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Relative Equilibria in Continuous Stellar Dynamics

Juan Campos[1], Manuel del Pino[2], Jean Dolbeault[1]

[1] Ceremade (UMR CNRS n° 7534), Université Paris-Dauphine, Place de Lattre de Tassigny, 75775 Paris Cédex 16, France. E-mail: campos@ceremade.dauphine.fr; dolbeaul@ceremade.dauphine.fr
[2] Departamento de Ingeniería Matemática and CMM, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile. E-mail: delpino@dim.uchile.cl

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Abstract: We study a three dimensional continuous model of gravitating matter rotating at constant angular velocity. In the rotating reference frame, by a finite dimensional reduction, we prove the existence of non-radial stationary solutions whose supports are made of an arbitrarily large number of disjoint compact sets, in the low angular velocity and large scale limit. At first order, the solutions behave like point particles, thus making the link with the *relative equilibria* in N-body dynamics.

1. Introduction and Statement of the Main Results

We consider the Vlasov-Poisson system

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = 0 \\ \phi = -\dfrac{1}{4\pi\,|\cdot|} * \rho, \quad \rho := \displaystyle\int_{\mathbb{R}^3} f\, dv \end{cases} \tag{1}$$

which models the dynamics of a cloud of particles moving under the action of a mean field gravitational potential ϕ solving the Poisson equation: $\Delta\phi = \rho$. Kinetic models like system (1) are typically used to describe gaseous stars or globular clusters. Here $f = f(t, x, v)$ is the so-called *distribution function,* a nonnegative function in $L^\infty(\mathbb{R}, L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ depending on *time* $t \in \mathbb{R}$, *position* $x \in \mathbb{R}^3$ and *velocity* $v \in \mathbb{R}^3$, which represents a density of particles in the *phase space,* $\mathbb{R}^3 \times \mathbb{R}^3$. The function ρ is the *spatial density* function and depends only on t and x. The total mass is conserved and hence

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} f(t, x, v)\, dx\, dv = \int_{\mathbb{R}^3} \rho(t, x)\, dx = M$$

does not depend on t.

The first equation in (1) is the *Vlasov equation,* also known as the *collisionless Boltzmann equation* in the astrophysical literature; see [5]. It is obtained by writing that the mass is transported by the flow of Newton's equations, when the gravitational field is computed as a mean field potential. Reciprocally, the dynamics of discrete particle systems can be formally recovered by considering empirical distributions, namely measure valued solutions made of a sum of Dirac masses, and neglecting the self-consistent gravitational terms associated to the interaction of each Dirac mass with itself.

It is also possible to relate (1) with discrete systems as follows. Consider the case of N gaseous spheres, far away from each other, in such a way that they weakly interact through gravitation. In terms of system (1), such a solution should be represented by a distribution function f, whose space density ρ is compactly supported, with several nearly spherical components. At large scale, the location of these spheres is governed at leading order by the N-body gravitational problem.

The purpose of this paper is to unveil this link by constructing a special class of solutions: we will build time-periodic, non radially symmetric solutions, which generalize to kinetic equations the notion of *relative equilibria* for the discrete N-body problem. Such solutions have a planar solid motion of rotation around an axis which contains the center of gravity of the system, so that the centrifugal force counter-balances the attraction due to gravitation. Let us give some details.

Consider N point particles with masses m_j, located at points $x_j(t) \in \mathbb{R}^3$ and assume that their dynamics is governed by Newton's gravitational equations

$$m_j \frac{d^2 x_j}{dt^2} = \sum_{j \neq k=1}^{N} \frac{m_j\, m_k}{4\pi} \frac{x_k - x_j}{|x_k - x_j|^3}, \quad j = 1, \dots N. \tag{2}$$

Let us write $x \in \mathbb{R}^3$ as $x = (x', x^3) \in \mathbb{R}^2 \times \mathbb{R} \approx \mathbb{C} \times \mathbb{R}$ where, using complex notations, $x' = (x^1, x^2) \approx x^1 + i\, x^2$ and rewrite system (2) in coordinates relative to a reference frame rotating at a constant velocity $\omega > 0$ around the x^3-axis. This amounts to carry out the change of variables

$$x = (e^{i\,\omega\, t}\, z',\, z^3), \quad z' = z^1 + i\, z^2.$$

In terms of the coordinates (z', z^3), system (2) then reads

$$\frac{d^2 z_j}{dt^2} = \sum_{j \neq k=1}^{N} \frac{m_k}{4\pi} \frac{z_k - z_j}{|z_k - z_j|^3} + \omega^2\, (z_j', 0) + 2\,\omega \left(i\, \frac{dz_j'}{dt}, 0 \right), \quad j = 1, \dots N. \tag{3}$$

We consider solutions which are stationary in the rotating frame, namely constant solutions $(z_1, \dots z_N)$ of system (3). Clearly all z_j's have their third component with the same value, which we assume zero. Hence, we have that

$$z_k = (\xi_k, 0), \quad \xi_k \in \mathbb{C},$$

where the ξ_k's are constants and satisfy the system of equations

$$\sum_{k \neq j=1}^{N} \frac{m_k}{4\pi} \frac{\xi_k - \xi_j}{|\xi_k - \xi_j|^3} + \omega^2\, \xi_j = 0, \quad j = 1, \dots N. \tag{4}$$

In the original reference frame, the solution of (2) obeys a rigid motion of rotation around the center of mass, with constant angular velocity ω. This solution is known as a *relative equilibrium*, thus taking the form

$$x_j^\omega(t) = (e^{i\,\omega\,t}\,\xi_j, 0), \quad \xi_j \in \mathbb{C}, \quad j = 1, \dots N.$$

System (4) has a variational formulation. In fact a vector $(\xi_1, \dots \xi_N)$ solves (4) if and only if it is a critical point of the function

$$\mathcal{V}_m^\omega(\xi_1, \dots \xi_N) := \frac{1}{8\pi} \sum_{j \neq k = 1}^{N} \frac{m_j\,m_k}{|\xi_k - \xi_j|} + \frac{\omega^2}{2} \sum_{j=1}^{N} m_j\,|\xi_j|^2.$$

Here m denotes $(m_j)_{j=1}^N$. A further simplification is achieved by considering the scaling

$$\xi_j = \omega^{-2/3}\,\zeta_j, \quad \mathcal{V}_m^\omega(\xi_1, \dots \xi_N) = \omega^{2/3}\,\mathcal{V}_m(\zeta_1, \dots \zeta_N), \tag{5}$$

where

$$\mathcal{V}_m(\zeta_1, \dots \zeta_N) := \frac{1}{8\pi} \sum_{j \neq k = 1}^{N} \frac{m_j\,m_k}{|\zeta_k - \zeta_j|} + \frac{1}{2} \sum_{j=1}^{N} m_j\,|\zeta_j|^2.$$

This function has in general many critical points, which are all *relative equilibria*. For instance, $\mathcal{V}_m$ clearly has a global minimum point.

Our aim is to construct solutions of gravitational models in continuum mechanics based on the theory of relative equilibria. We have the following result.

Theorem 1. *Given masses m_j, $j = 1, \dots N$, and any sufficiently small $\omega > 0$, there exists a solution $f_\omega(t, x, v)$ of Eq. (1) which is $\frac{2\pi}{\omega}$-periodic in time and whose spatial density takes the form*

$$\rho(t, x) := \int_{\mathbb{R}^3} f_\omega\,dv = \sum_{i=1}^{N} \rho_j(x - x_j^\omega(t)) + o(1).$$

Here $o(1)$ means that the remainder term uniformly converges to 0 as $\omega \to 0_+$ and identically vanishes away from $\cup_{j=1}^N B_R(x_j^\omega(t))$, for some $R > 0$, independent of ω. The functions $\rho_j(y)$ are non-negative, radially symmetric, non-increasing, compactly supported functions, independent of ω, with $\int_{\mathbb{R}^3} \rho_j(y)\,dy = m_j$ and the points $x_j^\omega(t)$ are such that

$$x_j^\omega(t) = \omega^{-2/3}\,(e^{i\,\omega\,t}\,\zeta_j^\omega, 0), \quad \zeta_j^\omega \in \mathbb{C}, \quad j = 1, \dots N$$

and

$$\lim_{\omega \to 0_+} \mathcal{V}_m(\zeta_1^\omega, \dots \zeta_N^\omega) = \min_{\mathbb{C}^N} \mathcal{V}_m, \quad \lim_{\omega \to 0_+} \nabla \mathcal{V}_m(\zeta_1^\omega, \dots \zeta_N^\omega) = 0.$$

The solution of Theorem 1 has a spatial density which is nearly spherically symmetric on each component of its support and these ball-like components rotate at constant, very small, angular velocity around the x^3-axis. The radii of these balls are very small compared with their distance to the axis. We shall call such a solution a *relative equilibrium* of (1), by extension of the discrete notion. The construction provides much more accurate information on the solution. In particular, the building blocks ρ_j are obtained as minimizers of an explicit reduced free energy functional, under suitable mass constraints.

It is also natural to consider other discrete relative equilibria, namely critical points of the energy $\mathcal{V}_m$ that may or may not be globally minimizing, and ask whether associated relative equilibria of system (1) exist. There are plenty of relative equilibria of the N-body problem. For instance, if all masses m_j are equal to some $m_* > 0$, a critical point is found by locating the ζ_j's at the vertices of a regular polygon:

$$\zeta_j = r\, e^{2\,i\,\pi(j-1)/N}, \quad j = 1, \dots N, \tag{6}$$

where r is such that

$$\frac{d}{dr}\left[\frac{a_N}{4\pi}\,\frac{m_*}{r} + \frac{1}{2}\,r^2\right] = 0 \quad \text{with} \quad a_N := \frac{1}{\sqrt{2}}\sum_{j=1}^{N-1}\frac{1}{\sqrt{1-\cos\left(2\pi j/N\right)}},$$

i.e. $r = (a_N\, m_*/(4\pi))^{1/3}$. This configuration is called the *Lagrange solution,* see [35]. The counterpart in terms of continuum mechanics goes as follows.

