1 - s^p/(p-1))^{1/p}} \tag{3.8}$$

and implicitly define the function w: $[0, \pi_p/2] \to [0, (p-1)^{1/p}]$ by

$$\int_0^{w(t)} \frac{ds}{(1 - s^p/(p-1))^{1/p}} = t. \tag{3.9}$$

We now extend w to $[-\pi_p, \pi_p]$ as follows. First we define $\tilde{w}(t) = w(t)$ for $t \in [0, \pi_p/2]$ and $\tilde{w}(t) = w(\pi_p - t)$ for $t \in [\pi_p/2, \pi_p]$ and then $\tilde{w}(t) = -\tilde{w}(-t)$ for $t \in [-\pi_p, 0]$. Finally we define $\sin_p$: $\mathbb{R} \to \mathbb{R}$ as the $2\pi_p$-periodic extension of $\tilde{w}$ to all of $\mathbb{R}$.

By a direct verification we can show that

$$u(t) = \alpha \lambda^{-1/p} \sin_p(t) \tag{3.10}$$

is a solution of (I_1) and hence

$$u(t) = \alpha \lambda^{-1/p} \sin_p(\lambda^{1/p} t) \tag{3.11}$$

is a solution of (I). Moreover, it can be proved that (3.11) is the unique solution to problem (I). See [3].

Now it is easy to solve problem (E). In fact, λ will be an eigenvalue of (E) if and only if

$$\sin_p(\lambda^{1/p}T)=0. \tag{3.12}$$

From $\sin_p(\mu)=0$ if and only if $\mu=n\pi_p$, where $n\in\mathbb{Z}$, and since the eigenvalues of (E) are positive, they finally turn out to be given by

$$\lambda_n(p)=\left(\frac{n\pi_p}{T}\right)^p, \tag{3.13}$$

where $n\in\mathbb{N}$, with corresponding eigenfunctions

$$u_n(t)=\alpha\lambda_n^{-1/p}\sin_p(\lambda_n^{1/p}t). \tag{3.14}$$

4. Homotopically Deforming along $p>1$ a Leray–Schauder Degree Result

For every fixed real number $p>1$ let T_p: $C^0\to C^0$ be the mapping defined by $T_p(u)=u-G_p(\lambda\phi_p(u))$, with $\lambda\in\mathbb{R}$. From the previous sections it is clear that for each fixed p, $T_p(u)=0$ has a nontrivial solution if and only if $\lambda=\lambda_n(p)$ as given in (3.13) with corresponding $u=u_n$ given by (3.14).

The following theorem is the main result of this section.

Theorem 4.1. *Let p be any fixed real number greater than one. Let $\lambda\in\mathbb{R}$ be such that $\lambda\neq\lambda_n(p)$ for each $n\in\mathbb{N}$. Then for every $r>0$, the Leray–Schauder degree $d(T_p, B(0,r), 0)$ is well defined and satisfies*

$$d(T_p, B(0,r), 0)=(-1)^\beta, \tag{4.1}$$

where β is the number of eigenvalues $\lambda_n(p)$ of problem (E) *less than λ.*

Remark 1. The above theorem is known to be true for $p=2$ as was established in Section 1. As a matter of fact, our proof of Theorem 4.1 will be carried out by constructing a suitable homotopic deformation along p from the case $p=2$.

Proof of Theorem 4.1. We will only prove the case $\lambda>\lambda_1(p)$ since the proof for the case $\lambda<\lambda_1(p)$ is similar. We also assume that $p\neq 2$.

Since $\lambda>\lambda_1(p)$ and $\lambda\neq\lambda_n(p)$ for all $n\in\mathbb{N}$, there exist $s\in(0,1)$ and $n\in\mathbb{N}$ such that $\lambda=((n+s)\pi_p/T)^p$. Hence, all we need to show is that for every $r>0$, $d(T_p, B(0,r), 0)=(-1)^n$.

Let p_0 denote the $\min\{p,2\}$ and let $\tilde{\lambda}$: $[p_0,+\infty)\to\mathbb{R}$ be defined by $\tilde{\lambda}(q)=((n+s)\pi_q/T)^q$. From the definition of π_q given in Section 3 it is easy to see that π_q seen as a function of $q \in (1, +\infty)$ is continuous and hence so is $\tilde{\lambda}$.

