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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO A NONLOCAL EQUATION WITH MONOSTABLE NONLINEARITY*

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Abstract. Let $J \in C(\mathbb{R})$, $J \geq 0$, $\int_{\mathbb{R}} J = 1$ and consider the nonlocal diffusion operator $\mathcal{M}[u] = J \star u - u$. We study the equation $\mathcal{M}u + f(x,u) = 0$, $u \geq 0$, in $\mathbb{R}$, where f is a KPP-type nonlinearity, periodic in x. We show that the principal eigenvalue of the linearization around zero is well defined and that a nontrivial solution of the nonlinear problem exists if and only if this eigenvalue is negative. We prove that if, additionally, J is symmetric, then the nontrivial solution is unique.

Key words. nonlocal dispersal, monostable, existence and uniqueness, convolution operator

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1. Introduction.

Reaction-diffusion equations have been used to describe a variety of phenomena in combustion theory, bacterial growth, nerve propagation, epidemiology, and spatial ecology [13, 12, 15, 19]. However, in many situations, such as in population ecology, dispersal is better described as a long range process rather than as a local one, and integral operators appear as a natural choice. Let us mention in particular the seminal work of Kolmogorov, Petrovsky, and Piskunov [16], who in 1937 introduced a model for the dispersion of gene fractions involving a nonlocal linear operator and a nonlinearity of the form $u(1-u)$, which many authors now call a KPP-type nonlinearity.

Nonlocal dispersal operators usually take the form $\mathcal{M}[u] = \int_{\mathbb{R}^N} k(x,y)u(y)dy - u(x)$, where $k \geq 0$ and $\int_{\mathbb{R}^N} k(y,x)dy = 1$ for all $x \in \mathbb{R}^N$. They have been mainly used in discrete time models [17], while continuous time versions have also been recently considered in population dynamics [14, 18]. Steady state and travelling wave solutions for single equations have been studied in the case $k(x,y) = J(x-y)$, with J even, for some specific reaction nonlinearities in [1, 10, 8, 2, 6, 21].

In this work we restrict ourselves to one dimension and take

$$k(x,y) = J(x-y).$$

We are interested in the existence/nonexistence and uniqueness of solutions of the following problem:

$$\mathcal{M}[u] + f(x,u) = 0 \quad \text{in } \mathbb{R}, \tag{1.1}$$

where $f(x,u)$ is a KPP-type nonlinearity, periodic in x, and

$$\mathcal{M}[u] := J \star u - u. \tag{1.2}$$

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We assume that J satisfies

$$J \in C(\mathbb{R}), \quad J \geq 0, \quad \int_{\mathbb{R}} J = 1, \tag{1.3}$$

$$\text{there exist } a < 0 < b \text{ such that } J(a) > 0,\ J(b) > 0. \tag{1.4}$$

On f we assume that

$$\begin{cases} f \in C(\mathbb{R} \times [0,\infty)) \text{ and is differentiable with respect to } u, \\ \text{for each } u,\ f(\cdot,u) \text{ is periodic with period } 2R, \\ f_u(\cdot,0) \text{ is Lipschitz}, \\ f(\cdot,0) \equiv 0 \text{ and } f(x,u)/u \text{ is decreasing with respect to } u, \\ \text{there exists } M > 0 \text{ such that } f(x,u) \leq 0 \text{ for all } u \geq M \text{ and all } x. \end{cases} \tag{1.5}$$

The model example of such a nonlinearity is

$$f(x,u) = u(a(x) - u),$$

where $a(x)$ is periodic and Lipschitz.

In a recent work, Berestycki, Hamel, and Roques [2] studied the analogue of (1.1) with a divergence operator in a periodic setting. More precisely, they considered

$$-\nabla \cdot (A(x)\nabla u) = f(x,u), \quad x \in \mathbb{R}^N, \quad u \geq 0, \tag{1.6}$$

where $A(x)$ is a symmetric matrix of class $C^{1,\alpha}$, periodic with respect to all variables and uniformly elliptic, and f is C^1 and satisfies (1.5). They showed existence of nontrivial solutions provided the linearization of the equation around zero has a negative first periodic eigenvalue.

We prove the following result.

THEOREM 1.1. *Assume J satisfies* (1.3), (1.4) *and f satisfies* (1.5). *Then there exists a nontrivial, periodic solution of* (1.1) *if and only if*

$$\lambda_1(\mathcal{M} + f_u(x,0)) < 0,$$

where λ_1 is the principal eigenvalue of the linear operator $-(\mathcal{M} + f_u(x,0))$ in the set of $2R$-periodic continuous functions. Moreover, if $\lambda_1 \geq 0$, then any nonnegative bounded solution is identically zero.

To prove Theorem 1.1, we first need to show that the principal periodic eigenvalue of $-(\mathcal{M} + f_u(x,0))$ is well defined. Let us introduce some notation:

$$C_{per}(\mathbb{R}) = \{u : \mathbb{R} \to \mathbb{R} \mid u \text{ is continuous and } 2R\text{-periodic}\},$$

$$C^{0,1}_{per}(\mathbb{R}) = \{u : \mathbb{R} \to \mathbb{R} \mid u \text{ is Lipschitz and } 2R\text{-periodic}\}.$$

THEOREM 1.2. *Suppose $a(x) \in C^{0,1}_{per}(\mathbb{R})$. Then the operator $-(\mathcal{M} + a(x))$ has a unique principal eigenvalue λ_1 in $C_{per}(\mathbb{R})$; that is, there is a unique $\lambda_1 \in \mathbb{R}$ such that*

$$\mathcal{M}[\phi_1] + a(x)\phi_1 = -\lambda_1\phi_1 \quad \text{in } \mathbb{R} \tag{1.7}$$

admits a positive solution $\phi_1 \in C_{per}(\mathbb{R})$. Moreover, λ_1 is simple, that is, the space of $C_{per}(\mathbb{R})$ solutions to (1.7) *is one dimensional.*

In [2] the authors proved that (1.6) has at most one nontrivial bounded solution, and that it has to be periodic. A similar result is true for the nonlocal problem (1.1), but this time we need J to be symmetric, that is,

$$J(x) = J(-x) \quad \text{for all } x \in \mathbb{R}. \tag{1.8}$$

Note, however, that for the existence result, Theorem 1.1, we do not need this condition.

THEOREM 1.3. *Assume J satisfies* (1.3), (1.4), (1.8) *and f satisfies* (1.5). *Let u be a nonnegative, bounded solution to* (1.1) *and let λ_1 be the principal eigenvalue of the operator $-(\mathcal{M} + f_u(x,0))$ with periodic boundary conditions.*

(a) *If $\lambda_1 < 0$, then either $u \equiv 0$ or $u \equiv p$, where p is the positive periodic solution of Theorem* 1.1.

(b) *If $\lambda_1 \geq 0$, then $u \equiv 0$.*

Part (b) of the preceding theorem is already covered in Theorem 1.1 and does not depend on the symmetry of J.

When f is independent of x and satisfies (1.5), the principal eigenvalue of $-(\mathcal{M} + f'(0))$ is given by $\lambda_1 = -f'(0)$ and ϕ_1 is just a constant. Thus in this case Theorem 1.1 says that a bounded, nonnegative, nontrivial solution exists if and only if $f'(0) > 0$, and this solution is just the constant u_0 such that $f(u_0) = 0$. Assuming that J is symmetric, Theorem 1.3 then implies that the constant u_0 is the unique solution in the class of nonnegative, bounded functions.

