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Infinitely Many T-Periodic Solutions for a Problem Arising in Nonlinear Elasticity*

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1. INTRODUCTION

In [12], the equation governing radial nonlinear vibrations of a radially forced thickwalled hollow sphere made of an elastic, homogeneous, isotropic, and incompressible material was derived.

The hollow sphere is forced with a time dependent radially symmetric pressure difference $p(t) = p_1(t) - p_2(t)$, $p_1(1)$, $p_2(t)$ being, respectively, the external pressure at the inner and outer surfaces of the sphere, as functions of the time t.

Let r denote the distance to the origin of a generic point of the sphere at its unstrained state. If r_1, r_2 are, respectively, the inner and outer radius of the sphere, then $r_1 \leqslant r \leqslant r_2$. Let $R(t, r)$ denote the radial position, at time t, of a point r of the sphere. The incompressibility condition tells us that

$$R^3(t, r) - R^3(t, r_1) = r^3 - r_1^3. \tag{1.1}$$

Hence the knowledge of $R_1(t) \equiv R(t, r_1)$ allows us to determine $R(t, r)$ for any $r \in [r_1, r_2]$.

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Let $u(t) = R_1(t)/r_1$ and assume that the material is characterized by an elastic potential $W(I_1, I_2)$ of the Mooney–Rivlin type, i.e.,

$$W(I_1, I_2) = \frac{\alpha^*}{2}(I_1 - 3) + \frac{B^*}{2}(I_2 - 3), \tag{1.2}$$

where I_1, I_2 are strain invariants and α^* and β^* are constants.

It follows from the results of [12] that under these conditions $u(t)$ satisfies the ordinary equation

$$\frac{d^2}{dt^2}(\Psi(u)) = \frac{u^2(p(t) - g(u))}{2\Psi'(u)}, \tag{1.3}$$

where

$$\Psi(s) = \int_0^s \left(1 - \frac{\tau}{(\mu + \tau^3)^{1/3}}\right)^{1/2} \tau^{3/2}\, d\tau, \tag{1.4}$$

$\mu = (r^2/r_1)^3 - 1$, and

$$g(s) = \int_{1/s}^{((\mu+1)/(\mu+s^2))^{1/3}} (1 + t^3)\left(\alpha^* + \frac{\beta^*}{t^2}\right) dt. \tag{1.5}$$

Clearly, Ψ defined by (1.4) is a diffeomorphism of $\mathbb{R}^+$. Hence, if we make the change of variables $x = \Psi(u)$ (1.3) becomes

$$\text{(E)} \quad \frac{d^2x}{dt^2} + F(t, x) = 0, \tag{1.6}$$

where

$$F(t, s) = \frac{(g(\Psi^{-1}(s)) - p(t))[\Psi^{-1}(s)]^2}{2\Psi'(\Psi^{-1}(s))}. \tag{1.7}$$

In Section 3 we will see that F satisfies the following asymptotic estimates:

(i) $F(t, s) = c_1 s^3 + o(s^3)$ for s near $+\infty$ and uniformly in t;

(ii) $F(t, s) = -c_2 s^{-7/5} + o(s^{-7/5})$ for s near 0^+ and uniformly in t

where c_1 and c_2 are certain positive constants. Hence F exhibits superlinear behaviour near $s = +\infty$ and a strong singularity at $s = 0^+$.

Let us assume that p is T-periodic. $T > 0$. We are interested in the problem of existence of T-periodic solutions of (E). We will actually study a more general problem which is motivated by (1.6). Thus, let $f: \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ be a continuous function satisfying the conditions

$$-\infty \leqslant \limsup_{s \to 0^+} sf(t, s) < 0 \tag{1.8}$$

and

$$\lim_{s \to +\infty} \frac{f(t, s)}{s} = +\infty, \tag{1.9}$$

both limits uniform in t. ote that (i) and (ii) imply that F indeed satisfies (1.8) and (1.9).

In Section 2 we will prove that under (1.8), (1.9) and some general additional conditions on f the problem

$$x'' + f(t, x) = 0 \tag{1.10}$$

has infinitely many positive T-periodic solutions. The proof of this result will make use of a useful version of the Poincaré–Birkhoff Theorem due to W.-Y. Ding [4] together with an analysis of some oscillatory properties of the solutions due to initial value problem associated with (1.10). Jacobowitz [8] has used the Poincaré–Birkhoff Theorem to prove the existence of infinitely many 2π-periodic solutions for a superlinear problem of the form (1.10), but with f defined through the whole $\mathbb{R} \times \mathbb{R}$, i.e., f free of singularities and under the hypothesis $f(t, 0) \equiv 0$. Using a different method, Fucik and Lovicar [5] have shown the existence of at least one T-periodic solution in a superlinear problem without this last assumption. See also Willem [11] for a result on infinitely many periodic solutions of a fixed period for a problem like the one considered in [5]. We also refer to [1, 2] for other applications of the Poincaré–Birkhoff Theorem to second order problems.

Related to the results of this paper are also the works of Lazer and Solimini [9], Solimini [10], and Gaete and Manásevich [6] where singularities like (1.8) were considered but for which

$$\limsup_{s \to +\infty} \frac{f(t, s)}{s} \leqslant 0. \tag{1.11}$$

Intermediate behaviours between (1.9) and (1.11) have been considered in a paper of the present authors and A. Montero [3].

This paper is organized as follows. In Section 2 we will prove our main results. We will show in Theorem 2.1 that (1.8), (1.9) and some general assumptions on f, lead to the existence of two T-periodic solutions of (1.10), $x_n^{\pm}$, such that $x_n^{\pm} - 1$ has exactly $2n$ zeros in $[0, T)$, for all sufficiently large n. Also we will see in Theorems 2.2 and 2.3 that more precise results can be obtained in case that we know a particular T-periodic solution of (1.10).

