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Solutions with multiple catenoidal ends to the Allen–Cahn equation in $\mathbb{R}^3$

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ABSTRACT

We consider the Allen–Cahn equation $\Delta u + u(1 - u^2) = 0$ in $\mathbb{R}^3$. We construct two classes of axially symmetric solutions $u = u(|x'|, x_3)$ such that the (multiple) components of the zero set look for large $|x'|$ like catenoids, namely $|x_3| \sim A \log |x'|$. In Theorem 1, we find a solution which is even in x_3, with Morse index one and a zero set with exactly two components, which are graphs. In Theorem 2, we construct a solution with a zero set with two or more nested catenoid-like components, whose Morse index become as large as we wish. While it is a common idea that nodal sets of the Allen–Cahn equation behave like minimal surfaces, these examples show that the nonlocal interaction between disjoint portions of the nodal set, governed in suitably asymptotic regimes by explicit systems of 2d PDE, induces richness and complex structure of the set of entire solutions, beyond the one in minimal surface theory.

RÉSUMÉ

On considère l'équation de Allen–Cahn $\Delta u + u(1 - u^2) = 0$ dans $\mathbb{R}^3$. On construit deux classes de solutions à symétrie axiale $u = u(|x'|, x_3)$ telles que les composantes (multiples) de l'ensemble des zéros ressemblent à des caténoïdes pour les grandes valeurs de $|x'|$, c'est-à-dire pour $|x'| \sim A \log |x'|$. Le Théorème 1 donne une solution paire en x_3, d'indice de Morse égal à 1 et un ensemble de zéros ayant exactement deux composantes qui sont des graphes. Dans le Théorème 2 on construit une solution avec un ensemble de zéros à deux ou plusieurs composantes imbriquées, semblables à des caténoïdes d'indice de Morse arbitrairement grand. Si on pense généralement que les ensembles nodaux de l'équation de Allen–Cahn se comportent comme des surfaces minimales, ces exemples montrent que l'interaction non locale entre les parties disjointes de l'ensemble nodal sont régies par des systèmes explicites de deux équations aux dérivées partielles. Ceci montre la richesse et la structure complexe de l'ensemble des solutions entières, bien au-delà de la théorie des surfaces minimales.

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1. Introduction

This paper deals with the discovery of new solutions to the classical Allen–Cahn equation

$$\Delta u + u\left(1-u^2\right) = 0, \quad \text{in } \mathbb{R}^N \tag{1.1}$$

when space dimension is $N = 3$. Eq. (1.1), introduced in [1] to model the allocation of binary mixtures, is a prototype for the continuous modeling of *phase transition phenomena*.

In the so-called gradient theory of phase transitions, the function u represents a continuous realization of the phase, with values making a transition between the pure states -1 and $+1$ along some thick interface. The most interesting solutions in that context are therefore those in which this transition wall takes an identifiable shape.

In the one-dimensional case, there is a standard solution connecting the states -1 and $+1$ namely

$$w(t) = \tanh\left(\frac{t}{\sqrt{2}}\right), \quad t \in \mathbb{R},$$

which is the unique solution, up to translations, of the problem

$$w'' + w\left(1-w^2\right) = 0, \quad \text{in } \mathbb{R}, \qquad w(\pm\infty) = \pm 1. \tag{1.2}$$

In 1978, E. De Giorgi raised in [10] a celebrated conjecture: solutions u to problem (1.1) which are monotone in one direction have the following rigidity property: its level sets $[u = \lambda]$ must be parallel hyperplanes, at least if $N = 8$. That is equivalent to saying that for some point p and a unit vector ν, u only depends on the normal coordinate to the hyperplane that passes through p with normal vector ν, namely

$$u(x) = w(t), \quad t = (x-p)\cdot\nu. \tag{1.3}$$

This rationale behind the conjecture is that level sets of an entire solution of (1.1) monotone in one direction should have the same rigidity of minimal surfaces that are graphs of entire functions of $N-1$ variables. The latter question is known as the *Bernstein Problem*, and it is known to be true precisely up to dimension 8, as established in the works [2,9,16,27] after the original result in 1917 by Bernstein in [4], proving it for $N = 3$. Bombieri, De Giorgi and Giusti [5] proved that Bernstein's statement is false in dimensions 9 or higher, by constructing a minimal graph in $\mathbb{R}^9$ which is not a hyperplane.

