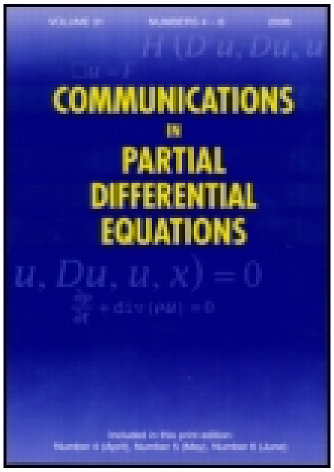

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Regularity of Radial Extremal Solutions for Some Non-Local Semilinear Equations

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We investigate stable solutions of elliptic equations of the type

$$\begin{cases} (-\Delta)^s u = \lambda f(u) & \textit{in } B_1 \subset \mathbb{R}^n \\ u = 0 & \textit{on } \partial B_1, \end{cases}$$

where $n \geq 2$, $s \in (0,1)$, $\lambda \geq 0$ and f is any smooth positive superlinear function. The operator $(-\Delta)^s$ stands for the fractional Laplacian, a pseudo-differential operator of order $2s$. According to the value of λ, we study the existence and regularity of weak solutions u.

Keywords Boundary reactions; Extremal solutions; Fractional operators.

Mathematics Subject Classification 35J25; 47G30; 35B45; 53A05.

1. Introduction

We are interested in the regularity properties of stable solutions satisfying the following semilinear problem involving the fractional Laplacian

$$\begin{cases} (-\Delta)^s u = \lambda f(u) & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases} \tag{1.1}$$

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Here, B_1 denotes the unit-ball in $\mathbb{R}^n$, $n \geq 2$, and $s \in (0, 1)$. The operator $(-\Delta)^s$ is defined as follows: let $\{\varphi_k\}_{k=1}^{\infty}$ denote an orthonormal basis of $L^2(B_1)$ consisting of eigenfunctions of $-\Delta$ in B_1 with homogeneous Dirichlet boundary conditions, associated to the eigenvalues $\{\mu_k\}_{k=1}^{\infty}$. Namely, $0 < \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \leq \mu_k \to +\infty$, $\int_{B_1} \varphi_j \varphi_k \, dx = \delta_{j,k}$ and

$$\begin{cases} -\Delta \varphi_k = \mu_k \varphi_k & \text{in } B_1 \\ \varphi_k = 0 & \text{on } \partial B_1. \end{cases}$$

The operator $(-\Delta)^s$ is defined for any $u \in C_c^\infty(B_1)$ by

$$(-\Delta)^s u = \sum_{k=1}^{\infty} \mu_k^s u_k \varphi_k, \tag{1.2}$$

where

$$u = \sum_{k=1}^{\infty} u_k \varphi_k, \quad \text{and} \quad u_k = \int_{B_1} u \varphi_k \, dx.$$

This operator can be extended by density for u in the Hilbert space

$$H = \left\{ u \in L^2(B_1) : \|u\|_H^2 = \sum_{k=1}^{\infty} \mu_k^s |u_k|^2 < +\infty \right\}. \tag{1.3}$$

Note that

$$H = \begin{cases} H^s(B_1) & \text{if } s \in (0, 1/2), \\ H_{00}^{1/2}(B_1) & \text{if } s = 1/2, \\ H_0^s(B_1) & \text{if } s \in (1/2, 1), \end{cases}$$

see Section 2 for further details. In all cases, $(-\Delta)^s : H \to H'$ is an isometric isomorphism from H to its topological dual H'. We denote by $(-\Delta)^{-s}$ its inverse, i.e., for $\psi \in H'$, $\varphi = (-\Delta)^{-s}\psi$ if φ is the unique solution in H of $(-\Delta)^s \varphi = \psi$.

The boundary condition $u = 0$ that appears in (1.1) has to be interpreted with some care if $0 < s < 1/2$, see the discussion in Section 2.

We will assume that the nonlinearity f is smooth, nondecreasing,

$$f(0) > 0, \quad \text{and} \quad \lim_{u \to +\infty} \frac{f(u)}{u} = +\infty. \tag{1.4}$$

In the spirit of [3], weak solutions for (1.1) are defined as follows: let $\varphi_1 > 0$ denote the eigenfunction associated to the principal eigenvalue of the operator $-\Delta$ with homogeneous Dirichlet boundary condition on B_1, normalized by $\|\varphi_1\|_{L^2(B_1)} = 1$.

Definition 1.1. A measurable function u in B_1 such that $\int_{B_1} |u| \varphi_1 \, dx < +\infty$ and $\int_{B_1} f(u) \varphi_1 \, dx < +\infty$, is a weak solution of (1.1) if

$$\int_{B_1} u \psi \, dx = \lambda \int_{B_1} f(u) (-\Delta)^{-s} \psi \, dx, \quad \text{for all } \psi \in C_c^\infty(B_1). \tag{1.5}$$

The right-hand side in (1.5) is well defined, since for every $\psi \in C_c^\infty(B_1)$, there exists a constant $C > 0$ such that $|(-\Delta)^{-s}\psi| \leq C\varphi_1$; see Lemma 3.1 and its proof.

The boundary condition $u = 0$ that appears in (1.1) is implicitly present in the weak formulation (1.5), similarly as in [3]. If $u \in C(\overline{B}_1)$ is a weak solution then one can deduce that u vanishes on the boundary.

We shall be interested in weak solutions of (1.1) having the following stability property.

Definition 1.2. A weak solution u of (1.1) is semi-stable if for all $\psi \in C_c^\infty(B_1)$ we have

$$\int_{B_1} |(-\Delta)^{\frac{s}{2}}\psi|^2\, dx \geq \int_{B_1} f'(u)\psi^2\, dx. \tag{1.6}$$

The following result gives the existence of solutions according to the values of λ.

Proposition 1.3. *Let $s \in (0, 1)$. There exists $\lambda^* > 0$ such that*

- *for $0 < \lambda < \lambda^*$, there exists a minimal solution $u_\lambda \in H \cap L^\infty(B_1)$ of* (1.1). *In addition, u_λ is semi-stable and increasing with λ.*
- *for $\lambda = \lambda^*$, the function $u^* = \lim_{\lambda \nearrow \lambda^*} u_\lambda$ is a weak solution of* (1.1). *We call λ^* the extremal value of the parameter and u^* the extremal solution.*
- *for $\lambda > \lambda^*$,* (1.1) *has no solution $u \in H \cap L^\infty(B_1)$.*

For the proof, see Section 3.

Remark 1.4. Proposition 1.3 remains true when B_1 is replaced by any smooth bounded domain.

Remark 1.5. For $0 < \lambda < \lambda^*$, the solution u_λ is minimal in the sense that $u_\lambda \leq u$ for any other weak solution u. In particular, u_λ and u^* are radial. In addition, u_λ and u^* are radially decreasing (see Section 4) and $u_\lambda \in C^\infty(B_1) \cap C^\alpha(\overline{B_1})$ for $\alpha \in (0, \min(2s, 1))$ (see Section 2). If u^* is bounded, then we also have $u^* \in C^\infty(B_1) \cap C^\alpha(\overline{B_1})$ for $\alpha \in (0, \min(2s, 1))$, using again Section 2.

Here is our main result, concerning the regularity of the extremal solution u^*.

Theorem 1.6. *Assume $n \geq 2$ and let u^* be the extremal solution of* (1.1). *We have that*:

(a) *If $n < 2(s + 2 + \sqrt{2(s+1)})$ then $u^* \in L^\infty(B_1)$.*
(b) *If $n \geq 2(s + 2 + \sqrt{2(s+1)})$, then for any $\mu > n/2 - 1 - \sqrt{n-1} - s$,*

there exists a constant $C > 0$ such that $u^*(x) \leq C|x|^{-\mu}$ for all $x \in B_1$.

Remark 1.7. In particular, for any $2 \leq n \leq 6$, any $s \in (0, 1)$, and any smooth nondecreasing f such that (1.4) holds, the extremal solution is always bounded.

Remark 1.8. We do not know if the bound $n < 2(s + 2 + \sqrt{2(s+1)})$ is optimal for the regularity of u^*. We note however that $\lim_{s \to 1^-} 2(s + 2 + \sqrt{2(s+1)}) = 10$, and that the extremal solution of

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.7}$$

is singular when $\Omega = B_1$, $f(u) = e^u$, and $n = 10$ (see e.g., [18]).

Nonlinear equations involving fractional powers of the Laplacian are currently actively studied. Caffarelli, Salsa and Silvestre studied free boundary problems for such operators in [11, 12]. Cabré and Tan [10] obtained several results in analogy with the classical Lane-Emden problem $-\Delta u = u^p$, posed on bounded domains and entire space, related to the role of the critical exponent. Previously, some authors considered elliptic equations with nonlinear Neumann boundary condition, which share some properties with semilinear equations of the form (1.1), see e.g., [9, 14].

Equation (1.1) is the fractional Laplacian version of the classical semilinear elliptic equation (1.7). When $f(u) = e^u$, (1.7) is known as the Liouville equation [21] or the Gelfand problem [16]. Joseph and Lundgren [18] showed in this case that if Ω is a ball, then the extremal solution u^* of (1.7) is bounded if and only if $n < 10$. Crandall and Rabinowitz [13] and Mignot and Puel [22] proved that if $f(u) = e^u$ and $n < 10$ then for any smoothly bounded domain Ω, u^* is bounded. Using Hardy's inequality, Brezis and Vázquez [4] provided a different proof that u^* is singular when $\Omega = B_1$ and $n \geq 10$. For some other explicit nonlinearities, such as $f(u) = (1+u)^p$ with $p > 1$ or $p < 0$, the critical dimension for the regularity of the extremal solution is known (for further details see the above mentioned references). For general nonlinearities, Nedev [23] proved that for any convex function f satisfying (1.4), and any smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n \leq 3$, u^* is bounded. This result has been extended by Cabré to the case $n = 4$ and Ω strictly convex [5]. Finally, Cabré and Capella [6] showed that if Ω is a ball and $n \leq 9$ then for any nonlinearity f satisfying (1.4), the extremal solution is bounded.

Theorem 1.6 is an extension to the fractional Laplacian defined by (1.2) of this result. There are other nonequivalent (see [24]) ways of defining the fractional Laplacian in B_1. Roughly speaking, interior regularity results for these operators are the same, but boundary regularity is different. As in [6], the proof of Theorem 1.6 uses the stability condition to deduce weighted integrability of a radial derivative of the solution, in a way which is independent of the nonlinearity. Since we work with radially decreasing solutions, this information is relevant near the origin, and therefore one can expect that for other definitions of fractional Laplacian Theorem 1.6 would also hold true with the same restriction on n and s. The optimal condition for n and s, can still depend on the definition of the fractional Laplacian.

2. Preliminaries

2.1. *Functional Spaces*

We start by recalling some functional spaces, see for instance [20, 25]. For $s \geq 0$, $H^s(\mathbb{R}^n)$ is defined as

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : |\xi|^s \hat{u}(\xi) \in L^2(\mathbb{R}^n)\}$$

where $\hat{u}$ denotes the Fourier transform of u, with norm

$$\|u\|_{H^s(\mathbb{R}^n)} = \|(1+|\xi|^s)\hat{u}(\xi)\|_{L^2(\mathbb{R}^n)}.$$

This norm is equivalent to

$$\|u\|_{L^2(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2}.$$

Given a smooth bounded domain $\Omega \subset \mathbb{R}^n$ and $0 < s < 1$, the space $H^s(\Omega)$ is defined as the set of functions $u \in L^2(\Omega)$ for which the following norm is finite

$$\|u\|_{H^s(\Omega)} = \|u\|_{L^2(\Omega)} + \left(\int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2}.$$

An equivalent construction consists of restrictions of functions in $H^s(\mathbb{R}^n)$. We define $H_0^s(\Omega)$ as the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^s(\Omega)}$. It is well known that for $0 < s \le \frac{1}{2}$, $H_0^s(\Omega) = H^s(\Omega)$, while for $1/2 < s < 1$ the inclusion $H_0^s(\Omega) \subseteq H^s(\Omega)$ is strict (see Theorem 11.1 in [20]).