In Section 3 we will apply these results to the elasticity problem (E). In particular, we will show that (E) possesses infinitely many T-periodic solutions if we assume the pressure $p(t)$ to be of class C^1. Furthermore we will "localize" a particular solution in case that the parameter β^* in the definition of g in (1.5) is sufficientlty small, so that Theorems 2.2 and 2.3 will be applicable to this situation.

2. Existence and Multiplicity of T-Periodic Solutions

In this section we consider the problem of finding T-periodic solutions of

$$x'' + f(t, x) = 0, \tag{2.1}$$

where $f: \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is assumed to be continuous, locally Lipschitz in x, T-periodic in t, and such that for $s, \beta \in \mathbb{R}$ and $\alpha > 0$ the local solution $x(t)$ of (2.1) satisfying

$$x(s) = \alpha, \qquad x'(s) = \beta \tag{2.2}$$

is continuable to the whole real line and $x(t) > 0$ for all $t \in \mathbb{R}$. We will denote this solution by $x(t, \alpha, \beta; s)$.

Our main purpose in this section is to prove the following result:

THEOREM 2.1. *Assume that f is as above and satisfies*

$$-\infty \leqslant \limsup_{s \to 0^+} sf(t, s) < 0 \tag{2.3}$$

$$\lim_{s \to +\infty} \frac{f(t, s)}{s} = +\infty \tag{2.4}$$

uniformly in t. Then there is a natural number n_0 such that for every $n \geqslant n_0$ there exist two T-periodic solutions $x_n^+(t)$, $x_n^-(t)$ of (2.1) such that $x_n^\pm(t) - 1$ has exactly $2n$ zeros in $[0, T)$. In particular, (2.1) possesses infinitely many T-periodic solutions.

We also show that more precise results can be obtained in case we know a particular T-periodic solutions of (2.1).

The proof of Theorem 2.1 will make use of some lemmas concerning properties of the solution to the I.V.P. (2.1)–(2.2) which we state and prove next.

In what remains of this section we will use the notations

$$r(a, b) \equiv \left(a^2 + \frac{1}{a^2} + b^2\right) \tag{2.5}$$

$$x(t, \alpha, \beta) \equiv x(t, \alpha, \beta; 0), \tag{2.6}$$

and will assume that f satisfies the hypotheses of Theorem 2.1.

LEMMA 2.1.

$$r(x(t, \alpha, \beta), x'(t, \alpha, \beta)) \to +\infty \qquad \textit{as} \quad r(\alpha, \beta) \to +\infty \tag{2.7}$$

uniformly in $t \in [0, T]$.

Proof. If (2.7) did not hold, there would exist sequence $\alpha_n > 0$, $\beta_n \in \mathbb{R}$, $t_n \in [0, T]$ such that $r(x(t_n, \alpha_n, \beta_n), x'(t_n, \alpha_n, \beta_n))$ is bounded, but $r(\alpha_n, \beta_n) \to +\infty$.

Observe that the application

$$(\tilde{\alpha}, \tilde{\beta}, t, s) \mapsto (x(t, \tilde{\alpha}, \tilde{\beta}, s), x'(t, \tilde{\alpha}, \tilde{\beta}, s)) \equiv M(\tilde{\alpha}, \tilde{\beta}, t, s)$$

is continuous on $\mathbb{R}^+ \times \mathbb{R}^3$ and maps into $\mathbb{R}^+ \times \mathbb{R}$.

Let $\tilde{\alpha}_n = x(t_n, \alpha_n, \beta_n)$, $\tilde{\beta}_n = x'(t_n, \alpha_n, \beta_n)$. Then the sequence $(\tilde{\alpha}_n, \tilde{\beta}_n, 0, t_n)$ lies on a compact subset of $\mathbb{R}^+ \times \mathbb{R}^3$. The continuity of M implies that the sequence $M(\tilde{\alpha}_n, \tilde{\beta}_n, 0, t_n) = (\alpha_n, \beta_n)$ lies on a compact subset of $\mathbb{R}^+ \times \mathbb{R}$ and hence $r(\alpha_n, \beta_n)$ is bounded, a contradiction concluding the proof. ∎

As an immediate consequence of Lemma 2.1 we obtain

LEMMA 2.2. *If* $r(\alpha, \beta)$ *is sufficiently large,* $\alpha < 0$, $\beta \in \mathbb{R}$, *then for all* $t \in [0, T]$, $x(t, \alpha, \beta) = 1$ *implies* $x'(t, \alpha, \beta) \neq 0$. *In other words, the zeros of* $x(t, \alpha, b) - 1$ *are simple.*

For a function $y \in C^1[0, T]$ having only simple zeros, we define its *rotation number* $\psi(y)$ as

$$\psi(y) = k\pi + \lim_{t \to 0_+} \tan^{-1} \frac{y'(t)}{y(t)} - \lim_{t \to T_-} \tan^{-1} \frac{y'(t)}{y(t)}, \tag{2.8}$$

where k is the number of zeros of $y(t)$ in $(0, T)$. Geometrically, $\psi(y)$ represents the total angle the vector from the origin to the point $(y(t), y'(t))$ in $\mathbb{R}^2$ describes as t goes from 0 to T, positive angles measured clockwise.

From Lemma 2.2, the quantity

$$\eta(\alpha, \beta) \equiv \psi(x(t, \alpha, \beta) - 1) \tag{2.9}$$

is well defined provided that $r(\alpha, \beta)$ is sufficiently large, $\alpha > 0$, $\beta \in \mathbb{R}$.