Theorem 2. *Let* $(\zeta_1, \dots \zeta_N)$ *be a regular polygon, namely with* ζ_j *given by* (6)*, and assume that all masses are equal. Then there exists a solution* f_ω *exactly as in Theorem* 1*, but with* $\lim_{\omega\to 0_+}(\zeta_1^\omega, \dots \zeta_N^\omega) = (\zeta_1, \dots \zeta_N)$.

Further examples of *relative equilibria* in the N-body problem can be obtained for instance by setting $N-1$ point particles of the same mass at the vertices of a regular polygon centered at the origin, then adding one more point particle at the center (not necessarily with the same mass), and finally adjusting the radius. Another family of solutions, known as the *Euler–Moulton solutions* is constituted by arrays of aligned points.

Critical points of the functional $\mathcal{V}_m$ are always degenerate because of their invariance under rotations: for any $\alpha \in \mathbb{R}$ we have

$$\mathcal{V}_m(\zeta_1, \dots \zeta_N) = \mathcal{V}_m(e^{i\,\alpha}\,\zeta_1, \dots e^{i\,\alpha}\,\zeta_N).$$

Let $\bar\zeta = (\bar\zeta_1, \dots \bar\zeta_N)$ be a critical point of $\mathcal{V}_m$ with $\bar\zeta_\ell \neq 0$. After a uniquely defined rotation, we may assume that $\bar\zeta_{\ell 2} = 0$. Moreover, we have a critical point of the function of $2N-1$ real variables,

$$\tilde{\mathcal{V}}_m(\zeta_1, \dots \zeta_{\ell 1}, \dots \zeta_N) := \mathcal{V}_m(\zeta_1, \dots (\zeta_{\ell 1}, 0), \dots \zeta_N).$$

We shall say that a critical point of $\mathcal{V}_m$ is *non-degenerate up to rotations* if the matrix $D^2\tilde{\mathcal{V}}_m(\bar\zeta_1, \dots \bar\zeta_{\ell 1}, \dots \bar\zeta_N)$ is non-singular. This property is clearly independent of the choice of ℓ.

Palmore in [30–34] has obtained classification results for the relative equilibria. In particular, it turns out that for *almost every choice of masses* m_j*,* all critical points of the functional $\mathcal{V}_m$ are non-degenerate up to rotations. Moreover, in such a case there exist at least $[2^{N-1}(N-2)+1]\,(N-2)\,!$ such distinct critical points. Many other results on *relative equilibria* are available in the literature. We have collected some of them in Appendix A with a list of relevant references. These results have a counterpart in terms of relative equilibria of system (1).

Theorem 3. *Let* $(\zeta_1, \ldots \zeta_N)$ *be a non-degenerate critical point of* $\mathcal{V}_m$ *up to rotations. Then there exists a solution* f_ω *as in Theorem 1, which satisfies, as in Theorem 2,* $\lim_{\omega \to 0_+}(\zeta_1^\omega, \ldots \zeta_N^\omega) = (\zeta_1, \ldots \zeta_N)$.

This paper is organized as follows. In the next section, we explain how the search for relative equilibria for the Vlasov-Poisson system can be reduced to the study of critical points of a functional acting on the gravitational potential. The construction of these critical points is detailed in Sect. 3. Sections 4 and 5 are respectively devoted to the linearization of the problem around a superposition of solutions of the problem with zero angular velocity, and to the existence of a solution of a nonlinear problem with appropriate orthogonality constraints depending on parameters $(\xi_j)_{j=1}^N$ related to the location of the N components of the support of the spatial density. Solving the original problem amounts to make all corresponding Lagrange multipliers equal to zero, which is equivalent to find a critical point of a function depending on $(\xi_j)_{j=1}^N$: this is the variational reduction described in Sect. 6. The proof of Theorems 1, 2 and 3 is given in Sect. 7 while known results on relative equilibria for the N-body, discrete problem are summarized in Appendix A.

2. The Setup

Guided by the representation (3) of the N-body problem in a rotating frame, we change variables in Eq. (1), replacing $x = (x', x^3)$ and $v = (v', v^3)$ respectively by

$$(e^{i\,\omega\,t}\,x', x^3) \quad \text{and} \quad (i\,\omega\,x' + e^{i\,\omega\,t}\,v', v^3).$$

Written in these new coordinates, Problem (1) becomes

$$\begin{cases} \frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f - \omega^2\,x' \cdot \nabla_{v'} f + 2\,\omega\,i\,v' \cdot \nabla_{v'} f = 0, \\ U = -\dfrac{1}{4\pi\,|\cdot|} * \rho, \quad \rho := \displaystyle\int_{\mathbb{R}^3} f\,dv. \end{cases} \tag{7}$$

The last two terms in the equation take into account the centrifugal and Coriolis force effects. System (7) can be regarded as the continuous version of problem (3). Accordingly, a *relative equilibrium* of System (1) will simply correspond to a stationary state of (7).

Such stationary solutions of (7) can be found by considering for instance critical points of the *free energy* functional

$$\mathcal{F}[f] := \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \beta(f)\,dx\,dv + \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \left(|v|^2 - \omega^2\,|x'|^2\right) f\,dx\,dv - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla U|^2\,dx$$

for some arbitrary convex function β, under the mass constraint

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} f\,dx\,dv = M.$$

A typical example of such a function is

$$\beta(f) = \frac{1}{q}\,\kappa_q^{q-1}\,f^q \tag{8}$$

for some $q \in (1, \infty)$ and some positive constant κ_q, to be fixed later. An additional restriction, $q > 9/7$, will come from the variational setting. The corresponding solution is known as the solution of the *polytropic gas model*, see [3–5,40,44].

When dealing with stationary solutions, it is not very difficult to rewrite the problem in terms of the potential. A critical point of $\mathcal{F}$ under the mass constraint $\iint_{\mathbb{R}^3\times\mathbb{R}^3} f\,dx\,dv = M$ is indeed given in terms of U by

$$f(x,v) = \gamma\left(\lambda + \frac{1}{2}\,|v|^2 + U(x) - \frac{1}{2}\,\omega^2\,|x'|^2\right), \tag{9}$$

where γ is, up to a sign, an appropriate generalized inverse of β'. In case (8), $\gamma(s) = \kappa_q^{-1}\,(-s)_+^{1/(q-1)}$, where $s_+ = (s + |s|)/2$ denotes the positive part of s. The parameter λ stands for the Lagrange multiplier associated to the mass constraint, at least if f has a single connected component. At this point, one should mention that the analysis is not exactly as simple as written above. Identity (9) indeed holds only component by component of the support of the solution, if this support has more than one connected component, and the Lagrange multipliers have to be defined for each component. The fact that

$$U(x) \underset{|x|\to\infty}{\sim} -\frac{M}{4\pi\,|x|}$$

is dominated by $-\frac{1}{2}\,\omega^2\,|x'|^2$ as $|x'| \to \infty$ is also a serious cause of trouble, which clearly discards the possibility that the free energy functional can be bounded from below if $\omega \neq 0$. This issue has been studied in [9], in the case of the so-called *flat systems.*

Finding a stationary solution in the rotating frame amounts to solving a non-linear Poisson equation, namely

$$\Delta U = g\left(\lambda + U(x) - \frac{1}{2}\,\omega^2\,|x'|^2\right) \quad \text{if } x \in \operatorname{supp}(\rho) \tag{10}$$

and $\Delta U = 0$ otherwise, where g is defined by

$$g(\mu) := \int_{\mathbb{R}^3} \gamma\left(\mu + \frac{1}{2}\,|v|^2\right) dv.$$

Hence, the problem can also be reduced to look for a critical point of the functional

$$\mathcal{J}[U] := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla U|^2\,dx + \int_{\cup_i K_i} G\left(\lambda + U(x) - \frac{1}{2}\,\omega^2\,|x'|^2\right) dx - \int_{\cup_i K_i} \lambda\,\rho\,dx,$$

where $\lambda = \lambda[x, U]$ is now a functional which is constant with respect to x, with value λ_i, on each connected component K_i of the support of $\rho(x) = g\,(\lambda[x,U] + U(x) - \frac{1}{2}\,\omega^2\,|x'|^2)$, $x \in \cup_i K_i$ and implicitly determined by the condition

$$\int_{K_i} g\left(\lambda_i + U(x) - \frac{1}{2}\,\omega^2\,|x'|^2\right) dx = m_i.$$

By G, we denote a primitive of g and the total mass is $M = \sum_{i=1}^N m_i$. Hence we can rewrite $\mathcal{J}$ as

$$\mathcal{J}[U] = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla U|^2\,dx + \sum_{i=1}^N\left[\int_{K_i} G\left(\lambda_i + U(x) - \frac{1}{2}\,\omega^2\,|x'|^2\right) dx - m_i\,\lambda_i\right]. \tag{11}$$

We may also observe that critical points of $\mathcal{F}$ correspond to critical points of the *reduced free energy* functional

$$\mathcal{G}[\rho] := \int_{\mathbb{R}^3} \left(h(\rho) - \frac{1}{2}\,\omega^2\,|x'|^2\,\rho \right) dx - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla U|^2\,dx$$

acting on the spatial densities if $h(\rho) = \int_0^\rho g^{-1}(-s)\,ds$. Also notice that, using the same function γ as in (9), to each distribution function f, we can associate a *local equilibrium*, or *local Gibbs state*,

$$G_f(x, v) = \gamma \left(\mu(\rho(x)) + \frac{1}{2}\,|v|^2 \right),$$

where μ is such that $g(\mu) = \rho$. This identity defines $\mu = \mu(\rho) = g^{-1}(\rho)$ as a function of ρ. Furthermore, by convexity, it follows that $\mathcal{F}[f] \geq \mathcal{F}[G_f] = \mathcal{G}[\rho]$ if $\rho(x) = \int_{\mathbb{R}^3} f(x, v)\; dv$, with equality if f is a local Gibbs state. See [11] for more details.