The space H defined in (1.3) is the interpolation space $(H_0^2(\Omega), L^2(\Omega))_{s,2}$, see for example [2, 20, 25]. Here we follow the notation from [25, Chap. 22]. Lions and Magenes [20] showed that $(H_0^2(\Omega), L^2(\Omega))_{s,2} = H_0^s(\Omega)$ for $0 < s < 1$, $s \neq 1/2$, while

$$(H_0^2(\Omega), L^2(\Omega))_{1/2,2} = H_{00}^{1/2}(\Omega)$$

where

$$H_{00}^{1/2}(\Omega) = \left\{u \in H^{1/2}(\Omega) : \int_\Omega \frac{u(x)^2}{d(x)}\,dx < +\infty\right\},$$

and $d(x) = \operatorname{dist}(x, \partial\Omega)$ for all $x \in \Omega$.

An important feature of the operator $(-\Delta)^s$ is its nonlocal character, which is best seen by realizing the fractional Laplacian as the boundary operator of a suitable extension in the half-cylinder $\Omega \times (0,\infty)$. Such an interpretation was demonstrated in [12] for the fractional Laplacian in $\mathbb{R}^n$. Their construction can easily be extended to the case of bounded domains as described below.

Let us define

$$\mathscr{C} = \Omega \times (0, +\infty),$$
$$\partial_L \mathscr{C} = \partial\Omega \times [0, +\infty).$$

We write points in the cylinder using the notation $(x, y) \in \mathscr{C} = \Omega \times (0, +\infty)$.

Given $s \in (0, 1)$, consider the space $H_{0,L}^1(y^{1-2s})$ of measurable functions $v : \mathscr{C} \to \mathbb{R}$ such that $v \in H^1(\Omega \times (s, t))$ for all $0 < s < t < +\infty$, $v = 0$ on $\partial_L\mathscr{C}$ and for which the following norm is finite

$$\|v\|^2_{H_{0,L}^1(y^{1-2s})} = \int_{\mathscr{C}} y^{1-2s}|\nabla v|^2\,dx\,dy.$$

Proposition 2.1. *There exists a trace operator from* $H^1_{0,L}(y^{1-2s})$ *into* $H^s_0(\Omega)$. *Furthermore, the space* H *given by* (1.3) *is characterized by*

$$H = \{u = tr_\Omega v : v \in H^1_{0,L}(y^{1-2s})\}.$$

Proof. For the case $s = 1/2$ see Proposition 2.1 in [10].

We consider now $s \neq 1/2$. Restating the results of Paragraph 5 of Lions [19], there exists a constant $C > 0$ such that

$$\|v(\cdot, 0)\|^2_{H^s(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^n \times (0,+\infty)} y^{1-2s} \left(v^2 + |\nabla v|^2\right) \, dx\, dy,$$

whenever the right-hand side in the above inequality is finite. Now for any $v \in H^1_{0,L}(y^{1-2s})$,

$$\int_{\mathscr{C}} y^{1-2s} v^2 \, dx\, dy \leq C \int_{\mathscr{C}} y^{1-2s} |\nabla v|^2 \, dx\, dy,$$

as follows from the standard Poincaré inequality in Ω. Hence, extending v by zero outside $\mathscr{C}$, we deduce that

$$\|v(\cdot, 0)\|_{H^s(\Omega)} \leq C \|v\|_{H^1_{0,L}(y^{1-2s})}.$$

This inequality shows that there exists a linear bounded trace operator

$$tr_\Omega : H^1_{0,L}(y^{1-2s}) \to H^s(\Omega).$$

This operator has its image contained in $H^s_0(\Omega)$. This is direct for $0 < s < 1/2$ because in this case $H^s_0(\Omega) = H^s(\Omega)$. If $1/2 < s < 1$ we argue that any $v \in H^1_{0,L}(y^{1-2s})$ can be approximated by functions in $H^1_{0,L}(y^{1-2s})$ that have support away from $\partial_L \mathscr{C}$. The trace of any such function has compact support in Ω and is therefore in $H^s_0(\Omega)$. In all cases, this implies that the image of the trace operator is contained in H.

Let us prove $tr_\Omega : H^1_{0,L}(y^{1-2s}) \to H$ is surjective. Take a function $u \in H$ and let us prove that there exists $v \in H^1_{0,L}(y^{1-2s})$ such that $tr_\Omega(v) = u$. Write its spectral decomposition $u(x) = \sum_{k=1}^{+\infty} b_k \varphi_k(x)$ and consider the function

$$v(x, y) = \sum_{k=1}^{+\infty} b_k \varphi_k(x) g_k(y), \tag{2.1}$$

where g_k satisfies

$$g_k'' + \frac{1-2s}{y} g_k' - \mu_k g_k = 0 \quad \text{in } (0, +\infty) \tag{2.2}$$

$$g_k(0) = 1 \qquad g_k(+\infty) = 0. \tag{2.3}$$

This ODE is a Bessel equation. Two independent solutions are given by $y^s I_s(\sqrt{\mu_k} y)$ and $y^s K_s(\sqrt{\mu_k} y)$, where I_s, K_s are the modified Bessel functions of the first and second kind, see [1]. Since I_s increases exponentially at infinity and K_s decreases exponentially, the solution we are seeking has the form

$$g_k(y) = c_k y^s K_s(\sqrt{\mu_k} y).$$

It is well-known that $K_s(t) = at^{-s} + o(t^{-s})$ as $t \to 0$, where $a > 0$. Therefore, one can choose c_k such that $g_k(0) = 1$ and one can see that g_k can be written in the form

$$g_k(y) = h(\sqrt{\mu_k} y),$$

for a fixed function h that verifies $h(0) = 1$ and $h'(t) = -ct^{2s-1} + o(t)$ as $t \to 0$, for some constant $c = c_{n,s} > 0$ depending only on s and n. This implies that

$$\lim_{y\to 0^+} -y^{1-2s} g_k'(y) = c_{n,s}\mu_k^s. \tag{2.4}$$

Since each of the functions g_k decreases exponentially at infinity we see that v defined by (2.1) is smooth for $y > 0$, $x \in \Omega$ and moreover satisfies

$$\operatorname{div}(y^{1-2s}\nabla v) = 0 \quad \text{in } \mathscr{C}.$$

Let us check that $v \in H^1_{0,L}(y^{1-2s})$. For any $y > 0$, by the properties of φ_k:

$$\int_\Omega |\nabla v(x, y)|^2\, dx = \sum_{k=1}^{\infty} b_k^2(\mu_k g_k(y)^2 + g_k'(y)^2).$$

Integrating with respect to y over $(\delta, +\infty)$ where $\delta > 0$:

$$\int_\delta^\infty \int_\Omega y^{1-2s}|\nabla v(x, y)|^2\, dx\, dy = \sum_{k=1}^{\infty} b_k^2(-y^{1-2s} g_k'(y) g_k(y))|_{y=\delta}. \tag{2.5}$$

From the ODE (2.2) we deduce that $g_k \geq 0$, $g_k' \leq 0$ and $g_k'(y)y^{1-2s}$ is non-decreasing. Thus, if $\delta_i \downarrow 0$, $i \to \infty$ is a decreasing sequence, $-\delta_i^{1-2s} g_k'(\delta_i) g_k(\delta_i)$ is increasing. By monotone convergence and thanks to (2.4) we deduce

$$\int_0^\infty \int_{\mathbb{R}^n} y^{1-2s}|\nabla v(x, y)|^2\, dx\, dy = c_{n,s} \sum_{k=1}^{+\infty} b_k^2 \mu_k^s.$$

This proves that $H \subseteq tr_\Omega(H^1_{0,L}(y^{1-2s}))$. $\square$

Let us remark that if $u \in H$, then the minimization problem

$$\min\left\{\int_{\mathscr{C}} y^{1-2s}|\nabla v|^2\, dx\, dy : v \in H^1_{0,L}(y^{1-2s}),\ tr_\Omega(v) = u\right\}$$

has a solution $v \in H^1_{0,L}(y^{1-2s})$, by the weak lower semi-continuity of the norm $\|\ \|_{H^1_{0,L}(y^{1-2s})}$ and continuity of tr_Ω. Moreover the minimizer v is unique, which follows e.g. from the strict convexity of the functional. By standard elliptic theory $v(x, y)$ is smooth for $y > 0$ and satisfies

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla v) = 0 & \text{in } \mathscr{C} \\ v = 0 & \text{on } \partial_L \mathscr{C} \\ v = u & \text{on } \Omega \times \{0\} \end{cases}$$

where the boundary condition on $\Omega \times \{0\}$ is in the sense of trace. For each $y > 0$ we may write $v(x, y) = \sum_{k=1}^{\infty} \varphi_k(x) g_k(y)$ where $g_k(y) = \int_\Omega \varphi_k(x) v(x, y)\, dx$. Since $v(\cdot, y) \to u$ in $L^2(\Omega)$ as $y \to 0$, $g_k(0)$ are the Fourier coefficients of u, that is $u = \sum_{k=1}^{\infty} g_k(0)\varphi_k$. Then we deduce that $g_k(y)$ is smooth for $y > 0$ and satisfies the ODE (2.2). One can check that $g_k(y) \to 0$ as $y \to +\infty$ and therefore $g_k(y) = c_k y^s K_s(\sqrt{\mu_k} y)$ for all $y > 0$ and some $c_k \in \mathbb{R}$. Then, similarly as in (2.5), we obtain for $\delta > 0$

$$\int_\delta^\infty \int_{\mathbb{R}^n} y^{1-2s} |\nabla v(x, y)|^2 \, dx\, dy = \sum_{k=1}^{\infty} (-y^{1-2s} g_k'(y) g_k(y))|_{y=\delta}. \tag{2.6}$$

Arguing as before, for each k

$$\lim_{y \downarrow 0} (-y^{1-2s} g_k'(y) g_k(y)) = c\mu_k^s g_k(0)^2.$$

We deduce from (2.6) that

$$\|u\|_H^2 = \sum_{k=1}^{\infty} \mu_k^s g_k(0) = c\|v\|^2_{H^1_{0,L}(y^{1-2s})}.$$

In what follows we will call v the *canonical extension* of u.

2.2. *Solvability for Data in $H^{-s}(\Omega)$*

This section is devoted to prove the following lemma:

Lemma 2.2. *Let $h \in H'$. Then, there is a unique solution to the problem*:

$$\textit{find } u \in H \textit{ such that} \quad (-\Delta)^s u = h. \tag{2.7}$$

Moreover u is the trace of $v \in H^1_{0,L}(y^{1-2s})$, where v is the unique solution to

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla v) = 0 & \text{in } \mathscr{C} \\ v = 0 & \text{on } \partial_L \mathscr{C} \\ -\lim_{y \to 0}(y^{1-2s} v_y) = c_{n,s} h & \text{on } \Omega \times \{0\} \end{cases} \tag{2.8}$$

where $c_{n,s} > 0$ is a constant depending on n and s only.

Remark 2.3. Equation (2.8) is understood in the sense that $v \in H^1_{0,L}(y^{1-2s})$ and

$$c_{n,s}\langle h, tr_\Omega(\zeta)\rangle_{H',H} = \int_{\mathscr{C}} y^{1-2s} \nabla v \nabla \zeta \, dx\, dy \quad \text{for all } \zeta \in H^1_{0,L}(y^{1-2s}), \tag{2.9}$$

where $\langle , \rangle_{H',H}$ is the duality pairing between H and H'. The constant $c_{n,s}$ is the same constant appearing in (2.4).

Remark 2.4. If $1/2 \leq s < 1$, instead of (2.7) we could use the notation

$$\begin{cases} (-\Delta)^s u = h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.10}$$

since for these values of s there is a trace operator from H to $L^2(\partial\Omega)$ and the boundary condition can be interpreted in this sense. For $0 < s < 1/2$ this interpretation is no longer possible. We will see later that if h is bounded then the solution u of (2.7) has a representative that is continuous up to the boundary, with zero boundary values, so the notation (2.10) is justified in this case. Note however that for $0 < s < 1/2$ and for arbitrary $h \in H'$, $u = 0$ on $\partial\Omega$ does not have a clear meaning. For instance, $h = (-\Delta)^s 1 \in H'$ and the solution to (2.7) is $u = 1$. For simplicity, from here on we use the notation (2.10) even if it is not entirely correct, and it will always mean (2.7).