Recently, considering a nonperiodic nonlinearity f, Berestycki, Hamel, and Rossi [3] analyzed the analogue of Theorem 1.3 for general elliptic operators in $\mathbb{R}^N$, finding sufficient conditions that ensure existence and uniqueness of a positive bounded solution. It is natural to ask whether the periodicity of f and the symmetry of J are crucial hypotheses in Theorem 1.3. We believe that this is the case, since a general nonlocal operator such as (1.2) may contain a transport term, and a standing wave connecting the steady states of the system could appear. We shall investigate further this issue in a forthcoming work.

Hypothesis (1.4) implies that the operator $\mathcal{M}$ satisfies the strong maximum principle. Suppose, for instance, that J satisfies (1.3), (1.4). If $u \in C(\mathbb{R})$ satisfies $\mathcal{M}[u] \geq 0$ in $\mathbb{R}$, then u cannot achieve a global maximum without being constant (see [9]). However, we will need the following version.

THEOREM 1.4. *Assume J satisfies* (1.3), (1.4) *and let $c \in L^\infty(\mathbb{R})$. If $u \in L^\infty(\mathbb{R})$ satisfies $u \leq 0$ a.e. and $\mathcal{M}[u] + c(x)u \geq 0$ a.e. in $\mathbb{R}$, then* $\operatorname{ess\,sup}_K u < 0$ *for all compact $K \subset \mathbb{R}$ or $u = 0$ a.e. in $\mathbb{R}$.*

If f satisfies the stronger hypothesis that, for any x, $f(x,u)$ is concave with respect to u, then actually the periodic solution p of Theorem 1.1 is continuous. To see this notice that from the strong maximum principle, Theorem 1.4, $J \star p > 0$ in $\mathbb{R}$. The concavity of f with respect to u implies that for any x the map $u \mapsto u - f(x,u)$ is strictly increasing whenever $u - f(x,u) > 0$. Then from the continuity of $J \star p$ and (1.1), which can be rewritten as in the form $J \star p = p - f(x,p)$, we deduce that p is continuous.

In section 2 we review some spectral theory and give the argument of Theorem 1.2. Then we prove Theorem 1.1 in section 3 and the uniqueness result, Theorem 1.3(a), in section 4. We leave for an appendix a proof of Theorem 1.4.

2. Some spectral theory. In this section we deal with the principal eigenvalue problem (1.7). Before stating our result, let us recall some basic spectral results for positive operators due to Edmunds, Potter, and Stuart [11] which are extensions of the Krein–Rutmann theorem for positive noncompact operators.

A cone in a real Banach space X is a nonempty closed set K such that for all $x, y \in K$ and all $\alpha \geq 0$ one has $x + \alpha y \in K$, and if $x \in K$, $-x \in K$, then $x = 0$. A cone K is called reproducing if $X = K - K$. A cone K induces a partial ordering in X by the relation $x \leq y$ if and only if $x - y \in K$. A linear map or operator $T : X \to X$ is called positive if $T(K) \subseteq K$. The dual cone K^* is the set of functionals $x^* \in X^*$ which are positive, that is, such that $x^*(K) \subset [0, \infty)$.

If $T : X \to X$ is a bounded linear map on a complex Banach space X, its essential spectrum (according to Browder [5]) consists of those λ in the spectrum of T such that at least one of the following conditions holds: (1) the range of $\lambda I - T$ is not closed, (2) λ is a limit point of the spectrum of T, (3) $\cup_{n=1}^{\infty} \ker((\lambda I - T)^n)$ is infinite dimensional. The radius of the essential spectrum of T, denoted by $r_e(T)$, is the largest value of $|\lambda|$ with λ in the essential spectrum of T. For more properties of $r_e(T)$ see [20].

THEOREM 2.1 (Edmunds, Potter, and Stuart [11]). *Let K be a reproducing cone in a real Banach space X, and let $T \in \mathcal{L}(X)$ be a positive operator such that $T^p(u) \geq cu$ for some $u \in K$ with $\|u\| = 1$, some positive integer p, and some positive number c. Then if $c^{\frac{1}{p}} > r_e(T)$, T has an eigenvector $v \in K$ with associated eigenvalue $\rho \geq c^{\frac{1}{p}}$ and T^* has an eigenvector $v^* \in K^*$ corresponding to the eigenvalue ρ.*

A proof of this theorem can be found in [11]. If the cone K has nonempty interior and T is strongly positive, i.e., $u \geq 0$, $u \neq 0$ implies $Tu \in \operatorname{int}(K)$, then ρ is the unique $\lambda \in \mathbb{R}$ for which there exists nontrivial $v \in K$ such that $Tv = \lambda v$ and ρ is simple; see [22].

Proof of Theorem 1.2. For convenience, in this proof we write the eigenvalue problem

$$\mathcal{M}[u] + a(x)u = -\lambda u$$

in the form

$$\mathcal{L}[u] + b(x)u = \mu u, \tag{2.1}$$

where

$$\mathcal{L}[u] = J \star u, \quad b(x) = a(x) + k, \quad \mu = -\lambda + 1 + k,$$

and $k > 0$ is a constant such that $\inf_{[-R,R]} b > 0$.

Observe that $\mathcal{L} : C_{per(\mathbb{R})} \to C_{per}(\mathbb{R})$ is compact ($C_{per}(\mathbb{R})$ is endowed with the norm $\|u\|_{L^\infty([-R,R])}$). Indeed, let $u_n \in C_{per}(\mathbb{R})$ be a bounded sequence, say $\|u_n\|_{L^\infty([-R,R])} \leq B$. Let $\epsilon > 0$ and let A be large enough so that $\int_{|x| \geq A} J \leq \epsilon$. Since J is uniformly continuous in $[-R - 2A, R + 2A]$ there is $\delta > 0$ such that $|J(z_1) - J(z_2)| \leq \frac{\epsilon}{2(A+R)}$ for $z_1, z_2 \in [-R - 2A, R + 2A]$ with $|z_1 - z_2| \leq \delta$. Then for $x_1, x_2 \in [-R, R]$,

$$\begin{aligned} |\mathcal{L}[u_n](x_1) - \mathcal{L}[u_n](x_2)| &\leq \int_{\mathbb{R}} |J(x_1 - y) - J(x_2 - y)|\, |u_n(y)|\, dy \\ &\leq 2B\epsilon + B \int_{-R-A}^{R+A} |J(x_1 - y) - J(x_2 - y)|\, dy \\ &\leq 3B\epsilon. \end{aligned}$$

This shows that $\mathcal{L}[u_n]$ is equicontinuous, and therefore by the Arzelà–Ascoli theorem, $\mathcal{L}[u_n]$ is relatively compact.

Let us now establish some useful lemma.

