Arch. Rational Mech. Anal. 137 (1997) 363–380. © Springer-Verlag 1997

On Nonlinear Parabolic Equations of Very Fast Diffusion

Panagiota Daskalopoulos & Manuel del Pino

Communicated by P. Rabinowitz

1. Introduction

Our aim in this paper is to provide both necessary and sufficient conditions for the solvability of the Cauchy problem

$$\frac{\partial u}{\partial t} = \operatorname{div}(u^{m-1}\nabla u) \quad \text{in } \mathbf{R}^N \times (0,T), \tag{1.1}$$

$$u(x,0) = f(x) \text{ in } \mathbf{R}^N, \tag{1.2}$$

where $T > 0$, $m < 0$ are given constants and f is a nonnegative, locally integrable function. Equation (1.1) can also be written as

$$\frac{\partial u}{\partial t} = \Delta\phi_m(u) \quad \text{in } Q_T, \tag{1.3}$$

where $\phi_m(u) = u^m/m$ and $Q_T = \mathbf{R}^N \times (0,T)$.

By a solution of (1.1), (1.2) we mean a nonnegative function u in $C([0,T); L^1_{\text{loc}}(\mathbf{R}^N))$ for which $\phi_m(u)$ belongs to $L^1_{\text{loc}}(\mathbf{R}^N \times [0,T))$ and which satisfies equation (1.3) in the distributional sense. Note that the assumption that $\phi_m(u)$ be locally integrable implies that u is nonzero almost everywhere in Q_T.

Equation (1.1) with $m < 0$ arises naturally in certain physical applications. For example, *superdiffusivities* of this type have been proposed by de Gennes [8] as a model for long-range Van der Waals interactions in thin films spreading on solid surfaces. This equation also appears in the study of cellular automata and interacting particle systems with *self-organized criticality*; see [6] and its references. Other physical applications are mentioned in the works [3] and [4].

In [11], Vázquez showed that if (1.1), (1.2) has a solution for some $T > 0$, then necessarily $f \notin L^1(\mathbf{R}^N)$. It is natural to ask what is the fastest possible decay of f at infinity if problem (1.1), (1.2) is solvable. A partial answer to this question has been recently provided in [7] where is shown the existence of a constant $C = C(m,N)$ such that if (1.1), (1.2) is solvable, then

$$\limsup_{R\to\infty} \frac{1}{R^{N-2/(1-m)}} \int_{B_R} f \geqq C T^{1/(1-m)}. \tag{1.4}$$

Here and in the sequel B_R denotes the ball of radius R centered at the origin. This condition is in some sense sharp, as seen from the explicit solution of (1.1) given by

$$v^T(x,t) = (2\alpha(T-t)_+|x|^{-2})^{1/(1-m)}, \qquad \alpha = (N - 2/(1-m)), \tag{1.5}$$

which exists exactly up to time T and which has initial data $f(x) = v^T(x,0)$ satisfying (1.4) with $C = C^*$ given by

$$C^* = \left[2\left(N - \frac{2}{1-m}\right)\right]^{1/(1-m)} \omega_N. \tag{1.6}$$

Here ω_N denotes the surface area of the unit sphere. It is interesting to observe that condition (1.4) is in correspondence with the *porous medium* or *slow diffusion* case $m > 1$. In fact, in that situation the solvability of (1.1), (1.2) implies that

$$\limsup_{R\to\infty} \frac{1}{R^{N-2/(1-m)}} \int_{B_R} f \leqq C T^{1/(1-m)}, \tag{1.7}$$

for a certain constant C depending only on m and N, as follows from the Harnack estimate established by Aronson & Caffarelli [1]. On the other hand, Bénilan, Crandall & Pierre [2] showed that a growth assumption like (1.7) is also *sufficient* for existence. It is tempting to guess that a condition of the form (1.4), possibly replacing the lim sup by lim inf, is sufficient for existence when $m < 0$. As we shall see, this is the case under *radial symmetry*. More precisely, if f is radial and if

$$\liminf_{R\to\infty} \frac{1}{R^{N-2/(1-m)}} \int_{B_R} f \geqq C^* T^{1/(1-m)}, \tag{1.8}$$

then (1.1), (1.2) is solvable. Here C^* is exactly the constant given by (1.6).

However, the general situation seems to be considerably more delicate than in the porous medium case. As an example, we will see in §3 that for any $0 < C \leqq +\infty$ we can find an f such that

$$\lim_{R\to\infty} \frac{1}{R^{N-2/(1-m)}} \int_{B_R} f = C$$

but with problem (1.1), (1.2) having no solutions for any $T > 0$.

The purpose of this paper is to advance the understanding of the mechanism leading to existence of solutions to problem (1.1), (1.2). Our main results, Theorems 1.2 and 1.3 below, provide optimal nonexistence and existence conditions for solvability of (1.1), (1.2).

As a motivation, we begin by stating our existence result in the radially symmetric, locally bounded case. To do this, we define the operator $\mathbf{N}^*$ on $L^\infty_{\text{loc}}(\mathbf{R}^N)$, as

$$\mathbf{N}^*(h)(r) = \int_0^r \frac{ds}{\omega_N s^{N-1}} \int_{B_s} h(x)dx, \qquad r > 0.$$

Theorem 1.1. *Assume that* $f \in L^\infty_{\text{loc}}(\mathbf{R}^N)$ *is radially symmetric and satisfies*

$$\liminf_{R\to\infty} R^{2m/(1-m)}\mathbf{N}^*(f)(R) \geqq E^* T^{1/(1-m)}, \tag{1.9}$$

where

$$E^* = -\frac{2mC^*}{1-m} > 0, \tag{1.10}$$

with C^* *the constant given by* (1.6). *Then* (1.1), (1.2) *possesses a solution.*

Observe that condition (1.9) is implied by (1.8). Concerning nonexistence, we have a result dual to the above which does not require radial symmetry and in particular improves condition (1.4) to the optimal constant $C = C^*$ given by (1.6).

Theorem 1.2. *Assume that* $f \in L^1_{\text{loc}}(\mathbf{R}^N)$ *is nonnegative and satisfies*

$$\limsup_{R\to\infty} R^{2m/(1-m)}\mathbf{N}^*(f)(R) < E^* T^{1/(1-m)}, \tag{1.11}$$

where E^* *is given by* (1.10). *Then* (1.1), (1.2) *admits no solutions.*

As we already mentioned, condition (1.8) or its weaker version (1.9), does not suffice for existence if f is not radial. Nevertheless, our next result provides a general existence condition which reduces to (1.9) in the case that f is radial. In order to state the result we will define a new operator, which equals $\mathbf{N}^*$ on radially symmetric functions.

For a number $\rho > 0$ we denote by G_ρ Green's function for the ball B_ρ. For a locally bounded function h, we set

$$G^*_\rho(h)(x) = \int_{B_\rho} [G_\rho(0,y) - G_\rho(x,y)]h(y)dy, \qquad x \in \overline{B}_\rho. \tag{1.12}$$

It is easy to verify that if h is radially symmetric, then

$$G^*_\rho(h)(x) = N(h)(|x|) \quad \text{for } x \in B_\rho,$$

and for all $\rho > 0$. An observation, crucial for our purposes, is that

$$G^*_\rho(\Delta w)(x) = w(x) - w(0) \tag{1.13}$$

for a function $w \in C^2(\overline{B}_\rho)$ which is constant on ∂B_ρ.