Proof of Lemma 2.2. The case $s = 1/2$ was treated in [10].

The space H' can be identified with the space of distributions $h = \sum_{k=1}^{\infty} h_k \varphi_k$ such that $\sum_{k=1}^{\infty} h_k^2 \mu_k^{-s} < \infty$. Then, it is straightforward to verify that for any $h \in H'$ there is a unique $u \in H$ such that $(-\Delta)^s u = h$. Fix now $h = \varphi_k$ for some $k \geq 1$ and let $u = \mu_k^{-s} \varphi_k$, so that $(-\Delta)^s u = h$. By the Lax–Milgram theorem, there is a unique $v \in H^1_{0,L}(y^{1-2s})$ such that (2.9) holds. Letting g_k denote the unique solution of (2.2)–(2.3), by a direct computation, we find that

$$v(x, y) = \mu_k^{-s} \varphi_k(x) g_k(y)$$

solves (2.9), with $h = \varphi_k$ and its trace is given by $\mu_k^{-s} \varphi_k = u$. This proves the lemma in the case $h = \varphi_k$. By linearity and density, the same holds true for any $h \in H'$. □

2.3. *Maximum Principles*

We say $h \in H'$ satisfies $h \geq 0$, if

$$\langle h, \varphi \rangle_{H',H} \geq 0 \quad \text{for all } \varphi \in H, \ \varphi \geq 0. \tag{2.11}$$

Lemma 2.5. *Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ any bounded open set. Take $h \in H'$ and let $u \in H$ be the corresponding solution of* (2.7). *Let also $v \in H^1_L(y^{1-2s})$ denote the canonical extension of u. If $h \geq 0$, then $u \geq 0$ a.e. in Ω and $v \geq 0$ in $\mathscr{C}$.*

Proof. Simply use v^- as a test function in (2.9). □

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^n$ denote any domain and take $R > 0$. Let v denote any locally integrable function on $\Omega \times (0, R)$ such that*

$$\int_{\Omega \times (0,R)} y^{1-2s} |\nabla v|^2 \, dx \, dy < +\infty.$$

Assume in addition that

$$-\nabla \cdot (y^{1-2s}\nabla v) = 0 \quad in\ \Omega \times (0, R),$$

$v \geq 0$ *in* $\Omega \times (0, R)$, *and* $-y^{1-2s}v_y|_{y=0} \geq 0$ *in* Ω *in the sense that*

$$\int_{\Omega\times(0,R)} y^{1-2s}\nabla v \cdot \nabla\zeta\, dx\, dy \geq 0$$

for all $\zeta \in H^1(y^{1-2s}, \Omega \times (0, R))$ *such that* $\zeta \geq 0$ *a.e. in* $\Omega \times (0, R)$ *and* $\zeta = 0$ *on* $\partial\Omega \times (0, R) \cup \Omega \times \{R\}$.

Then, either $v \equiv 0$, *or for any compact subset* K *of* $\Omega \times [0, R)$,

$$ess\ inf\ v|_K > 0.$$

Proof. Let $\tilde{v}$ denote the even extension of v with respect to the y variable, defined in $\Omega \times (-R, R)$ by

$$\tilde{v}(x, y) = \begin{cases} v(x, y) & \text{if } y > 0, \\ v(x, -y) & \text{if } y < 0. \end{cases}$$

Then,

$$\int_{\Omega\times(-R,R)} y^{1-2s}\nabla\tilde{v}\nabla\zeta\, dx\, dy \geq 0,$$

for all $\zeta \in H^1(y^{1-2s}, \Omega \times (-R, R))$, such that $\zeta \geq 0$ a.e. in $\Omega \times (-R, R)$ and $\zeta = 0$ on $\partial(\Omega \times (-R, R))$. By the results of Fabes, Kenig, and Serapioni (see Theorem 2.3.1 and the second line of equation (2.3.7) in [15]), either $\tilde{v} \equiv 0$, or ess inf $\tilde{v}|_K > 0$ for any compact set K of $\Omega \times (-R, R)$. □

Lemma 2.7. *Let* $\Omega \subset \mathbb{R}^n$ *denote an open set satisfying an interior sphere condition at some point* $x_0 \in \partial\Omega$, *that is,* $x_0 \in \partial B_r(x_1)$ *for some ball* $B_r(x_1) \subset \Omega$.*Let* $R > 2$ *and let* v *denote any measurable function on* $\Omega \times (0, R)$, $v \geq 0$, $v \not\equiv 0$, *such that*

$$\int_{\Omega\times(0,R)} y^{1-2s}|\nabla v|^2\, dx\, dy < +\infty.$$

Assume in addition that

$$-\nabla \cdot (y^{1-2s}\nabla v) = 0 \quad in\ \Omega \times (0, R),$$

and $-y^{1-2s}v_y|_{y=0} \geq 0$ *in* Ω *in the sense that*

$$\int_{\Omega\times(0,R)} y^{1-2s}\nabla v \cdot \nabla\zeta\, dx\, dy \geq 0$$

for all $\zeta \in H^1(y^{1-2s}, \Omega \times (0, R))$ *such that* $\zeta \geq 0$ *a.e. in* $\Omega \times (0, R)$ *and* $\zeta = 0$ *on* $\partial\Omega \times (0, R) \cup \Omega \times \{R\}$.

Then, there exists $\epsilon > 0$ and a constant $c = c(R) > 0$ such that

$$v(x, y) \geq c|x - x_0|$$

for all x in the line segment from x_1 to x_0 with $|x - x_0| < \epsilon$ and all $y \in [0, R-2)$.

Proof. Take an interior sphere B which is tangent to $\partial\Omega$ at x_0. Translating and dilating Ω if necessary, we may always assume that B is the unit ball centered at the origin. Take $\alpha > n-2$ to be fixed later and consider $z = z(x, y)$ the function defined by

$$z(x, y) = (1 + y^{2s})(e^{-y^2} - e^{-(R-1)^2})(|x|^{-\alpha} - 1) \quad \text{for } x \neq 0 \quad \text{and} \quad y \in [0, R-1].$$

We compute

$$\begin{aligned} \Delta_x z &= (1 + y^{2s})(e^{-y^2} - e^{-(R-1)^2})\alpha(\alpha - (N-2))|x|^{-\alpha-2}, \\ \lim_{y\to 0^+}(-y^{1-2s}z_y) &= -2s(1 - e^{-(R-1)^2})(|x|^{-\alpha} - 1) \quad \text{for } x \neq 0, \\ z_{yy} + \frac{1-2s}{y}z_y &= -4e^{-y^2}[(1-s) + (1+s)y^{2s} - y^2 - y^{2s+2}](|x|^{-\alpha} - 1). \end{aligned}$$

If $y^2 \geq (1+s)$, then $z_{yy} + \frac{1-2s}{y}z_y \geq 0$ and $\nabla \cdot (y^{1-2s}\nabla z) \geq 0$. If $y^2 < (1+s)$, then $z_{yy} + \frac{1-2s}{y}z_y \leq C(|x|^{-\alpha} - 1)$. Choosing α large enough, we deduce that

$$\nabla \cdot (y^{1-2s}\nabla z) \geq 0 \quad \text{for all } x \neq 0, \quad y \in [0, R-1].$$

Now, let v be as in the statement of the lemma. By Lemma 2.6, ess inf $v|_K > 0$, on $K = \partial B_{1/2} \times [0, R-1]$. Choose $\delta > 0$ so small that $v \geq \delta z$ a.e. on K. By the maximum principle, applied in the region $(B_1 \backslash B_{1/2}) \times (0, R-1)$, we deduce that $v \geq \delta z$ in this region. $\square$

Lemma 2.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Let v denote a measurable function on $\Omega \times (0, +\infty)$, such that*

$$\int_{\Omega\times(0,R)} y^{1-2s}|\nabla v|^2 \, dx\, dy < +\infty \quad \textit{for all } R > 0.$$

Assume that $v \geq 0$ on $\partial\Omega \times (0, +\infty)$, that

$$-\nabla \cdot (y^{1-2s}\nabla v) \geq 0 \quad \textit{in } \Omega \times (0, R),$$

and $-y^{1-2s}v_y|_{y=0} \geq 0$ in Ω in the sense that

$$\int_{\Omega\times(0,R)} y^{1-2s}\nabla v \cdot \nabla\zeta \, dx\, dy \geq 0$$

for all $\zeta \in H^1(y^{1-2s}, \Omega \times (0, +\infty))$ with compact support in $\overline{\Omega} \times [0, +\infty)$ such that $\zeta \geq 0$ and $\zeta = 0$ on $\partial\Omega \times (0, R) \cup \Omega \times \{R\}$.

If there exist $C > 0$ *and* $m > 0$ *such that*

$$|v(x, y)| \leq C(1 + |y|^m) \quad \text{for all } (x, y) \in \Omega \times (0, +\infty), \tag{2.12}$$

then $v \geq 0$ *in* $\Omega \times (0, +\infty)$.

Proof. Take $R > 0$ such that $\Omega \subseteq B_R(0)$. Let φ_R denote the first eigenfunction of $-\Delta$ in $B_R(0)$ with zero Dirichlet boundary condition and let $\mu_R > 0$ be its corresponding eigenvalue. Let $\lambda > 0$ to be chosen and set

$$z(x, y) = \varphi_R(x)(e^{\lambda y} - \lambda y).$$

We compute

$$\nabla \cdot (y^{1-2s}\nabla z) = y^{1-2s}\left[-\mu_R + \lambda^2 + \lambda^2(1-2s)e^{-\lambda y}\frac{e^{\lambda y} - 1}{\lambda y}\right]\varphi_R(x)e^{\lambda y}.$$

By choosing $\lambda > 0$ small we have $\nabla \cdot (y^{1-2s}\nabla z) < 0$ in $B_R(0) \times (0, +\infty)$. Let $\epsilon > 0$. By (2.12) there exists $L > 0$ such that $v + \epsilon z \geq 0$ for $x \in \Omega$ and $y \geq L$. Using the maximum principle in the form of Lemma 2.5 we deduce that $v + \epsilon z \geq 0$ in $\Omega \times (0, L)$. Letting $L \to \infty$ we conclude that $v + \epsilon z \geq 0$ in $\Omega \times (0, +\infty)$. Finally, by letting $\epsilon \to 0$ we obtain the stated result. $\square$

2.4. *Interior Regularity*

In this section, we study the extension problem (2.8), when h is bounded or belongs to a Hölder space. The proof of the next lemma can be found in [8, Lemma 4.4].

Lemma 2.9. *Let* $h \in H'$ *and* $v \in H^1_{0,L}(y^{1-2s})$ *denote the solution of* (2.8)*. Then, for any* $\omega \subset\subset \Omega$, $R > 0$, *we have*

(i) *If* $h \in L^\infty(\Omega)$, *then* $v \in C^\beta(\omega \times [0, R])$, *for any* $\beta \in (0, \min(1, 2s))$,
(ii) *If* $h \in C^\beta(\Omega)$ *then*

(1) $v \in C^{\beta+2s}(\omega \times [0, R])$ *if* $\beta + 2s < 1$,
(2) $\frac{\partial v}{\partial x_i} \in C^{\beta+2s-1}(\omega \times [0, R])$ *if* $1 < \beta + 2s < 2$, $i = 1, \ldots, n$,
(3) $\frac{\partial^2 v}{\partial x_i \partial x_j} \in C^{\beta+2s-2}(\omega \times [0, R])$ *if* $2 < \beta + 2s$, $i, j = 1, \ldots, n$.

2.5. *Boundary Regularity*

Lemma 2.10. *Let* $u \in H$ *be the solution of*

$$\begin{cases} (-\Delta)^s u = h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.13}$$

where $h \in L^\infty(\Omega)$. *Then* $u \in C^\alpha(\overline{\Omega})$ *for all* $\alpha \in (0, \min(2s, 1))$.