In [17,3], De Giorgi's conjecture was established in dimensions $N = 2, 3$. In [25], it was proved to hold true in dimensions $4 \leqslant N \leqslant 8$ under the additional assumption

$$\lim_{x_N \to \pm\infty} u\left(x', x_N\right) = \pm 1.$$

On the other hand, in [14] a counterexample was built in dimension $N = 9$ in the following way: a nontrivial minimal graph Γ as built in [5] is fixed, then a large dilation of it is taken, $\Gamma_\alpha = \alpha^{-1}\Gamma$, where α is a very small positive number. Since Γ_α is nearly flat around each of this points, then the quantity $w(t)$, where $t = t(x)$ is a choice of normal coordinate (signed distance) from x to Γ_α, is a good approximation to a solution of Eq. (1.1). This approximation turns out to have an extra order of accuracy in α thanks to the fact that Γ is a minimal surface. In [14] it is proven that there exists an actual solution u_α to (1.1) which is monotone in one direction and such that

$$u_\alpha(x) = w(t) + o(\alpha)$$

The zero level set $[u_\alpha = 0]$ is then a manifold close to Γ_α, therefore non-flat.

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The method in [14] actually applies to more general minimal surfaces. Recently in [12] this approach was used in dimension $N = 3$ to construct, for a general embedded minimal surface M with finite total curvature, that satisfies certain non-degeneracy assumptions, and all small $\alpha > 0$, a solution u_α with $u_\alpha(x) \approx w(z)$ where z denotes a choice of normal coordinate to $M_\alpha = \alpha^{-1}M$. The setting of a compact manifold was considered in [23,20]. See also [22,21,24] for the earlier connection discovered between this problem and minimal surfaces.

A notable example of such a M is given by **a catenoid**. In that case, there exists an axially symmetric solution with zero set constituted by a smooth surface close to M_α. Another example are the Costa surface, and more generally the Costa–Hoffman–Meeks surfaces [19,7,18]. These solutions have *finite Morse index.*

For a bounded, entire solution u to (1.1), its Morse index $m(u)$ is defined as the maximal dimension of a vector space E of compactly supported smooth functions such that

$$\mathcal{B}(\psi,\psi) := \int_{\mathbb{R}^N} |\nabla\psi|^2 + \left(3u^2 - 1\right)\psi^2 < 0, \quad \forall\psi \in E - \{0\}.$$

For the solutions in [12], we have that $m(u_\alpha)$ coincides with the number of negative eigenvalues in $L^\infty(\mathbb{R}^3)$ of the linearized operator $\Delta + (1 - 3u_\alpha^2)$. Moreover, it is found that $m(u_\alpha) = i(M)$, the index of M, which under the assumptions in [12] corresponds to the number of negative eigenvalues in $L^\infty(M)$ of the Jacobi operator $\Delta_M + |A_M|^2$. That number is indeed finite, because of the finite total curvature assumption. In particular $i(M) = 1$ for a catenoid, and $i(M) = 2\ell + 3$ for the Costa–Hoffmann–Meeks surface genus $\ell \geqslant 1$.

The results in [12] provide a connection between a large class of minimal surfaces in $\mathbb{R}^3$ and families of solutions to the Allen–Cahn equation, where even Morse index is transmitted. The purpose of this paper is to show that **more richness** is present in solutions to Allen–Cahn with transition layers. A big difference between Allen–Cahn and the minimal surface problem, is that two disjoint surfaces **do not interact** in the latter problem, while they do as components of the zero set of solutions to the Allen–Cahh equations. These nodal sets are actually solving a form of nonlocal minimal surface problem, which is interesting in its own sake, not just regarding Allen–Cahn as a sort of regularization of the minimal surface problem. We will show in this paper two results showing that in the simple setting of axially symmetric solutions in $\mathbb{R}^3$, very interesting phenomena in going on.