Our main existence result is

Theorem 1.3. *Let* E^* *be the constant defined in* (1.10). *Assume that there exists a nonnegative, locally bounded function* $\tilde{f}$ *for which* $f \geqq \tilde{f}$ *and a sequence* $\rho_n \uparrow +\infty$ *such that*

$$|x|^{2m/(1-m)} G^*_{\rho_n}(\tilde{f})(x) \geqq E^* T^{1/(1-m)} + \theta(x), \tag{1.15}$$

for all $|x| < \rho_n$. *Here* $\theta(x)$ *is a function such that* $\theta(x)|x|^{-2m/(1-m)}$ *is locally bounded and* $\theta(x) \to 0$ as $|x| \to \infty$. *Then, problem* (1.1), (1.2) *is solvable.*

This result clearly reduces to Theorem 1.1 in the case that f is radial and locally bounded. Moreover, we have the validity of the following result.

Corollary 1.1. *Assume that there are nonnegative, locally bounded functions* g *and* h *with* $f \geqq g - h$, *such that* g *is radial and satisfies the growth assumption* (1.9), $|y|^{2-N} h(y)$ *is a function in* $L^1(\mathbf{R}^N)$ *and*

$$\lim_{R\to\infty} R^{\frac{2}{1-m} - \frac{N}{p}} \left(\int_{B_R} |h|^p dx \right)^{1/p} = 0$$

for some $p > N/2$. *Then problem* (1.1), (1.2) *is solvable.*

The rest of the paper will be devoted to the proof of these results. In § 2 we prove some results concerning solvability of (1.1), (1.2) in a cylinder $\Omega \times (0, T)$, with infinite boundary values, which will be fundamental in the later sections. In §3 we deal with the counterexample mentioned before, while §4 will be fully devoted to the proof of Theorem 1.3. The proof of Theorem 1.2 is carried out in §5.

2. Preliminary results

In this section we state and prove some facts, basic tools in the proofs of Theorems 1.2 and 1.3. We begin by solving the following initial-boundary-value problem on a bounded, smooth domain Ω in $\mathbf{R}^N$.

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta \phi_m(u) && \text{in } \Omega \times [0, \infty), \\ u(x,t) &= +\infty && \text{on } \partial\Omega \times (0, \infty), \\ u(x,0) &= f(x), && x \in \Omega. \end{aligned} \tag{2.1}$$

Here $f \in L^\infty(\Omega)$ and is nonnegative.

By a solution of (2.1) we mean a nonnegative function $u(x,t)$ continuous in $Q = \Omega \times (0, \infty)$, for which $u(x,t) \to +\infty$ as x approaches $\partial\Omega$ for each $t > 0$, and which satisfies (2.1) in the weak sense, namely, $u^m \in L^1_{\text{loc}}(\Omega \times [0, \infty))$ and

$$\int\int_{\Omega\times[0,T]} \left\{ \phi_m(u)\Delta\eta + u \frac{\partial \eta}{\partial t} \right\} dx\, dt = \int_\Omega u(x,T)\eta(x,T)dx - \int_\Omega f(x)\eta(x,0)dx, \tag{2.2}$$

for all $T > 0$ and $\eta \in C_c^\infty(\Omega \times [0,\infty))$.

Lemma 2.1. *There exists a solution u to the boundary-value problem* (2.1) *for each nonnegative $f \in L^\infty(\Omega)$.*

Proof. Let us fix numbers $1 < M < \infty$ and $\varepsilon \in (0,1)$. We denote by $u_{\varepsilon,M}$ the unique solution to the boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta\phi_m(u) && \text{in } \Omega \times [0,\infty), \\ u(x,t) &= M && \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) &= f(x) + \varepsilon, && x \in \Omega. \end{aligned} \tag{2.3}$$

Existence and uniqueness of $u_{\varepsilon,M}$ follow from the standard theory of non-degenerate quasilinear parabolic equations; see for example [10]. Moreover, the maximum principle implies that

$$\varepsilon \leqq u_{\varepsilon,M}(x,0) \leqq ||f||_\infty + M.$$

We let $\varepsilon \to 0$ and $M \to \infty$ to obtain a solution of (2.1). In order to control this limiting process, we need uniform barriers from above and below for $u_{\varepsilon,M}$.

Given a time $T > 0$, we next construct a lower barrier for $u_{\varepsilon,M}$ on $\Omega \times (0,T)$. Let ϕ denote a positive first eigenfunction of the Laplacian in Ω under Dirichlet boundary conditions. Choose a number $0 < \eta < -1/m$ such that $\eta < 2/(1-m)$. For $\delta \geqq 0$ and $C > 0$ we set $z_\delta = (\delta + C\phi)^{-\eta}$. The following fact follows from a direct computation: There is a constant C sufficiently large, uniform in all $0 \leqq \delta \leqq 1$, such that z_δ satisfies in Ω the elliptic inequality

$$\Delta z_\delta^m + z_\delta \geqq 0.$$

Next notice that if we define the function v on $\Omega \times [0,\infty)$ as

$$v_\delta(x,t) = t^{\frac{1}{1-m}} z_\delta(x),$$

then v_δ satisfies

$$\frac{\partial v_\delta}{\partial t} - \Delta\left(\frac{v_\delta^m}{m}\right) = \frac{1}{1-m} t^{\frac{m}{1-m}} z_\delta - \frac{1}{m} t^{\frac{m}{1-m}} \Delta z_\delta^m \leqq \frac{1}{(1-m)m} t^{\frac{m}{1-m}} z_\delta \leqq 0. \tag{2.4}$$

Now observe that if we choose $\delta = T^{\frac{1}{\eta(1-m)}} M^{-\frac{1}{\eta}}$ sufficiently small, then we get $v_\delta \leqq M$ on $\partial\Omega \times (0,T)$. Since $v_\delta = 0$ when $t = 0$, (2.4) and the maximum principle gives us that $u_{\varepsilon,M} \geqq v_\delta$ for $t \in (0,T)$ and for this choice of δ. In short, we have established that

$$t^{\frac{1}{1-m}} (T^{\frac{1}{\eta(1-m)}} M^{-\frac{1}{\eta}} + C\phi(x))^{-\eta} \leqq u_{\varepsilon,M}(x,t) \qquad \text{in } \Omega \times (0,T) \tag{2.5}$$

for some fixed $C > 0$.