Summarizing, the heuristics are now as follows. The various components K_i of the support of the spatial density ρ of a critical point are assumed to be far away from each other so that the dynamics of their center of mass is described by the N-body point particles system, at first order. On each component K_i, the solution is a perturbation of an isolated minimizer of the free energy functional $\mathcal{F}$ (without angular rotation) under the constraint that the mass is equal to m_i. In the spatial density picture, on K_i, the solution is a perturbation of a minimizer of the reduced free energy functional $\mathcal{G}$.

To further simplify the presentation of our results, we shall focus on the model of *polytropic gases* corresponding to (8). In such a case, with $p := \frac{1}{q-1} + \frac{3}{2}$, g is given by

$$g(\mu) = (-\mu)_+^p$$

if the constant κ_q is fixed so that $\kappa_q = 4\pi\sqrt{2}\int_0^{+\infty} \sqrt{t}\,(1+t)^{\frac{1}{q-1}}\,dt = (2\pi)^{\frac{3}{2}}\,\frac{\Gamma(\frac{q}{q-1})}{\Gamma(\frac{3}{2}+\frac{q}{q-1})}$. For compactness reasons, we shall further restrict p to be subcritical, so that the range covered by our aproach is $p \in 3/2, 5)$.

Free energy functionals have been very much studied over the last years, not only to characterize special stationary states, but also because they provide a framework to deal with *orbital stability,* which is a fundamental issue in the mechanics of gravitation. The use of a free energy functional, whose entropy part, $\iint_{\mathbb{R}^3\times\mathbb{R}^3} \beta(f)\;\,dx\,dv$ is sometimes also called the *Casimir energy functional,* goes back to the work of V.I. Arnold (see [1,2,45]). The variational characterization of special stationary solutions and their orbital stability have been studied by Y. Guo and G. Rein in a series of papers [16–19,36–39] and by many other authors, see for instance [9,10,22–25,40,41,44].

The main drawback of such approaches is that stationary solutions which are characterized by these techniques are in some sense trivial: radial, with a single simply connected component support. Here we use a different approach to construct the solutions, which goes back to [13] in the context of Schrödinger equations. We are not aware of attempts to use dimensional reduction coupled to power-law non-linearities and Poisson force fields except in the similar case of a nonlinear Schrödinger equation with power law nonlinearity and repulsive Coulomb forces (see [8]), or in the case of an attractive Hartree-Fock model (see [21]). Technically, our results turn out to be closely related to the ones in [6,7].

Compared to previous results on gravitational systems, the main interest of our approach is to provide a much richer set of solutions, which is definitely of interest in astrophysics for describing complex patterns like binary gaseous stars or even more complex objects. The need of such an improvement was pointed out for instance in [20]. An earlier attempt in this direction has been done in the framework of Wasserstein's distance and mass transport theory in [27]. The point of this paper is that we can take advantage of the knowledge of special solutions of the N-body problem to produce solutions of the corresponding problem in continuum mechanics, which are still reminiscent of the discrete system.

3. Construction of Relative Equilibria

3.1. Some notations. We denote by $x = (x', x^3) \in \mathbb{R}^2 \times \mathbb{R}$ a generic point in $\mathbb{R}^3$. We may reformulate Problem (10) in terms of the potential $u = -U$ as follows. Given N positive numbers $\lambda_1, \dots \lambda_N$ and a small positive parameter ω, we consider the problem of finding N non-empty, compact, disjoint, connected subsets K_i of $\mathbb{R}^3$, $i = 1, 2 \dots N$, and a positive solution u of the problem

$$-\Delta u = \sum_{i=1}^{N} \rho_i \quad \text{in } \mathbb{R}^3 , \quad \rho_i := \left(u - \lambda_i + \frac{1}{2}\, \omega^2\, |x'|^2 \right)_+^p \chi_i , \tag{12}$$

$$\lim_{|x| \to \infty} u(x) = 0, \tag{13}$$

where χ_i denotes the characteristic function of K_i. We define the mass and the center of mass associated to each component by

$$m_i^\omega := \int_{\mathbb{R}^3} \rho_i \; dx \quad \text{and} \quad x_i^\omega := \frac{1}{m_i} \int_{\mathbb{R}^3} x\, \rho_i \; dx .$$

In our construction, when $\omega \to 0$, the sets K_i are asymptotically balls centered around x_i. It is crucial to localize the support of ρ_i since $u - \lambda_i + \frac{1}{2}\, \omega^2\, |x'|^2$ is always positive for large values of $|x'|$. We shall find a solution of (12) as a critical point u of the functional

$$\mathsf{J}\,[u] = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \; dx - \frac{1}{p+1} \sum_{i=1}^{N} \int_{\mathbb{R}^3} \left(u - \lambda_i + \frac{1}{2}\, \omega^2\, |x'|^2 \right)_+^{p+1} \chi_i \; dx , \tag{14}$$

so that $-u$ is a critical point of $\mathcal{J}$ in (11) in the case (8), namely $G(-s) = \frac{1}{p+1}\, s_+^{p+1}$.

Heuristically, our method goes as follows. We first consider the so-called *basic cell* problem: we characterize the solution with a single component support, when $\omega = 0$ and then build an *ansatz* by considering approximate solutions made of the superposition of basic cell solutions located close to relative equilibrium points, when they are far apart from each other. This can be done using the scaling invariance, in the low angular velocity limit $\omega \to 0_+$. The proof of our main results will be given in Sects. 4–7. It relies on a dimensional reduction of the variational problem: we shall prove that for a well chosen u, $\mathsf{J}\,[u] = \sum_{i=1}^N \lambda_i^{5-p}\, \mathsf{e}_* - \mathcal{V}_m^\omega(\xi_1, \dots \xi_N) + o(1)$ for some constant e_*, up to $o(1)$ terms, which are uniform in $\omega > 0$, small. Hence finding a critical point for J will be reduced to look for a critical point of $\mathcal{V}_m^\omega$ as a function of $(\xi_1, \dots \xi_N)$.

3.2. The basic cell problem. Let us consider the following problem:

$$-\Delta w = (w-1)_+^p \quad \text{in } \mathbb{R}^3. \tag{15}$$

Lemma 1. *Under the condition* $\lim_{|x|\to\infty} w(x) = 0$*, Eq.* (15) *has a unique solution, up to translations, which is positive and radially symmetric if* $p \in (1,5)$.

Proof. Since p is subcritical, it is well known that the problem

$$-\Delta Z = Z^p \quad \text{in } B_1(0)$$

with homogeneous Dirichlet boundary conditions, $Z = 0$, on $\partial B_1(0)$, has a unique positive solution, which is also radially symmetric (see [15]). For any $R > 0$, the function $Z_R(x) := R^{-2/(p-1)}\, Z(x/R)$ is the unique radial, positive solution of

$$-\Delta Z_R = Z_R^p \quad \text{in } B_R(0)$$

with homogeneous Dirichlet boundary conditions on $\partial B_R(0)$. According to [14,15], any positive solution of (15) is radially symmetric, up to translations. Finding such a solution w of (15) is equivalent to finding numbers $R > 0$ and $m_* > 0$ such that the function, defined by pieces as $w = Z_R + 1$ in B_R and $w(x) = m_*/(4\pi\,|x|)$ for any $x \in \mathbb{R}^3$ such that $|x| > R$, is of class C^1. These numbers are therefore uniquely determined by

$$w(R^-) = 1 = \frac{m_*}{4\pi\, R} = w(R^+)\,,\;\; w'(R^-) = R^{-\frac{2}{p-1}-1}\, Z'(1) = -\frac{m_*}{4\pi\, R^2} = w'(R^+)\,,$$

which uniquely determines the solution of (15). □

Now let us consider the slightly more general problem

$$-\Delta w^\lambda = (w^\lambda - \lambda)_+^p \quad \text{in } \mathbb{R}^3$$

with $\lim_{|x|\to\infty} w^\lambda(x) = 0$. For any $\lambda > 0$, it is straightforward to check that it has a unique radial solution given by

$$w^\lambda(x) = \lambda\, w\left(\lambda^{(p-1)/2}\, x\right) \quad \forall x \in \mathbb{R}^3.$$

Let us observe, for later reference, that

$$\int_{\mathbb{R}^3} (w^\lambda - \lambda)_+^p \, dx = \lambda^{(3-p)/2} \int_{\mathbb{R}^3} (w-1)_+^p \, dx =: \lambda^{(3-p)/2}\, m_*. \tag{16}$$

Moreover, w^λ is given by

$$w^\lambda(x) = \frac{m_*}{4\pi\,|x|}\, \lambda^{(3-p)/2} \quad \forall x \in \mathbb{R}^3 \text{ such that } |x| > R\,\lambda^{-(p-1)/2}.$$

3.3. The ansatz. We consider now a first approximation of a solution of (12)–(13), built as a superposition of the radially symmetric functions w^{λ_i} translated to points ξ_i, $i = 1, \dots N$ in $\mathbb{R}^2 \times \{0\}$, far away from each other:

$$w_i(x) := w^{\lambda_i}(x - \xi_i), \quad W_\xi := \sum_{i=1}^N w_i .$$

Recall that we are given the masses $m_1, \dots m_N$. We choose, according to formula (16), the positive numbers λ_i so that

$$\int_{\mathbb{R}^3} (w_i - \lambda_i)_+^p \, dx \; = \; m_i \quad \text{for all } i = 1, \dots N.$$

By ξ we denote the array $(\xi_1, \xi_2, \dots \xi_N)$.