LEMMA 2.3. *Fix a positive number* ε *and, according to* (2.3)–(2.4), *choose positive numbers* δ, c, M *such that for all* $t \in \mathbb{R}$

$$f(t, s) < -\frac{c}{s} \qquad \textit{for} \quad 0 < s < \delta \tag{2.10}$$

and

$$f(t,s) > \left(\frac{\pi}{\varepsilon}\right)^2 s \qquad \textit{for} \quad s > M. \tag{2.11}$$

Then there is an $R > 0$ such that for $r(\alpha, \beta) > R$ and $t_0 \in [0, T]$:

(i) *If $x(t_0, \alpha, \beta) \leqslant \delta$ and $x'(t_0, \alpha, \beta) < 0$ (> 0), then there is a number t_1, $t_1 > t_0$ ($t_1 < t_0$), such that $|t_1 - t_0| < \varepsilon$, $x(t_1, \alpha, \beta) = \delta$, and $x'(t_1, \alpha, \beta) >$ ($<$) 0.*

(ii) *If $x(t_0, \alpha, \beta) \geqslant M$ and $x'(t_0, \alpha, \beta) > 0$ (<0), then there is a number t_1, $t_1 > t_0$ ($t_1 < t_0$) such that $|t_1 - t_0| < \varepsilon$, $x(t_1, \alpha, \beta) = M$, and $x'(t_1, \alpha, \beta) <$ ($>$) 0.*

Proof. Write, for brevity, $x(t) = x(t, \alpha, \beta)$ and assume that $x(t_0) \leqslant \delta$ and $x'(t_0) = -\lambda < 0$. Since $x'' = -f(t, x)$, from (2.10) we have that on any interval $[t_0, a)$ where $x(t) \leqslant \delta$ there holds:

$$x(t) \geqslant x(t_0) - \lambda(t - t_0) + \frac{c}{2\delta}(t - t_0)^2 \qquad \text{for} \quad t \in [t_0, a). \tag{2.12}$$

It follows that x must take the value δ at some later time t_1. Choose t_1 to be the first of those instants. Note that x is strictly convex on $[t_0, t_1]$ and that $x'(t_1, \alpha, \beta) < 0$. Let t^* be the unique number in (t_0, t_1) such that $x'(t^*) = 0$ and denote by x^{-1} the inverse of the restriction of x to $(t^*, t_1]$. Multiplying (2.1) by x' and integrating between t^* and $t \in [t^*, t_1]$ we obtain

$$\frac{x'(t)^2}{2} = \int_{\bar{x}}^{x(t)} f(x^{-1}(s), s)\, ds,$$

where $\bar{x} = x(t^*)$. From this and (2.10) it follows that

$$t_1 - t^* = \int_{\bar{x}}^{\delta} \frac{d\tau}{\left(-2\int_{\bar{x}}^{\tau} f(x^{-1}(s), s)\, ds\right)^{1/2}} < \int_{\bar{x}}^{\delta} \frac{d\tau}{(2c\log(\tau/\bar{x}))^{1/2}}$$
$$= \bar{x}\int_1^{\delta/\bar{x}} \frac{d\tau}{(2c\log\tau)^{1/2}}.$$

The last integral is less than $\varepsilon/2$ if $\bar{x}$ is less than some $\rho > 0$ sufficiently smalll. But this necessarily if $r(\alpha, \beta)$ is large enough, as follows from Lemma 2.1. Similarly, $t^* - t_0 < \varepsilon/2$ and we obtain $t_1 - t_0 < \varepsilon$ for every large $r(\alpha, \beta)$, as desired.

Now assume $x(t_0) \geqslant M$, $x'(t_0) > 0$. As above, using now (2.11), we see that $x(t_1) = M$ for some $t_1 < t_0$ with $x(t) > M$, $t \in (t_0, t_1)$. If $t^* \in (t_0, t_1)$ is such that $x(t^*) = 0$, setting again $\bar{x} = x(t^*)$ and reasoning as above we obtain

$$t_1 - t_0 < \frac{2\varepsilon}{\pi} \int_M^{\bar{x}} \frac{ds}{(\bar{x}^2 - s^2)^{1/2}} < \varepsilon,$$

as desired. Also, necessarily $x'(t_1) < 0$. The other cases follow by changing the variable t by $T - t$. ∎

LEMMA 2.4. *Given $\varepsilon > 0$, there is an $R > 0$ such that for every $s \in [0, T]$ there exists an s^* with $0 < |s^* - s| < \varepsilon$ and $x(s^*, \alpha, \beta) = 1$, provided $r(\alpha, \beta) > R$.*

Proof. Let δ, M be as in Lemma 2.3, and assume $\delta < 1 < M$. From Lemma 2.1, we can choose $R > 0$ such that $r(\alpha, \beta) > R$, $\delta < x(t, \alpha, \beta) < M$, $t \in [0, T]$, imply

$$|x'(t, \alpha, \beta)| > \frac{3(M - \delta)}{\varepsilon}. \tag{2.13}$$

Assume $\delta < x(s, \alpha, \beta) < M$, $x'(s, \alpha, \beta) > 0$, and $r(\alpha, b) > R$. Inequality (2.13) implies the existence of a point $t_0 > s$ with $x(t_0, \alpha, \beta) = M$, $x'(t_0, \alpha, \beta) > 0$, and $t_0 - s < \varepsilon/3$. From Lemma 2.3, there is a $t_1 > t_0$ with $t_1 - t_0 < \varepsilon/3$ and $x(t_1, \alpha, b) = M$, $x'(t_1, \alpha, \beta) < 0$. Inequality (2.13) yields the existence of $s^* > t_1$ with $s^* - t_1 < \varepsilon/3$ and $x(s^*, \alpha, \beta) = 1$. Then $|s^* - s| < \varepsilon$ as required.