We next find an upper barrier for $u_{\varepsilon,M}$. Let us consider the function w defined on $\Omega \times [0,\infty)$ as

$$w(x,t) = (1+t)^{\frac{1}{1-m}}(c\phi(x))^{\frac{1}{m}},$$

where ϕ again denotes a positive first eigenfunction of the Laplacian under Dirichlet boundary conditions. Again, a direct computation shows that an appropriate choice of c, this time sufficiently small, makes w satisfy

$$\frac{\partial w}{\partial t} - \Delta\left(\frac{w^m}{m}\right) \geqq 0.$$

Noting that w is $+\infty$ on $\partial\Omega \times (0,\infty)$ and reducing c if necessary, we also have

$$w(x,0) \geqq ||f||_\infty + \varepsilon.$$

It follows from the maximum principle that

$$u_{\varepsilon,M}(x,t) \leqq w(x,t) = (1+t)^{\frac{1}{1-m}}(C\phi(x))^{\frac{1}{m}} \quad \text{in } \Omega \times [0,\infty). \tag{2.6}$$

Equations (2.5) and (2.6) provide the necessary control on the sequence to take limits. First observe that the maximum principle implies that the sequence $\{u_{M,\varepsilon}\}$ is monotonic in ε and therefore the limit

$$u_M(x,t) := \lim_{\varepsilon\to 0} u_{\varepsilon,M}(x,t) \tag{2.7}$$

exists for all $(x,t) \in \Omega \times [0,\infty)$. Clearly u_M still satisfies the bounds (2.5) and (2.6) and satisfies (2.1) with initial data f in the weak sense. Moreover, u_M defines an increasing sequence in M. Finally letting $M \uparrow \infty$, and using the uniformity of the bounds found, we conclude that

$$u(x,t) := \lim_{M\to\infty} u_M(x,t)$$

satisfies

$$t^{\frac{1}{1-m}}(C\phi(x))^{-\eta} \leqq u(x,t) \leqq (t+1)^{\frac{1}{1-m}}(c\phi(x))^{\frac{1}{m}}, \tag{2.8}$$

for all $(x,t) \in \Omega \times (0,\infty)$. Therefore $u(\cdot,t) \in L^1_{\text{loc}}(\Omega)$, for all t, $u^m \in L^1_{\text{loc}}(\Omega \times [0,\infty))$ and u satisfies (2.2) for all $\eta \in C_c^\infty(\Omega \times [0,\infty))$. Also, if Ω' is a domain such that $\overline{\Omega'} \subset \Omega$ and $0 < \tau_1 < \tau_2 < \infty$, it follows from the bounds (2.8) that

$$\mu_1 \leqq u(x,t) \leqq \mu_2 \quad \text{on } \Omega' \times [\tau_1, \tau_2] \tag{2.9}$$

for some positive constants μ_1 and μ_2. Thus, from classical results on nondegenerate parabolic equations we can conclude that u is continuous on $\Omega' \times [\tau_1,\tau_2]$. Hence u is continuous in $\Omega \times (0,\infty)$ and it follows from (2.8) that $u(x,t) \to +\infty$ as x approaches $\partial\Omega$ for all $t > 0$. We have shown that u is the desired solution of problem (2.1). □

The following two Remarks, easy consequences of the maximum principle and the construction in Lemma 2.1, are going to be used in Sections 3 and 4.

Remark 2.1. For $0 < R_1 < R_2 < \infty$ and $f \in L^\infty(B_{R_2})$, let u_{R_i}, $i = 1,2$ be the solutions to problems (2.1) on $Q_{R_i} = B_{R_i} \times (0,\infty)$ with initial data f, constructed in Lemma 2.1. Then

$$u_{R_1} \geqq u_{R_2} \quad \text{on } Q_{R_1}. \tag{2.10}$$

To see this we just have to observe that if $u^{R_i}_{\varepsilon,M}$, $i = 1, 2$, denotes the solution to the problem (2.3) in Q_{R_i}, then as a direct consequence of the maximum principle applied on Q_{R_1}, we have $u^{R_1}_{\varepsilon,M} \geqq u^{R_2}_{\varepsilon,M}$, for all $\varepsilon \in (0,1)$ and $M > 1 + \|f\|_{L^\infty(B_{R_2})}$. Letting $\varepsilon \downarrow 0$ and $M \uparrow \infty$, we obtain the inequality (2.10).

Remark 2.2. For $0 < R < \infty$ and $f_1, f_2 \in L^\infty(B_R)$ let u_1, u_2 be the solutions to the problem (2.1) in $Q_R = B_R \times (0, \infty)$ with initial data f_1, f_2 respectively, constructed in Lemma 2.1. Assume that $f_1 \leqq f_2$ in B_R. Then

$$u_1 \leqq u_2 \quad \text{in } Q_R. \tag{2.11}$$

To show this we observe again that if $u^1_{\varepsilon,M}, u^2_{\varepsilon,M}$ are the solutions to problems (2.3) on Q_R with initial data f_1, f_2 respectively, then the classical maximum principle implies that $u^1_{\varepsilon,M} \leqq u^2_{\varepsilon,M}$ in Q_R and thus (2.11) follows by taking limits $\varepsilon \downarrow 0$ and $M \uparrow \infty$.

The following basic a priori estimate, known as the Aronson-Bénilan inequality, will be used in the proofs of the existence and nonexistence results.

Lemma 2.2. *Let u be a solution of* (2.3) *with* $f \in L^\infty(\Omega)$. *Then u satisfies*

$$u_t \leqq \frac{1}{1-m}\,\frac{1}{t}\,u. \tag{2.12}$$

As a consequence, the solution u to the problem (2.1) *constructed in Lemma* 2.1 *satisfies* (2.12) *in the sense of distributions.*

Proof. To prove the inequality (2.12) we apply the maximum principle to the equation satisfied by $\omega = 1/(1-m)tu_t - u$. We refer the reader to [5] for the details of this simple computation. □

We next state a nonlinear version of the weak Harnack inequality. Its proof can be found in [11].

Lemma 2.3. *Let* Ω *be a domain in* $\mathbf{R}^N$, $N \geqq 2$, *and for* $m < 0$ *let u be a positive, bounded and smooth solution of equation* $\partial u/\partial t = \Delta u^m/m$ in $Q_T = \Omega \times (0, T)$ *for some* $T > 0$. *Then for every* $0 < \tau_1 < \tau_2 < T$ *and* $x_0 \in \Omega$, $\rho > 0$ *such that* $B_\rho(x_0) \subset \Omega$,

$$\frac{1}{\rho^N} \int_{B_\rho(x_0)} u(x, \tau_1)^m dx \leqq \left(\frac{\tau_2}{\tau_1}\right)^{\frac{-m}{1-m}} u^m(x_0, \tau_2) + c\left(\frac{\tau_2}{\tau_1}\right)^{\frac{-m}{1-m}} \frac{\Lambda \rho^2}{\tau_2 - \tau_1}, \tag{2.13}$$

where $B_\rho(x_0)$ *denotes the ball with center* x_0 *and radius* ρ, $\Lambda = \|u(\cdot, \tau_1)\|_\infty$ *and c is a positive constant depending only on N and m.*

The following lemma gives an a priori pointwise estimate for u^m in terms of its spatial averages.