As remarked in [8], the Morse index is a natural quantity to consider in the classification of entire solutions to (1.1). It is easy to see that a solution u monotone in one direction is stable, in the sense that $m(u) = 0$. For instance, it is natural step beyond De Giorgi's conjecture, to understand "mountain pass solutions" namely those with $m(u) = 1$. The only example available of such a solution seems to be the *catenoidal* axially symmetric solution in [12]. More precisely, for $x = (x', x_3)$ we write

$$r = |x'| = \sqrt{x_1^2 + x_2^2}.$$

We have that

$$u_\alpha(x) = u_\alpha(r, x_3),$$

and $u_\alpha(x) = w(z) + O(\alpha)$ where z is a choice of signed distance to $M_\alpha = \alpha^{-1}M$. In precise terms, M is parametrized as

$$|x_3| = \log\left(r + \sqrt{r^2 - 1}\right), \quad r > 1, \tag{1.4}$$

so that M_α becomes

$$|x_3| = \alpha^{-1}\log\left(\alpha r + \sqrt{\alpha^2 r^2 - 1}\right). \tag{1.5}$$

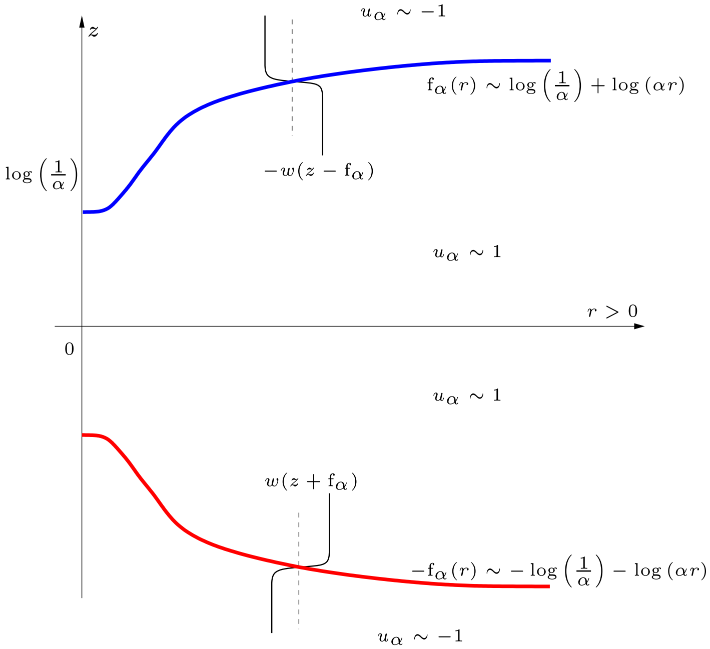

Fig. 1. Solutions for Theorem 1. Morse index 1 and two logarithmical sheets as nodal set.

For minimal surfaces, a famous result by Schoen [26] asserts that a minimal surface with embedded ends and Morse index 1 must be a catenoid. Our first result shows that the structure of Morse index one solutions of (1.1) is more complicated than dilations of a catenoid: there exists an axially symmetric solution of Morse index one whose zero set is *disconnected*.

Theorem 1. *For all sufficiently small $\alpha > 0$ there exists an smooth axially symmetric bounded solution $u_\alpha(r, x_3)$ to Eq. (1.1) for $N = 3$, with Morse index $m(u_\alpha) = 1$ and*

$$u_\alpha(r, x_3) = w\big(x_3 + \mathrm{q}_\alpha(r)\big) - w\big(x_3 - \mathrm{q}_\alpha(r)\big) - 1 + O(\alpha), \quad \text{uniformly as } \alpha \to 0, \tag{1.6}$$

where

$$\mathrm{q}_\alpha(r) = \frac{\sqrt{2}}{2}\big(1 + o(\alpha)\big)\log\big(1 + \alpha^2 r^2\big) + b_0 + \frac{\sqrt{2}}{2}\log\frac{1}{\alpha}. \tag{1.7}$$

uniformly, as $\alpha \to 0$. Here b_0 is an explicit constant.