LEMMA 2.2. *Suppose* $b(x) \in C^{0,1}(\mathbb{R})$ *is* $2R$*-periodic,* $b(x) > 0$*, and let* $\sigma := \max_{[-R,R]} b(x)$*. Then there exist* $p \in \mathbb{N}, \delta > 0$*, and* $u \in C_{per}(\mathbb{R})$*,* $u \geq 0$*,* $u \not\equiv 0$*, such that*

$$\mathcal{L}^p u + b(x)^p u \geq (\sigma^p + \delta)u.$$

Observe that the proof of Theorem 1.2 will then easily follow from the above lemma. Indeed, if the lemma holds, then since u and b are nonnegative and $\mathcal{L}$ is a positive operator, we easily see that

$$(\mathcal{L} + b(x))^p[u] \geq \mathcal{L}^p[u] + b(x)^p u \geq (\sigma^p + \delta)u.$$

Using the compactness of the operator $\mathcal{L}$, we have $r_e(\mathcal{L} + b(x)) = r_e(b(x))=\sigma$, and thus $(\sigma^p + \delta)^{\frac{1}{p}} > r_e(\mathcal{L} + b(x))$ and Theorem 2.1 applies. Finally, we observe that the principal eigenvalue is simple since the cone of positive $2R$-periodic functions has nonempty interior and, for a sufficiently large p, the operator $(\mathcal{L} + b)^p$ is strongly positive. □

Let us now turn our attention to the proof of the above lemma.

Proof of Lemma 2.2. Recall that for $p \in \mathbb{N} \setminus \{0\}$, $J \star^p u := J \star (J \star^{p-1} u)$ is well defined by induction and satisfies $J \star^p u = \mathcal{J}_p \star u$ with $\mathcal{J}_p$ defined as follows:

$$\mathcal{J}_p := \underbrace{J \star J \star \cdots \star J \star J}_{p \text{ times}}.$$

By (1.4) it follows that there exists $p \in \mathbb{N}$ such that $\inf_{(-2R-1,2R+1)} \mathcal{J}_p > 0$. Using the definition of $\mathcal{L}$, a short computation shows that

$$\mathcal{L}^p[u] := \int_{-R}^{R} \widetilde{\mathcal{J}}_p(x,y)u(y)\,dy$$

with $\widetilde{\mathcal{J}}_p(x,y) = \sum_{k\in\mathbb{Z}} \mathcal{J}_p(x + 2kR - y)$. Following the idea of Hutson et al. [14], consider now the following function:

$$v(x) := \begin{cases} \frac{\eta(x)}{b^p(x_0)-b^p(x)+\gamma} & \text{in } \Omega_{2\epsilon} := (x_0 - 2\epsilon, x_0 + 2\epsilon), \\ 0 & \text{elsewhere,} \end{cases}$$

where $x_0 \in (-R,R)$ is a point of maximum of $b(x)$, $\epsilon > 0$ is chosen such that $(x_0 - 2\epsilon, x_0 + 2\epsilon) \subset (-R,R)$, γ is a positive constant that we will define later on, and η is a smooth function such that $0 \leq \eta \leq 1$, $\eta(x) = 1$ for $|x - x_0| \leq \epsilon$, $\eta(x) = 0$ for $|x - x_0| \geq 2\epsilon$. Let us compute $\mathcal{L}^p[v] + b^p(x)v - \sigma^p v$:

$$\begin{aligned}
\mathcal{L}^p[v] + b^p(x)v - \sigma^p v &= \int_{x_0-\epsilon}^{x_0+\epsilon} \widetilde{\mathcal{J}}_p(x,y)\frac{dy}{b^p(x_0) - b^p(y) + \gamma} + \int_{\Omega_{2\epsilon}\setminus\Omega_\epsilon} \widetilde{\mathcal{J}}_p(x,y)v(y)\,dy \\
&\quad + (b^p(x) - b^p(x_0))v \\
&\geq \int_{x_0-\epsilon}^{x_0+\epsilon} \widetilde{\mathcal{J}}_p(x,y)\frac{dy}{b^p(x_0) - b^p(y) + \gamma} + (b^p(x) - b^p(x_0))v \\
&\geq \int_{x_0-\epsilon}^{x_0+\epsilon} \widetilde{\mathcal{J}}_p(x,y)\frac{dy}{b^p(x_0) - b^p(y) + \gamma} - 1.
\end{aligned}$$

Using that $\inf_{(-2R-1,2R+1)} \mathcal{J}_p > 0$, it follows that $\widetilde{\mathcal{J}}_p(x,y) \geq c > 0$ for $x, y \in (-R, R)$. Hence

$$\int_{x_0-\epsilon}^{x_0+\epsilon} \widetilde{\mathcal{J}}_p(x,y) \frac{dy}{b^p(x_0) - b^p(y) + \gamma} \geq c \int_{x_0-\epsilon}^{x_0+\epsilon} \frac{dy}{k|x_0 - y| + \gamma},$$

where k is the Lipschitz constant for b^p. Using this inequality in the above estimate yields

$$\mathcal{L}^p[v] + b^p(x)v - \sigma^p v \geq c \int_{x_0-\epsilon}^{x_0+\epsilon} \frac{dy}{k|x_0 - y| + \gamma} - 1.$$

Therefore we have

$$\begin{aligned}\mathcal{L}^p[v] + b^p(x)v - (\sigma^p + \delta)v &\geq \frac{2c}{k} \log\left(1 + \frac{k\epsilon}{\gamma}\right) - 1 - \delta v \\ &\geq \frac{2c}{k} \log\left(1 + \frac{k\epsilon}{\gamma}\right) - 1 - \frac{\delta}{\gamma}.\end{aligned}$$

Choosing now $\gamma > 0$ small so that $\frac{2c}{k} \log(1 + \frac{k\epsilon}{\gamma}) - 1 > \frac{1}{2}$ and $\delta = \frac{\gamma}{4}$, we end up with

$$\mathcal{L}^p[v] + b^p(x)v - (\sigma^p + \delta)v \geq \frac{1}{4} > 0. \qquad \square$$

3. Existence of solutions.

Proof of Theorem 1.1. We follow the argument developed by Berestycki, Hamel, and Roques in [2].

First assume that $\lambda_1 < 0$. From Theorem 1.2 there exists a positive eigenfunction ϕ_1 such that

$$\mathcal{M}[\phi_1] + f_u(x,0)\phi_1 = -\lambda_1 \phi_1 \geq 0.$$

Computing $\mathcal{M}[\epsilon\phi_1] + f(x, \epsilon\phi_1)$, it follows that

$$\begin{aligned}\mathcal{M}[\epsilon\phi_1] + f(x, \epsilon\phi_1) &= f(x, \epsilon\phi_1) - f_u(x,0)\epsilon\phi_1 - \lambda_1\epsilon\phi_1 \\ &= -\lambda_1 \epsilon\phi_1 + o(\epsilon\phi_1) > 0.\end{aligned}$$

Therefore, for $\epsilon > 0$ small, $\epsilon\phi_1$ is a periodic subsolution of (1.1). By definition of f, any constant M sufficiently large is a periodic supersolution of the problem. Choosing M so large that $\epsilon\phi_1 \leq M$ and using a basic iterative scheme yields the existence of a positive periodic solution u of (1.1).