Let us next define T: $[p_0, +\infty) \times C^0 \to C^0$ by

$$T(q, u) = u - G(q, \tilde{\lambda}(q)\, \phi(q, u)) \equiv u - \tilde{G}(q, u). \tag{4.2}$$

From the continuity of $\tilde{\lambda}$, Proposition 2.1, and Corollary 2.3 we get that $\tilde{G}$ is a completely continuous mapping. Also from the very definition of $\tilde{\lambda}$ and since $T(q, \cdot) = T_q$ for $\lambda = \tilde{\lambda}(q)$, we find that for $u \neq 0$, $T(q, u) \neq 0$ for all $q \in [p_0, +\infty)$. Hence, from the invariance of the Leray–Schauder degree under homotopies and (1.5) we obtain

$$d(T_p, B(0, r), 0) = d(T_2, B(0, r), 0) = (-1)^n, \tag{4.3}$$

for any $r > 0$. This concludes the proof of Theorem 4.1. ∎

5. Existence of Solutions to Problem (D)

The purpose of the first part of this section is to show two existence results for problem (D) under certain conditions on f. Throughout this section p will denote a fixed real number greater than one. We recall from Section 2 that showing existence for problem (D) is equivalent to solving the abstract equation (2.10).

Our existence results will be based upon Theorem 4.1 of the last section and on the following proposition which is a direct consequence of Theorem 1.1 of [4].

Proposition 5.1. *Let c be a measurable real valued function defined on $[0, T]$. Let k be a positive integer and let $\lambda_k(p)$, $\lambda_{k+1}(p)$ be defined by (3.13). Suppose that*

$$\lambda_k(p) \leqslant c(t) \leqslant \lambda_{k+1}(p) \qquad a.e. \quad on\ [0, T], \tag{5.1}$$

the strict inequalities holding true in some subsets of positive measure in $[0, T]$. Then the problem

$$(\phi_p(u'))' + c(t)\, \phi_p(u) = 0 \tag{5.2}$$

$$u(0) = u(T) = 0 \tag{5.3}$$

does not have a nontrivial solution.

Our first result is the following.

THEOREM 5.2. *Suppose that k is a positive integer and that f in problem* (D) *satisfies*

$$\lambda_k(p) \leqslant a(t) \equiv \liminf_{|s| \to +\infty} \frac{f(t,s)}{\phi_p(s)} \leqslant \limsup_{|s| \to +\infty} \frac{f(t,s)}{\phi_p(s)}$$
$$\equiv b(t) \leqslant \lambda_{k+1}(p), \tag{5.4}$$

uniformly on $[0, T]$, *the first and last inequalities being strict in some subsets of positive measure in* $[0, T]$. *Then problem* (D) *has a solution.*

Proof. Let $v \in (\lambda_k, \lambda_{k+1})$ and consider the completely continuous homotopy H: $[0, 1] \times C^0 \to C^0$ defined by

$$H(\tau, u) = G_p(\tau v \phi_p(u) + (1-\tau) F(u)). \tag{5.5}$$

We claim that for a big enough $r > 0$, $u - H(\tau, u) \neq 0$ holds for all $u \in \partial B(0, r)$ and for all $\tau \in [0, 1]$. Suppose the claim is not true. Then there exists a sequence $\{u_n\}_{n=1}^{\infty}$ in C^0 and a sequence $\{\tau_n\}_{n=1}^{\infty}$ in $[0, 1]$ such that $|u_n|_0 \to \infty$ and $\tau_n \to \tau_0 \in [0, 1]$ as $n \to \infty$ and such that

$$v_n = G_p(\tau_n v \phi_p(v_n) + (1-\tau_n) F(u_n)/|u_n|_0^{p-1}), \tag{5.6}$$

where $v_n = u_n/|u_n|_0^{p-1}$.