We begin with the following estimate.

Lemma 2.11. *Let $u \in H$ be the solution of* (2.13), *where $h \in L^\infty(\Omega)$. Then there is constant C such that*

$$\text{if } 0 < s < 1/2, \quad |u(x)| \le C dist(x, \partial\Omega)^{2s} \|h\|_{L^\infty(\Omega)} \text{ for all } x \in \Omega,$$

and

$$\text{if } 1/2 \le s < 1, \quad |u(x)| \le C dist(x, \partial\Omega) \|h\|_{L^\infty(\Omega)} \text{ for all } x \in \Omega.$$

Proof. We use a suitable barrier to prove the estimate. To construct it, we write $x = (x_1, \dots, x_n)$ and define

$$\bar{h}(x) = \begin{cases} 1 & \text{if } x \in B_2, \ x_1 < 0 \\ -1 & \text{if } x \in B_2, \ x_1 > 0 \\ 0 & \text{if } x \notin B_2. \end{cases}$$

We construct a solution $\bar{v}$ of the problem

$$\begin{cases} \mathrm{div}(y^{1-2s}\nabla v) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ v(z) \to 0 & \text{as } |z| \to \infty \\ -y^{1-2s} v_y = \bar{h}(x) & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$

as

$$\bar{v}(x, y) = C_{n,s} \int_y^\infty t \int_{\mathbb{R}^n} \frac{\bar{h}(\tilde{x})}{(t^2 + |x - \tilde{x}|^2)^{\frac{n+2-2s}{s}}} \, d\tilde{x} \, dt. \tag{2.14}$$

This implies

$$\bar{v}(x, 0) = C'_{n,s} \int_{\mathbb{R}^n} \frac{\bar{h}(\tilde{x})}{|x - \tilde{x}|^{n-2s}} \, d\tilde{x} \quad x \in \mathbb{R}^n,$$

where $C'_{n,s} = \frac{C_{n,s}}{n-2s}$. By our choice of $\bar{h}$ we can write for $x \in \mathbb{R}^n$

$$\bar{v}(x, 0) = -C'_{n,s}(I(x) - I(-x))$$

where

$$I(x) = \int_{B_2^+} \frac{1}{|x - \tilde{x}|^{n-2s}} \, d\tilde{x}$$

and $B_2^+ = \{(x_1, \dots, x_n) \in B_2(0) : x_1 > 0\}$. From this formula we see that if $0 < s < 1/2$ then

$$|I(x) - I(0)| \le C|x|^{2s} \quad \text{for all } x \in \mathbb{R}^n,$$

and if $1/2 \le s < 1$ then

$$|I(x) - I(0)| \le C|x| \quad \text{for all } x \in \mathbb{R}^n.$$

These estimates imply that if $0 < s < 1/2$

$$|\bar{v}(x)| \le C|x|^{2s} \quad \text{for all } x \in \mathbb{R}^n, \tag{2.15}$$

and if $1/2 \le s < 1$

$$|\bar{v}(x)| \le C|x| \quad \text{for all } x \in \mathbb{R}^n. \tag{2.16}$$

Now let $u \in H$ be the solution to (2.13) with $h \in L^\infty(\Omega)$ and let v denote its canonical extension. Take a point $x_0 \in \partial\Omega$. By the smoothness of $\partial\Omega$ we can find $x_1 \in \mathbb{R}^n \setminus \Omega$ and $R > 0$ such that $B_R(x_1) \subseteq \mathbb{R}^n \setminus \Omega$ and $x_0 \in \partial B_R(x_1)$. We can choose R bounded and bounded below. By suitable translation and rescaling, we can assume that $x_1 = 0$, $R = 1$ and $|x_0| = 1$. After a further rotation we can also assume $x_0 = (1, 0, \dots, 0) \in \mathbb{R}^n$.

We will then define a comparison function w as the Kelvin transform of a translate of $\bar{v}$ as defined by (2.14). Let $\tilde{v}(x, y) = \bar{v}(x - x_0, y)$. We write points in $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ as $X = (x, y)$ and $|X|^2 = |x|^2 + y^2$. We also write $\mathbb{R}^{n+1}_+$ for the set of points $X = (x, y) \in \mathbb{R}^n \times \mathbb{R}$ with $y > 0$. Let

$$w(X) = |X|^{2s-n}\tilde{v}\left(\frac{X}{|X|^2}\right) \quad X \in \mathbb{R}^{n+1}_+, \quad X \neq 0.$$

A direct calculation shows that

$$\operatorname{div}(y^{1-2s}\nabla w) = 0 \quad \text{in } \mathbb{R}^{n+1}_+$$

and

$$\lim_{y\to 0^+}(-y^{1-2s}w_y(x, y)) = |x|^{-2s-n}\bar{h}\left(\frac{x}{|x|^2} - x_0\right)s \quad \text{for all } x \in \mathbb{R}^n, \quad x \neq 0.$$

For $x \in \mathbb{R}^n \setminus B_1(0)$ we have $x/|x|^2 \in B_1(0)$ and so $\bar{h}(x/|x|^2 - x_0) = 1$. Since Ω is bounded and contained in $\mathbb{R}^n \setminus B_1(0)$, we see that there is some constant $c > 0$ (bounded uniformly from below with respect to the parameters x_0, x_1, R with R bounded from below) such that

$$\lim_{y\to 0^+}(-y^{1-2s}w_y(x, y)) \ge c \quad \text{for all } x \in \Omega.$$

Since $\tilde{v} > 0$ in $B_1(0) \times (0, +\infty)$ we have $w > 0$ in $\Omega \times (0, +\infty)$. Then, there is a constant $c > 0$ (uniformly bounded from below as x_0, x_1 and R vary) such that $w(x, 1) \ge c$ for all $x \in \Omega$. Since $w \ge 0$ on $\partial\Omega \times (0, +\infty)$ and v vanishes there, by the maximum principle we have

$$v \le C\|h\|_{L^\infty(\Omega)} w \quad \text{in } \Omega \times (0, 1)$$

for some $C > 0$. From this, (2.15) and (2.16) we deduce the stated estimates. □

Proof of Lemma 2.10. We use a standard scaling argument combined with interior regularity estimates from Lemma 2.9 and Lemma 2.11. Let v denote the canonical extension of u and let us concentrate on the case $0 < s < 1/2$.

Take $x_0, y_0 \in \Omega$. If x_0, y_0 and satisfy $|x_0 - y_0| \geq dist(x_0, \partial\Omega)/2$ and $|x_0 - y_0| \geq dist(y_0, \partial\Omega)/2$ from Lemma 2.11

$$\begin{aligned}|v(x_0, 0) - v(y_0, 0)| \leq |v(x_0, 0)| + |v(y_0, 0)| &\leq C\|h\|_{L^\infty(\Omega)}|x_0 - y_0|^{2s} \\ &\leq C\|h\|_{L^\infty(\Omega)}|x_0 - y_0|^{\beta}.\end{aligned}$$

Now suppose that $|x_0 - y_0| \leq dist(x_0, \partial\Omega)/2$ and let $r = dist(x_0, \partial\Omega)/2$. Consider the function $\tilde{v}(x, y) = v(x_0 + rx, ry)$ defined for $x \in B(0, 1)$ and $y > 0$. Thus

$$\operatorname{div}(y^{1-2s}\nabla\tilde{v}) = 0 \quad \text{in } B_1(0) \times (0, +\infty)$$

and

$$\lim_{y\to 0^+}(-y^{1-2s}\tilde{v}(x, y)) = \tilde{h}(x) \quad x \in B_1(0),$$

where $\tilde{h}(x) = r^{2s}h(rx)$. By Lemma 2.11 we find

$$\sup_{B_1(0)} |\tilde{v}| \leq Cr^{2s}\|h\|_{L^\infty(\Omega)}.$$

Let $0 < \beta < 2s$. Using the interior estimate (Lemma 2.9)

$$\|\tilde{v}\|_{C^\beta(\overline{B}_{1/2})} \leq C(\sup_{B_1}|\tilde{v}| + \sup_{B_1}|\tilde{h}|) \leq Cr^{2s}\|h\|_{L^\infty(\Omega)}$$

we deduce

$$|v(x_0, 0) - v(y_0, 0)| \leq C\|h\|_{L^\infty(\Omega)}|x_0 - y_0|^\beta r^{2s-\beta} \leq C\|h\|_{L^\infty(\Omega)}|x_0 - y_0|^\beta.$$

The proof in the case $1/2 \leq s < 1$ follows analogously. □

3. Proof of Proposition 1.3

Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ denote a smooth bounded domain. We begin by adapting Lemma 1 in [3]:

Lemma 3.1. *Take $f \in L^1(\Omega, \varphi_1 dx)$. Then, there exists a unique $u \in L^1(\Omega, \varphi_1 dx)$ such that*

$$\begin{cases}(-\Delta)^s u = f & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega,\end{cases} \tag{3.1}$$

in the sense that

$$\int_\Omega u\psi\, dx = \int_\Omega f(-\Delta)^{-s}\psi\, dx, \quad \text{for all } \psi \in C_c^\infty(\Omega). \tag{3.2}$$

In addition, letting $\mu_1 > 0$ denote the principal eigenvalue of the Laplace operator with homogeneous Dirichlet boundary condition on $\partial\Omega$, we have

$$\int_\Omega |u|\varphi_1 \, dx \leq \frac{1}{\mu_1} \int_\Omega |f|\varphi_1 \, dx. \tag{3.3}$$

Moreover, if $f \geq 0$ a.e., then $u \geq 0$ a.e. in Ω.

Proof. Take $\psi \in C_c^\infty(\Omega)$. Then, there exists a constant $C > 0$ such that $|\psi| \leq C\varphi_1$. By the maximum principle (Lemma 2.5), it follows that $\varphi = (-\Delta)^{-s}\psi$ satisfies $|\varphi| \leq \frac{C}{\mu_1}\varphi_1$. In particular, (3.2) makes sense for any $\psi \in C_c^\infty(\Omega)$.

Let $f \in L^\infty(\Omega) \subset H'$. Then, equation (3.1) has a unique solution $u \in H$, i.e., for any $\zeta \in H$,

$$\sum_{k=1}^{+\infty} \mu_k^s u_k \zeta_k = \sum_{k=1}^{+\infty} f_k \zeta_k,$$

where $u_k = \int_\Omega u\varphi_k \, dx$, and ζ_k, f_k are similarly defined. Take now $\zeta = (-\Delta)^{-s}\psi$, $\psi \in C_c^\infty(\Omega)$. Then, $\zeta_k = \mu_k^{-s}\psi_k$ and

$$\sum_{k=1}^{+\infty} u_k \psi_k = \sum_{k=1}^{+\infty} f_k \mu_k^{-s} \psi_k,$$

which is equivalent to (3.2). We prove next that (3.3) holds. To see this, write $f = f^+ - f^-$, where f^+ is the positive part of f and f^- its negative part. Without loss of generality, we may always assume that $f \geq 0$ a.e. Then, by the maximum principle (Lemma 2.5), $u \geq 0$ a.e. and using (3.2) with $\psi = \varphi_1$, we deduce (3.3). The rest of the proof is the same as that of Lemma 1 in [3], so we skip it. □

The method of sub and supersolutions can be applied in the context of solutions of (1.1) belonging to $H \cap L^\infty(\Omega)$. We call $\bar{u} \in H \cap L^\infty(\Omega)$ a supersolution of (1.1) if

$$(-\Delta)^s u \geq \lambda f(\bar{u})$$

where the inequality is in the sense of (2.11). A subsolution $\underline{u}$ is defined be reversing the inequality. If $\underline{u}, \bar{u} \in H \cap L^\infty(\Omega)$ are a subsbolution and a supersolution respectively, and $\underline{u} \leq \bar{u}$, then a solution can be constructed by the monotone iteration method. This works thanks to the maximum principle (Lemma 2.5) and the estimates given by Lemmas 2.9 and 2.10.