Let us now turn our attention to the nonexistence setting and assume that $\lambda_1 \geq 0$. Let u be a bounded nonnegative solution of (1.1). Observe that $\gamma\phi_1$ is a periodic supersolution for any positive γ. Indeed,

$$\begin{aligned}\mathcal{M}[\gamma\phi_1] + f(x, \gamma\phi_1) &< \mathcal{M}[\gamma\phi_1] + f_u(x,0)\gamma\phi_1 \\ &\leq -\lambda_1 \gamma\phi_1 \leq 0.\end{aligned}$$

Since $\phi_1 \geq \delta$ for some positive δ we may define the following quantity:

$$\gamma^* := \inf\{\gamma > 0 | u \leq \gamma\phi_1\}.$$

We have the following claim.

CLAIM 3.1. $\gamma^* = 0$.

Observe that we end the proof of the theorem by proving the above claim.

Proof of the claim. Assume that $\gamma^* > 0$. Since $v := u - \gamma^*\phi_1$ satisfies $v \le 0$ in $\mathbb{R}$ and

$$\mathcal{M}[v] + c(x)v \ge 0 \quad \text{in } \mathbb{R},$$

where $c(x) = \frac{f(x,u)-f(x,\gamma^*\phi_1)}{v}$ by the strong maximum principle, Theorem 1.4, we have the following possibilities:

- either $u \equiv \gamma^*\phi_1$, or
- there exists a sequence of points $(x_n)_{n\in\mathbb{N}}$ such that $|x_n| \to +\infty$ and $\lim_{n\to+\infty} \gamma^*\phi_1(x_n) - u(x_n) = 0$.

In the first case we get the following contradiction:

$$0 = \mathcal{M}[\gamma^*\phi_1] + f(x, \gamma^*\phi_1) < \mathcal{M}[\gamma^*\phi_1] + f_u(x, 0)\gamma^*\phi_1 \le 0.$$

Hence $\gamma^* = 0$.

In the second case we argue as follows. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence of points satisfying, for all n, $y_n \in [-R, R]$ and $x_n - y_n \in 2R\mathbb{Z}$. Up to extraction of a subsequence, $y_n \to \bar{y}$. Now consider the following sequence of functions $u_n := u(. + x_n)$, $\phi_n := \phi_1(. + x_n)$, and $w_n := \gamma^*\phi_n - u_n$ so that $w_n > 0$ in $\mathbb{R}$. Since $\mathcal{M}$ is translation invariant and f is periodic, u_n and $\phi_n > 0$ satisfy

$$\mathcal{M}[u_n] + f(x + y_n, u_n) = 0 \quad \text{in } \mathbb{R},$$

$$\mathcal{M}[\gamma^*\phi_n] + f_u(x + y_n, 0)\gamma^*\phi_n \le 0 \quad \text{in } \mathbb{R}.$$

It follows that

$$J \star w_n \le a_n(x)w_n,$$

where

$$a_n(x) = 1 - \frac{\gamma^* f_u(x + y_n, 0)\phi_n - f(x + y_n, u_n)}{\gamma^*\phi_n - u_n}.$$

Since $w_n > 0$ we see that a_n is well defined and $a_n \ge 0$. Using that $f(x, u)/u$ is nonincreasing with respect to u we have $f(x, \gamma^*\phi_n) \le \gamma^* f_u(x, 0)\phi_n$. This implies

$$\frac{\gamma^* f_u(x + y_n, 0)\phi_n - f(x + y_n, u_n)}{\gamma^*\phi_n - u_n} \ge \frac{f(x + y_n, \gamma^*\phi_n) - f(x + y_n, u_n)}{\gamma^*\phi_n - u_n} \ge -C.$$

Thus

$$0 \le a_n \le C + 1 \quad \text{in } \mathbb{R} \quad \text{for all } n,$$

with C independent of n. Observe that

$$J \star w_n(0) = a_n(0)(\gamma^*\phi_1(x_n) - u(x_n)) \to 0,$$

which implies

$$\int_{\mathbb{R}} J(-y)w_n(y)\, dy \to 0 \quad \text{as } n \to +\infty.$$

Similarly,

$$J \star J \star w_n(0) = J \star (a_n w_n)(0) = \int_{\mathbb{R}} J(-y) a_n(y) w_n(y)\, dy,$$

but

$$\int_{\mathbb{R}} J(-y) a_n(y) w_n(y)\, dy \le \|a_n\|_{L^\infty} \int_{\mathbb{R}} J(-y) w_n(y)\, dy \to 0.$$

Hence

$$J \star J \star w_n(0) = \int_{\mathbb{R}} (J \star J)(-y) w_n(y)\, dy \to 0 \quad \text{as } n \to +\infty.$$

Defining

$$\mathcal{J}_k := \underbrace{J \star \cdots \star J}_{k \text{ times}},$$

we see that for any fixed $k \in \mathbb{N}$,

$$\int_{\mathbb{R}} \mathcal{J}_k(-y) w_n(y)\, dy \to 0 \quad \text{as } n \to +\infty.$$

By (1.4) the support of $\mathcal{J}_k$ increases to all of $\mathbb{R}$ as $k \to +\infty$. Thus we may find a new subsequence such that $w_n \to 0$ a.e. in $\mathbb{R}$ as $n \to +\infty$. Since ϕ_1 is periodic and continuous, $\phi_n(x) \to \bar{\phi}(x)$ uniformly with respect to x, where $\bar{\phi}(x) = \phi(x + \bar{y})$. Hence $\bar{u}(x) = \lim_{n\to+\infty} u_n(x)$ exists a.e. and is given by $\bar{u}(x) = \gamma^* \bar{\phi}$. By dominated convergence, $\bar{u}$ is a solution to

$$\mathcal{M}[\bar{u}] + f(x + \bar{y}, \bar{u}) = 0,$$

while by uniform convergence

$$\mathcal{M}[\gamma^* \bar{\phi}] + f_u(x + \bar{y}, 0)\gamma^* \bar{\phi} \le 0 \quad \text{in } \mathbb{R}.$$

Since $\bar{u} = \gamma^* \bar{\phi}$ it follows that $f(x + \bar{y}, \gamma^* \bar{\phi}) \equiv f_u(x + \bar{y}, 0)\gamma^* \bar{\phi}$. This contradicts the fact that $f(x, u)/u$ is decreasing in u. Hence, $\gamma^* = 0$. ∎

4. Uniqueness when *J* is symmetric. Throughout this section we assume that J is symmetric. For the proof of Theorem 1.3 we follow the ideas in [2].

Proof of Theorem 1.3. Part (b) of this theorem is contained in Theorem 1.1 so we concentrate on part (a).

Let p denote the positive periodic solution to (1.1) constructed in Theorem 1.1 and let $u \ge 0$, $u \not\equiv 0$ be a bounded solution. We will prove that $u \equiv p$.