Let us define the sequence $\{h_n\}_{n=1}^{\infty}$ by $h_n = F(u_n)/|u_n|_0^{p-1}$. From (5.4) it clearly follows that $\{h_n\}_{n=1}^{\infty}$ is a bounded sequence in C^0. This, (5.6), and the fact that G_p: $C^0 \to C^0$ is completely continuous imply that $\{v_n\}_{n=1}^{\infty}$ possesses a convergent subsequence in C^0. Let us denote this subsequence again by $\{v_n\}_{n=1}^{\infty}$ and let $v \equiv \lim_{n \to \infty} v_n$. Then $|v|_0 = 1$. Calling again on the fact that $\{h_n\}_{n=1}^{\infty}$ is bounded on C^0 and hence on L^q, $q > 1$, and reasoning as in the proof of lemma 4.2 of [1] (see also [3]), we find that $\{h_n\}_{n=1}^{\infty}$ possesses a weakly convergent subsequence $\{h_{n_j}\}_{j=1}^{\infty}$ in L^q such that $h_{n_j} \rightharpoonup \theta \phi_p(v)$ as $j \to \infty$. Here θ is a real valued measurable function defined on $[0, T]$ such that

$$a(t) \leqslant \theta(t) \leqslant b(t) \qquad \text{a.e.} \quad \text{on } [0, T]. \tag{5.7}$$

From (5.6)

$$v_{n_j} = G_p(\tau_{n_j} v \phi_p(v_{n_j}) + (1-\tau_{n_j}) h_{n_j}) \tag{5.8}$$

for $j \in \mathbb{N}$. From Proposition (2.2) we can let j go to ∞ in (5.8) to obtain

$$v = G_p(c\phi_p(v)), \tag{5.9}$$

where

$$c(t) = \tau_0 \nu + (1 - \tau_0)\, \theta(t). \tag{5.10}$$

Equation (5.9) implies that v satisfies

$$(\phi_p(v'))' + c(t)\, \phi_p(v) = 0 \tag{5.11}$$

$$v(0) = v(T) = 0. \tag{5.12}$$

From (5.11), (5.12), and since $c(t)$ as defined in (5.10) satisfies (5.1) of Proposition 5.1, it follows from the latter that $v(t) = 0$ for all $t \in [0, T]$. This contradicts $|v|_0 = 1$, and the claim follows.

Next, from the invariance of Leray–Schauder degree under homotopies (see [9]), we find that for a big enough $r > 0$

$$d(I - H(0, \cdot), B(0, r), 0) = d(I - H(1, \cdot), B(0, r), 0). \tag{5.13}$$

Since $u - H(1, u) = u - G_p(\nu\phi_p(u)) = T_p(u)$, $\nu \in (\lambda_k(p),\ \lambda_{k+1}(p))$, and $H(0, \cdot) = G_p \circ F$ it follows from Theorem 4.1 and (5.13) that

$$d(I - G_p \circ F, B(0, r), 0) = (-1)^k \neq 0. \tag{5.14}$$

In accordance with the Leray–Schauder degree existence result (see [9]), we conclude from (5.14) that there exists a $u \in B(0, r)$ such that

$$u - G_p(F(u)) = 0. \tag{5.15}$$

Hence the theorem follows. ∎

Remark 2. Condition (5.4) of Theorem 5.2 generalizes (E_l) of Theorem 2.1 of [1]. Furthermore, (E_0) of that theorem can be generalized to

$$\limsup_{|s| \to \infty} \frac{f(t, s)}{\phi_p(s)} \leqslant \lambda_1 \tag{5.16}$$

uniformly on $[0, T]$, and the strict inequality holding true in a subset of positive measure in $[0, T]$, and still problem (D) will have a solution.

Remark 3. Conditions (5.4) and (5.16) can be thought of as non-uniform nonresonant conditions at the eigenvalues for solutions of problem (D). In addition, (5.16) can be replaced by the weaker condition

$$\limsup_{|s| \to \infty} \frac{p}{|s|^p} \int_0^s f(t, w)\, dw \leqslant \lambda_1 \tag{5.17}$$

uniformly on $[0, T]$, the strict inequality holding true in a subset of positive measure in $[0, T]$, without affecting the existence of at least one solution to problem (D). Conditions (5.4) and (5.17) generalize for such problems some of the main ideas used in [7, 8] for the pde case and $p = 2$.

Theorem 5.2 does not guarantee the existence of nontrivial solutions when $f(t, 0) = 0$, for all $t \in [0, T]$. It is the purpose of our next theorem to provide, in the spirit of our work, additional conditions to the ones in Theorem 5.2 for problem (D) to have nontrivial solutions.