Proof of Proposition 1.3. Since $\zeta = 0$ is always a subsolution, we begin by showing that there exists a positive supersolution of (1.1) for small $\lambda > 0$. Take ζ_0 to be the solution of

$$\zeta_0 \in H, \quad (-\Delta)^s \zeta_0 = 1. \tag{3.4}$$

By Lemma 2.10, $\zeta_0 \in C(\overline{\Omega})$ and

$$(-\Delta)^s \zeta_0 = 1 \geq \lambda f(\zeta_0), \quad \text{for } \lambda \leq 1/\|f(\zeta_0)\|_{L^\infty(\Omega)}.$$

Hence,

$$\lambda^* = \sup\{\lambda > 0 : (1.1) \quad \text{has a solution in } H \cap L^\infty(\Omega)\}$$

is positive and well-defined. Multiplying (1.1) by φ_1 and using that f is superlinear, we easily deduce that $\lambda^* < +\infty$. It is also clear by the method of sub and supersolutions that (1.1) has a minimal positive solution $u_\lambda \in H \cap L^\infty(\Omega)$, for all $\lambda \in (0, \lambda^*)$. The solution u_λ is also semi-stable, which can be proved in a similar way as for the second order case (see [6]).

We note that for $\lambda \in [0, \lambda^*)$, $f(u_\lambda) \in L^\infty(\Omega)$ and hence u_λ is continuous up to the boundary and satisfies the boundary condition in the classical sense. Lemma 3.1 implies that u_λ also satisfies the weak formulation (1.5). We will see now that we can take the limit in (1.5). By minimality, u_λ increases with λ. We claim that $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ is a weak solution of (1.1) for $\lambda = \lambda^*$. Take $\lambda < \lambda^*$, $u = u_\lambda$ and multiply (1.1) by φ_1. Then,

$$\mu_1 \int_\Omega u\varphi_1 \, dx = \lambda \int_\Omega f(u)\varphi_1 \, dx. \tag{3.5}$$

Since f is superlinear, for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that, for all $t \geq 0$ $f(t) \geq \frac{1}{\epsilon}t - C_\epsilon$. Hence,

$$\lambda^* C_\epsilon \geq \left(\frac{\lambda}{\epsilon} - \mu_1\right) \int_\Omega u\varphi_1 \, dx.$$

Choosing $\epsilon = \frac{\lambda}{2\mu_1}$, we obtain that

$$\int_\Omega u_\lambda \varphi_1 \, dx \leq C,$$

for some constant C independent of λ. By (3.5), we also have

$$\int_\Omega f(u_\lambda)\varphi_1 \, dx \leq C, \tag{3.6}$$

and, by monotone convergence, we may pass to the limit as $\lambda \to \lambda^*$ in (1.5). □

Remark 3.2. Observe that for $s \geq 1/2$, we have the stronger estimate

$$\|u_\lambda\|_{L^1(\Omega)} \leq C, \tag{3.7}$$

as follows from multiplying (1.1) by ζ_0 (defined in (3.4)) and using Lemma 2.11, giving the estimate

$$\zeta_0 \leq C\varphi_1. \tag{3.8}$$

Note also that (3.8) fails for $s < 1/2$. Due to radial monotonicity (see Lemma 4.1), estimate (3.7) remains however true if $\Omega = B_1$ and $s \in (0, 1)$ is arbitrary.

4. Radial Symmetry

Lemma 4.1. *Let $u \in H \cap L^\infty(B)$, $u \geq 0$ denote a solution of* (1.1)*. Then, u is radially decreasing, i.e., $u(x) = u(\rho)$ whenever $|x| = \rho$, u is smooth in B, and*

$$\frac{\partial u}{\partial \rho} < 0 \quad in\ B \backslash \{0\}. \tag{4.1}$$

In addition, the canonical extension v of u is smooth in $\mathscr{C}$, $v(x, y) = v(\rho, y)$, and

$$\frac{\partial v}{\partial \rho} < 0 \quad in\ \mathscr{C} \backslash \{\rho = 0\}. \tag{4.2}$$

We remark that since we assume $f(t) > 0$ for all $t \geq 0$, if $\lambda > 0$ then u in the statement of this Lemma is positive in B, thanks to the strong maximum principle Lemma 2.6.

Proof. By Lemmata 2.9 and 2.10, u and v are Hölder continuous up to the boundary. One can get higher interior regularity by applying the argument in [8]. In few words, the idea is to differentiate the equation of the canonical extension in x variables, to use the explicit kernel in the entire space to get an homogeneous problem for a new function, to extend this new function to $\mathbb{R}^{n+1}$ as an even function, and finally apply the estimates of [15].

To prove radial symmetry, (4.1), and (4.2), we apply the moving plane method [17]. Thus, it suffices to show that

$$\frac{\partial v}{\partial x_1} < 0 \quad \text{in } \{(x, y) \in B_1 \times [0, +\infty) : x_1 > 0\}.$$

Now we show the last statement. Given $\mu \in (0, 1]$, let $T_\mu = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^+ : x_1 = \mu\}$ and $\Sigma_\mu = \{(x, y) \in B_1 \times [0, +\infty) : x_1 > \mu\}$. Let also $v_\mu(x, y) = v(2\mu - x_1, x', y)$ for $(x, y) \in \Sigma_\mu$ and $w_\mu = v_\mu - v$. We claim that $w_\mu \geq 0$ in Σ_μ, for μ close to 1. To prove this, observe that $w = w_\mu$ solves

$$\begin{cases} \operatorname{div}(y^{1-2s} \nabla w) = 0 & \text{in } \Sigma_\mu \\ w \geq 0 & \text{on } \partial_L \Sigma_\mu \\ -y^{1-2s} w_y - a(x) w = 0 & \text{on } \{x \in B_1 : x_1 > \mu\} \times \{0\}, \end{cases}$$

where

$$a(x) = \begin{cases} \dfrac{f(u_\mu) - f(u)}{u_\mu - u} & \text{whenever } u_\mu \neq u, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Now multiply the above equation by w^- and integrate over Σ_μ. Then,

$$\int_{\Sigma_\mu} y^{1-2s} |\nabla w^-|^2 \, dx \, dy = \int_{\{x \in B_1 : x_1 > \mu\}} a(x) (w^-)^2 \, dx.$$

We extend w^- by 0 outside Σ_μ, so that $w^- \in H^1(y^{1-2s}; \mathbb{R}^n)$. By the trace theorem (Proposition 2.1), there exists a constant $C_{tr} > 0$ such that

$$\|w^-\|^2_{H^s(\mathbb{R}^n)} \le C_{tr} \int_{\mathbb{R}^n} y^{1-2s} |\nabla w^-|^2 \, dx \, dy,$$

and by the Sobolev imbedding of $H^s(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$, with

$$\frac{1}{p} = \frac{1}{2} - \frac{s}{n}, \tag{4.4}$$

we have

$$\|w^-\|^2_{L^p(\mathbb{R}^n)} \le C_S \|w^-\|^2_{H^s(\mathbb{R}^n)}.$$

Hence, by Hölder's inequality

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |w^-|^p \, dx\right)^{2/p} &\le C_{tr} C_S \int_{\{x \in B_1 : x_1 > \mu\}} |a(x)| (w^-)^2 \, dx \\ &\le C_{tr} C_S \left(\int_{\{x \in B_1 : x_1 > \mu\}} (w^-)^p \, dx\right)^{2/p} \left(\int_{\{x \in B_1 : x_1 > \mu\}} |a|^{\frac{p}{p-2}} \, dx\right)^{1-2/p}.\end{aligned}$$

Since a is uniformly bounded, $\int_{\{x \in B_1 : x_1 > \mu\}} |a|^{\frac{p}{p-2}} \, dx \to 0$, as $\mu \to 1^-$. Therefore, for μ sufficiently close to 1, we conclude that $w^- \equiv 0$, and the claim.

Consider now

$$\mu_0 = \inf\{\mu \in (0, 1) : w_\mu \ge 0 \text{ in } \Sigma_\mu\}.$$

The above argument shows that μ_0 is well-defined and $\mu_0 < 1$. We want to prove that $\mu_0 = 0$. Assume by contradiction that $\mu_0 > 0$. By continuity, $w_{\mu_0} \ge 0$ in Σ_{μ_0}, and by the strong maximum principle (Lemma 2.6), $w_{\mu_0} > 0$ in Σ_{μ_0}. Fix now $\epsilon > 0$ small, $\mu = \mu_0 - \epsilon$ and choose a compact set $K \subset \{x \in B_1 : x_1 > \mu_0\}$ such that

$$C_{tr} C_S \left(\int_{\{x \in B_1 : x_1 > \mu\} \setminus K} |a|^{\frac{p}{p-2}} \, dx\right)^{1-2/p} < \frac{1}{2}.$$

Taking $\epsilon > 0$ smaller if necessary, we can assume that $w_\mu > 0$ in K. Arguing as before, we can prove that $w_\mu^- \equiv 0$ in $\Sigma_\mu \setminus K$, and thus $w_\mu \ge 0$ everywhere in Σ_μ, contradicting the definition of μ_0.

We have just proved that $w_\mu \ge 0$ in Σ_μ for all $\mu \in (0, 1)$, and by the strong maximum principle (Lemma 2.6) we find that $w_\mu > 0$ in Σ_μ. Finally, by the boundary point lemma (Lemma 2.7), we conclude

$$2\frac{\partial v_\mu}{\partial x_1}(\mu, x', y) = -\frac{\partial w_\mu}{\partial x_1}(\mu, x', y) < 0 \quad \text{for all } (\mu, x', y) \in B_1 \times [0, +\infty),$$

as desired. $\square$

5. Weighted Integrability

We will use the following notation. Given a point $(x, y) \in \mathscr{C} = B_1 \times (0, +\infty)$, we let $\rho = |x|$ and $v_\rho = \frac{\partial v}{\partial \rho}$ for any C^1 function v defined on $\mathscr{C}$, which depends only on ρ and y.

In what follows, for $\lambda \in [0, \lambda^*)$, u_λ denotes the minimal solution of (1.1) and v_λ its canonical extension, which satisfies

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla v) = 0 & \text{in } \mathscr{C} \\ v = 0 & \text{on } \partial_L \mathscr{C} \\ -\lim_{y\to 0}(y^{1-2s}v_y) = \lambda f(v) & \text{on } B_1 \times \{0\}. \end{cases} \tag{5.1}$$

For $\lambda \in [0, \lambda^*)$, $u_\lambda \in C^\infty(B_1) \cap C(\overline{B_1})$, and v_λ is smooth in $\mathscr{C}$. Indeed, by Lemmata 2.9 and 2.10 we obtain Hölder regularity of u_λ up to the boundary. One can get higher interior regularity by applying the argument in [8]. In a similar way, we also deduce that $v_\lambda \in C^\alpha(K \times [0, R])$ for every compact $K \subset B_1$ and $R > 0$. Moreover, any of the derivatives of v_λ with respect to the x variables belongs to $C^\alpha(K \times [0, R])$ for every compact set $K \subset B_1$ and $R > 0$.

The main result in this section is the following. It is an estimate for the L^2-norm of $y^{\frac{1-2s}{2}}\rho^{-\alpha}u_\rho$ for certain exponent α that depends on the dimension n. This estimate, which is independent of λ and holds for stable solutions only, is the key ingredient in the proof of the regularity Theorem 1.6.

Proposition 5.1. *Assume $n \geq 2$. Let $\lambda \in (0, \lambda^*)$, $u = u_\lambda$ be the minimal solution of* (1.1) *and v its canonical extension. Let α satisfy*

$$1 \leq \alpha < 1 + \sqrt{n-1}. \tag{5.2}$$

Then

$$\int_{[\rho \leq 1/2]} y^{1-2s} v_\rho^2 \rho^{-2\alpha} dx\, dy \leq C \tag{5.3}$$

where C is a constant independent of λ, and $[\rho \leq 1/2]$ denotes the set $\{(x, y) \in \mathscr{C} : |x| \leq 1/2\}$.

Before proving Proposition 5.1 we need two preliminary results.