We show first that $u \le p$. Set

$$\gamma^* := \inf\{\gamma > 0 \mid u \le \gamma p\}.$$

Note that γ^* is well defined because u is bounded and p is bounded below by a positive constant. We claim that

$$\gamma^* \le 1.$$

Suppose that $\gamma^* > 1$ and note that $u \le \gamma^* p$. By Theorem 1.4 either $u \equiv \gamma^* p$ or $\operatorname{ess\,inf}_K(\gamma^* p - u) > 0$ for all compact $K \subset \mathbb{R}$. The first possibility leads to $f(x, \gamma^* p) = \gamma^* f(x, p)$ for all $x \in \mathbb{R}$, which is not possible if $\gamma^* > 1$. In the second case there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $|x_n| \to +\infty$ and $\lim_{n\to+\infty} \gamma^* p(x_n) - u(x_n) = 0$. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence satisfying $y_n \in [-R, R]$ and $x_n - y_n = k_n 2R$ for some $k_n \in \mathbb{Z}$. We may assume that $y_n \to \bar{y}$. Let $u_n := u(. + x_n)$, which satisfies

$$\mathcal{M}[u_n] + f(x + y_n, u_n) = 0.$$

Let $w_n = \gamma^* p(. + y_n) - u_n \ge 0$. Then $w_n > 0$ in $\mathbb{R}$ and

$$J \star w_n = a_n(x) w_n,$$

where

$$a_n(x) = 1 - \frac{\gamma^* f(x + y_n, p(x + y_n)) - f(x + y_n, u_n(x))}{\gamma^* p(x + y_n) - u_n(x)}.$$

Since $w_n > 0$ we deduce that a_n is well defined and $a_n \ge 0$. Using that $f(x, u)/u$ is nonincreasing with respect to u and the fact that $\gamma^* > 1$, we have $f(x, \gamma^* p) \le \gamma^* f(x, p)$. This implies

$$\frac{\gamma^* f(x, p) - f(x, u)}{\gamma^* p - u} \ge \frac{f(x, \gamma^* p) - f(x, u)}{\gamma^* p - u} \ge -C.$$

Thus

$$0 \le a_n \le C + 1 \quad \text{in } \mathbb{R} \quad \text{for all } n,$$

with C independent of n. Observe that

$$J \star w_n(0) = a_n(0)(\gamma^* p(y_n) - u(x_n)) = a_n(0)(\gamma^* p(x_n) - u(x_n)) \to 0,$$

which implies

$$\int_{\mathbb{R}} J(-y) w_n(y)\, dy \to 0 \quad \text{as } n \to +\infty.$$

Similarly,

$$J \star J \star w_n(0) = J \star (a_n w_n)(0) = \int_{\mathbb{R}} J(-y) a_n(y) w_n(y)\, dy,$$

but

$$\int_{\mathbb{R}} J(-y) a_n(y) w_n(y)\, dy \le \|a_n\|_{L^\infty} \int_{\mathbb{R}} J(-y) w_n(y)\, dy \to 0.$$

Hence

$$J \star J \star w_n(0) = \int_{\mathbb{R}} (J \star J)(-y) w_n(y)\, dy \to 0 \quad \text{as } n \to +\infty.$$

Defining

$$\mathcal{J}_k := \underbrace{J \star \cdots \star J}_{k \text{ times}},$$

we see that for all $k \in \mathbb{N}$,

$$\int_{\mathbb{R}} \mathcal{J}_k(-y) w_n(y)\, dy \to 0 \quad \text{as } n \to +\infty.$$

Hypothesis (1.4) implies that the support of $\mathcal{J}_k$ converges to all of $\mathbb{R}$ as $k \to +\infty$. Therefore, for a subsequence, $w_n \to 0$ a.e. in $\mathbb{R}$ as $n \to +\infty$. Since p is periodic, for possibly a new subsequence $p(x + y_n) \to p(x + \bar{y})$ a.e. Hence, $\bar{u}(x) = \lim_{n\to+\infty} u_n(x)$ exists a.e. and by dominated convergence, $\bar{u}$ is a solution to

$$\mathcal{M}[\bar{u}] + f(x + \bar{y}, \bar{u}) = 0. \tag{4.1}$$

But since $w_n \to 0$ a.e. we have $\bar{u} = \gamma^* p(\cdot + \bar{y})$. Thus $\gamma^* p(\cdot + \bar{y})$ is a solution to (4.1), which is impossible for $\gamma^* > 1$ as argued before.

The proof that $p \le u$ is analogous, but a key point is to prove first that under the conditions of Theorem 1.3 any nontrivial, nonnegative solution is bounded below by a positive constant. This is the content of Proposition 4.1. ∎

PROPOSITION 4.1. *Assume that J satisfies* (1.3), (1.4), *and* (1.8), *f satisfies* (1.5), *and that the operator $-(\mathcal{M} - f_u(x, 0))$ has a negative principal periodic eigenvalue. Suppose that u is a nonnegative, bounded solution to* (1.1). *Then $u \equiv 0$ or there exists a constant $c > 0$ such that*

$$u(x) \ge c \quad \textit{for all } x \in \mathbb{R}.$$

The basic tool to prove Proposition 4.1, following an idea in [2], is to study the principal eigenvalue of the linearized operator in bounded domains. More precisely, let $\Omega = (-r, +r)$ and $a\colon \Omega \to \mathbb{R}$ be Lipschitz. We consider the eigenvalue problem in Ω with "Dirichlet boundary condition" in the following sense:

$$\begin{cases} \mathcal{M}[\varphi] + a(x)\varphi = -\lambda\varphi & \text{in } \Omega, \\ \varphi(x) = 0 \quad \text{for all } x \notin \Omega, \\ \varphi|_{\overline{\Omega}} \text{ is continuous.} \end{cases} \tag{4.2}$$

We show that the principal eigenvalue for (4.2) exists and converges to the principal periodic eigenvalue as $r \to +\infty$. The first step is to establish variational characterizations of these eigenvalues, which is the argument that requires the symmetry of J.

LEMMA 4.2. *Let $\Omega \subset \mathbb{R}$ be a bounded open interval. Assume that J satisfies* (1.3), (1.4), *and* (1.8), *and let $a\colon \Omega \to \mathbb{R}$ be Lipschitz. Then there exists a smallest λ_1 such that* (4.2) *has a nontrivial solution. This eigenvalue is simple and the eigenfunctions are of constant sign in Ω. Moreover,*

$$\lambda_1 = \min_{\varphi \in C(\overline{\Omega})} -\frac{\int_\Omega (\mathcal{M}[\widetilde{\varphi}] + a(x)\varphi)\varphi}{\int_\Omega \varphi^2}, \tag{4.3}$$

where $\widetilde{\varphi}$ denotes the extension by 0 *of φ to $\mathbb{R}$ and the minimum is attained.*

The statement and the proof are analogous to those of Theorem 3.1 in [14] except that here we do not assume that $J(0) > 0$. A different formula for the principal eigenvalue with a Dirichlet boundary condition appears in [7], where it is used to characterize the rate of decay of solutions to a linear evolution equation.