The proof of the remaining cases is similar, using Lemma 2.3 in its full strength. ∎

Lemmas 2.3 and 2.4 imply, in particular, that the number of zeros of $x(t, \alpha, \beta) - 1$ on $[0, T]$ becomes arbitrarily large as $r(\alpha, \beta)$ grows. This implies the validity of the following

Proposition 2.1. *$\eta(\alpha, \beta) \to +\infty$ as $r(\alpha, \beta) \to +\infty$.*

This fact is the key ingredient needed to apply Ding's version of the Poincaré–Birkhoff Theorem. For the convenience of the reader we state Ding's result in the following lemma.

LEMMA 2.5 [4]. *Let A denote an annular region whose inner boundary C_1 and outer boundary C_2 of A are simple curves. Denote by D_i the open region bounded by C_i, $i = 1, 2$. Let $W: A \to W(A) \subset \mathbb{R}^2 \setminus \{0\}$ be an area-preserving homeomorphism. Suppose that*

(1) *The inner boundary curve C_1 is star shaped about the origin.*

(2) *W has a lifting $\tilde{W}$ to the polar coordinates plane, that is, $\tilde{W}$ satisfies $P \circ \tilde{W} = W \circ P$, where $P(\rho, \theta) = (\rho \cos \theta, \ \rho \sin \theta)$ such that if $\tilde{W}(\rho, \theta) = (R(\rho, \theta), \Theta(\rho, \theta))$ then $\Theta(\rho, \theta) - \theta > 0$ (<0) on $P^{-1}(C_1)$* and $\Theta(\rho, \theta) - \theta < 0$ (>0) on *$P^{-1}(C_2)$. The functions R and Θ are continuous and 2π periodic in θ.*

(3) *W can be extended as an area-preserving homeomorphism $W: \bar{D}_2 \to \mathbb{R}^2$ so that $0 \in W(D_1)$.*

Then $\tilde{W}$ has at least two fixed points such that their images under P are two different fixed points of W.

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Consider the operator $W: (-1, \infty) \times \mathbb{R} \to (-1, \infty) \times \mathbb{R}$ defined by

$$W(\alpha, \beta) \equiv (x(T, 1+\alpha, \beta) - 1, x'(T, \alpha+1, \beta)). \tag{2.14}$$

Standard arguments show that W is an area-preserving homeomorphism. Observe also that $x(t, \ 1+\alpha, \beta)$ is a T-periodic solution of (2.1) if and only if (α, β) is a fixed point of W.

Fix a positive number r_0 so large that $\eta(1+\alpha, \beta)$ is well defined for all (α, β) with $\alpha > -1$ and $r(1+\alpha, \beta) \geqslant r_0$. From the continuity of η, there exists a positive integer n_0 such that $\eta(\alpha+1, \beta) < 2n_0\pi$ whenever $r(1+\alpha, \beta) = r_0$. Fix any integer N with $N \geqslant n_0$. Proposition 2.1 implies then that we can find a real number $r_1 > r_0$ such that

$$\begin{aligned} &\inf\{\eta(\alpha+1, \beta) \mid r(\alpha+1, \beta) = r_1\} > 2N\pi \\ &\quad \geqslant 2n_0\pi > \sup\{\eta(\alpha+1, \beta) \mid r(\alpha+1, \beta) = r_0\}. \end{aligned} \tag{2.15}$$

Let us define the curves C_1 and C_2 in $\mathbb{R}^2$ by

$$\begin{aligned} C_1 &= \{(\alpha, \beta) \in (-1, +\infty) \times \mathbb{R} \mid r(\alpha+1, \beta) = r_0\} \\ C_2 &= \{(\alpha, \beta) \in (-1, +\infty) \times \mathbb{R} \mid r(\alpha+1, \beta) = r_1\}. \end{aligned} \tag{2.16}$$

C_1 and C_2 are closed simple curves and, in particular, C_1 is star shaped around the origin. Thus define A to be the annular region between these two curves. It is standard that the restriction of W to A can be lifted to the polar coordinate plane through the usual covering map $P(\rho, \theta) = (\rho \cos \theta, \rho \sin \theta)$ rto a map $\tilde{W}$ satisfying the periodicidity condition

$$\tilde{W}(\rho, \theta + 2\pi) = \tilde{W}(\rho, \theta) + (0, 2\pi). \tag{2.17}$$

Then, if we write $\tilde{W}(\rho, \theta) = (R(\rho, \theta), \Theta(\rho, \theta))$, the definitions of W and the rotation number η, together with the continuity of $\tilde{W}$, imply the existence of an integer $\tilde{n}$ so that

$$\Theta(\rho, \theta) - \theta = 2\tilde{n}\pi - \eta(\alpha + 1, \beta), \tag{2.18}$$

for all $(\rho, \theta) \in P^{-1}(A)$, where $(\alpha, \beta) = P(\rho, \theta)$.

Now, for each integer n such that $n_0 \leqslant n \leqslant N$ define

$$\tilde{W}_n(\rho, \theta) \equiv \tilde{W}(\rho, \theta) + (0, 2(n - \tilde{n})\pi), \tag{2.19}$$

and observe that $\tilde{W}_n$ is still a lifting of W via the covering map P. From this definition, (2.15), and (2.18), it is clear that $\tilde{W}_n$ satisfies condition (2) of Lemma 2.5. The only thing left is to verify that $0 \in W(D_1)$. But this is equivalent to saying that if (α, β) is such that $x(T, \alpha, \beta) = 1$, $x'(T, \alpha, \beta) = 0$ then $r(\alpha, \beta) < r_0$. This obviously holds if we choose r_0 sufficiently large.