We shall assume in what follows that the points ξ_i are such that for a large, fixed $\mu > 0$, and all small $\omega > 0$ we have

$$|\xi_i| < \mu\, \omega^{-2/3}, \quad |\xi_i - \xi_j| > \mu^{-1}\, \omega^{-2/3}. \tag{17}$$

Equivalently,

$$|\zeta_i| < \mu\ , \quad |\zeta_i - \zeta_j| > \mu^{-1} \quad \text{where} \quad \zeta_i := \omega^{-2/3}\, \xi_i . \tag{18}$$

We look for a solution of (12) of the form

$$u = W_\xi + \phi$$

for a convenient choice of the points ξ_i, where ϕ is globally uniformly small when compared with W_ξ. For this purpose, we consider a fixed number $R > 1$ such that

$$\operatorname{supp}\, (w^{\lambda_i} - \lambda_i)_+ \subset B_{R-1}(0) \quad \forall i = 1,\, 2 \dots N$$

and define the functions

$$\chi(x) = \begin{cases} 1 & \text{if } |x| < R \\ 0 & \text{if } |x| \ge R \end{cases} \quad \text{and} \quad \chi_i(x) = \chi(x - \xi_i).$$

Thus we want to find a solution to the problem

$$\Delta(W_\xi + \phi) + \sum_{i=1}^N \left(W_\xi - \lambda_i + \phi + \frac{1}{2}\, \omega^2\, |x'|^2 \right)_+^p \chi_i = 0 \quad \text{in } \mathbb{R}^3$$

with $\lim_{|x| \to \infty} \phi(x) = 0$, that is we want to solve the problem

$$\Delta\phi + \sum_{i=1}^N p \left(W_\xi - \lambda_i + \frac{1}{2}\, \omega^2\, |x'|^2 \right)_+^{p-1} \chi_i\, \phi = -\mathsf{E} - \mathsf{N}[\phi], \tag{19}$$

where

$$\mathsf{E} := \Delta W_\xi + \sum_{i=1}^N \left(W_\xi - \lambda_i + \frac{1}{2}\,\omega^2\,|x'|^2 \right)_+^p \chi_i\,,$$

$$\mathsf{N}[\phi] := \sum_{i=1}^N \Bigg[\left(W_\xi - \lambda_i + \frac{1}{2}\,\omega^2\,|x'|^2 + \phi \right)_+^p - \left(W_\xi - \lambda_i + \frac{1}{2}\,\omega^2\,|x'|^2 \right)_+^p - p \left(W_\xi - \lambda_i + \frac{1}{2}\,\omega^2\,|x'|^2 \right)_+^{p-1} \phi \Bigg] \chi_i\,.$$

4. A Linear Theory

The purpose of this section is to develop a solvability theory for the operator

$$\mathsf{L}[\phi] = \Delta\phi + \sum_{i=1}^N p \left(W_\xi - \lambda_i + \frac{1}{2}\,\omega^2\,|x'|^2 \right)_+^{p-1} \chi_i\,\phi.$$

To this end we introduce the norms

$$\|\phi\|_* = \sup_{x\in\mathbb{R}^3} \left(\sum_{i=1}^N |x-\xi_i| + 1 \right) |\phi(x)|, \quad \|h\|_{**} = \sup_{x\in\mathbb{R}^3} \left(\sum_{i=1}^N |x-\xi_i|^4 + 1 \right) |h(x)|.$$

We want to solve problems of the form $\mathsf{L}[\phi] = h$ with h and ϕ having the above norms finite. Rather than solving this problem directly, we consider a *projected problem* of the form

$$\mathsf{L}[\phi] = h + \sum_{i=1}^N \sum_{j=1}^3 c_{ij}\,Z_{ij}\,\chi_i\,, \tag{20}$$

$$\lim_{|x|\to\infty} \phi(x) = 0, \tag{21}$$

where $Z_{ij} := \partial_{x_j} w_i$, subject to orthogonality conditions

$$\int_{\mathbb{R}^3} \phi\,Z_{ij}\,\chi_i\,dx = 0 \quad \forall i = 1,\,2\ldots N, \quad j = 1,\,2,\,3. \tag{22}$$

Equation (20) involves the coefficients c_{ij} as Lagrange multipliers associated to the constraints (22). If we can solve the equations $\mathsf{L}[\psi] = h$ and $\mathsf{L}[Y_{ij}] = Z_{ij}$, and if we define c_{ij} such that $\int_{\mathbb{R}^3} \psi\,Z_{ij}\,dx + c_{ij} \int_{\mathbb{R}^3} Y_{ij}\,Z_{ij}\,dx = 0$, then we observe that $\phi = \psi + \sum_{i,\,j} c_{ij}\,Y_{ij}$ solves (20) and satisfies (22). However, for existence, we will rather reformulate the question as a constrained variational problem; see Eq. (25) below.

Lemma 2. *Assume that* (17) *holds. Given h with $\|h\|_{**} < +\infty$, Problem* (20)–(22) *has a unique solution $\phi =: \mathsf{T}[h]$ and there exists a positive constant C, which is independent of ξ such that, for $\omega > 0$ small enough,*

$$\|\phi\|_* \le C\,\|h\|_{**}. \tag{23}$$

Proof. In order to solve (20)–(22), we first establish (23) as an *a priori* estimate. Assume by contradiction the existence of sequences $\omega_n \to 0$, ξ_i^n satisfying (17) for $\omega = \omega_n$, of functions ϕ_n, h_n and of constants c_{ij}^n for which

$$\|\phi_n\|_* = 1, \quad \lim_{n\to\infty} \|h_n\|_{**} = 0,$$
$$\int_{\mathbb{R}^3} \phi_n\, Z_{ij}\, \chi_i\, dx = 0 \quad \forall i, j \quad \text{and} \quad L[\phi_n] = h_n + \sum_{i=1}^N \sum_{j=1}^3 c_{ij}^n\, Z_{ij}\, \chi_i .$$

Testing the equation against $Z_{k\ell}$, we obtain, after an integration by parts,

$$\int_{\mathbb{R}^3} p \left[\sum_{i=1}^N \left(W_{\xi^n} - \lambda_i + \frac{1}{2}\, \omega_n^2\, |x'|^2 \right)_+^{p-1} \chi_i - (w_k - \lambda_k)_+^{p-1} \right] \phi_n\, Z_{k\ell}\, dx$$
$$= \int_{\mathbb{R}^3} h_n\, Z_{k\ell}\, dx + \sum_{i=1}^N \sum_{j=1}^3 c_{ij}^n \int_{\mathbb{R}^3} Z_{k\ell}\, Z_{ij}\, \chi_i\, dx. \tag{24}$$

Here we have used $\Delta Z_{k\ell} = p\, (w_k - \lambda_k)_+^{p-1}\, Z_{k\ell}$. The integrals in the sum can be estimated as follows:

$$\int_{\mathbb{R}^3} |Z_{k\ell}\, Z_{ij}\, \chi_i|\, dx = \int_{B(0,R)} |\partial_{x_\ell} w^{\lambda_k}(x)\, \partial_{x_j} w^{\lambda_i}(x + \xi_i^n - \xi_k^n)|\, dx$$
$$\leq \int_{B(0,R)} |\partial_{x_\ell}\, w^{\lambda_k}(x)|\, \frac{C}{|x + \xi_i^n - \xi_k^n|^2}\, dx = O(\omega_n^{4/3})$$

for some generic constant $C > 0$ which will change from line to line. Now we turn our attention to the left-hand side of (24). Since $p - 1 > 0$, we first notice that

$$\int_{\mathbb{R}^3} \left| \sum_{i=1}^N \left(W_{\xi^n} - \lambda_i + \frac{1}{2}\, \omega_n^2\, |x'|^2 \right)_+^{p-1} \chi_i - (w_k - \lambda_k)_+^{p-1} \right| |\phi_n\, Z_{k\ell}|\, dx$$
$$\leq \int_{\mathbb{R}^3} \sum_{i=1, i\neq k}^N \left(W_{\xi^n} - \lambda_i + \frac{1}{2}\, \omega_n^2\, |x'|^2 \right)_+^{p-1} \chi_i\, |\phi_n\, Z_{k\ell}|\, dx$$
$$+ C \int_{\mathbb{R}^3} \left(\frac{1}{2}\, \omega_n^2\, |x'|^2 + \sum_{i=1, i\neq k}^N w_i \right) \chi_k\, |\phi_n\, Z_{k\ell}|\, dx.$$

It is not hard to check that

$$\int_{\mathbb{R}^3} \sum_{i=1, i\neq k}^N \left(W_{\xi^n} - \lambda_i + \tfrac{1}{2}\, \omega_n^2\, |x'|^2 \right)_+^{p-1} \chi_i\, |\phi_n\, Z_{k\ell}|\, dx$$
$$\leq C\, \|\phi_n\|_* \left(\sum_{i=1, i\neq k}^N \int_{B(0,R)} |\partial_{x_\ell} w^{\lambda_k}(x + \xi_i - \xi_k)|\, dx \right) = O(\omega_n^{4/3})$$

and

$$\int_{\mathbb{R}^3} \left(\frac{1}{2}\, \omega_n^2\, |x'|^2 + \sum_{i=1, i\neq k}^{N} w_i \right) \chi_k\, |\phi_n\, Z_{k\ell}|\, dx$$
$$\leq C\, \|\phi_n\|_* \left(\omega_n^2 + \sum_{i=1, i\neq k}^{N} \int_{B(0,R)} \frac{|\partial_{x_\ell} w^{\lambda_k}|}{|x+\xi_k-\xi_i|} \right) dx = O(\omega_n^{2/3}).$$

Summarizing, we have found that, for each $k = 1,\ 2 \ldots N$,

$$O(\omega_n^{2/3})\, \|\phi_n\|_* = \int_{\mathbb{R}^3} h_n\, Z_{k\ell}\, dx + c_{k\ell}^n \int_{\mathbb{R}^3} |Z_{k\ell}|^2\, \chi_k\, dx + O(\omega_n^{4/3}) \sum_{(i,j)\neq(k,\ell)} |c_{ij}^n|,$$

from which we deduce that $c_{k\ell}^n = O(\omega_n^{2/3}) + O(\|h_n\|_{**}) \to 0$ for all $k,\ \ell$. We may indeed notice that for ω_n small enough, the above equations define an almost diagonal system, so that the coefficients c_{ij} are uniquely determined.