Lemma 2.4. *Under the same hypotheses and notation as in Lemma 2.3, the estimate*

$$u^m(x_0,t) \leqq \frac{1}{\rho^N} \int\limits_{B_\rho(x_0)} u^m(x,t)dx + c\frac{\Lambda\rho^2}{t} \tag{2.14}$$

holds for all $0 < t < T$, *where* $\Lambda = ||u(\cdot,t)||_\infty$ *and c is a positive constant depending only on m and N.*

Proof. The proof is standard so we omit the details. Fix a number $t \in (0,T)$, and notice that it follows from the Aronson-Bénilan inequality that $u(\cdot,t)$ satisfies

$$\Delta u^m - \frac{m}{1-m}\frac{1}{t}u \geqq 0 \tag{2.15}$$

in Ω. Let $G_\rho(r) = G_\rho(|x-x_0|)$ denote the elliptic Green function defined in a ball $B_\rho(x_0)$ by

$$G_\rho(r) = \begin{cases} r^{2-N} - \rho^{2-N} + \frac{N-2}{2}\rho^{-N}(r^2-\rho^2) & \text{if } N > 2, \\ \log\rho - \log r + \frac{1}{2}\rho^{-2}(r^2-\rho^2) & \text{if } N = 2. \end{cases}$$

Then G_ρ satisfies $\Delta G_\rho = N(N-2)\rho^{-N} - (N-2)\omega_N\delta_{x_0}$ in $B_\rho(x_0)$ and $G_\rho(r) = G'_\rho(r) = 0$ for $r = \rho$. Therefore, testing the equation (2.10) against G and integrating by parts, we obtain

$$\frac{1}{\rho^N}\int\limits_{B_\rho(x_0)} u^m(x,t)dx - u^m(x_0,t) + C(m,N)\frac{1}{t}\int\limits_{B_\rho(x_0)} u(x,t)G_\rho(x)dx \geqq 0.$$

Since $G_\rho(r) = \rho^{2-N}G_1(r/\rho)$, the desired inequality easily follows by rescaling. □

3. A counterexample

From the analogy made with the porous medium equation, and in light of the fact that a condition of the form (1.8) suffices for existence in the radial case, it is natural to ask whether the same condition still suffices for existence in the general case.

The purpose of this section is to show via an example that the answer to this question is negative. In fact, we will see that for any $0 < C \leqq +\infty$ one can find an f such that

$$\lim_{R\to\infty} \frac{1}{R^{N-2/(1-m)}} \int\limits_{B_R} f = C \tag{3.1}$$

but with problem (1.1), (1.2) having no solution for any $T > 0$.

We restrict our attention to the case $N = 2$; however the construction we will make can be extended to any dimension. Let us consider, for a number $k > 0$, the region bounded between two logarithmic spirals S_k given in polar coordinates by

$$\theta \in (-\infty, \infty), \quad \theta < \log r < \theta + \frac{\pi}{k}. \tag{3.2}$$

Then the following result holds.

Theorem 3.1. *Assume that* $N = 2$. *Then there exists a number* $k_0 = k_0(m) > 0$ *such that if* $0 < k < k_0$ *and* f *is in* $L^\infty(\mathbf{R}^N)$ *and vanishes in* A_k, *then problem* (1.1), (1.2) *is not solvable for any* $T > 0$.

Since this result allows arbitrary values of f outside A_k, the existence of an f satisfying (3.1) for which no local solution to (1.1), (1.2) exists follows.

Proof of Theorem 3.1. For $k > 0$ let us consider the region A_k defined by (3.2), and assume that $f \equiv 0$ on A_k and that (1.1), (1.2) has a solution $u(x,t)$ for some $T > 0$. We consider the bounded open set A_k^R defined as consisting of the $(r, \theta) \in A_k$ with $r < R$. The proof we present consists of the construction of smooth positive functions $w_R(x,t)$ defined on $A_k^R \times (0, \infty)$ and satisfying $w_R(x,t) \to +\infty$ as $x \to \partial A_k^R$ for all $t > 0$ so that

$$\frac{\partial w_R}{\partial t} - \Delta\left(\frac{w_R^m}{m}\right) \geqq 0 \quad \text{in } A_k^R \times (0, \infty). \tag{3.3}$$

The maximum principle then implies that such functions satisfy

$$u(x,t) \leqq w_R(x,t) \quad \text{in } A_k^R \times (0, T), \tag{3.4}$$

for all $R > 0$. We will see that if k is chosen sufficiently small, then along a sequence $R_n \to +\infty$, we have

$$\lim_{n \to \infty} w_{R_n}(x,t) = 0 \quad \text{for all } (x,t) \in A_k \times (0, \infty). \tag{3.5}$$

But this is impossible by (3.4) since $u > 0$ almost everywhere. Thus the problem is reduced to finding the desired functions w_R. Our starting point in the construction is that the following function is positive and harmonic in A_k and vanishes on its boundary:

$$v(x) = e^{k\theta} r^k \sin[k(\log r - \theta)].$$

Let η be a positive, smooth cut-off function on B_1 so that $\eta \equiv 1$ on $B_{1/2}$ and $\eta = 0$ on ∂B_1. For a small number $\delta > 0$ to be fixed, we set

$$w_1(x,t) = (\delta\eta(x)v(x))^{1/m}(t+1)^{1/(1-m)}, \quad x \in A_1, \ t > 0.$$

Using the fact that v is harmonic, we immediately check that if δ is chosen sufficiently small, then

$$\frac{\partial w_1}{\partial t} - \Delta\left(\frac{w_1^m}{m}\right) \geqq 0$$

so that w_1 satisfies the desired requirements. Next we define w_R by just setting

$$w_R(x,t) = R^{-2/(1-m)} w_1(x/R, t).$$

The function w_R clearly satisfies (3.3) and approaches $+\infty$ if x approaches the boundary of A_k^R. Moreover $w_R(x, 0) \geqq c(R) > 0$, for all $x \in A_k^R$. It follows then by the maximum principle and a standard approximation argument that $u \leqq w_R$ in $A_k^R \times (0, T)$. We finally check the assertion (3.5) for a sufficiently small $k > 0$. Let x be a point in A_k with polar coordinates (r_0, θ_0) and set $\alpha = k(\log r_0 - \theta_0)$, so that $\alpha \in (0, \pi)$. Choosing $R_n = \exp(2n\pi/k)$, we have

$$\begin{aligned} w_{R_n}(x,t) &= R_n^{-2/(1-m)} v^{1/m}(x/R_n, t)(t+1)^{1/(1-m)} \\ &= R_n^{-k/m-2/(1-m)} (e^{k\theta_0} \sin\alpha)^{1/m} (t+1)^{1/(1-m)}. \end{aligned}$$

Therefore, if we choose $0 < k < -2m/(1-m)$, we have that $w_{R_n}(x,t) \to 0$ as $n \to \infty$, as desired. This concludes the proof. □

4. Existence of solutions

In this section we carry out the proof of our main existence result, Theorem 1.3. The procedure we follow roughly consists in solving the initial-boundary-value problem on a sequence of expanding cylinders with infinite values on the lateral boundary, using the results of the previous section. The associated sequence of solutions turns out to be decreasing, and the assumptions of the theorem permit us to obtain an appropriate "control from below" of the sequence, whose limit will be the desired solution. The key step in obtaining such control is contained in the following lemma.