Proof. Define the operator $X[\varphi] = \int_\Omega J(x-y)\,\varphi(y)\,dy$ for $\varphi \in C(\overline{\Omega})$. Then $X : C(\overline{\Omega}) \to C(\overline{\Omega})$ is compact. Let $c_0 > 0$ be such that $\inf_\Omega a(x) + c_0 > 0$ and define $\widetilde{a} = a + c_0$. The eigenvalue problem (4.2) is equivalent to the following: find $\varphi \in C(\overline{\Omega})$ and $\lambda \in \mathbb{R}$ such that

$$X[\varphi] + \widetilde{a}\varphi = (-\lambda + 1 + c_0)\varphi \quad \text{in } \Omega.$$

A calculation similar to Lemma 2.2 shows that there exists an integer p, $u \in C(\overline{\Omega})$, and $\delta > 0$ such that

$$(X + \widetilde{a})^p u \geq \left(\left(\max_{\overline{\Omega}} \widetilde{a} \right)^p + \delta \right) u \quad \text{in } \Omega. \tag{4.4}$$

Using Theorem 2.1 we deduce that the operator $X + \widetilde{a}$ has a unique principal eigenvalue $\rho > 0$ and a principal eigenvector $\varphi_1 \in C(\overline{\Omega})$. Let $\lambda = 1 + c_0 - \rho$ so that $X[\varphi_1] + a(x)\varphi_1 = (1-\lambda)\varphi_1$. From (4.4) we deduce that σ_+ defined by

$$\sigma_+ = \sup_{\varphi \in C(\overline{\Omega})} \frac{\int_\Omega (X[\varphi] + a(x)\varphi)\varphi}{\int_\Omega \varphi^2} \tag{4.5}$$

satisfies

$$\sigma_+ \geq 1 - \lambda > \max_{\overline{\Omega}} a. \tag{4.6}$$

Now, using the same argument as in [14] we deduce that the supremum in (4.5) is achieved. Indeed, it is standard [4] that the spectrum of $\hat{X} + a(x)$ is to the left of σ_+ and that there exists a sequence $\varphi_n \in C(\overline{\Omega})$ such that $\|\varphi_n\|_{L^2(\Omega)} = 1$ and $\|(X + a(x) - \sigma_+)\varphi_n\|_{L^2(\Omega)} \to 0$ as $n \to +\infty$. By compactness of $X : L^2(\Omega) \to C(\overline{\Omega})$ for a subsequence, $\lim_{n\to+\infty} X[\varphi_n]$ exists in $C(\overline{\Omega})$. Then, using (4.6), we see that $\varphi_n \to \varphi$ in $L^2(\Omega)$ for some φ and $(X + a)\varphi = \sigma_+\varphi$. This equation implies $\varphi \in C(\overline{\Omega})$, and hence σ_+ is a principal eigenvalue for the operator X and by uniqueness of this eigenvalue we have $\sigma_+ = 1 - \lambda$. ∎

LEMMA 4.3. *Assume that J satisfies* (1.3), (1.4), *and* (1.8) *and that $a : \mathbb{R} \to \mathbb{R}$ is a $2R$-periodic, Lipschitz function. Then the principal eigenvalue of the operator $-(\mathcal{M} + a(x))$ in $C_{per}(\mathbb{R})$ is given by*

$$\lambda_1(a) = \inf_{\|\varphi\|_{L^2(\mathbb{R})}=1} - \int_{\mathbb{R}} (\mathcal{M}[\varphi] + a(x)\varphi)\varphi \tag{4.7}$$

$$= \min_{\varphi \in C_{per}(\mathbb{R})} - \frac{\int_{-R}^{R} (\mathcal{M}[\varphi] + a(x)\varphi)\varphi}{\int_{-R}^{R} \varphi^2}. \tag{4.8}$$

Proof. By Theorem 1.2 we know that there exists a unique principal eigenvalue $\lambda_1(a)$ of the operator $-(\mathcal{M} + a)$ in $C_{per}(\mathbb{R})$. Let $\phi_1 \in C_{per}(\mathbb{R})$ denote a positive eigenfunction associated with $\lambda_1(a)$. We normalize ϕ_1 such that

$$\int_{-R}^{R} \phi_1^2 = 2R. \tag{4.9}$$

On the other hand, the quantity

$$\widetilde{\lambda}_1(a) = \inf_{\varphi \in C_{per}(\mathbb{R})} - \frac{\int_{-R}^{R} (\mathcal{M}[\varphi] + a(x)\varphi)\varphi}{\int_{-R}^{R} \varphi^2}$$

is also an eigenvalue of $-(\mathcal{M}+a)$ on $C_{per}(\mathbb{R})$ with a positive eigenfunction. By uniqueness of the principal eigenvalue, $\lambda_1(a)=\widetilde{\lambda}_1(a)$.

We claim that

$$\inf_{\|\varphi\|_{L^2(\mathbb{R})}=1} -\int_{\mathbb{R}} (\mathcal{M}[\varphi]+a(x)\varphi)\varphi \le \lambda_1(a).$$

Indeed, for $r>0$ let $\eta_r \in C_0^\infty(\mathbb{R})$ be such that $0\le \eta_r \le 1$, $\eta_r(x)=1$ for $|x|\le r$, $\eta_r(x)=0$ for $|x|\ge r+1$. It will be sufficient to show that

$$\lim_{r\to+\infty} \frac{\int_{\mathbb{R}}(\mathcal{M}[\phi_1\eta_r]+a\phi_1\eta_r)\phi_1\eta_r}{\int_{\mathbb{R}}(\phi_1\eta_r)^2} = -\lambda_1(a). \tag{4.10}$$

By (4.9) we have

$$\int_{\mathbb{R}} (\phi_1\eta_r)^2 = 2r + O(1) \quad \text{ as } r\to+\infty. \tag{4.11}$$

Let $0<\theta<1$. Then

$$\begin{aligned} |\mathcal{M}[\phi_1](x) - \mathcal{M}[\phi_1\eta_r]| &\le \|\phi_1\|_{L^\infty} \int_{|x-z|\ge r} |J(z)|\,dz \\ &\le \|\phi_1\|_{L^\infty} \int_{|z|\ge(1-\theta)r} |J(z)|\,dz \quad \text{for all } |x|\le\theta r \\ &= o(1) \quad \text{uniformly for all } |x|\le \theta r. \end{aligned} \tag{4.12}$$

We split the integral

$$\int_{\mathbb{R}} (\mathcal{M}[\phi_1\eta_r]+a\phi_1\eta_r)\phi_1\eta_r = \int_{|x|\le\theta r}\dots dx + \int_{|x|\ge\theta r}\dots dx. \tag{4.13}$$

Using $\eta_r(x)=1$ for $|x|\le\theta r$ and (4.12) we see that

$$\begin{aligned} \int_{|x|\le\theta r} (\mathcal{M}[\phi_1\eta_r]+a\phi_1\eta_r)\phi_1\eta_r &= \int_{|x|\le\theta r} (\mathcal{M}[\phi_1\eta_r]+a\phi_1)\phi_1 \\ &= \int_{|x|\le\theta r} (\mathcal{M}[\phi_1]+a\phi_1+o(1))\phi_1 \\ &= -2\theta\lambda_1(a)r + o(r) \quad \text{ as } r\to+\infty. \end{aligned}$$

The second integral in (4.13) is bounded by

$$\left| \int_{|x|\ge\theta r} (\mathcal{M}[\phi_1\eta_r]+a\phi_1\eta_r)\phi_1\eta_r \right| \le C(1-\theta)r. \tag{4.14}$$

Thus from (4.11)–(4.14) we conclude that

$$\left| \frac{\int_{\mathbb{R}}(\mathcal{M}[\phi_1\eta_r]+a\phi_1\eta_r)\phi_1\eta_r}{\int_{\mathbb{R}}(\phi_1\eta_r)^2} + \lambda_1(a) \right| \le C(1-\theta)+o(1),$$

which proves (4.10).