Let us prove that $\lim_{n\to\infty} \|\phi_n\|_{L^\infty(\mathbb{R}^3)} = 0$. If not, since $\|\phi_n\|_* = 1$, we may assume that there is an index i and a sufficiently large number $R > 0$ for which

$$\liminf_{n\to\infty} \|\phi_n\|_{L^\infty(B_R(\xi_i))} > 0.$$

Using elliptic estimates, and defining $\psi_n(x) = \phi_n(\xi_i^n + x)$, we may assume that ψ_n uniformly converges in the C^1 sense over compact subsets of $\mathbb{R}^3$ to a bounded, non-trivial solution ψ of the equation

$$\Delta\psi + p\, \left(w^{\lambda_i} - \lambda_i\right)_+^{p-1} \psi = 0,$$
$$\int_{\mathbb{R}^3} \psi\, \partial_{x_j} w^{\lambda_i}\, \chi\, dx = 0 \quad \forall j = 1,\ 2,\ 3.$$

According to [14, Lemma 5], ψ must be a linear combination of the functions $\partial_{x_j} w^{\lambda_i}$, $j = 1, 2, 3$. The latter orthogonality conditions yield $\psi \equiv 0$. This is a contradiction and the claim is proven. Finally, let

$$\tilde{h}_n := h_n + \sum_{i=1}^{N} \sum_{j=1}^{3} c_{ij}^n\, Z_{ij}\, \chi_i.$$

Then we have that

$$|\tilde{h}_n(x)| \leq \left(O(\omega_n^{2/3}) + O(\|h_n\|_{**}) \right) \sum_{i=1}^{k} \frac{1}{1+|x-\xi_i^n|^4},$$

and hence $\tilde{\phi}_n$, the unique solution in $\mathbb{R}^3$ of

$$\Delta\tilde{\phi}_n = \tilde{h}_n, \quad \lim_{|x|\to\infty} \tilde{\phi}_n(x) = 0,$$

satisfies

$$|\tilde{\phi}_n(x)| \leq \left(O(\omega_n^{2/3}) + O(\|h_n\|_{**}) \right) \sum_{i=1}^{k} \frac{1}{|x-\xi_i^n|}.$$

Now, since $\phi_n - \tilde{\phi}_n$ is harmonic in $\mathbb{R}^3 \setminus \cup_i B_R(\xi_i^n)$, it tends to zero as $|x| \to \infty$ and gets uniformly small on the boundary of this set. By the maximum principle, we get the estimate

$$|\phi_n(x)| \le \left(O(\omega_n^{2/3}) + O(\|h_n\|_{**}) \right) \sum_{i=1}^{k} \frac{1}{|x - \xi_i^n|} \quad \text{on } \mathbb{R}^3 \setminus \cup_i B_R(\xi_i^n).$$

This shows that $\lim_{n\to\infty} \|\phi_n\|_* = 0$, a contradiction with $\|\phi_n\|_* = 1$, and (23) follows.

Now, for existence issues, we observe that problem (20)-(22) can be set up in variational form in the Hilbert space

$$\mathcal{H} = \left\{ \phi \in \mathcal{D}^{1,2}(\mathbb{R}^3) \ : \ \int_{\mathbb{R}^3} \phi \, Z_{ij} \, \chi_i \, dx = 0 \quad \forall i = 1, 2 \dots N, \ j = 1, 2, 3 \right\}$$

endowed with the inner product $\langle \phi, \psi \rangle = \int_{\mathbb{R}^3} \nabla \psi \cdot \nabla \phi \, dx$, as

$$\int_{\mathbb{R}^3} \nabla \phi \cdot \nabla \psi \, dx - \int_{\mathbb{R}^3} \sum_{i=1}^{N} p \left(W_\xi - \lambda_i + \frac{1}{2} \omega^2 \, |x'|^2 \right)_+^{p-1} \chi_i \, \phi \, \psi \, dx$$
$$+ \int_{\mathbb{R}^3} \psi \, h \, dx = 0 \tag{25}$$

for all $\psi \in \mathcal{H}$. Since the potential defined by the second term of the above equality is compactly supported and h decays sufficiently fast, this equation takes the form $\phi + \mathsf{K}[\phi] = \tilde{h}$, where K is a compact linear operator of $\mathcal{H}$. The equation for $\tilde{h} = 0$ has just the trivial solution in view of estimate (23). Fredholm's alternative thus applies to yield existence. This concludes the proof of Lemma 2. □

Notice that the convergence in the norm $\|\cdot\|_{**}$-norm guarantees that there is no issue with the localization of the support of the components of the spatial density.

We conclude this section with some considerations on the differentiability of the solution with respect to the parameter ξ. Let us assume that $h = h(\cdot, \xi)$ defines a continuous operator into the space of functions with finite $\|\cdot\|_{**}$-norm. We also assume that $\|\partial_\xi h(\cdot, \xi)\|_{**} < +\infty$. Let $\phi = \phi(\cdot, \xi)$ be the unique solution of Problem (20)–(22) for that right hand side, with corresponding constants $c_{ij}(\xi)$. Then ϕ is differentiable in ξ. Moreover $\partial_\xi \phi$ can be decomposed as

$$\partial_\xi \phi = \psi + \sum_{ij} d_{ij} \, Z_{ij} \, \chi_j ,$$

where ψ solves

$$\mathsf{L}[\psi] = \partial_\xi h \ - \sum_{i=1}^{N} p \, \partial_\xi \left[\left(W_\xi - \lambda_i + \frac{1}{2} \omega^2 \, |x'|^2 \right)_+^{p-1} \chi_i \right] \phi$$
$$+ \sum_{i=1}^{N} \sum_{j=1}^{3} \left[c_{ij} \, \partial_\xi \left(Z_{ij} \, \chi_i \right) + b_{ij} \, Z_{ij} \, \chi_i \right] ,$$
$$\lim_{|x|\to\infty} \psi(x) = 0, \quad \int_{\mathbb{R}^3} \psi \, Z_{ij} \, \chi_i \, dx = 0 \quad \forall i = 1, 2 \dots N, \ j = 1, 2, 3,$$

and the constants d_{ij} are chosen so that $\eta := \sum_{i=1}^{N} \sum_{j=1}^{3} d_{ij}\, Z_{ij}$ satisfies

$$\int_{\mathbb{R}^3} \chi_{ij}\, Z_{ij}\, \eta\, dx = - \int_{\mathbb{R}^3} \partial_\xi \left(\chi_{ij}\, Z_{ij} \right) \phi\, dx \quad \forall i = 1,\, 2 \ldots N,\;\; j = 1,\, 2,\, 3.$$

Lemma 3. *With the same notations and conditions as in Lemma 2, we have*

$$\| \partial_\xi \phi(\cdot, \xi) \|_* \;\le\; C \; \left(\| h(\cdot, \xi) \|_{**} + \| \partial_\xi h(\cdot, \xi) \|_{**} \right).$$

5. The Projected Nonlinear Problem

Next we want to solve a *projected version* of the nonlinear problem (19) using Lemma 2. Thus we consider the problem of finding ϕ with $\|\phi\|_* < +\infty$, the solution of

$$\mathsf{L}[\phi] = -\mathsf{E} - \mathsf{N}[\phi] + \sum_{i=1}^{N} \sum_{j=1}^{3} c_{ij}\, Z_{ij}\, \chi_i \tag{26}$$

$$\lim_{|x| \to +\infty} \phi(x) = 0, \tag{27}$$

where the coefficients c_{ij} are Lagrange multipliers associated to the orthogonality conditions

$$\int_{\mathbb{R}^3} \phi\, Z_{ij}\, \chi_i\, dx = 0 \quad \forall i = 1,\, 2 \ldots N,\, j = 1,\, 2,\, 3. \tag{28}$$

In other words, we look for a critical point of the functional J defined by (14) under the constraints (28).