Lemma 4.1. *Assume that* $N \geqq 2$ *and that* $g \in L^\infty(B_\rho)$ *is a nonnegative function satisfying*

$$G_\rho^*(g)(x) \geqq E^*|x|^{-2m/(1-m)} T^{1/(1-m)} - l, \tag{4.1}$$

where G_ρ^* *is the operator defined in* (1.12), E^* *is the constant given by* (1.10) *and* l *is a positive constant. Let* w *be the solution to the boundary-value problem* (2.1) *on* $B_\rho \times (0, \infty)$ *with* $w(x, 0) = g(x)$, *constructed in Lemma 2.1. Then,*

$$\int_0^r \frac{ds}{\omega_N s^{N-1}} \int_{B_s} w(x,t)dx \geqq E^* r^{-2m/(1-m)} (T-t)_+^{1/(1-m)} - l \tag{4.2}$$

for all $0 < r < \rho$ *and* $0 < t < T$.

Proof. Let v^T be the explicit solution of equation (1.1), defined by (1.5). An easy computation shows that

$$G_\rho^*(v^T(\cdot,t))(x) = N(v^T(\cdot,t))(|x|) = E^*|x|^{-2m/(1-m)}(T-t)_+^{1/(1-m)},$$

and therefore the proof of the lemma is reduced to showing that if

$$G_\rho^*(g(\cdot) - v^T(\cdot, 0))(x) \geqq -l$$

for all $x \in B_\rho$, then

$$G_\rho^*(w(\cdot,t) - v^T(\cdot,t))(x) \geqq -l \tag{4.3}$$

for all $(x,t) \in B_\rho \times (0,T)$. This inequality will follow from an application of the maximum principle.

For $\varepsilon \in (0,1)$ and $M > 1$, let $w_{M,\varepsilon}$ denote the unique classical solution to the problem (2.3) on $B_\rho \times [0,\infty)$, with $w_{M,\varepsilon}(x,0) = g(x) + \varepsilon$. Let us fix a small number $\delta > 0$. We will show that

$$G_\rho^*(w_{M,\varepsilon}(\cdot,t) - v^T(\cdot,t))(x) + l \geqq 0 \tag{4.4}$$

for all sufficiently large M and for $x \in B_R$ and $0 \leqq t \leqq T - \delta$. To simplify the notation, let us set

$$\tilde{w} = w_{M,\varepsilon}, \quad W(x,t) = |m|[G_\rho^*(w_{M,\varepsilon}(\cdot,t) - v^T(\cdot,t))(x) + l].$$

Using (1.11) and the fact that $(v^T)^m(0,t) = 0$, we obtain

$$\frac{\partial W}{\partial t} = G_\rho^*(\Delta[(\tilde{w})^m - (v^T)^m]) = [-(\tilde{w})^m(x,t) + (\tilde{w})^m(0,t) + (v^T)^m(x,t)]. \tag{4.5}$$

Therefore W satisfies

$$\frac{\partial W}{\partial t} \geqq A(x,t)\Delta W, \tag{4.6}$$

with

$$A(x,t) = |m|^{-1}\frac{-(\tilde{w})^m(x,t) + (v^T)^m(x,t)}{\tilde{w}(x,t) - v^T(x,t)}.$$

Observe that $W(0,t) = l|m| > 0$ and that W is uniformly continuous on $B_\rho \times (0, T-\delta)$. Hence there is a number $\sigma > 0$, such that $W(x,t) > 0$ for $x \in B_\sigma$ and all t. On the other hand, W is in $C^2(Q) \cap C(\bar{Q})$, where $Q = \{B_\rho \backslash B_\sigma\} \times (0, T-\delta)$, so that we can apply the maximum principle as long as we verify that $W \geqq 0$ on the parabolic boundary of Q. We will see that this is the case if M is sufficiently large. For $(x,t) \in \partial B_\rho \times (0, T-\delta)$ we obtain, using (4.5),

$$\frac{\partial W}{\partial t} \geqq \delta^{m/(1-m)}\rho^{-2m/(1-m)} - M^m,$$

and thus for large M we have $W \geqq 0$ on $\partial B_\rho \times (0, T-\delta)$. Since we also have $W(x,0) \geqq 0$ by assumption, we conclude from the maximum principle that $W(x,t) \geqq 0$. Hence, (4.4) holds, so that

$$G_\rho^*(w_{M,\varepsilon}(\cdot,t)(x)) \geqq E^*|x|^{-2m/(1-m)}(T-t)^{1/(1-m)} - l.$$

Taking spherical averages in this inequality we obtain

$$\int_0^r \frac{ds}{\omega_N s^{N-1}} \int_{B_s} w_{M,\varepsilon}(x,t)dx \geqq E^* r^{-2m/(1-m)}(T-t)_+^{1/(1-m)} - l \tag{4.7}$$

for $0 < r < \rho$ and $0 < t < T - \delta$. To conclude the lemma, we just take the limits $\varepsilon \to 0$ and $M \to \infty$ in (4.7) as we did in the proof of Lemma 2.1, and use the fact that δ is arbitrary. We then obtain that (4.3) holds, and the proof is complete. □

Now we are in a position to prove Theorem 1.3.

Proof of Theorem 1.3. We will first construct a solution $\tilde{u}$ to (1.1), with initial data $\tilde{f}$. We begin by observing that (1.14) and the assumption (1.15) gives us the existence of an increasing sequence $\rho_n \uparrow \infty$ with the property that for a fixed $\delta > 0$ there are numbers n_0 and l, such that for all $n \geqq n_0$ and $x \in B_{\rho_n}$, we have

$$G^*_{\rho_n}(\tilde{f})(x) \geqq E^*|x|^{-2m/(1-m)}(T-\delta)^{1/(1-m)} - l. \tag{4.8}$$

Let $\tilde{u}_n$ be the solution of problem (2.1) on $B_{\rho_n} \times (0,\infty)$ with $\tilde{u}_n(x,0) = \tilde{f}(x)$ constructed in Lemma 2.1. It follows from (4.8) and Lemma 3.1 that

$$\int_0^r \frac{ds}{\omega_N s^{N-1}} \int_{B_s} \tilde{u}_n(x,t)dx \geqq E^* r^{-2m/(1-m)}(T-\delta-t)_+^{1/(1-m)} - l \tag{4.9}$$

for all $0 < r < \rho_n$ and $0 < t < T - \delta$. On the other hand, as we have mentioned in Remark 2.1, the sequence $\{\tilde{u}_n\}$ is decreasing in n and therefore the limit

$$\tilde{u}(x,t) = \lim_{n\to\infty} \tilde{u}_n(x,t) \tag{4.10}$$

exists for all $(x,t) \in \mathbf{R}^N \times (0,\infty)$. Our aim is to show that $\tilde{u}$ is a solution to problem (1.1), (1.2) with initial data $\tilde{f}$. We first observe that we can take the limit $n \to \infty$ in (4.9) and use monotone convergence to conclude that

$$\int_0^r \frac{ds}{\omega_N s^{N-1}} \int_{B_s} \tilde{u}(x,t)dx \geqq E^* r^{-2m/(1-m)}(T-\delta-t)_+^{1/(1-m)} - l \tag{4.11}$$

for all $r > 0$ and $0 < t < T - \delta$. Since l is a fixed number and $r^{-2m/(1-m)} \to \infty$ as $r \to \infty$, this estimate implies that there exists a point $x_0 \in \mathbf{R}^N$ such that

$$\tilde{u}(x_0, T-2\delta) = \lim_{n\to\infty} \tilde{u}_n(x_0, T-2\delta) > 0. \tag{4.12}$$