It follows from Lemma 2.5 that $\tilde{W}_n$ possesses two fixed points yielding two distinct fixed points $(\alpha_n^{\pm}, \beta_n^{\pm})$ of W. Thus $x(t, \alpha_n^{\pm} + 1, \beta_n^{\pm})$ are two different T-periodic solutions of (2.1). Observing that $\eta(\alpha_n^{\pm} + 1, \beta_n^{\pm}) = 2n\pi$ we find that $x(t, \alpha_n^{\pm} + 1, \beta_n^{\pm}) - 1$ has exactly $2n$ zeros in $[0, T)$.

Since this happens for $n = n_0, ..., N$ and N can be chosen arbitrarily large, the theorem follows. ∎

Next assume that we know a particular T-periodic positive solution $z(t)$ of (2.1). In this case the result of Theorem 2.1 can be strengthened as we will show next.

First, defining $h(t) \equiv u(t) - z(t)$, we see that (2.1) is equivalent to

$$h'' + \tilde{f}(t, h) = 0, \tag{2.20}$$

where $\tilde{f}(t, h) = f(t, z(t) + h) - f(t, z(t))$. Then $h \equiv 0$ is a trivial solution of (2.20). Observe that, by local uniqueness, the zeros of a nontrivial solution of (2.20) must be simple. In particular, calling $h(t, \alpha, \beta)$ the solution of the I.V.P. associated to (2.20) such that $h(0) = \alpha > -z(0)$, $h'(0) = \beta$, from (2.8) we cen define its rotation

$$\tilde{\eta}(\alpha, \beta) \equiv \psi(h(\cdot, \alpha, \beta)). \tag{2.21}$$

Now define

$$\tilde{r}(\alpha, \beta) = r\left(\frac{\alpha}{z(0)} + 1, \beta\right) - r(1, 0).$$

By slightly modifying the arguments from which Proposition 2.1 was obtained, it follows

PROPOSITION 2.2. *$\tilde{\eta}(\alpha, \beta) \to +\infty$ as $\tilde{r}(\alpha, \beta) \to +\infty$.*

If we apply the Poincaré–Birkhoff Theorem as in the proof of Theorem 2.1, but now to the operator

$$V(\alpha, \beta) \equiv (h(T, \alpha, \beta), h'(T, \alpha, \beta)) \tag{2.22}$$

defined on the annular region

$$B = \{(\alpha, \beta) \in [-z(0), +\infty) \times \mathbb{R} \mid \varepsilon \leqslant \tilde{r}(\alpha, \beta) \leqslant r_1\} \tag{2.23}$$

for a large r_1 and a small $\varepsilon > 0$, we obtain the existence of pairs of T-periodic solutions of (2.23) $h_n^{\pm}(t)$ such that $\psi(h_n^{\pm}) = 2n\pi$, for every $n \geqslant n_0$, where n_0 is any integer such that

$$\sup\{\tilde{\eta}(\alpha, \beta) \mid \tilde{r}(\alpha, \beta) = \varepsilon\} < 2n_0\pi. \tag{2.24}$$

Assuming further that f in (2.1) is of class C^1 and linearizing (2.20) around the trivial solution, we see that for (α, β) near $(0, 0)$, $\tilde{\eta}(\alpha, \beta)$ must be close to the rotation of a nontrivial solution of

$$v'' + \frac{\partial f}{\partial s}(t, z(t))v = 0. \tag{2.25}$$

Hence, if for instance

$$\frac{\partial f}{\partial s}(t, z(t)) < \left\{\frac{2\pi n_0}{T}\right\}^2 \quad \text{for all} \quad t \in [0, T], \tag{2.26}$$

$n_0 \geqslant 1$, and since nontrivial solutions of $v'' + \{2\pi n_0/T\}^2 v = 0$ have rotation $2n_0\pi$ on $(0, T]$, we can take this n_0 in (2.24).

The above arguments can be summarized in the following

THEOREM 2.2. *Assume that f in* (2.1) *is of class C^1. If $z(t)$ is a particular T-periodic solution of* (2.1) *and $n_0 \in \mathbb{N}$, $n_0 \geqslant 1$, is such that* (2.26) *holds, then for all $n \geqslant n_0$ there exists a pair of T-periodic solutions of* (2.1), *$x_n^{\pm}(t)$ such that $x_n^{\pm}(t) - z(t)$ possesses exactly $2n$ zeros in* $[0, T)$.

We note that in case we can take $n_0 = 0$ in (2.26), Theorem 2.2 does not predict the existence of solutions $x_0^{\pm}$ such that $x_0^{\pm} - z$ does not have zeros. We will see, however. that we can still deal with this case, but now via degree-theoretical arguments. Moreover, the continuation assumption is not necessary for the next result.

THEOREM 2.3. *Assume the hypotheses of Theorem* 2.2 *hold except the continuation assumption, and that* (2.26) *holds with $n_0 = 0$. Then there exist two T-periodic solutions of* (2.1) *$x^+(t)$, $x^-(t)$ such that $x^+(t) - z(t)$ is strictly positive and $x^-(t) - z(t)$ is strictly negative.*

Proof. We have to show that (2.20) possesses a negative and a positive solution.

Denote by $C_T(\mathbb{R})$ the Banach space of all the real valued, continuous T-periodic real functions endowed with the sup norm.

Let $g \in C_T(\mathbb{R})$ and $R(g)$ be the unique T-periodic solution of the problem

$$v'' - v + g = 0.$$

It is well known that R defines a compact linear operator of $C_T(\mathbb{R})$ into itself. Then the problem of finding T-periodic solutions of (2.20) is equivalent to the fixed point problem:

$$h = R(\tilde{f}(\cdot, h) + h) \equiv G(h). \tag{2.27}$$

Let $B(0, \delta)$ be the ball center 0 radius δ in $C_T(\mathbb{R})$. Since $(\partial\tilde{f}/\partial s)(t, 0)$ for all t, we see that for δ small enough

$$\deg(I - T, B(0, \delta), 0) = 1. \tag{2.28}$$

Here and henceforth deg denotes the Leray–Schauder degree.