THEOREM 5.3. *Suppose that besides the conditions of Theorem 5.2 for the function f, there exists a positive integer j such that $(j-k)$ is an odd integer and*

$$\lambda_j \leqslant \liminf_{s \to 0} \frac{f(t, s)}{\phi_p(s)} \leqslant \limsup_{s \to 0} \frac{f(t, s)}{\phi_p(s)} \leqslant \lambda_{j+1} \tag{5.18}$$

uniformly on $[0, T]$, the first and the last inequalities being strict in some subsets of positive measure in $[0, T]$. Then problem (D) possesses a nontrivial solution.

Proof. We have to prove that (2.10) possesses a nontrivial solution. Since the proof of this fact is very much like that of the last theorem we will only sketch it.

Reasoning as in the proof of Theorem 5.2, we find that for a small enough $\varepsilon > 0$

$$d(I - G_p \circ F, B(0, \varepsilon), 0) = (-1)^j. \tag{5.19}$$

Since for a big enough $r > 0$ (5.14) holds, then from the additivity and excision properties of the Leray–Schauder degree (see [9]), we obtain

$$d(I - G_p \circ F, B(0, r) \backslash \overline{B(0, \varepsilon)}, 0) = (-1)^k - (-1)^j \neq 0. \tag{5.20}$$

Hence, (2.10) has a solution belonging to $B(0, r) \backslash \overline{B(0, \varepsilon)}$, and the theorem follows. ∎

We conclude this section showing by means of an example that, concerning the existence of solutions to problem (D), the case $p \neq 2$, $p > 1$, may be quite different from the case $p = 2$. In fact, let us consider the problem

$$-(\phi_p(u'))' = \lambda \phi_p(u) + h \tag{5.21}$$

$$u(0) = u(T) = 0, \tag{5.22}$$

where $p > 1$ and $h \in C^0$. For $p = 2$, (5.21) reduces to

$$-u'' = \lambda u + h. \tag{5.23}$$

It is well known that (5.23) subjected to (5.22) has a unique solution u for each $h \in C^0$ when λ is less than the first eigenvalue $\lambda_1(2) = (\pi/T)^2$. Furthermore, at this solution the strictly convex functional

$$\Phi_2(z) = \frac{1}{2}\int_0^T |z'|^2 - \frac{\lambda}{2}\int_0^T |z|^2 - \int_0^T hz \tag{5.24}$$

reaches its minimum over $W_0^{1,2}$.

Similarly, for each $\lambda < \lambda_1(p)$ and each $h \in C^0$, (5.21) subjected to (5.22) possesses a solution at which the functional

$$\Phi_p(z) = \frac{1}{p}\int_0^T |z'|^p - \frac{\lambda}{p}\int_0^T |z|^p - \int_0^T hz \tag{5.25}$$

reaches its minimum over $W_0^{1,p}$. Nevertheless, if $0 < \lambda < \lambda_1(p)$ and $p > 2$ we can always find an $h \in C^0$ such that (5.21) subjected to (5.22) possesses at least two solutions. To prove this, let $u_0 \in C^2$ be a function which is equal to a constant different from zero on $[\varepsilon, T-\varepsilon]$, for some small $\varepsilon > 0$ and such that $u(0) = u(T) = 0$.

Define $h \in C^0$ by

$$h = -(\phi_p(u_0'))' - \lambda\phi_p(u_0). \tag{5.26}$$

Then u_0 is a solution to (5.21), (5.22) for this h. We claim that at u_0, Φ_p does not reach its minimum in $W_0^{1,p}$ and therefore there are two solutions of (5.21), (5.22). To prove the claim we note that for $p > 2$, Φ_p is twice Frechet differentiable. If we denote the second derivative of Φ_p at u_0 by $\Phi_p''(u_0)$ then

$$\langle \Phi_p''(u_0)v, v\rangle = (p-1)\left(\int_0^T |u_0'|^{p-2}\, v'^2 - \frac{\lambda}{2}\int_0^T |u_0|^{p-2}\, v^2\right) \tag{5.27}$$

for all $v \in W_0^{1,p}$, and where $\langle\ ,\ \rangle$ denotes the duality pairing between $W_0^{1,p}$ and its dual.

Next let $z \in C_0^\infty$ be such that $\operatorname{supp} z \subset (\varepsilon, T-\varepsilon)$. We find from (5.27) and the definition of u_0 that

$$\langle \Phi_p''(u_0)z, z\rangle = -(p-1)\frac{\lambda}{2}\int_0^T |u_0|^{p-2} z^2 < 0, \tag{5.28}$$

which shows that at u_0 Φ_p does not reach its minimum.

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