In the next lemma we collect some basic estimates expressing that v_λ and its derivatives have exponential decay for $y \geq 1$, which is uniform up to $\lambda < \lambda^*$, and that for fixed $\lambda < \lambda^*$, $v_\rho(\rho, y) = O(\rho)$ as $\rho \to 0$, uniformly as $y \to 0$.

Lemma 5.2.

a) *There are $\gamma > 0$, $C > 0$ such that*

$$v_\lambda(x, y) \leq Ce^{-\gamma y}\varphi_1(x) \quad \textit{for all } y \geq 1, \quad x \in B_1, \quad \lambda \in [0, \lambda^*). \tag{5.4}$$

Moreover, for any $k \geq 0$ there is $C_k > 0$ such that

$$|D^k v_\lambda(x, y)| \leq C_k e^{-\gamma y} \quad \textit{for all } y \geq 1, \quad x \in B_1, \quad \lambda \in [0, \lambda^*). \tag{5.5}$$

The constants γ and C are independent of λ.

b) *Given $\lambda \in [0, \lambda^*)$ and K a compact subset of B_1 there exists $C > 0$ such that*

$$|\partial_\rho v_\lambda(|x|, y)| \le C|x| \quad \forall x \in K, \quad y \ge 0. \tag{5.6}$$

The constant in (5.6) may blow up as $\lambda \to \lambda^*$, but we will use this inequality for fixed $\lambda \in [0, \lambda^*)$.

Proof. a) Define $w(x, y) = \varphi_1(x) y^{2s} e^{-\gamma y}$. A straight forward computation shows that

$$\nabla \cdot (y^{1-2s} \nabla w) = \varphi_1(x) e^{-\gamma y}[(\gamma^2 - \lambda_1) y - \gamma(1 + 2s)]$$

and

$$y^{1-2s} w_y|_{y=0} = \lim_{y\to 0} y^{1-2s} \varphi_1(x) e^{-\gamma y}(-\gamma y^{2s} + 2s y^{2s-1}) = 2s\varphi_1(x).$$

Multiplying equation (5.1) by w and integrating by parts twice gives

$$\lambda \int_{B_1} f(u_\lambda) w \, dx + \int_{B_1} y^{1-2s} w_y v_\lambda \, dx + \int_{\mathscr{C}} \nabla(y^{1-2s} \nabla w) v_\lambda = 0.$$

Recalling that $w(x, 0) = 0$, we find

$$2s \int_{B_1} \varphi_1 u_\lambda \, dx = \int_{\mathscr{C}} v_\lambda \varphi_1(x) e^{-\gamma y} \left[(\lambda_1 - \gamma^2) y + \gamma(1 + 2s)\right] dx \, dy.$$

Now, we choose $0 < \gamma < \sqrt{\lambda_1}$ and use estimate $\int_{B_1} \varphi_1 u_\lambda \, dx \le C$ derived in (3.6), to find

$$\int_{\mathscr{C}} v_\lambda \varphi_1(x) e^{-\gamma y} dx \, dy \le C \tag{5.7}$$

for all $0 \le \lambda < \lambda^*$.

Let z be the solution to

$$\begin{cases} -\Delta z = 1 & \text{in } B_1 \\ z = 0 & \text{on } \partial B_1. \end{cases}$$

For $\tau \ge t > 0$ define $\varphi(x, y) = z(x)(\tau - y)(y - t)$. We compute

$$\nabla \cdot (y^{1-2s} \nabla \varphi) = y^{1-2s} \left[-(\tau - y)(y - t) + z(x)\left(-2 + (1 - 2s)\left(-2 + \frac{\tau + t}{y} \right)\right)\right]$$

Assume that $0 < t \le \tau \le 3t/2$. We find

$$\nabla \cdot (y^{1-2s} \nabla \varphi) \le -y^{1-2s}(\tau - y)(y - t).$$

Multiplying (5.1) by φ and integrating over $B_1 \times (t, \tau)$ we obtain

$$\begin{aligned} &\tau^{1-2s}(\tau - t) \int_{B_1} v_\lambda(x, \tau) z(x) \, dx + t^{1-2s}(\tau - t) \int_{B_1} v_\lambda(x, t) z(x) \, dx \\ &\quad = -\int_{B_1 \times (t,\tau)} y^{1-2s} v_\lambda \nabla(y^{1-2s} \nabla \varphi) \, dx \, dy \ge \int_{B_1 \times (t,\tau)} y^{1-2s} v_\lambda (\tau - y)(y - t) \, dx \, dy. \end{aligned}$$

Thus, for $t \geq 6$ we deduce

$$\int_{B_1\times(t+1,t+2)} y^{1-2s} v_\lambda \, dx\, dy \leq Ct^{1-2s} \int_{B_1} v_\lambda(x,t) z(x)\, dx + C(t+3)^{1+2s} \int_{B_1} v_\lambda(x,t+3) z(x)\, dx.$$

Integrating this inequality with respect to $t \in [6, 13]$, recalling that $z \leq C\varphi_1$ for some $C > 0$, and using (5.7) we obtain

$$\int_{B_1\times[8,11]} v_\lambda \, dx\, dy \leq C$$

with a constant independent of λ as $\lambda \to \lambda^*$.

This inequality and standard elliptic estimates imply

$$v_\lambda(x,y) \leq Ce^{-\gamma y}\varphi_1(x) \quad \text{for all } y \in [8,10], \quad x \in B_1, \quad \text{and} \quad \lambda \in [0,\lambda^*). \tag{5.8}$$

Now let $\bar{w}(x,y) = C\varphi_1(x)e^{-\gamma y}$. For $0 < \gamma < \sqrt{\lambda_1}$, this is a supersolution of the equation in (5.1) and by comparison in $B_1 \times (1,+\infty)$, using (5.8), we deduce (5.4). Inequality (5.5) is a consequence of (5.4) and elliptic estimates.

b) This part follows from the fact that for $\lambda < \lambda^*$, u_λ is smooth in B_1 and hence v_λ and its derivatives with respect to the x variables are in $C^\alpha(K \times [0,R])$ for any compact $K \subset B_1$ and $R > 0$. □

Now, we show a result on the form of the second variation of the energy for radially symmetric solutions of (1.1). An analogous result was showed in Lemma 1 of [6] for the Laplacian, and the ideas of its proof where inspired by the Simons Theorem on the existence of minimal cones (see [7] for further details). We also point out that, the expression for the second variation of the energy in the following lemma is two dimensional, in the sense that it depends on both ρ and y. This is in contrast to the previous results of this kind where all the expression depends on the radial coordinate only (see [7].)

Lemma 5.3. *Given $\lambda \in (0,\lambda^*)$, let $u = u_\lambda \in H \cap L^\infty(B_1)$ denote the minimal solution of* (1.1)*, and let $v \in H^1_{0,L}(y^{1-2s})$ denote its canonical extension. Then, for every $\eta \in C^1(B_1 \times [0,+\infty))$ with compact support in $\mathscr{C}$, but not necessarily vanishing on $B_1 \times \{0\}$, we have*

$$\int_{\mathscr{C}} y^{1-2s} v_\rho^2 |\nabla\eta|^2 \, dx\, dy \geq (n-1) \int_{\mathscr{C}} y^{1-2s} \frac{v_\rho^2}{\rho^2}\eta^2 \, dx\, dy. \tag{5.9}$$

Proof. Inequality (1.6) implies that for all $\xi \in H^1_{0,L}(y^{1-2s})$, there holds

$$\int_{\mathscr{C}} y^{1-2s} |\nabla\xi|^2 \, dx\, dy \geq \lambda \int_{B_1} f'(u)\xi^2 \, dx, \tag{5.10}$$

where in the right-hand-side integral we identified ξ with its trace.

Let $\eta \in C^1(B_1 \times [0, +\infty))$ as in the statement of the lemma and take $\xi = \eta v_\rho$. By Lemma 5.2, $\xi \in H^1_{0,L}(y^{1-2s})$ and from (5.10) we obtain

$$\begin{aligned}\lambda \int_{B_1} f'(u) v_\rho^2 \eta^2 dx &\leq \int_{\mathscr{C}} y^{1-2s} |\nabla(v_\rho \eta)|^2 dx\, dy \\ &= \int_{\mathscr{C}} y^{1-2s} \{|\nabla v_\rho|^2 \eta^2 + v_\rho^2 |\nabla \eta|^2 + v_\rho \nabla v_\rho \cdot \nabla \eta^2\}\, dx\, dy \\ &= \int_{\mathscr{C}} y^{1-2s} \{v_\rho^2 |\nabla \eta|^2 + \nabla(\eta^2 v_\rho) \cdot \nabla v_\rho\}\, dx\, dy. \qquad (5.11)\end{aligned}$$

Since by Lemma 4.1, u is radially symmetric, by differentiation of (5.1) with respect to ρ, one gets

$$\nabla \cdot (y^{1-2s} \nabla v_\rho) = y^{1-2s} \frac{n-1}{\rho^2} v_\rho \quad \text{in } \mathscr{C}. \qquad (5.12)$$

Next, we differentiate the Neumann boundary condition in (5.1) with respect to ρ to obtain

$$-y^{1-2s} \partial_y v_\rho = \lambda f'(v) v_\rho \quad \text{for } 0 \leq \rho < 1. \qquad (5.13)$$

Now, we multiply (5.12) by $\eta^2 v_\rho$, and integrate by parts and use (5.13) to find

$$\int_{\mathscr{C}} y^{1-2s} \nabla(\eta^2 v_\rho) \cdot \nabla v_\rho\, dx\, dy = \lambda \int_{B_1} f'(u)(v_\rho \eta)^2\, dx - (n-1) \int_{\mathscr{C}} y^{1-2s} \frac{(v_\rho \eta)^2}{\rho^2}\, dx\, dy.$$

Combining the last equation with (5.11) yields (5.9). $\square$

Now we give:

Proof of Proposition 5.1. Given $\varepsilon > 0$ let $\zeta_\varepsilon \in C^\infty(\mathbb{R})$ be such that $\zeta_\varepsilon(t) = 0$ for $t \leq \varepsilon$ and $t \geq 3/4$, $\zeta_\varepsilon(t) = 1$ for $t \in [2\varepsilon, 1/2]$, and $\zeta'_\varepsilon(t) \leq C/\varepsilon$ for $t \in [\varepsilon, 2\varepsilon]$. Given $R > 0$ we let ϕ_R denote a function $C^\infty(\mathbb{R})$ such that $\psi_R(y) = 1$ for all $r \leq R$ and $\psi_R(y) = 0$ for all $y \geq R + 1$.