Next, for $\rho > 0$, $\varepsilon > 0$, $\rho < \min_{t \in [0, T]} z(t)$ define

$$\Omega_{\rho,\varepsilon} = \{h \in C_T(\mathbb{R}) \mid -z(t) + \rho < h(t) < \varepsilon \text{ for all } t \in [0, T]\}. \tag{2.29}$$

Then $\Omega_{\rho,\varepsilon}$ is an open and bounded subset of $C_T(\mathbb{R})$. We will see that for some suitable ρ, ε,

$$\deg(I - G, \Omega_{\rho,\varepsilon}, 0) = 0. \tag{2.30}$$

To do this, we consider the compact homotopy

$$G_\lambda(h) \equiv R(\tilde{f}(\cdot, h) + h - \lambda C), \qquad \lambda \in [0, 1], \tag{2.31}$$

where C is a positive constant, to be chosen later.

Let ε be any positive number such that $\tilde{f}(t, s) < 0$ for $0 < s < \varepsilon$ and all $t \in [0, T)$. Suppose $h \in C_T(\mathbb{R})$ is such that $-z(t) < h(t) \leqslant \varepsilon$ and for some $\lambda \in [0, 1]$ satisfies the equation

$$h = G_\lambda(h). \tag{2.32}$$

Then

$$h'' + \tilde{f}(t, h) - \lambda C = 0. \tag{2.33}$$

Let t_+ be a point where h attains its maximum. Then $h''(t_+) \leqslant 0$, and from (2.33) $\tilde{f}(t_+, h(t_+)) \geqslant 0$. This implies $h(t_+) \leqslant 0$, max $h \leqslant 0$.

Let us write $x = z + h$. Then x satisfies

$$x'' + f(t, x) - \lambda C = 0. \tag{2.34}$$

Let t_- be a point where x attains its minimum. Thus $x''(t_-) \geqslant 0$. If $x''(t_-) = 0$, then $f(t_-, x(t_-)) \geqslant 0$, and hence $x(t_-) \geqslant m_-$, where

$$m_- = \inf\{s < 0 \mid f(t, s) = 0 \text{ for some } t \in [0, T]\}. \tag{2.35}$$

Note that, from (2.3), $m_- > 0$.

Assume next $x''(t_-) \geqslant 0$. We have $x'(t_-) = 0$. Let $t^* > t_-$ be the next point where x' vanishes. Then necessarily $x(t^*) > x(t_-)$. Multiplying (2.34) by x' and integrating between t_- and t^*,

$$0 = \int_{t_-}^{t^*} (f(t,x) - \lambda C)\, x'\, dt = \int_{x(t_-)}^{x(t^*)} (f(x^{-1}(s), s) - \lambda C)\, ds. \tag{2.36}$$

This implies

$$0 \leqslant \int_{x(t_-)}^{m_-} f(x^{-1}(s), s)\, ds + \int_{m_-}^{\|z\|_\infty} |f(x^{-1}(s), s)|\, ds. \tag{2.37}$$

since $\max(x - z) = \max h \leqslant 0$. Now, we have that $-f(t, s) \geqslant c/s - k$ for all $0 < s < m_-$ and $t \in [0, T]$ and some $c, k > 0$. Inequality (2.37) implies

$$c \log\left(\frac{m_-}{x(t_-)}\right) \leqslant km_- + A\, \|z\|_\infty, \tag{2.38}$$

where $A = \max\{|f(t, s)| \mid t \in [0, T],\ s \in [m_-, \|z\|_\infty]\}$. Hence, $x(t_-) > \rho$ for some positive constant ρ depending only of f and z. It follows that $-z(t) + \rho < h(t) < \varepsilon$ for all t, that is, $h \in \Omega_{\rho,\varepsilon}$. We conclude that any solution of (2.32) in $\bar{\Omega}_{\rho,\varepsilon}$ must be in $\Omega_{\rho,\varepsilon}$ for all $\lambda \in [0, 1]$. It follows that $\deg(I - G_\lambda, \Omega_{\rho,\varepsilon}, 0)$ is well defined and it is constant for $\lambda \in [0, 1]$. Note that $G_0 = G$ and that a zero of $I - G$ is a T-periodic solution of

$$h'' = C - \tilde{f}(t, h). \tag{2.39}$$

Choose $C > \max\{|\tilde{f}(t, s)| \mid 0 \leqslant s \leqslant \varepsilon\}$. Then

$$0 = \int_0^T h'' = \int_0^T (C - \tilde{f}(t, h)) > 0,$$

a contradiction showing that (2.31) cannot have any solution in $\Omega_{\rho,\varepsilon}$ and hence

$$0 = \deg(I - G_1, \Omega_{\rho,\varepsilon}, 0) = \deg(I - G, \Omega_{\rho,\varepsilon}, 0)$$

and (2.30) holds. It follows from (2.28), (2.30), and the excision property of the degree that (2.27) possesses a solution h not identically zero in $\Omega_{\rho,\varepsilon}$ which must satisfy $h \leqslant 0$. Local uniqueness shows that actually $h<0$ and the first half of the theorem follows.