For this purpose, we first have to measure the error E. We recall that

$$\begin{aligned}
\mathsf{E} &= \Delta W_\xi + \sum_{i=1}^{N} \left(W_\xi - \lambda_i + \frac{1}{2}\, \omega^2\, |x'|^2 \right)_+^p \chi_i \\
&= \sum_{i=1}^{N} \left[\left(w_i + \sum_{j \ne i} w_j - \lambda_i + \frac{1}{2}\, \omega^2\, |x'|^2 \right)_+^p - (w_i - \lambda_i)_+^p \right] \chi_i \\
&= \sum_{i=1}^{N} p \left[w_i - \lambda_i + t \left(\sum_{j \ne i} w_j + \frac{1}{2}\, \omega^2\, |x'|^2 \right) \right]_+^{p-1} \left(\sum_{j \ne i} w_j + \frac{1}{2}\, \omega^2\, |x'|^2 \right) \chi_i
\end{aligned}$$

for some function t taking values in $(0, 1)$. It follows that

$$|\mathsf{E}| \le C \sum_{i=1}^{N} \left[\sum_{j \ne i} \frac{1}{|\xi_i - \xi_j|} + \frac{1}{2}\, \omega^2\, |\xi_i|^2 \right] \chi_i \;\le C\, \omega^{2/3} \sum_{i=1}^{N} \chi_i\,,$$

from which we deduce the estimate

$$\|\mathsf{E}\|_{**} \le C\, \omega^{2/3}.$$

As for the operator $\mathsf{N}[\phi]$, we easily check that for $\|\phi\|_* \le 1$,

$$|\,\mathsf{N}[\phi]\,| \le C \sum_{i=1}^{N} |\phi|^\gamma \, \chi_i \quad \text{with } \gamma := \min\{p, 2\},$$

which implies

$$\|\,\mathsf{N}[\phi]\,\|_{**} \le C\,\|\phi\|_*^\gamma .$$

Let T be the linear operator defined in Lemma 2. Equation (26) can be rewritten as

$$\phi = \mathsf{A}[\phi] := -\mathsf{T}\left[\mathsf{E} + \mathsf{N}[\phi]\right].$$

Clearly the operator A maps the region

$$\mathcal{B} = \left\{\phi \in L^\infty(\mathbb{R}^3) \;:\; \|\phi\|_* \le K\,\omega^{2/3}\right\}$$

into itself if the constant K is fixed, large enough. It is straightforward to check that $\mathsf{N}[\phi]$ satisfies in this region a Lipschitz property of the form

$$\|\,\mathsf{N}[\phi_1] - \mathsf{N}[\phi_2]\,\|_{**} \le \kappa_\omega\,\|\phi_1 - \phi_2\|_*$$

for some positive κ_ω such that $\lim_{\omega\to 0}\kappa_\omega = 0$, and hence existence of a unique fixed point ϕ of A in $\mathcal{B}$ immediately follows for ω small enough. We have then solved the projected nonlinear problem.

Since the error E is even with respect to the variable x^3, uniqueness of the solution of (26)–(28) implies that this symmetry is also valid for ϕ itself, and besides, the numbers c_{i3} are automatically all zero. Summarizing, we have proven the following result.

Lemma 4. *Assume that $\xi = (\xi_1, \xi_2, \ldots \xi_N) \in \mathbb{R}^{2N}$ is given and satisfies* (17). *Then Problem* (26)–(28) *has a unique solution ϕ_ξ which depends continuously on ξ and ω for the $\|\ \|_*$-norm and satisfies $\|\phi_\xi\|_* \le C\,\omega^{2/3}$ for some positive C, which is independent of ω, small enough. Besides, the numbers c_{i3} are all equal to zero for $i = 1,\ 2\ldots N$.*

It is important to mention that ϕ_ξ also defines a continuously differentiable operator in its parameter. Indeed, combining its fixed point characterization with the implicit function theorem and the result of Lemma 3, we find in fact that

$$\|\partial_\xi \phi_\xi\|_* \le C\,\omega^{2/3}.$$

We leave the details to the reader.

With the complex notation of Sect. 1, let us consider the rotation $e^{i\,\alpha}$ of an angle α around the x^3-axis and let $e^{i\,\alpha}\,\xi = (e^{i\,\alpha}\,\xi_1, \ldots e^{i\,\alpha}\,\xi_N)$. By construction, there is a rotational symmetry around the x^3-axis, which is reflected at the level of Problem (26)–(28) as follows.

Lemma 5. *Consider the solution ϕ found in Lemma* 4. *For any $\alpha \in \mathbb{R}$ and any $(x', x^3) \in \mathbb{C}\times\mathbb{R}$, we have that*

$$\phi_{e^{i\,\alpha}\xi}(x', x^3) = \phi_\xi(e^{-i\,\alpha}\,x', x^3).$$

The proof is a direct consequence of uniqueness and rotation invariance of the equation satisfied by ϕ_ξ.

6. The Variational Reduction

We consider the functional J defined in (14). Our goal is to find a critical point satisfying (28), of the form $u = W_\xi + \phi_\xi$. We estimate $\mathsf{J}[W_\xi]$ by computing first

$$\int_{\mathbb{R}^3} |\nabla W_\xi|^2\, dx = \int_{\mathbb{R}^3} \left|\sum_{i=1}^N \nabla w_i\right|^2 dx = \sum_{i=1}^N \int_{\mathbb{R}^3} |\nabla w_i|^2\, dx + \sum_{i\neq j} \int_{\mathbb{R}^3} \nabla w_i \cdot \nabla w_j\, dx.$$

The last term of the right hand side can be estimated by

$$\begin{aligned}\int_{\mathbb{R}^3} \nabla w_i \cdot \nabla w_j\, dx &= -\int_{\mathbb{R}^3} \Delta w_i\, w_j\, dx = \int_{\mathbb{R}^3} (w_i - \lambda_i)_+^p\, w_j\, dx\\ &= \int_{\mathbb{R}^3} \left(w^{\lambda_i} - \lambda_i\right)_+^p\, w^{\lambda_j}(x + \xi_i - \xi_j)\, dx\\ &= \int_{\mathbb{R}^3} \left(w^{\lambda_i} - \lambda_i\right)_+^p\, \frac{m_j}{4\pi\, |x + \xi_i - \xi_j|}\, dx.\end{aligned}$$

If we Taylor expand $x \mapsto |x + \xi_i - \xi_j|^{-1}$ around $x = 0$, we obtain by (18)

$$\begin{aligned}&\int_{\mathbb{R}^3} \left(w^{\lambda_i} - \lambda_i\right)_+^p\, \frac{m_j}{4\pi\, |x + \xi_i - \xi_j|}\, dx\\ &= \int_{B(0,R)} \left(w^{\lambda_i} - \lambda_i\right)_+^p\, \frac{m_j}{4\pi} \left(\frac{1}{|\xi_i - \xi_j|} - \frac{(\xi_i - \xi_j)\cdot x}{|\xi_i - \xi_j|^3} + O(\omega^2\, |x|^2)\right) dx\\ &= \frac{m_i\, m_j}{4\pi\, |\xi_i - \xi_j|} + O(\omega^{4/3}),\end{aligned}$$

where $m_i = \int_{\mathbb{R}^3} \left(w^{\lambda_i} - \lambda_i\right)_+^p dx = m_*\, \lambda_i^{(3-p)/2}$. Next we find that

$$\begin{aligned}&\int_{\mathbb{R}^3} \left(w_i + \sum_{j\neq i} w_j - \lambda_i + \frac{1}{2}\,\omega^2\, |x'|^2\right)_+^{p+1} \chi_i\, dx\\ &= \int_{\mathbb{R}^3} (w_i - \lambda_i)_+^{p+1}\, dx\\ &\quad + (p+1) \int_{\mathbb{R}^3} (w_i - \lambda_i)_+^p \left(\sum_{j\neq i} w_j + \frac{1}{2}\,\omega^2\, |x'|^2\right) dx \;+\; O(\omega^{4/3})\\ &= \int_{\mathbb{R}^3} (w_i - \lambda_i)_+^{p+1}\, dx + (p+1) \left(\sum_{j\neq i} \frac{m_i\, m_j}{4\pi\, |\xi_i - \xi_j|} + \frac{1}{2}\,\omega^2 m_i\, |\xi_i|^2\right) \;+\; O(\omega^{4/3}).\end{aligned}$$

Let us define

$$\mathsf{e}_* := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla w|^2\, dx - \frac{1}{p+1} \int_{\mathbb{R}^3} (w - 1)_+^{p+1}\, dx.$$

Combining the above estimates, we obtain that

$$\mathsf{J}[W_\xi] = \sum_{i=1}^N \lambda_i^{5-p}\, \mathsf{e}_* \;-\; \mathcal{V}_m^\omega(\xi_1, \dots \xi_N) \;+\; O(\omega^{4/3}), \tag{29}$$

where $\mathcal{V}_m^\omega(\xi) = \sum_{i \neq j} \frac{m_i\, m_j}{8\pi\, |\xi_i - \xi_j|} + \frac{1}{2}\,\omega^2 \sum_{i=1}^N m_i\, |\xi_i|^2$ has been introduced in Sect. 1. Here the $O(\omega^{4/3})$ term is uniform as $\omega \to 0$ on the set of ξ satisfying the constraints (17). This approximation is also uniform in the C^1 sense. Indeed, we directly check that

$$\nabla_\xi \mathsf{J}\,[W_\xi] = -\nabla_\xi \mathcal{V}_m^\omega(\xi) + O(\omega^{4/3}). \tag{30}$$

According to (5), we have $\mathcal{V}_m^\omega(\xi) = \omega^{2/3}\, \mathcal{V}_m(\zeta)$ for $\zeta = \omega^{2/3}\,\xi$. We get a solution of Problem (12)–(13) as soon as all constants c_{ij} are equal to zero in (26).

Lemma 6. *With the above notations, $c_{ij} = 0$ for all $i = 1,\, 2 \ldots N,\ j = 1,\, 2,\, 3$ if and only if ξ is a critical point of the functional $\xi \mapsto \Lambda(\xi) := \mathsf{J}\,[W_\xi + \phi_\xi]$.*

Proof. We have already noticed in Lemma 4 that the numbers c_{i3} are all equal to zero. On the other hand, we have that

$$\begin{aligned}\partial_{\xi_{ij}} \Lambda &= D\mathsf{J}\,[W_\xi + \phi_\xi] \cdot \partial_{\xi_{ij}} (W_\xi + \phi_\xi) = \sum_{k,\,\ell} c_{k\ell} \int_{\mathbb{R}^3} \partial_{\xi_{ij}} (W_\xi + \phi_\xi)\, Z_{k\ell}\, \chi_\ell\, dx \\ &= c_{ij} \left(\int_{\mathbb{R}^3} |Z_{ij}|^2\, \chi_i\, dx \right) + \left(\sum_{(k,\,\ell) \neq (i,j)} c_{k\ell} \right) O(\omega^{2/3}).\end{aligned} \tag{31}$$

From here the assertion of the lemma readily follows, provided that ω is sufficiently small. □

Remark 1. An important observation that follows from the rotation invariance of the equation is the following. Assume that the point ξ is such that $\xi_\ell = (\xi_{\ell 1}, 0) \neq (0, 0)$ for some $\ell \in \{1, \ldots N\}$. Then if

$$\partial_{\xi_{kj}} \Lambda(\xi) = 0 \text{ for all } k = 1, \ldots N\,,\ \ j = 1, 2, \quad (k, j) \neq (\ell, 2),$$

it follows that actually ξ is a critical point of Λ. Indeed, differentiating in α the relation $\Lambda(e^{i\,\alpha}\,\xi) = \Lambda(\xi)$ we get

$$0 = \sum_{k=1}^N \partial_{\xi_k} \Lambda(\xi) \cdot i\,\xi_k \;=\; \partial_{\xi_\ell} \Lambda(\xi) \cdot i\,\xi_\ell \;=\; -\,\xi_{\ell 1}\, \partial_{\xi_{\ell 2}} \Lambda(\xi),$$

and the result follows.