Let α satisfy (5.2) and for $\varepsilon > 0$, $R > 0$ define $\eta(\rho, y) = \rho^{1-\alpha} \zeta_\varepsilon(\rho) \psi_R(y)$. Given $\delta > 0$ we estimate

$$|\nabla \eta|^2 \leq ((1-\alpha)^2 + \delta) \rho^{-2\alpha} \zeta_\varepsilon(\rho)^2 \psi_R(y)^2 + C_\delta \rho^{2-2\alpha} |\nabla(\zeta_\varepsilon \psi_R)|^2$$

for some $C_\delta > 0$. Then by (5.9)

$$\begin{aligned}(n-1) \int_{\mathscr{C}} y^{1-2s} v_\rho^2 \rho^{-2\alpha} (\zeta_\varepsilon \psi_R)^2 dx\, dy &\leq ((1-\alpha)^2 + \delta) \int_{\mathscr{C}} y^{1-2s} v_\rho^2 \rho^{-2\alpha} (\zeta_\varepsilon \psi_R)^2 dx\, dy \\ &+ C \int_{\mathscr{C}} y^{1-2s} \rho^{2-2\alpha} v_\rho^2 |\nabla(\zeta_\varepsilon \psi_R)|^2 dx\, dy.\end{aligned}$$

Choosing $\delta > 0$ small enough

$$\int_{\mathscr{C}} y^{1-2s} v_\rho^2 \rho^{-2\alpha} (\zeta_\varepsilon \psi_R)^2 dx\, dy \leq C \int_{\mathscr{C}} y^{1-2s} \rho^{2-2\alpha} v_\rho^2 (|\nabla \zeta_\varepsilon|^2 \psi_R^2 + \zeta_\varepsilon^2 |\nabla \psi_R|^2) dx\, dy$$

where $C > 0$. Thanks to (5.6) we have

$$\int_{\mathscr{C}} y^{1-2s}\rho^{2-2\alpha}v_\rho^2|\nabla\zeta_\varepsilon|^2\psi_R^2 dx\,dy \le \frac{C}{\varepsilon^2}\int_{[\varepsilon\le\rho\le 2\varepsilon, 0\le y\le R+1]} y^{1-2s}\rho^{4-2\alpha}dx\,dy$$
$$\le C(R+1)^{2-2s}\varepsilon^{2-2\alpha+n}. \tag{5.14}$$

Because of (5.2) we have that $2-2\alpha+n > 0$. Letting $\varepsilon \to 0$ we find

$$\int_{[\rho\le 1/2, y\le R]} y^{1-2s}v_\rho^2\rho^{-2\alpha}dx\,dy \le C\int_{[1/2\le\rho\le 3/4]\cup[R\le y\le R+1]} y^{1-2s}\rho^{2-2\alpha}v_\rho^2 dx\,dy \le C$$

where the last inequality follows from (5.5). Finally, letting $R \to \infty$ we conclude (5.3). Note that the constant C in (5.14) depends on λ, but we let $\varepsilon \to 0$ with λ fixed, so estimate (5.3) does not depend on λ in the end. □

6. Proof of Theorem 1.6

Before giving the proof of Theorem 1.6 we need the following result, where we show an explicit point-wise bound for the solution of the linear problem.

Lemma 6.1. *Let $h \in L^\infty(B_1)$ and $u \in H$ be the unique solution of*

$$(-\Delta)^s u = h \quad \textit{in } B_1.$$

Then

$$|u(x)| \le C_{n,s}\int_{B_1}\frac{|h(\tilde{x})|}{|x-\tilde{x}|^{n-2s}}d\tilde{x} \quad \textit{for every } x \in B_1. \tag{6.1}$$

Proof. Writing $h = h^+ - h^-$ with $h^+, h^- \ge 0$ we see that it is sufficient to prove the result in the case $h \ge 0$, so that also $u \ge 0$.

Let v be the canonical extension of u. Since $v(x,\infty) = 0$, for every x, we can write

$$v(x,0) = -\int_0^\infty v_y(x,y)dy \quad \text{for all } x \in B_1. \tag{6.2}$$

Let $g(x)$ be equal to $h(x)$ extended by 0 in $\mathbb{R}^n\backslash B_1$, and denote by $\tilde{v}$ the solution of

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla\tilde{v}) = 0 & \text{in } \mathbb{R}^n\times(0,+\infty)\\ \tilde{v}(z)\to 0 & \text{as } |z|\to\infty\\ -y^{1-2s}\tilde{v}_y = g(x) & \text{on } \mathbb{R}^n\times\{0\}. \end{cases} \tag{6.3}$$

By the Green's representation formula for (6.3), we have

$$-\tilde{v}_y(x,y) = C_{n,s}\,y\int_{\mathbb{R}^n}\frac{g(\tilde{x})}{(|x-\tilde{x}|^2+y^2)^{\frac{n+2-2s}{2}}}\,d\tilde{x}. \tag{6.4}$$

Consider the functions $w = -y^{1-2s} v_y$ and $\tilde{w} = -y^{1-2s} \tilde{v}_y$. Then, w and $\tilde{w}$ satisfy

$$\nabla(y^{2s-1} \nabla w) = 0 \quad \text{in } \mathscr{C}.$$

Since $-\tilde{v}_y \geq 0$ in $\mathbb{R}^n \times [0, +\infty)$ in particular we have

$$\tilde{w} \geq 0 = w \quad \text{on } \partial_L \mathscr{C}.$$

Furthermore

$$w \leq \tilde{w} \quad \text{in } B_1 \times \{0\}$$

and for $z \in \mathscr{C}$, $w(z), \tilde{w}(z) \to 0$ as $|z| \to +\infty$. Then, the maximum principle (Lemma 2.8) implies that

$$-v_y \leq -\tilde{v}_y \quad \text{in } \mathscr{C}. \tag{6.5}$$

Combining (6.2), (6.5) together with (6.4) we find

$$\begin{aligned} v(x, 0) &\leq C_{n,s} \int_0^\infty y \int_{\mathbb{R}^n} \frac{g(\tilde{x})}{(|x - \tilde{x}|^2 + y^2)^{\frac{n+2-2s}{2}}} \, d\tilde{x} \, dy \\ &= C_{n,s} \int_{\mathbb{R}^n} g(\tilde{x}) \left(\int_0^\infty \frac{y}{(|x - \tilde{x}|^2 + y^2)^{\frac{n+2-2s}{2}}} \, dy \right) d\tilde{x} \end{aligned}$$

for all $x \in B_1$, where we have used Fubini's theorem in the last line. Claim (6.1) follows by performing the integration over the y variable in the last expression, and recalling the definition of $g(x)$. □

Now we give:

Proof of Theorem 1.6. We denote points in $\mathscr{C} = B_1 \times (0, +\infty)$ as $(x, y) \in \mathscr{C}$, where $x \in B_1$, $y \in (0, +\infty)$, and $\rho = |x|$.

Step 1. Take α such that (5.2) holds. We claim that for $\beta > 0$ such that $2(\beta + s - \alpha) < n$ we have

$$\int_{B_1} f(u_\lambda) \rho^{-\beta} \, dx \leq C \tag{6.6}$$

with C independent of λ as $\lambda \to \lambda^*$.

To prove the claim, let $\varepsilon > 0$, $R > 0$ and multiply (2.8) by $(\rho^2 + y^2 + \varepsilon)^{-\beta/2}$ and integrate over $[\rho \leq 1/2, 0 \leq y \leq R]$ to get

$$0 = \int_{[\rho \leq 1/2, 0 \leq y \leq R]} \nabla \cdot (y^{1-2s} \nabla v)(\rho^2 + y^2 + \varepsilon)^{-\beta/2} \, dx \, dy.$$

Integrating by parts we find

$$\lambda \int_{B_{1/2}} f(u_\lambda)(\rho^2 + \varepsilon)^{-\beta/2} \, dx = -I_1 - I_2 + I_3 \tag{6.7}$$

where

$$I_1 = \int_{[\rho\le 1/2]} R^{1-2s} v_y(\rho, R)(\rho^2 + R^2 + \varepsilon)^{-\beta/2}\, dx$$
$$I_2 = \int_{[0\le y\le R]} y^{1-2s} v_\rho(1/2, y)(1/4 + y^2 + \varepsilon)^{-\beta/2}\, dy$$
$$I_3 = -\beta \int_{[\rho\le 1/2, 0\le y\le R]} y^{1-2s}(\rho^2 + y^2 + \varepsilon)^{-\beta/2-1}(v_\rho \rho + v_y y)\, dx\, dy.$$

By (5.4) and (5.5), I_1 and I_2 remain uniformly bounded as $\varepsilon \to 0$ and $\lambda \to \lambda^*$. We decompose further

$$I_3 = I_\rho + I_y$$

where

$$I_\rho = -\beta \int_{[\rho\le 1/2, 0\le y\le R]} y^{1-2s}(\rho^2 + y^2 + \varepsilon)^{-\beta/2-1} v_\rho \rho\, dx\, dy,$$
$$I_y = -\beta \int_{[\rho\le 1/2, 0\le y\le R]} y^{1-2s}(\rho^2 + y^2 + \varepsilon)^{-\beta/2-1} v_y y\, dx\, dy.$$

Now we estimate I_y. Let $g(x)$ be equal to $\lambda f(u_\lambda(x))$ extended by 0 in $\mathbb{R}^n \backslash B_1$, and denote by $\tilde{v}$ the solution of

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla \tilde{v}) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ \tilde{v}(z) \to 0 & \text{as } |z| \to \infty \\ -y^{1-2s}\tilde{v}_y = g(x) & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{6.8}$$

By the Green representation formula for (6.8), we have

$$-\tilde{v}_y(x, y) = C_{n,s}\, y \int_{\mathbb{R}^n} \frac{g(\tilde{x})}{(|x - \tilde{x}|^2 + y^2)^{\frac{n+2-2s}{2}}}\, d\tilde{x}. \tag{6.9}$$

Consider the functions $w = -y^{1-2s} v_y$ and $\tilde{w} = -y^{1-2s}\tilde{v}_y$. Then, w and $\tilde{w}$ satisfy

$$\nabla \cdot (y^{2s-1}\nabla w) = 0 \quad \text{in } \mathscr{C}.$$

Since $-\tilde{v}_y \ge 0$ in $\mathbb{R}^n \times [0, +\infty)$ we have in particular

$$\tilde{w} \ge 0 = w \quad \text{on } \partial_L \mathscr{C}.$$

Furthermore

$$w \le \tilde{w} \quad \text{in } B_1 \times \{0\}$$

and for $z \in \mathscr{C}$, $w(z), \tilde{w}(z) \to 0$ as $|z| \to +\infty$. Then, the maximum principle (Lemma 2.8) implies that

$$-v_y \le -\tilde{v}_y \quad \text{in } \mathscr{C}.$$

It follows that

$$I_y \le -\beta \int_{[\rho\le 1/2, 0\le y\le R]} y^{1-2s}(\rho^2+y^2+\varepsilon)^{-\beta/2-1}\tilde{v}_y y\, dx\, dy$$

and by (6.9)

$$\begin{aligned} I_y &\le \beta C_{n,s} \int_{[\rho\le 1/2, 0\le y\le R]} \int_{\mathbb{R}^n} \frac{y^{3-2s} g(\tilde{x})}{(\rho^2+y^2+\varepsilon)^{\beta/2+1}(|x-\tilde{x}|^2+y^2)^{\frac{n+2-2s}{2}}}\, d\tilde{x}\, dx\, dy \\ &\le \beta C_{n,s} \int_{\mathbb{R}^n} g(\tilde{x}) \left(\int_{[\rho\le 1/2, 0\le y\le R]} \frac{y^{3-2s}}{(\rho^2+y^2+\varepsilon)^{\beta/2+1}(|x-\tilde{x}|^2+y^2)^{\frac{n+2-2s}{2}}}\, dx\, dy \right) d\tilde{x}. \end{aligned} \tag{6.10}$$

For $0<\beta<n$ let us introduce the number

$$A_{n,s,\beta} = \int_{\mathbb{R}^n\times(0,+\infty)} \frac{y^{3-2s}}{(|x|^2+y^2)^{\frac{\beta+2}{2}}(y^2+|x-e|^2)^{\frac{n+2-2s}{2}}}\, dx\, dy$$

where e is any unit vector in $\mathbb{R}^n$. Note that

$$\begin{aligned} &\int_{[\rho\le 1/2, 0\le y\le R]} \frac{y^{3-2s}}{(\rho^2+y^2+\varepsilon)^{\beta/2+1}(|x-\tilde{x}|^2+y^2)^{\frac{n+2-2s}{2}}}\, dx\, dy \\ &\quad\le \int_{\mathbb{R}^n\times(0,+\infty)} \frac{y^{3-2s}}{(\rho^2+y^2)^{\beta/2+1}(|x-\tilde{x}|^2+y^2)^{\frac{n+2-2s}{2}}}\, dx\, dy = |\tilde{x}|^{-\beta} A_{n,s,\beta}. \end{aligned}$$

From (6.10) we get

$$I_y \le \beta C_{n,s} A_{n,s,\beta} \int_{\mathbb{R}^n} g(\tilde{x})|\tilde{x}|^{-\beta}\, d\tilde{x} = \beta C_{n,s} A_{n,s,\beta}\lambda \int_{B_1} f(u_\lambda)|x|^{-\beta}\, dx. \tag{6.11}$$

Combining (6.7) with (6.11) we obtain

$$\lambda \int_{B_{1/2}} f(u_\lambda)(\rho^2+\varepsilon)^{-\beta/2}\, dx \le -I_1 - I_2 + \beta C_{n,s} A_{n,s,\beta}\lambda \int_{B_1} f(u_\lambda)|x|^{-\beta}\, dx + I_\rho.$$

Letting $\varepsilon \to 0$ we deduce

$$\begin{aligned} (1-\beta C_{n,s} A_{n,s,\beta})\lambda \int_{B_1} f(u_\lambda)\rho^{-\beta}\, dx &\le C\lambda \int_{B_1\setminus B_{1/2}} f(u_\lambda)\, dx \\ &\quad + \limsup_{\varepsilon\to 0}(|I_1|+|I_2|+|I_\rho|). \end{aligned} \tag{6.12}$$

We postpone for later the proof of the following:

Lemma 6.2. *We have* $1-\beta C_{n,s} A_{n,s,\beta} > 0$, *where* $C_{n.s}$ *is the constant in the representation formula* (2.14).