The existence of a strictly positive solution of (2.27) follows similarly. In fact, for $0<\varepsilon<B$, denote

$$\Lambda_{\varepsilon,B} = \{h \in C_T(\mathbb{R}) - \varepsilon < h(t) < B\}.$$

Choose ε such that $\tilde{f}(t,s)>0$ if $-\varepsilon<s<0$. Arguing as above we can prove that if $h \geqslant -\varepsilon$ solves (2.27), then $h \geqslant 0$ and $h<m_+ + k$ for some $k>0$ independent of h, where

$$m_+ = \max\{s>0 \mid f(t,s)=0 \text{ for some } t\}.$$

Then, if $B=m_+ + k$, we obtain that

$$\deg(I-G, \Lambda_{\varepsilon,B}, 0)=0 \tag{2.40}$$

and the existence of a nontrivial positive solution of (2.27) follows from (2.28) and (2.40) fisnishing the proof of the theorem. ▮

3. Problem (E)

In this section we apply the results of Section 2 to problem (E). We start by showing that (E) satisfies the hypotheses of Theorem 2.1.

Lemma 3.1. *The function $F(t,s)$ in* (1.7) *satisfies*

(i) $F(t,s)=c_1 s^{-7/5}+o(s^{-7/5})$ *near* $s=0$,

(ii) $F(t,s)=c_2 s^3+o(s^3)$ *near* $s=\infty$

for some positive constants c_1, c_2 and uniformly in $t \in [0,T]$.

Proof. By direct integration, we find that g in (1.5) satisfies

$$g(s)=\begin{cases}\theta s+o(s) & \text{near } \quad s=+\infty \\ -\alpha s^4+o(s^4) & \text{near } \quad s=0,\end{cases} \tag{3.1}$$

where $\theta=\beta(1-1/(\mu+1)^{1/3})$. Wer see also that Ψ defined by (1.4) has the following properties:

(i) $\Psi(s)$ is strinctly increasing;

(ii) $\Psi'(s)=s^{3/2}+o(s^{3/2})$ near $s=0$;

(iii) $\lim_{s\to\infty} \Psi'(s)=(\mu/3)^{1/2}$.

It follows that

$$\Psi^{-1}(s)=\begin{cases}(3/\mu)^{1/2}\,s+o(s) & \text{near}\quad s=+\infty\\ (5/2)\,s^{2/5}+o(s^{2/5}) & \text{near}\quad s=0.\end{cases} \tag{3.2}$$

Therefore,

$$\frac{\Psi^{-1}(s)^2}{\Psi'(\Psi^{-1}(s))}=\begin{cases}(3/\mu)^{3/2}\,s^2+o(s^2) & \text{near}\quad s=+\infty\\ (5/2)^{1/2}\,s^{1/5}+o(s^{1/5}) & \text{near}\quad s=0.\end{cases} \tag{3.3}$$

Also,

$$g(\Psi^{-1}(s))=\begin{cases}\theta(3/\mu)^{1/2}\,s+o(s) & \text{near}\quad s=+\infty\\ -\alpha(2/5)^4\,s^{-8/5}+o(s^{-8/5}) & \text{near}\quad s=0.\end{cases} \tag{3.4}$$

Hence,

$$2F(t,s)=\frac{\Psi^{-1}(s)^2}{\Psi'(\Psi^{-1}(s))}\,(g(\Psi^{-1}(s))-p(t))$$

$$=\begin{cases}\theta(3/\mu)\,s^3+o(s^3) & \text{near}\quad s=+\infty\\ -\alpha(2/5)^{7/2}\,s^{-7/5}+o(s^{-7/5}) & \text{near}\quad s=0\end{cases} \tag{3.5}$$

as desired. ∎

From this lemma we see that F satisfies (2.3) and (2.4).

Next we will show that the hypothesis of continuation of the solutions to the initial value problem (2.1)–(2.2) with F in the place of f is satisfied if we assume the pressure $p(t)$ to be a C^1 function. This will be a direct consequence of the following general lemma.

LEMMA 3.2. *If f in (2.1) is of class C^1 and satisfies*

(i) $\lim_{s\to\infty}\int_1^s f(t,\tau)\,d\tau=+\infty$ *uniformly in* $t\in[0,T]$.

(ii) $\lim_{s\to 0_+}\int_1^s f(t,\tau)\,d\tau=+\infty$ *uniformly in* $t\in[0,T]$.

(iii) *There exists a locally integrable, nonnegative function $a(t)$ such that*

$$\int_1^s \frac{\partial f}{\partial t}(t,\tau)\,d\tau\leqslant\alpha(t)\left\{\int_1^s f(t,\tau)\,d\tau+1\right\}$$

for all $s>0$. Then the local solutions of the initial value problem (2.1)–(2.3) *can be continued to the whole real line.*

Proof. Define

$$c(t)=\frac{x'(t)^2}{2}+\int_1^{x(t)} f(t,\tau)\,d\tau, \tag{3.6}$$

where x is the local solution of (2.1) with $x(0)=\alpha$, $x'(0)=\beta$.

Assume that x cannot be continued after a finite time $t=t^*$. Observe then that $c(t)$ becomes unbounded as t approaches t^*. Now,

$$\int_1^{x(t)} \frac{\partial f}{\partial t}(t,\tau)\,d\tau \leqslant a(t)\{1+c(t)\} \leqslant a(t)\{M+c(t)\}, \tag{3.7}$$

where $M > -\inf c$, $M>1$ (observe that c is bounded below). But (3.7) implies that

$$c(t)\leqslant (M+c(0))\exp\left(\int_0^t a(\tau)\,d\tau\right)-M \tag{3.8}$$

which contradicts the unboundedness of $c(t)$ near t^*, concluding the proof. ∎

Note. Lemma 3.2 has been adapted from a similar result proved by Jacobowitz [8].