7. Proofs of Theorems 1–3

Let us consider the solution ϕ_ξ of (26)–(28), i.e. of the problem

$$\mathsf{L}[\phi_\xi] = -\mathsf{E} - \mathsf{N}[\phi_\xi] + \sum_{i=1}^N \sum_{j=1}^3 c_{ij}(\xi)\, Z_{ij}\, \chi_i$$
$$\lim_{|x| \to +\infty} \phi_\xi(x) = 0$$

given by Lemma 4. We will then get a solution of Problem (12)–(13), of the desired form $u = W_\xi + \phi_\xi$, inducing the ones for Theorems 1 and 3, if we can adjust ξ in such a way that

$$c_{ij}(\xi) = 0 \quad \text{for all } \ i = 1,\, 2 \ldots N,\ \ j = 1,\, 2,\, 3.$$

According to Lemma 6, this is equivalent to finding a critical point of the functional

$$\Lambda(\xi) \ := \ \mathsf{J}\,[W_\xi + \phi_\xi].$$

We expand this functional as follows:

$$\mathsf{J}\,[W_\xi] = \mathsf{J}\,[W_\xi + \phi_\xi] - D\mathsf{J}\,[W_\xi + \phi_\xi] \cdot \phi_\xi + \frac{1}{2}\int_0^1 D^2\mathsf{J}\,[W_\xi + (1-t)\,\phi_\xi] \cdot (\phi_\xi\ ,\ \phi_\xi)\ dt.$$

By definition of ϕ_ξ we have that $D\mathsf{J}\,[W_\xi + \phi_\xi] \cdot \phi_\xi = 0$. On the other hand, using Lemma 4, we check directly, out of the definition of ϕ_ξ, that

$$D^2\mathsf{J}\,[W_\xi + (1-t)\,\phi_\xi] \cdot (\phi_\xi\ ,\ \phi_\xi) = O(\omega^{4/3})$$

uniformly on points ξ_i satisfying constraints (17). Hence, from expansion (29) we obtain that

$$\Lambda(\xi) = \sum_{i=1}^N \lambda_i^{5-p}\,\mathsf{e}_* \ - \ \mathcal{V}_m^\omega(\xi) \ + \ O(\omega^{4/3}).$$

We claim that this expansion also holds in the C^1 sense. Let us first observe that

$$\int_{\mathbb{R}^3} \mathsf{E}\,\partial_\xi W_\xi \ dx = \nabla_\xi \mathsf{J}\,[W_\xi] \quad \text{and} \quad \partial_{\xi_{ij}} W_\xi = Z_{ij}.$$

Then, testing Eq. (26) against Z_{ij}, we see that

$$\int_{\mathbb{R}^3} \big(\mathsf{N}[\phi]\,Z_{ij} + \mathsf{L}[Z_{ij}]\big)\ \phi\ dx = -\int_{\mathbb{R}^3} \mathsf{E}\,Z_{ij}\ dx + \sum_{k\ell} c_{k\ell} \int_{\mathbb{R}^3} Z_{ij}\,Z_{k\ell}\,\chi_i\ dx.$$

Next we observe that

$$\|\,\mathsf{L}[Z_{ij}]\,\|_{**} = O(\omega^{2/3}) \quad \text{and} \quad \int_{\mathbb{R}^3} Z_{ij}\,Z_{k\ell}\,\chi_i\ dx = O(\omega^{2/3}) \quad \text{if } (i,\,j) \neq (k,\,\ell).$$

By Lemma 4, $\|\phi_\xi\|_* = O(\omega^{2/3})$, and so we get

$$\Big(1 + O(\omega^{2/3})\Big)\ c_{ij} \ = \ O(\omega^{4/3}) \ + \ \partial_{\xi_{ij}}\mathsf{J}\,[W_\xi].$$

Hence, according to relation (31), we obtain

$$\Big(\mathrm{I}_{3N} + O(\omega^{2/3})\Big)\ \nabla_\xi \Lambda(\xi) \ = \ \nabla_\xi \mathsf{J}\,[W_\xi] \ + \ O(\omega^{4/3}),$$

where $\nabla_\xi \mathsf{J}\,[W_\xi]$ has been computed in (30). Summarizing, we have found that

$$\nabla_\xi \Lambda(\xi) = -\nabla_\xi \mathcal{V}_m^\omega(\xi_1, \ldots \xi_N) \ + \ O(\omega^{2/3}).$$

Therefore, setting $\xi = \omega^{2/3}\,\zeta$ with $\zeta = (\zeta_1, \ldots \zeta_N)$ and defining $\Gamma(\zeta) := \Lambda(\xi)$ on $\mathfrak{B}_\mu := \big\{\zeta \in \mathbb{R}^{2N} \ : \ (18) \text{ holds}\big\}$, we have shown the following result.

Proposition 1. *With the above notations, we have that*

$$\Gamma(\zeta) = \sum_{i=1}^N \lambda_i^{5-p}\,\mathsf{e}_* \ - \ \omega^{2/3}\,\mathcal{V}_m(\zeta) \ + \ O(\omega^{4/3}),$$

$$\nabla\Gamma(\zeta) = -\,\nabla\mathcal{V}_m(\zeta) \ + \ O(\omega^{2/3}),$$

uniformly on ζ satisfying (18). *Here the terms $O(\cdot)$ are continuous functions of ζ defined on $\mathfrak{B}_\mu$.*

7.1. Proof of Theorem 1. If $\mu > 0$ is fixed large enough, we have that

$$\inf_{\mathfrak{B}_\mu} \mathcal{V}_m < \inf_{\partial \mathfrak{B}_\mu} \mathcal{V}_m .$$

Fixing such a μ, we get from Proposition 1 that, for all sufficiently small ω,

$$\sup_{\mathfrak{B}_\mu} \Gamma > \sup_{\partial \mathfrak{B}_\mu} \Gamma ,$$

so that the functional Λ has a maximum value somewhere in $\omega^{2/3}\,\mathfrak{B}_\mu$, which is close to a maximum value of $\mathcal{V}_m^\omega$. This value is achieved at critical point of Λ, and hence a solution with the desired features exists. The construction is concluded. $\square$

7.2. Proof of Theorem 2. When $(\zeta_1, \dots \zeta_N)$ is a regular polygon with ζ_j given by (6) and all masses are equal, the system is invariant under the rotation defined by

$$x = \underbrace{(x^1, x^2, x^3)}_{\in \mathbb{R}^3} \approx \underbrace{((x^1 + i\,x^2),\ x^3)}_{\in \mathbb{C}\times\mathbb{R}} \mapsto \left(e^{2\,i\,\pi/N}(x^1 + i\,x^2),\ x^3\right) =: \mathcal{R}_N\,x .$$

We can therefore pass to the quotient with respect to this group of invariance and look for solutions u which are invariant under the action of $\mathcal{R}_N$ and moreover symmetric with respect to the reflections $(x^1, x^2, x^3) \mapsto (x^1, -x^2, x^3)$ and $(x^1, x^2, x^3) \mapsto (x^1, x^2, -x^3)$. Here we assume that $(\zeta_1, \dots \zeta_N)$ is contained in the plane $\{x^3 = 0\}$ and $\zeta_1 = (r, 0, 0)$. Altogether this amounts to look for critical points of the functional

$$\mathsf{J}_1[u] = \frac{1}{2}\int_{\Omega_1} |\nabla u|^2\,dx - \frac{1}{p+1}\int_{\Omega_1}\left(u - \lambda_1 + \frac{1}{2}\,\omega^2\,|x'|^2\right)_+^{p+1} \chi_1\,dx ,$$

where $\Omega_1 = \{x = (x', x^3) \in \mathbb{R}^3 \approx \mathbb{C}\times\mathbb{R} \,:\, x' = r\,e^{i\,\theta}$ s.t. $-\frac{\pi}{N} < \theta < \frac{\pi}{N}\}$ and $u \in H^1(\Omega)$ is invariant under the above two reflections and such that $\nabla u \cdot n = 0$ on $\partial\Omega\setminus\{0\}\times\mathbb{R}$ and χ_1 is the characteristic function of the support of $\rho_1 = \left(u - \lambda_1 + \frac{1}{2}\,\omega^2\,|x'|^2\right)_+^{p+1} \chi_1$ in Ω_1. Here $n = n(x)$ denotes the unit outgoing normal vector at $x \in \partial\Omega_1$. With J defined by (14), it is straightforward to see that $\mathsf{J}\,[u] = N\,\mathsf{J}_1[u]$ if u is extended to $\mathbb{R}^3$ by assuming that $u(\mathcal{R}_N\,x) = u(x)$. With these notations, we find that

$$\mathcal{V}_m(\zeta_1, \dots \zeta_N) = N\,m_*\left(\frac{a_N}{4\pi}\,\frac{m_*}{r} + \frac{1}{2}\,r^2\right).$$

The proof goes the same as for Theorem 1. Because of the symmetry assumptions, $c_{1j} = 0$ if $j = 2$ or 3. Details are left to the reader. $\square$

7.3. Proof of Theorem 3. We look for a critical point of the functional Γ of Proposition 1 in a neighborhood of a critical point ζ of $\mathcal{V}_m$, which is nondegenerate up to rotations. With no loss of generality, we may assume that $\zeta_1 \neq 0$, $\zeta_{12} = 0$ and denote by $\tilde{\mathcal{V}}_m$ the restriction of $\mathcal{V}_m$ to $(\mathbb{R}\times\{0\})\times(\mathbb{R}^2)^{N-1} \ni \zeta$. Similarly, we denote by $\tilde{\Gamma}$ the restriction of Γ to $(\mathbb{R}\times\{0\})\times(\mathbb{R}^2)^{N-1}$.