Now let $\rho > 0$ and let n_0 be sufficiently large so that $B_\rho(x_0)$ is strictly contained in B_{ρ_n} for all $n \geqq n_0$. Applying the Harnack estimate (2.13) for each of the u_n's, we conclude that for all $n \geqq n_0$ we have

$$\rho^{-N} \int_{B_\rho(x_0)} \tilde{u}_n^m(x, T-3\delta)dx \leqq C(T)[\tilde{u}_n^m(x_0, T-2\delta) + c\delta\Lambda(\rho)\rho^2], \tag{4.13}$$

where $C(T)$ is a constant which depends only on T, N and m, and $\Lambda(\rho)$ is an upper bound for $u_{n_0}(\cdot, T-3\delta)$ on $B_\rho(x_0)$. We wish to estimate the spatial averages of $\tilde{u}^m(\cdot,t)$ for all $0 < t \leqq T - 3\delta$. For this we use the Aronson-Bénilan inequality (2.12). Indeed, integrating (2.12) in time we obtain the estimate

$$\tilde{u}_n(x,t) \geqq \left(\frac{t}{T-3\delta}\right)^{1/(1-m)} \tilde{u}_n(x, T-3\delta), \tag{4.14}$$

and therefore as a combination of (4.13) and (4.14) we have

$$\rho^{-N} \int_{B_\rho(x_0)} \tilde{u}_n^m(x,t)dx \leqq C(T)t^{m/(1-m)}[\tilde{u}_n^m(x_0, T-2\delta) + c\delta\Lambda(\rho)\rho^2].$$

It follows by monotone convergence and (4.12) that

$$\int_{B_\rho(x_0)} \tilde{u}^m(x,t)dx \leqq C(\rho, T, \delta, \tilde{u})t^{m/(1-m)}, \tag{4.15}$$

where the constant $C(\rho, T, \delta, \tilde{u})$ is independent of t. This in particular implies that $\tilde{u} \in L^1_{\text{loc}}(\mathbf{R}^N \times [0, T-3\delta])$. It remains to show that $\tilde{u}$ satisfies (1.1) in the distributional sense and that $\tilde{u}(\cdot,\tau) \to \tilde{f}$ in $L^1_{\text{loc}}(\mathbf{R}^N)$, as $\tau \to 0$. This follows from the estimate (4.15). Indeed, let $\eta \in C_c^\infty(\mathbf{R}^N \times [0,T))$ be a nonnegative test function and $0 < \tau < T - 3\delta$. Then, if n is sufficiently large so that $B_{\rho_n} \times [0,T)$ contains the support of η, we have

$$\int \tilde{u}_n(x,\tau)\eta(x,\tau)dx - \int \tilde{f}(x)\eta(x,0)dx = \int_0^\tau\!\!\int \tilde{u}_n \frac{\partial\eta}{\partial t} + \frac{\tilde{u}_n{}^m}{m}\Delta\eta \; dx\; dt. \tag{4.16}$$

It follows from (4.10) and (4.15) that we can pass to the limit $n \to \infty$ in (4.16) to conclude that the same integral equality holds for $\tilde{u}$. Hence, $\tilde{u}$ satisfies $\partial\tilde{u}/\partial t = \Delta\tilde{u}^m/m$ in the distributional sense. Also combining (4.15) and (4.16) we obtain

$$\left|\int \tilde{u}(x,\tau)\eta(x,\tau)dx - \int \tilde{f}(x)\eta(x,0)dx\right| \leqq C(\eta, T, \delta)\tau = o(\tau)$$

as $\tau \to 0$. We conclude then that $\tilde{u}$ is a solution of the problem $\partial\tilde{u}/\partial t = \Delta u^m/m$, $\tilde{u}(x,0) = \tilde{f}(x)$, on $\mathbf{R}^N \times (0, T-3\delta)$. Since $\delta > 0$ can be chosen arbitrarily small, we finally obtain that $\tilde{u}$ is a solution of (1.1), (1.2) with initial data $\tilde{f}$.

We now construct a solution u of the problem (1.1), (1.2) with initial data the given function $f \in L^1_{\text{loc}}(\mathbf{R}^N)$ and such that

$$u \geqq \tilde{u} \qquad \text{on } \mathbf{R}^N \times (0,T). \tag{4.17}$$

Let $\rho_n \to \infty$ and $\tilde{u}_n$ be chosen as at the beginning of the proof, and for $k \in \mathbf{N}$, let $u_{n,k}$ be the solution of the problem (2.1) with $\Omega = B_{\rho_n}$ and $u_{n,k}(\cdot,0) = f_k \equiv \min(f,k)$. Then it follows from the Remark 2.2 that

$$u_{n,k+1} \geqq u_{n,k} \geqq \tilde{u}_n \geqq \tilde{u} \qquad \text{in } B_{\rho_n} \times (0,T) \tag{4.18}$$

for all $k \geqq k_0$ if k_0 is chosen so that $f_{k_0} \geqq \tilde{f}$ in B_{ρ_n}. Moreover, if $\eta \in C_c^\infty(B_{\rho_n})$ is a nonnegative test function, each $u_{n,k}$ satisfies the integral identity

$$\int_{B_{\rho_n}} u_{n,k}(x,\tau)\eta(x)dx - \int_{B_{\rho_n}} f_k(x)\eta(x)dx = m^{-1}\int_0^\tau\!\!\int_{B_{\rho_n}} u_{n,k}^m\Delta\eta(x)\; dx\; dt \tag{4.19}$$

for all $\tau \in (0, T)$. By using (4.18) the right-hand side of (4.19) can be estimated by

$$\left| \int_0^\tau \int_{B_{\rho_n}} u_{n,k}^m \Delta\eta(x)\, dx\, dt \right| \leqq C(\eta) \int_0^\tau \int_{B_{\rho_n}} \tilde{u}^m dx\, dt < \infty.$$

Since the sequence $\{u_{n,k}\}$ is increasing in k, the limit

$$u_n(x,t) := \lim_{k\to\infty} u_{n,k}(x,t) \tag{4.20}$$

exists for all (x,t), and it follows from (4.19), (4.20) and the monotone convergence theorem that $u_n(\cdot,t) \in L^1_{\text{loc}}(B_{\rho_n})$ for all $t \in (0,T)$ and $u_n^m \in L^1_{\text{loc}}(B_{\rho_n} \times [0,T))$. It is then easy to conclude that u_n satisfies the equation $\partial u_n/\partial t = \Delta u_n^m/m$ in the sense of distributions on $B_{\rho_n} \times (0,T)$. Also combining (4.15) with (4.19) and (4.20) we conclude that for all $\eta \in C_c^\infty(B_{\rho_n})$

$$\left| \int u_n(x,\tau)\eta(x)dx - \int f(x)\eta(x)dx \right| \leqq C(\eta,T)\tau = o(\tau), \tag{4.21}$$

as $\tau \to 0$.