To establish (4.7) it remains to verify that

$$\lambda_1(a) \le -\frac{\int_{\mathbb{R}}(\mathcal{M}[\varphi]+a(x)\varphi)\varphi}{\int_{\mathbb{R}}\varphi^2} \quad \text{for all } \varphi\in C_c(\mathbb{R}). \tag{4.15}$$

By uniqueness of the principal eigenvalue we have

$$\lambda_1(a) = \inf_{\varphi\in C_{per}(\Omega_k)} -\frac{\int_{-kR}^{kR}(\mathcal{M}[\varphi]+a(x)\varphi)\varphi}{\int_{-kR}^{kR}\varphi^2}, \tag{4.16}$$

where

$$\Omega_k = (-kR, kR) \quad \text{ for } k\ge 1$$

and $C_{per}(\Omega_k)$ is the set of continuous $2kR$-periodic functions on $\mathbb{R}$.

Fix $\varphi\in C_c(\mathbb{R})$ and consider k large enough so that $\text{supp}(\varphi)\subseteq\Omega_k$. Consider now φ_k the $4kR$-periodic extension of φ. Since $\varphi_k\in C_{per}(\Omega_{2k})$, (4.16) yields

$$\lambda_1(a) \le -\frac{\int_{-2kR}^{2kR}(\mathcal{M}[\varphi_k]+a(x)\varphi_k)\varphi_k}{\int_{-2kR}^{2kR}\varphi_k^2} = -\frac{\int_{\mathbb{R}}(\mathcal{M}[\varphi_k]+a(x)\varphi)\varphi}{\int_{\mathbb{R}}\varphi^2}. \tag{4.17}$$

For $|x|\le kR$ we have

$$|\mathcal{M}[\varphi_k](x)-\mathcal{M}[\varphi](x)| \le \|\varphi\|_{L^\infty}\int_{|y|\ge 2kR}|J(x-y)|\,dy \le \|\varphi\|_{L^\infty}\int_{|z|\ge kR}|J(z)|\,dz.$$

Hence

$$\lim_{k\to+\infty}\int_{\mathbb{R}}(\mathcal{M}[\varphi_k]+a(x)\varphi)\varphi = \int_{\mathbb{R}}(\mathcal{M}[\varphi]+a(x)\varphi)\varphi. \tag{4.18}$$

Thanks to (4.17) and (4.18), we conclude the validity of (4.15). □

LEMMA 4.4. *Assume J satisfies* (1.3), (1.4), *and* (1.8) *and that $a\colon\mathbb{R}\to\mathbb{R}$ is a $2R$-periodic, Lipschitz function. Let $\lambda_{r,y}$ be the principal eigenvalue of* (4.2) *for*

$$\Omega_{r,y} = B_r(y)$$

and let $\lambda_1(a)$ denote the principal eigenvalue of $-(\mathcal{M}+a(x))$ in $C_{per}(\mathbb{R})$. Then

$$\lim_{r\to+\infty}\lambda_{r,y} = \lambda_1(a).$$

Moreover, the applications $y\mapsto\lambda_{r,y}$ and $y\mapsto\varphi_{r,y}$ are periodic. The periodicity of the application $y\mapsto\varphi_{r,y}$ is understood as follows:

$$\varphi_{r,y+2R}(x) = \varphi_{r,y}(x-2R).$$

Proof. For convenience we write

$$\lambda_r = \lambda_{r,y}$$

and let φ_r be a positive eigenfunction of (4.2) in Ω_r.

By the variational characterization (4.3) we see that $r \mapsto \lambda_r$ is nonincreasing, and hence $\lim_{r\to+\infty} \lambda_r$ exists. Moreover, using (4.7) we have

$$\lambda_r \geq \lambda_1(a) \quad \text{for all } r > 0. \tag{4.19}$$

Let $\phi_1 \in C_{per}(\mathbb{R})$ be a positive eigenfunction of $-(\mathcal{M} + a(x))$ with eigenvalue $\lambda_1(a)$ normalized such that

$$\int_{-R}^{R} \phi_1^2 = 2R.$$

Let $\eta_r \in C_0^\infty(\mathbb{R})$ be such that $0 \leq \eta \leq 1$,

$$\eta_r(x) = 1 \text{ for } |x-y| \leq r-1, \qquad \eta_r(x) = 0 \text{ for } |x-y| \geq r$$

and such that $\|\eta_r\|_{C^2(\mathbb{R})} \leq C$ with C independent of r. Arguing in the same way as in the proof of Lemma 4.3 we obtain

$$\lim_{r\to+\infty} \frac{\int_{\mathbb{R}}(\mathcal{M}[\phi_1\eta_r] + a\phi_1\eta_r)\phi_1\eta_r}{\int_{\mathbb{R}}(\phi_1\eta_r)^2} = -\lambda_1(a).$$

Since

$$\lambda_r \leq -\frac{\int_{\mathbb{R}}(\mathcal{M}[\phi_1\eta_r] + a\phi_1\eta_r)\phi_1\eta_r}{\int_{\mathbb{R}}(\phi_1\eta_r)^2}$$

we conclude that

$$\lim_{r\to+\infty} \lambda_r \leq \lambda_1(a).$$

This and (4.19) prove the desired result.

Let us now show the periodicity of the applications $y \mapsto \lambda_{r,y}$ and $y \mapsto \varphi_{r,y}$. Replace y by $y+2R$ in the above problem (4.2) and let us denote by $\lambda_{r,y+2R}$ and $\varphi_{r,y+2R}$ the corresponding principal eigenvalue and the associated positive eigenfunction:

$$\mathcal{M}[\varphi_{r,y+2R}] + a(x)\varphi_{r,y+2R} = -\lambda_{r,y+2R}\varphi_{r,y+2R} \quad \text{ in } B_r(y+2R).$$

We take the following normalization:

$$\int_{\Omega_{r,y+2R}} \varphi_{r,y+2R}^2(x)\,dx = 1.$$

Let us defined $\psi(x) := \varphi_{r,y+2R}(x+2R)$ for any $x \in B_r(y)$. A short computation shows that

$$\mathcal{M}[\psi](x) = \mathcal{M}[\varphi]_{r,y+2R}(x+2R).$$

Therefore, using the periodicity of $a(x)$, we have

$$\mathcal{M}[\psi](x) + a(x+2R)\psi(x) = \lambda_{r,y+2R}\psi \quad \text{in } B_r(y),$$
$$\mathcal{M}[\psi](x) + a(x)\psi(x) = \lambda_{r,y+2R}\psi \quad \text{in } B_r(y).$$

Thus, $\lambda_{r,y+2R}$ is a principal eigenvalue of the problem (4.2) with $\Omega_{r,y} = B_r(y)$. Hence, by uniqueness of the principal eigenvalue we have $\lambda_{r,y} = \lambda_{r,y+2R}$ and $\psi = \gamma\varphi_{r,y}$ for some positive γ. Using the normalization, it follows that $\gamma = 1$. Therefore, $\varphi_{r,y}(x) = \varphi_{r,y+2R}(x + 2R)$; in other words

$$\varphi_{r,y+2R}(x) = \varphi_{r,y}(x - 2R). \qquad \square$$

Remark 4.5. The proof of Lemma 4.4 yields the slightly stronger conclusion that the convergence

$$\lim_{r\to+\infty} \lambda_{r,y} = \lambda_1(a)$$

is uniform with respect to $y \in \mathbb{R}$, since $\lambda_{r,y}$ is continuous in y.