By assumption, ζ is a non-degenerate critical point of $\tilde{\mathcal{V}}_m$, i.e. an isolated zero of $\nabla\tilde{\mathcal{V}}_m$. Besides, its local degree is non-zero. It follows that on an arbitrarily small neighborhood of that point, the degree for $\nabla\tilde{\Gamma}$ is non-zero for all sufficiently small ω. Hence there exists a zero $\zeta^\omega \in (\mathbb{R}\times\{0\})\times(\mathbb{R}^2)^{N-1}$ of $\nabla\tilde{\Gamma}$ as close to é1ζ as we wish. From the rotation invariance, it follows that ζ^ω is also a critical point of Γ. The proof of Theorem 3 is concluded. □

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Appendix A. Facts on Relative Equilibria

In this Appendix we have collected some results on the N-body problem introduced in Sect. 1 which are of interest for the proofs of Theorems 1-3, with a list of relevant references.

Non-degeneracy of relative equilibria in a standard form. Relative equilibria are by definition critical points of the function $\mathcal{V}_m : \mathbb{R}^{2N} \to \mathbb{R}$ defined by

$$\mathcal{V}_m(\zeta_1, \zeta_2, \dots \zeta_N) = \frac{1}{8\pi}\sum_{i\neq j=1}^{N}\frac{m_i\, m_j}{|\zeta_j - \zeta_i|} + \frac{1}{2}\sum_{i=1}^{N} m_i\,|\zeta_i|^2.$$

Here we assume that $N \geq 2$, and $m_i > 0$, $i = 1, \dots N$ are given parameters.

Following Smale in [42], we can rewrite this problem as follows. Let us consider the $(2N-3)$-dimensional manifold

$$S_m := \left\{ q = (q_1, \dots q_N) \in \mathbb{R}^{2N} \ : \ \sum_{i=1}^{N} m_i\,(q_i, \frac{1}{2}\,|q_i|^2) = (0, 1)\,,\ q_i \neq q_j \text{ if } i \neq j \right\}.$$

The problem of finding critical points of the functional

$$U_m(q_1, \dots q_N) = \frac{1}{8\pi}\sum_{i\neq j=1}^{N}\frac{m_i\, m_j}{|q_j - q_i|}$$

on S_m is equivalent to that of relative equilibria; see for instance [12]. Let us give some details. Let $\bar{q}$ be a critical point of U_m on S_m. Then by definition, there are Lagrange multipliers $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}^2$ for which

$$-\frac{1}{8\pi}\sum_{i\neq j}^{N}\frac{m_i\, m_j}{|\bar{q}_i - \bar{q}_j|^3}\,(\bar{q}_i - \bar{q}_j) = \lambda\, m_j\,\bar{q}_j + m_j\,\mu \quad \forall j = 1, \dots N.$$

First, adding in j the above relations and using that $M = \sum_{j=1}^{N} m_j > 0$ we obtain that $\mu = 0$. Second, taking the scalar product of $\mathbb{R}^2$ against $\bar{q}_j$ and then adding in j, we easily obtain that $U_m(\bar{q}) = \lambda$. From here it follows that the point $\bar{\zeta} = \lambda^{1/3}\,\bar{q}$ is a critical point of the functional $\mathcal{V}_m$, hence a relative equilibrium.

With the reparametrization of $\mathbb{R}^{2N}$ given by

$$\zeta(\alpha, p, q) = (\zeta_1, \dots \zeta_N) = (\alpha\, q_1 + p, \dots \alpha\, q_N + p), \quad (\alpha, p, q) \in \mathbb{R}\times\mathbb{R}^2\times S_m,$$

the Hessian matrix of $\mathcal{V}_m$ at the critical point $\bar{\zeta} = \zeta(\lambda^{1/3}, 0, \bar{q})$ found above is represented as the block matrix

$$D^2\mathcal{V}_m(\bar{\zeta}) = \begin{pmatrix} 2 & & \\ & M\,\mathrm{I}_2 & \\ & & \lambda^{-1/3}\, D^2_{S_m} U_m(\bar{q}) \end{pmatrix},$$

where I_2 is the 2×2 identity matrix and $D^2_{S_m}$ represents the second covariant derivative on S_m. Reciprocally, we check that a critical point $\bar{\zeta} = (\zeta_j)_{j=1}^N$ of $\mathcal{V}_m$ necessarily satisfies $\sum_{j=1}^N m_j\,\zeta_j = 0$. Defining $\bar{q} = \left(\frac{1}{2}\sum_{j=1}^N m_j\,|\zeta_j|^2\right)^{-1/2}\bar{\zeta}$, we readily check that $\bar{q}$ is a critical point of U_m in S_m.

Any rotation $e^{i\,\alpha}\,\bar{q}$ of a critical point $\bar{q}$ of U_m on S_m is also a critical point. We say that two such critical points are *equivalent* in S_m. Let us denote by $\mathcal{S}_m$ the quotient manifold of S_m by this equivalence relation. On $\mathcal{S}_m$, critical points of the potential U_m yield critical points of U_m on S_m and hence equivalence classes of critical points $e^{i\,\alpha}\,\zeta$ for $\mathcal{V}_m$ using the reparametrization.

A critical point $\tilde{q}$ of U_m on $\mathcal{S}_m$ is said to be *non-degenerate* if the second variation of U_m at $\tilde{q}$ is non-singular. Let us assume that $\tilde{q}_\ell \neq 0$, with either $\ell = 1$, or $\ell = 2$ if $\tilde{q}_1 = 0$. Then there is a unique representative $\bar{q}$ of this class of equivalence for which $\bar{q}_{\ell 2} = 0$. It is a routine verification to check that $\bar{q}$ is then a critical point of U_m on the $(2N-4)$-dimensional manifold

$$\mathsf{S}_m := \{q \in S_m \;:\; q_\ell \neq 0 \text{ as above}\,,\; q_{\ell 2} = 0\}\,.$$

Moreover, the second derivative of U_m on $\mathcal{S}_m$ at $\tilde{q}$ is non-degenerate if and only if $D^2_{\mathsf{S}_m} U_m(\bar{q})$ is non-singular. Because of the expression of $D^2\mathcal{V}_m(\bar{\zeta})$, we see that $\bar{\zeta}$ is non-degenerate as a critical point of $\mathcal{V}_m$ on the space of $\zeta \in \mathbb{R}^{2N}$ with $q_{\ell 2} = 0$, which is the notion of *non-degeneracy up to rotations* of a relative equilibrium that we have used in this paper. Finally we define the *index* of a non-degenerate relative equilibrium $\bar{\zeta}$ as the number of negative eigenvalues of $D^2_{\mathsf{S}_m} U_m(\bar{q})$.

Some results on classification of relative equilibria. For simplicity, we will assume that masses are all different: for any $i, j = 1, \ldots N$, if $m_i = m_j$, then $i = j$. This is the generic case.

The cases $N = 2, 3$ are well known; see for instance [28]. For $N = 2$, the only class of critical points is such that

$$|\zeta_1 - \zeta_2| = \left(\frac{M}{4\pi}\right)^{1/3} \quad \text{and} \quad m_1\,\zeta_1 + m_2\,\zeta_2 = 0 \quad \text{with } M = m_1 + m_2.$$

For $N = 3$, there are two types of solutions, the Lagrange and the Euler solutions. The *Lagrange solutions* are such that their center of mass is fixed at the origin, the masses are located at the vertices of an equilateral triangle, and the distance between each point is $(M/(4\pi))^{1/3}$ with $M = m_1 + m_2 + m_3$. They give rise to two classes of solutions corresponding to the two orientations of the triangle when labeled by the masses. The *Euler solutions* are made of aligned points and provide three classes of critical points, one for each ordering of the masses on the line.

In the case $N \geq 4$, the classes of solutions for which all points are collinear still exist (see [29]) and are known as the *Moulton solutions*. But the configuration of relative equilibria where all particles are located at the vertices of a regular N-polygon exists if and only if all masses are equal; see [12,26,35,43,46]. Various classification results which have been obtained by Palmore are summarized below.

Theorem 4 ([30–34]). *We have the following multiplicity results*:

(a) *For $N \geq 3$, the index of a relative equilibrium is always greater than or equal to $N-2$. This bound is achieved by Moulton's solutions.*

(b) *For $N \geq 3$, there are at least $\mu_i(N) := \binom{N}{i}(N-1-i)\,(N-2)\,!$ distinct relative equilibria in $\mathcal{S}_m$ of index $2N-4-i$ if U_m is a Morse function. As a consequence, there are at least*

$$\sum_{i=0}^{N-2} \mu_i(N) = [2^{N-1}(N-2)+1]\,(N-2)\,!$$

distinct relative equilibria in $\mathcal{S}_m$ if U_m is a Morse function.

(c) *For every $N \geq 3$ and for almost all masses $m \in \mathbb{R}_+^N$, U_m is a Morse function.*

(d) *There are only finitely many classes of relative equilibria for every $N \geq 3$ and for almost all masses $m = (m_i)_{i=1}^N \in \mathbb{R}_+^N$.*

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