We next observe that

$$u_{n+1} \leqq u_n \quad \text{on } B_{\rho_n} \times (0,T),$$

since it follows from the Remark 2.2 that $u_{n+1,k} \leqq u_{n,k}$ for all k sufficiently large. Therefore the limit

$$u(\cdot,t) = \lim_{n\to\infty} u_n(\cdot,t)$$

exists and $u(\cdot,t) \in L^1_{\text{loc}}(\mathbf{R}^N)$ for all $t \in (0,T)$. Moreover, (4.17) holds. It is then easy to see that $u^m \in L^1_{\text{loc}}(\mathbf{R}^N \times [0,T))$ and that u satisfies (1.1) in the distributional sense. Finally taking the limit $n \to \infty$ in (4.21) we conclude that $u(\cdot,\tau) \to f$ in $L^1_{\text{loc}}(\mathbf{R}^N)$, as $\tau \to 0$. Hence u is a solution of the problem (1.1), (1.2) and the proof of the theorem is complete. □

5. A nonexistence result

In this section we prove the nonexistence result Theorem 1.2. Our proof is based on comparison arguments similar to those used in the proof of Theorem 1.3. We also use many of the results in [7], so we refer the reader to that paper for some of the details. For $\varepsilon \in (0,1)$ and $M > 1$, we denote by $u_{\varepsilon,M}$ the unique classical solution of the initial-value problem $\partial u/\partial t = \Delta u^m/m$, on $\mathbf{R}^N \times (0,\infty)$, with $u(\cdot,0) = \min(f,M) + \varepsilon$. We also define

$$u_\varepsilon := \lim_{M\to\infty} u_{\varepsilon,M}, \quad w := \lim_{\varepsilon\to 0} u_\varepsilon, \tag{5.1}$$

where the limits exist, as was shown in [7]. Moreover, it was proved in [7] that if u is a solution of (1.1), (1.2) with initial data f, then

$$u(\cdot,t) \leqq w(\cdot,t) \quad \text{for all } t \in (0,T). \tag{5.2}$$

The important step in our proof is to show that if f satisfies (1.11), then

$$N(w(\cdot,t))(|x|) \leqq N(v^{T-\delta}(\cdot,t))(|x|) + l, \tag{5.3}$$

for all $0 < t \leqq T - \delta$. The desired result then easily follows as a combination of (5.2), (5.3) and Theorem 1.1 in [7].

Set $\tau = T - \delta$ and fix a number $R > 1$. Let $\tilde{v}_R$ be the solution of the boundary value problem (2.1) with $\Omega = B_R$, $\tilde{v}(x,0) = (2\alpha\tau)^{1/(1-m)} \min(|x|^{-2/(1-m)}, 1)$, $\alpha = N - 2/(1-m)$, as constructed in Lemma 2.1. It is easy to see that $\tilde{v}_R$ is radially symmetric in x. The proof of the inequality (5.3) will be based upon the following lemma.

Lemma 5.1. *Let f be a nonnegative locally bounded function satisfying* (1.13) *and assume that $u_\varepsilon(0,\tau) < \infty$ and $w(0,\tau) = \mu > 0$. Then, there exists a positive constant l independent of R, such that*

$$N(w(\cdot,t))(|x|) \leqq N(\tilde{v}_R(\cdot,t))(|x|) + l \quad \text{for } |x| < R, \tag{5.4}$$

for all $0 \leqq t \leqq \tau$.

Proof. To simplify the notation set $\tilde{v} = \tilde{v}_R$ and $r = |x|$. We begin by noticing that condition (1.13) implies the existence of a positive constant l_0 depending only on N, m and τ, such that $N(f - \tilde{v}_R(\cdot,0))(r) \leqq l_0$. In particular, if we define

$$W(r,t) = |m|[N(u_{M,\varepsilon}(\cdot,t))(r) - N(\tilde{v}_R(\cdot,t))(r)], \tag{5.5}$$

then $W(r,0) \leqq l_0 + \varepsilon R^2$ for all $r \leqq R$. Denote by $\overline{u_{M,\varepsilon}}$ the spherical averages of $u_{M,\varepsilon}$. since $\partial\overline{u_{M,\varepsilon}}/\partial t = \Delta\overline{u^m_{M,\varepsilon}}/m$ and $\overline{u^m_{M,\varepsilon}} \geqq (\overline{u_{M,\varepsilon}})^m$, the function W satisfies the differential inequality

$$\frac{\partial W}{\partial t} \leqq A(r,t)\Delta W + u^m_{M,\varepsilon}(0,t), \tag{5.6}$$

with $A(r,t) = |m|^{-1}\{[-(\overline{u_{M,\varepsilon}})^m + \tilde{v}^m]/[\overline{u_{M,\varepsilon}} - \tilde{v}]\}$. It follows from the Aronson-Bénilan inequality that $u^m_{M,\varepsilon}(0,t) \leqq C_0 t^{m/(1-m)}$ with $C_0 = C_0(\tau,\mu)$ independent of ε and M. Therefore the function

$$\tilde{W} = W - (1-m)C_0 t^{1/(1-m)}$$

satisfies $\partial\tilde{W}/\partial t \leqq A(r,t)\Delta\tilde{W}$ with $\tilde{W}(r,0) \leqq l_0 + \varepsilon R^2$ for all $0 < r \leqq R$, and $\partial\tilde{W}/\partial t = -(\overline{u_{M,\varepsilon}})^m(R,t) \leqq 0$ on $\partial B_R \times (0,\tau)$. The maximum principle then implies that $\tilde{W}(r,t) \leqq l_0 + \varepsilon R^2$ on $B_R \times (0,\tau)$. In particular,

$$N(u_{\varepsilon,M}(\cdot,t))(r) - N(\tilde{v}_R(\cdot,t))(r) \leqq l_0 + (1-m)C_0 t^{1/(1-m)} \leqq l,$$

with $l = l_0 + (1-m)C_0\tau^{1/(1-m)}$, and therefore taking limits $M \to \infty$ and $\varepsilon \to 0$, we finally obtain the desired inequality (5.4). □

It follows from the Remark 2.1 that the sequence $\{\tilde{v}_R\}$ is decreasing in R, and therefore the limit

$$\tilde{v} := \lim_{R\to\infty} \tilde{v}_R \tag{5.7}$$

exists. It follows from the proof of Theorem 1.3 that $\tilde{v}$ is a solution of $\partial\tilde{v}/\partial t = \Delta\tilde{v}^m/m$ on $\mathbf{R}^N \times (0, \tau)$ with $\tilde{v}(x, 0) = (2\alpha\tau)^{1/(1-m)} \min(|x|^{-2/(1-m)}, 1)$, $\alpha = N - 2/(1-m)$.