Proof of Proposition 4.1. Let $u \geq 0$ be a bounded solution to (1.1) such that $u \not\equiv 0$. By the strong maximum principle (Theorem 1.4) we must have $\inf_K u > 0$ for compact sets $K \subset \mathbb{R}$.

Given $y \in \mathbb{R}$ and $r > 0$ we write $\Omega_{r,y} = (y-r, y+r)$, $\lambda_{r,y}$ the principal eigenvalue of $-(\mathcal{M} + f_u(x, 0))$ with Dirichlet boundary condition in $\Omega_{r,y}$ as in (4.2), and $\varphi_{r,y}$ a positive Dirichlet eigenfunction normalized so that

$$\int_{\Omega_{r,y}} \varphi_{r,y}^2 = 1.$$

Since the principal eigenvalue $\lambda_1 := \lambda_1(f_u(x, 0))$ of $-(\mathcal{M} + f_u(x, 0))$ with periodic boundary conditions is negative by hypothesis, by Lemma 4.4 and Remark 4.5 we may fix $r > 0$ large enough so that

$$\lambda_{r,y} < \lambda_1/2 \quad \text{for all } y \in \mathbb{R}.$$

Note that for $x \in \Omega_{r,y}$,

$$\begin{aligned}\mathcal{M}[\gamma\varphi_{r,y}] + f(x, \gamma_{r,y}) &= -\lambda_{r,y}\gamma\varphi_{r,y} - f_u(x, 0)\gamma\varphi_{r,y} + f(x, \gamma\varphi_{r,y})\\ &\geq -\lambda_1/2\gamma\varphi_{r,y} - f_u(x, 0)\gamma\varphi_{r,y} + f(x, \gamma\varphi_{r,y})\\ &\geq 0\end{aligned}$$

if $0 \leq \gamma \leq \gamma_0$ with γ_0 fixed suitably small. For $x \notin \Omega_{y,r}$ we have $\varphi_{y,r}(x) = 0$ and $\mathcal{M}[\varphi_{r,y}] \geq 0$. Thus

$$\mathcal{M}[\gamma\varphi_{r,y}] + f(x, \gamma\varphi_{r,y}) \geq 0 \quad \text{in } \mathbb{R} \tag{4.20}$$

for all $0 < \gamma < \gamma_0$.

We claim that

$$\gamma_0\varphi_{r,y} \leq u \quad \text{in } \mathbb{R} \quad \text{for all } y \in \mathbb{R}. \tag{4.21}$$

This proves the proposition because there is a positive constant c such that $\varphi_{r,y}(y) \geq c$ for all $y \in \mathbb{R}$ since the application $y \mapsto \varphi_{r,y}$ is periodic and $\varphi_{r,y}(y) > 0$ for any $y \in [-2R, 2R]$.

Now, to prove (4.21) fix $y \in \mathbb{R}$ and set

$$\gamma^* = \sup\{\, \gamma > 0 \,/\, \gamma\varphi_{r,y} \leq u \text{ in } \mathbb{R}\}.$$

Since $\inf_K u > 0$ for compact sets $K \subset \mathbb{R}$ and $\varphi_{r,y}$ has compact support we see that $\gamma^* > 0$. Assume that $\gamma^* < \gamma_0$. Then by (4.20), $\gamma^* \varphi_{r,y}$ is a subsolution of (1.1) while u is a solution. By the strong maximum principle (Theorem 1.4) either $\gamma^* \varphi_{r,y} \equiv u$ in $\mathbb{R}$ or $\inf_K (u - \gamma^* \varphi_{r,y}) > 0$ for compact sets $K \subset \mathbb{R}$. The former case is impossible because u is strictly positive, while the latter case yields a contradiction with the definition of γ^*. It follows that $\gamma^* \geq \gamma_0$ as desired. ∎

Appendix. In this appendix we give a short proof of Theorem 1.4. We assume that J satisfies (1.3), (1.4), $c \in L^\infty(\mathbb{R})$, and $u \in L^\infty(\mathbb{R})$ satisfies

$$\begin{aligned} u \leq 0 \quad &\text{a.e. in } \mathbb{R}, \\ \mathcal{M}[u] + cu \geq 0 \quad &\text{a.e. in } \mathbb{R}. \end{aligned} \tag{A.1}$$

For $\epsilon > 0$ define

$$u_\epsilon(x) = \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} u.$$

Then u_ϵ is continuous in $\mathbb{R}$, $u_\epsilon \leq 0$, and $u_\epsilon \to u$ a.e. as $\epsilon \to 0$. There are two cases:

(1) for any closed interval I one has $\limsup_{\epsilon \to 0} \sup_I u_\epsilon < 0$, or

(2) for some closed interval I one has $\limsup_{\epsilon \to 0} \sup_I u_\epsilon = 0$.

If case (1) occurs, we see that for all closed intervals I we have ess $\sup_I u < 0$. Assume case (2) holds. Let I be a closed interval and $\epsilon_n \to 0$ be such that $\lim_{n \to +\infty} u_{\epsilon_n}(x_n) = 0$, where $x_n \in I$ is such that $\sup_I u_{\epsilon_n} = u_{\epsilon_n}(x_n)$. Integrating (A.1) from $x_n - \epsilon_n$ to $x_n + \epsilon_n$ and dividing by $2\epsilon_n$, we have

$$J \star u_{\epsilon_n}(x_n) \geq u_{\epsilon_n}(x_n) - \frac{1}{2\epsilon_n} \int_{x_n - \epsilon_n}^{x_n + \epsilon_n} cu.$$

But, since $u \leq 0$ a.e.,

$$\left| \frac{1}{2\epsilon_n} \int_{x_n - \epsilon_n}^{x_n + \epsilon_n} cu \right| \leq -\|c\|_{L^\infty} u_{\epsilon_n}(x_n) \to 0.$$

Hence

$$\liminf_{n \to +\infty} J \star u_{\epsilon_n}(x_n) \geq 0.$$

We may assume that $x_n \to x \in I$. Then by dominated convergence,

$$J \star u_{\epsilon_n}(x_n) = \int_{\mathbb{R}} J(x_n - y) u_{\epsilon_n}(y)\, dy \to \int_{\mathbb{R}} J(x - y) u(y)\, dy.$$

This shows that $u = 0$ a.e. in $x - \mathrm{supp}(J)$. Now, for any x_1 in the interior of $x - \mathrm{supp}(J)$ we have $J \star u(x_1) \geq 0$, which shows that $u = 0$ a.e. in $x - 2\mathrm{supp}(J)$, where $2\mathrm{supp}(J) = \mathrm{supp}(J) + \mathrm{supp}(J)$. Note that assumption (1.4) implies that $k\,\mathrm{supp}(J)$ covers all of $\mathbb{R}$ as $k \to +\infty$, where $k\,\mathrm{supp}(J)$ is defined inductively as $(k-1)\,\mathrm{supp}(J) + \mathrm{supp}(J)$. Repeating the previous argument we deduce that $u = 0$ a.e. in $\mathbb{R}$. ∎

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