Proof of Theorem 1.6 *Continued.* Thanks to the previous lemma we can obtain (6.6) from (6.12) by estimating the terms in the right hand side of this inequality. Recall that by (5.4) and (5.5),

$$|I_1| \le C, \quad |I_2| \le C \tag{6.13}$$

for some C independent $\varepsilon \to 0$ and $\lambda \to \lambda^*$.

By the Cauchy–Schwarz inequality

$$|I_\rho| \le \beta \left(\int_{[\rho\le 1/2, 0\le y\le R]} y^{1-2s} v_\rho^2 \rho^{-2\alpha}\, dx\, dy \right)^{1/2} \left(\int_{[\rho\le 1/2, 0\le y\le R]} \frac{y^{1-2s}\rho^{2+2\alpha}}{(\rho^2+y^2+\varepsilon)^{\beta+2}}\, dx\, dy \right)^{1/2}$$

The last integral can be estimated by

$$\int_{[\rho\le 1/2, 0\le y\le R]} \frac{y^{1-2s}\rho^{2-2\alpha}}{(\rho^2+y^2+\varepsilon)^{\beta+2}}\, dx\, dy \le \int_0^\infty \int_{[\rho\le 1/2]} \frac{y^{1-2s}\rho^{2\alpha}}{(\rho^2+y^2)^{\beta+1}}\, dx\, dy.$$

We change variables $y = \rho t$ for $\rho > 0$. Since $\beta > 0$, we have

$$\int_{[\rho\le 1/2, 0\le y\le R]} \frac{y^{1-2s}\rho^{2+2\alpha}}{(\rho^2+y^2+\varepsilon)^{\beta+2}}\, dx\, dy \le \int_{[\rho\le 1/2]} \rho^{2\alpha-2\beta-2s}\, dx \int_0^\infty \frac{t^{1-2s}}{(1+t^2)^{\beta+1}}\, dt$$

and this integral is finite if $2(\beta+s-\alpha) < n$. The integral $\int_{[\rho\le 1/2, 0\le y\le R]} y^{1-2s} v_\rho^2 \rho^{-2\alpha}\, dx\, dy$ remains bounded as $\varepsilon \to 0$ and $\lambda \to \lambda^*$ by (5.3), provided α satisfies (5.2).

Thus, if α satisfies (5.2) and $2(\beta+s-\alpha) < n$ we deduce that

$$|I_\rho| \le C \tag{6.14}$$

with C independent of $\varepsilon > 0$ and $\lambda \in [0, \lambda^*)$.

Therefore, from (6.12), (6.13) and (6.14), and using a uniform bound for u_λ in $B_1 \setminus B_{1/2}$ we deduce (6.6).

Step 2. Conclusion.

(a) Assume first that $n < 2(s+2+\sqrt{2(s+1)})$. Then, $n/2 - s < 1 + \sqrt{n-1}$ and we can choose α satisfying $n/2 - s < \alpha < 1 + \sqrt{n-1}$. Thus, $n - 2s < n/2 + \alpha - s$ and we may choose $\beta = n - 2s$ in (6.6), which implies that $\int_{B_1} f(u_\lambda)\rho^{-n+2s}\, dx \le C$ with a constant independent of λ. By (6.1) we have

$$u_\lambda(0) \le C_n \int_{B_1} \rho^{-n+2s} f(u_\lambda(\rho)) dx \le C.$$

Since u_λ is radially decreasing, we conclude that u_λ is uniformly bounded in B_1 as $\lambda \to \lambda^*$.

(b) Now assume that $n \ge 2(s+2+\sqrt{2(s+1)})$. Suppose $1 < \alpha < 1 + \sqrt{n-1}$, $\beta > 0$, and $2(\beta+s-\alpha) < n$. Then, using that $f' > 0$, that u_λ is radially decreasing, as well as the estimate (6.6), we have for $\rho \le 1/2$

$$c\rho^{n-\beta} f(u_\lambda(\rho)) = f(u_\lambda(\rho)) \int_{B_{2\rho}\setminus B_\rho} |x|^{-\beta}\, dx \le \int_{B_1} f(u_\lambda)|x|^{-\beta}\, dx \le C$$

where $c > 0$. This yields

$$f(u_\lambda(\rho)) \le C\rho^{\beta-n} \quad \text{for } 0 < \rho \le 1$$

where C is independent of λ. Using (6.1), this implies that if additionally $\beta < n - 2s$, then

$$u_\lambda(x) \le \frac{C}{|x|^{n-\beta-2s}} \quad \text{for all } x \in B_1.$$

Since we have the restrictions $\beta < n/2 + \alpha - s$ and $\alpha < 1 + \sqrt{n-1}$, we see that for any $\mu > n/2 - s - 1 - \sqrt{n-1}$, there is C independent of λ such that

$$u_\lambda(x) \le \frac{C}{|x|^{\mu}} \quad \text{for all } x \in B_1.$$

By letting $\lambda \to \lambda^*$ in the last expression we conclude the proof. □

Finally, it only rest to give:

Proof of Lemma 6.2. Let $h \in L^\infty(\mathbb{R}^n)$ be radial and have compact support, and $u(x, y) = u(\rho, y)$ be a solution of

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla u) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ u(x, y) \to 0 & \text{as } |(x, y)| \to \infty \\ -y^{1-2s}u_y = h(x) & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{6.15}$$

Now, we claim that, for any $0 < \beta < n$

$$0 = (1 - \beta C_{n,s}A_{n,s,\beta}) \int_{\mathbb{R}^n} h(x)\rho^{-\beta}\, dx + \beta \int_{\mathbb{R}^n\times(0,+\infty)} y^{1-2s}r^{-\beta-2}\rho u_\rho\, dx\, dy. \tag{6.16}$$

Assuming the claim for a moment we prove the lemma. Choose a smooth radially decreasing function $h \ge 0$, $h \not\equiv 0$ with compact support. Let u be the solution of (6.15). By (2.14), u can be explicitly given by a convolution kernel. In turn, this shows that u is radial with respect to x and non-increasing in $|x|$. Hence

$$\int_{\mathbb{R}^n\times(0,+\infty)} y^{1-2s}r^{-\beta-2}\rho u_\rho\, dx\, dy < 0$$

and

$$\int_{\mathbb{R}^n} h(x)\rho^{-\beta}\, dx > 0.$$

This shows that $1 - \beta C_{n,s}A_{n,s,\beta} > 0$.

Now we give the argument for (6.16). Let $\varepsilon > 0$, $\beta \in (0, n + 2 - 2s)$ and multiply equation (6.15) by $(\rho^2 + y^2 + \varepsilon)^{-\beta/2}$ to get

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n\times(0,+\infty)} \operatorname{div}(y^{1-2s}\nabla u)(\rho^2 + y^2 + \varepsilon)^{-\beta/2}\, dx\, dy \\ &= -\int_{\mathbb{R}^n} y^{1-2s}u_y(\rho^2 + \varepsilon)^{-\beta/2}\, dx \end{aligned}$$

$$+\beta\int_{\mathbb{R}^n\times(0,+\infty)} y^{1-2s}(\rho^2+y^2+\varepsilon)^{-\beta/2-1}(x\cdot\nabla_x u + y\ u_y)\,dx\,dy$$
$$=\int_{\mathbb{R}^n} h(x)(\rho^2+\varepsilon)^{-\beta/2}\,dx+\beta\int_{\mathbb{R}^n\times(0,+\infty)} y^{1-2s}(\rho^2+y^2+\varepsilon)^{-\beta/2-1}\rho u_\rho\,dx\,dy$$
$$+\beta\int_{\mathbb{R}^n\times(0,+\infty)} y^{2-2s}(\rho^2+y^2+\varepsilon)^{-\beta/2-1}u_y\,dx\,dy.$$

Using the representation formula

$$-y^{1-2s}u_y(x,y)=C_{n,s}y^{2-2s}\int_{\mathbb{R}^n}\frac{h(\tilde{x})}{(y^2+|x-\tilde{x}|^2)^{\frac{n+2-2s}{2}}}\,d\tilde{x}$$

we find

$$0=\int_{\mathbb{R}^n} h(x)(\rho^2+\varepsilon)^{-\beta/2}\,dx+\beta\int_{\mathbb{R}^n\times(0,+\infty)} y^{1-2s}(\rho^2+y^2+\varepsilon)^{-\beta/2-1}\rho u_\rho\,dx\,dy$$
$$-\beta C_{n,s}\int_{\mathbb{R}^n\times(0,+\infty)}\int_{\mathbb{R}^n} y^{3-2s}(\rho^2+y^2+\varepsilon)^{-\beta/2-1}\frac{h(\tilde{x})}{(y^2+|x-\tilde{x}|^2)^{\frac{n+2-2s}{2}}}\,d\tilde{x}\,dx\,dy.$$

By Fubini, the last integral becomes

$$\int_{\mathbb{R}^n} h(\tilde{x})\int_{\mathbb{R}^n\times(0,+\infty)}\frac{y^{3-2s}}{(|x|^2+y^2+\varepsilon)^{\beta/2+1}(y^2+|x-\tilde{x}|^2)^{\frac{n+2-2s}{2}}}\,dx\,dy\,d\tilde{x},$$

and by the change variables: $y=|\tilde{x}|y'$, $y>0$, $x=|\tilde{x}|x'$, $x'\in\mathbb{R}^n$, we find

$$\int_{\mathbb{R}^n\times(0,+\infty)}\frac{y^{3-2s}}{(|x|^2+y^2+\varepsilon)^{\beta/2+1}(y^2+|x-\tilde{x}|^2)^{\frac{n+2-2s}{2}}}\,dx\,dy=|\tilde{x}|^{-\beta}A_{n,s,\beta}\left(\frac{\varepsilon}{|\tilde{x}|^2}\right)$$

where

$$A_{n,s,\beta}(t)=\int_{\mathbb{R}^n\times(0,+\infty)}\frac{y^{3-2s}}{(|x|^2+y^2+t)^{\frac{\beta+2}{2}}(y^2+|x-\frac{\tilde{x}}{|\tilde{x}|}|^2)^{\frac{n+2-2s}{2}}}\,dx\,dy.$$

Therefore, from the above computations we get

$$0=\int_{\mathbb{R}^n} h(x)(\rho^2+\varepsilon)^{-\beta/2}(1-\beta C_{n,s}A_{n,s,\beta}(\varepsilon/|\tilde{x}|^2))\,dx$$
$$+\beta\int_{\mathbb{R}^n\times(0,+\infty)} y^{1-2s}(\rho^2+y^2+\varepsilon)^{-\beta/2-1}\rho u_\rho\,dx\,dy. \tag{6.17}$$

Notice that

$$\lim_{\varepsilon\to 0}A_{n,s,\beta}(\varepsilon/|\tilde{x}|^2)=A_{n,s,\beta}\quad\text{for all }\tilde{x}\in\mathbb{R}^n$$

and that this limit is finite for $0<\beta<n+2-2s$. Moreover $A_{n,s,\beta}$ is independent of $\tilde{x}$. Since $\beta<n$ and h is bounded with compact support the function $h(\rho)\rho^{-\beta}$ is integrable. Hence, by letting $\varepsilon\to 0$ in (6.17) we obtain (6.16). □

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