Next we check that by setting $f=F$ in Lemma 3.2, then (i), (ii), and (iii) are satisfied. Indeed, (i) and (ii) are immediate from Lemma 3.1. Now, from (3.3) and (3.5)

$$2\frac{\partial F}{\partial t}(t,s)=-p'(t)\frac{\Psi^{-1}(s)^2}{\Psi'(\Psi^{-1}(s))}\leqslant F(t,s) \tag{3.9}$$

for large s. Also,

$$\frac{\partial F}{\partial t}(t,s)\geqslant -C\geqslant F(t,s) \qquad \text{near} \quad s=0, \tag{3.10}$$

and hence (iii) follows from (3.9) and (3.10).

We conclude from the above discussion that Theorem 2.1 is applicable to problem (E).

Next we will see that if the parameter β in the definition of g in (1.5) is sufficiently small and the pressure $p(t)$ lies on a certain range, we can identify a particular T-periodic solution of (E), so that we can apply Theorem 2.2 to problem (E).

In what follows we will write $g(s)$ defined in (1.5) as $g(s,\beta)$.

LEMMA 3.3. (i) *$g(s,0)>0$ if $s>1$, $g(s,0)<0$ if $s<1$.*

(ii) *$\lim_{s\to 0_+} g(s,0) = -\infty$, $\lim_{s\to +\infty} g(s,0)=0$.*

(iii) *$g(s,0)$ has a unique critical point $\tilde{s}>0$ and $g(\tilde{s},0) = \max\{g(s,0) \mid s>0\}$.*

Proof. (i) and (ii) follow immediately from the definition of $g(s,0)$. By direct differentiation, we see that $(\partial g/\partial s)(s,0)=0$ is equivalent to

$$\frac{(1+((\mu+1)/(\mu+s^3)))}{(1+s^{-3})}\frac{s^4}{(\mu+s^3)^{4/3}} = \frac{1}{(\mu+1)^{1/3}}. \tag{3.11}$$

Call $h(s)$ the left-hand side of (3.11) and observe that

$$(\log h(s))' = s^2\mu\left\{\frac{7}{s^3(\mu+s^3)} - \frac{6}{(s^3+1)(1+2\mu+s^2)}\right\} \tag{3.12}$$

and that this last quantity is strictly positive for all $s>0$. This implies that (3.11) possesses at most one solution $\tilde{s}$, which is necessarily equal to the point where $g(s,0)$ attains its maximum value. ∎

Let $M=g(\tilde{s},0)$ and assume that $0<p(t)<M$ for all t. Observe that for $\beta>0$ sufficiently small, we can find points $s^-(\beta)<s^+(\beta)$ corresponding respectively to a local maximum and a local minimum of $g(\cdot,\beta)$, satisfying

$$\lim_{\beta\to 0} g(s^-(\beta),\beta)=M, \qquad \lim_{\beta\to 0} g(s^+(\beta),\beta)=0, \tag{3.13}$$

and $(\partial g/\partial s)(s,\beta)<0$ for $s\in(s^-(\beta),s^+(\beta))$. Thus, for $\beta>0$ sufficiently small we have,

$$g(s^+(\beta),\beta)<p(t)<g(s^-(\beta),\beta) \tag{3.14}$$

for all $t\in[0,T]$. It follows that

$$F(t,s^+(\beta))<0<F(t,s^-(\beta)) \tag{3.15}$$

for all $t\in[0,T]$, and hence $s^+(\beta)$, $s^-(\beta)$ represent respectively a supersolution and a subsolution of the T-periodic problem (E). From [7], we obtaion the existence of a T-periodic solution $z(t)$ of (E) such that $s^-(\beta)<z(t)<s^+(\beta)$ and we can apply Theorem 2.2 to this particular solution.

Finally, observe that if we assume small amplitude in the oscillation of $p(t)$, namely $|p(t)-\bar{p}|$ uniformly small for some constant $\bar{p}$, then we can replace $s^\pm(\beta)$ in (3.15) by numbers $\tilde{s}^\pm(\beta)$ still satisfying (3.15) and such that $(\partial F/\partial s)(t,s)<0$ for $s\in(\tilde{s}^-(\beta)(,\tilde{s}^+(\beta))$ so that $(\partial F/\partial s)(t,z(t))<0$ making Theorem 2.3 applicable to this case.

We summarize the results of this section in the following existence theorwem for problem (E).

THEOREM 3.1. *Assume that $p(t)$ is of Class C^1. Then*

(i) *There exists a natural number n_0 such that for every integer $n\geqslant n_0$ there exist two T-periodic solutions $x_n^\pm(t)$ of* (E) *such that $x_n^\pm(t)-1$ have exactly 2n zeros in $[0,T)$.*

(ii) *If $0<p(t)<\max\{g(s,0)\mid s>0\}$, then for every $\beta>0$ sufficiently small and numbers $s^-(\beta)<s^-(\beta)$ satisfying*

$$F(t,s^+(\beta))<0<F(t,s^-(\beta))$$

for all $t\in[0,T]$, there is a T-periodic solution $z(t)$ with $s^-(\beta)<z(t)<s^+(\beta)$ and solutions $x_n^\pm(t)$ of (E) *such that $x_n^\pm(t)-z(t)$ have exactly 2n zeros in $[0,T)$ for every $n\geqslant n_0$, where n_0 is any integer such that $(\partial f/\partial s)(t,s)<\{2\pi n_0/T\}^2$ for $t\in[0,T]$, $s\in(s^-(\beta),s^+(\beta))$.*

(iii) *There is an $\varepsilon>0$ such that if $0<\bar{p}<\max\{g(s,0)\mid s>0\}$ is such that $|p(t)-\bar{p}|<\varepsilon$ for all $t\in(0,T]$, then for all $\beta>0$ sufficiently small we can take $n_0=0$ in* (ii), *so that we can find solutions $x^\pm(t)$ of* (E) *such that $x_n^\pm(t)-z(t)$ are respectively positive and negative on $[0,T)$.*

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