We have the following comparison lemma.

Lemma 5.2. *If $\tilde{v}$ is the solution defined by* (5.7), *then $\tilde{v} \leqq v^\tau$ on $\mathbf{R}^N \times (0, \tau)$, $\tau = T - \sigma$.*

Proof. Set $z = \tilde{v} - v^\tau$, and let χ denote the characteristic function of the set where $\tilde{v}(x, t) > \tilde{u}(x, t)$. Fix a number $0 < t < \tau$ and pick a test function $\eta \in C_c^\infty(\mathbf{R}^N)$, $\eta \geqq 0$. Then, since $z(x, 0) \leqq 0$, we have

$$\int z^+(x, t)\eta(x)dx \leqq \int_0^t\!\!\int \chi(x, s)[\phi_m(\tilde{v}(x, s)) - \phi_m(v^\tau(x, s))]\Delta\eta(x)\, dxds, \quad (5.8)$$

where $\phi_m(u) = u^m/m$. To derive (5.8) one combines KATO's inequality [9] and an approximation argument to conclude that

$$\Delta[\phi_m(\tilde{v}) - \phi_m(v^\tau)]^+ \geqq \chi\Delta[\phi_m(\tilde{v}) - \phi_m(v^\tau)] \quad (5.9)$$

in the distributional sense. Also, since $\partial z^+/\partial t = \chi\partial z/\partial t$ again in the distributional sense, (5.8) follows. Let $A(x, s) = [\phi_m(\tilde{v}) - \phi_m(v^\tau)]/[\tilde{v} - v^\tau]$ whenever $\tilde{v} > v^\tau$ and zero elsewhere. It follows from the exact formula defining v^τ that $A(x, s) \leqq (\tilde{v}^\tau)^{m-1}(x, s) \leqq c(\tau)|x|^2$. Hence, if $\eta \in C_c^\infty(B_{2R})$ is chosen so that $0 \leqq \eta \leqq 1$, $\eta \equiv 1$ on B_R and $|\Delta\eta| \leqq CR^{-2}$, we obtain

$$\int_{B_R} z^+(x, t)\, dx \leqq C \int_0^t \int_{B_{2R}} z^+(x, s)\, dxds. \quad (5.10)$$

Also, notice that with the same of choice of η in (5.8) we can also conclude that

$$\int_{B_R} z^+(x, t)\, dx \leqq CR^{-2} \int_0^t \int_{B_{2R}} |\phi^m(\tilde{v}^\tau)|\, dxds \leqq C(\tau)R^{N-2/(1-m)} \quad (5.11)$$

for all $R > 1$. It follows that

$$l(t) := \sup_{R \geqq 1} \frac{1}{R^{N-2/(1-m)}} \int_{B_r} z^+(x, t)\, dx \leqq C(\tau) < \infty.$$

Now choose $R > 1$ such that $\int_{B_R} z^+(x, t) \geqq \frac{1}{2} l(t) R^{N-2/(1-m)}$ and apply (5.10) to conclude that

$$l(t) \leqq C \int_0^t l(s)\, ds$$

for some constant $C = C(N, m, \tau)$. Hence, $l(t) = 0$, which implies that $\tilde{v}(\cdot, t) \leqq v^\tau(\cdot, t)$, for all $0 < t < \tau$. □

We are now in position to prove Theorem 1.2.

Proof of Theorem 1.2. We show that for the solution w defined in (5.1) we have $w(\cdot, t) = 0$, for all $t > T - \delta$. Since, as we have proved in [7], any solution u of the problem (1.1), (1.2) satisfies $u \leqq w$, this would imply the desired nonexistence result.

Assume that $w(\cdot, T - \delta) \neq 0$; otherwise the desired result follows from Theorem 1.2 in [7]. It follows then from the Lemmas 2.3 and 2.4 that $w(0, T - \delta) = \mu > 0$ and therefore as a combination of Lemmas 5.1 and 5.2 we have

$$\int_0^R \frac{1}{r^{N-1}\omega_N} \int_{B_r} w(x,t)\, dxdr \leqq l, \tag{5.12}$$

for all $t \geqq T - \delta$ and $R > 0$.

The proof of Theorem 2.1 in [7] then implies that $w(\cdot, t) = 0$, for all $t > T - \delta$. □

Acknowledgement. We are indebted to Carlos Kenig for his encouragement and valuable suggestions in the course of this work. The work of P. Daskalopoulos was partially supported by NSF grant 445860–21170. The work of M. Del Pino was partially supported by grants CI1CT93–0323 CCE and Fondecyt-Chile 1950303.

References

1. Aronson, D. G., & Caffarelli, L. A., The initial trace of a solution of the porous medium equation, *Trans. Amer. Math. Soc.*, **280**, 1983, 351–366.
2. Bénilan, P., Crandall, M. G., & Pierre, M., Solutions of the porous medium equation under optimal conditions on the initial values, *Indiana Univ. Math. J.*, **33**, 1984, 51–87.
3. Berryman, J. G., & Holland, C. J., Asymptotic behaviour of the nonlinear differential equation $n_t = (n^{-1} n_x)_x$, *J. Math. Phys.*, **23**, 1982, 983–987.
4. Bluman, G., & Kumei, S., On the remarkable nonlinear diffusion equation $\partial/\partial x[a(u+b)^{-2}[\partial u/\partial x] - (\partial u/\partial x) = 0$, *J. Math. Phys.* **21**, 1980, 1019–1023.
5. Caffarelli, L. A., & Friedman, A., Regularity of the free boundary of a gas flow in an n-dimensional porous medium, *Indiana Univ. Math. J.*, **29**, 1980, 361–389.
6. Chayes, J. T., Osher, S. J., & Ralston, J. V., On singular diffusion equations with applications to self-organized criticality, *Comm. Pure Appl. Math.*, **46**, 1993, 1363–1377.
7. Daskalopoulos, P., & Del Pino, M. A., On fast diffusion nonlinear heat equations and a related singular elliptic problem, *Indiana Univ. Math. J.*, **43**, 1994, 703–728.
8. De Gennes, P. G., Wetting: statics and dynamics, *Reviews of Modern Physics*, **57**, 1985, 827–863.
9. Kato, T., Schrödinger operators with singular potentials, *Israel J. Math.*, **13**, 1973, 135–148.
10. Ladyženskaja, O. A., Solonnikov, V. A., & Ural'ceva, N. N., *Linear and quasilinear equations of parabolic type*, *American Mathematical Society*, 1968.

11. VÁZQUEZ, J. L., Nonexistence of solutions for nonlinear heat equations of fast-diffusion type, *J. Math. Pures Appl.*, **71**, 1992, 503–526.

Department of Mathematics
University of California
Irvine, California 92717

and

Departamento Ingeniería
Matemática
Universidad de Chile
Casilla 170, Correo 3
Santiago, Chile

(Accepted November 17, 1995)