The solution of the above theorem is in addition even in the x_3-coordinate. The zero level set of u_α of this result is the union of the graph of a positive radially symmetric function which asymptotically behaves logarithmically, and its reflection through the plane $x_3 = 0$. We can actually think of this solution as having a parallel with minimal surfaces: If we take two planes $x_3 = \pm A$, their union is a (disconnected) minimal surface. For no solution of the Allen–Cahn equation the zero set can be close to this two-plane object. Instead, the Allen–Cahn equation produces (for A large) a nonlocal interaction between the corresponding components of the nodal set, which can be quantified, making them diverge logarithmically (Fig. 1).

As we shall see, the law governing the interaction of the two components, assumed to be graphs, $x_3 = \pm\mathrm{q}_\alpha(r)$, is a perturbation of the Liouville equation

$$\Delta \mathrm{q}_\alpha - a_0 e^{-2\sqrt{2}\mathrm{q}_\alpha} = 0, \quad \text{in } \mathbb{R}^2. \tag{1.8}$$

Then all radial solutions of (1.8) are given by the one-parameter family of functions

$$\mathrm{q}_\alpha(r) = q(\alpha r) + \frac{\sqrt{2}}{2}\log\left(\frac{1}{\alpha}\right).$$

where q is given by

$$q(r) = \frac{1}{2\sqrt{2}}\log\left(\frac{\sqrt{2}a_0}{4}\left(1+r^2\right)^2\right). \tag{1.9}$$

This is how expression (1.7) comes into play.

Until now, two families of Morse index 1 axially symmetric solutions have become known: That with a connected, catenoidal zero set constructed in [12], and the two-component constructed in Theorem 1. We believe these solutions correspond to limiting situations of a single one-parameter family of solutions, in a similar sense to how two-sheet and one sheet revolution hyperboloids are connected by a parameter, where the family becomes singular in the form of two opposite cones with same vertex for some special value.

The examples found suggest that the richness of the topology of the zero level set may be in accordance to the size of the Morse index. Our second result exhibits a surprising phenomenon: *this is not the case.* In fact we can find axially symmetric solutions whose zero set is the union of any given catenoid-like nested surfaces, whose Morse index *becomes arbitrarily large.*

Let M be the catenoid described in (1.4), and its dilation $M_\alpha = \alpha^{-1}M$ parametrized by (1.5). We choose a normal vector field $\nu(y)$ for $y \in M$, so that $\nu_\alpha(y) = \nu(\alpha y)$ is the corresponding normal for $y \in M_\alpha$.

Theorem 2. *Let $N = 3$ and M be the catenoid in $\mathbb{R}^3$. Then for any $m \in \mathbb{N}$, $m \geqslant 2$ and for all sufficiently small $\alpha > 0$ there exists a bounded solution u_α to problem (1.1) such that as $\alpha \to 0$,*

$$u_\alpha(x) = \sum_{j=1}^{m}(-1)^{j-1}w\big(z - h_j(\alpha y)\big) + \frac{(-1)^{m-1}-1}{2} + o(1), \qquad x = y + z\nu(\alpha y), \quad y \in M_\alpha.$$

Here for $y = (y', y_3) \in M$ and $r = |y'|$,

$$h_l(y) = \left(l - \frac{m+1}{2}\right)\left[2\log\frac{1}{\alpha} - \log\left(\log\frac{1}{\alpha}\right) + 4\log(1+r) + O(1)\right] + o(1)\log(1+r), \quad l = 1,\dots,m. \tag{1.10}$$

The Morse index of u_α, $m(u_\alpha)$ goes to $+\infty$ as $\alpha \to 0$. In fact, it satisfies that

$$m(u_\alpha) \geqslant c_0 \log\left(\frac{1}{\alpha}\right).$$

The location of the interfaces $z = h_\ell(\alpha y)$ (see Fig. 2) is governed by a *Jacobi–Toda system*, described in (3.1). Entire solutions with multiple transition layers to (1.1) in $\mathbb{R}^2$ were found in [11]. In this case the nodal set of the solutions consists on multiple asymptotically straight lines, not intersecting themselves, whose locations are governed by a Toda system of ODEs. A Jacobi–Toda system was introduced in [13] to find multiple interfaces on a compact manifold, and in [15] in a related traveling wave problem.

This paper is structured as follows. Sections 2 through 8 are concerned with the construction of the solutions predicted in Theorems 2 and 1, while Section 9 sketches the estimates and computations regarding information about the Morse index of these families of solutions.

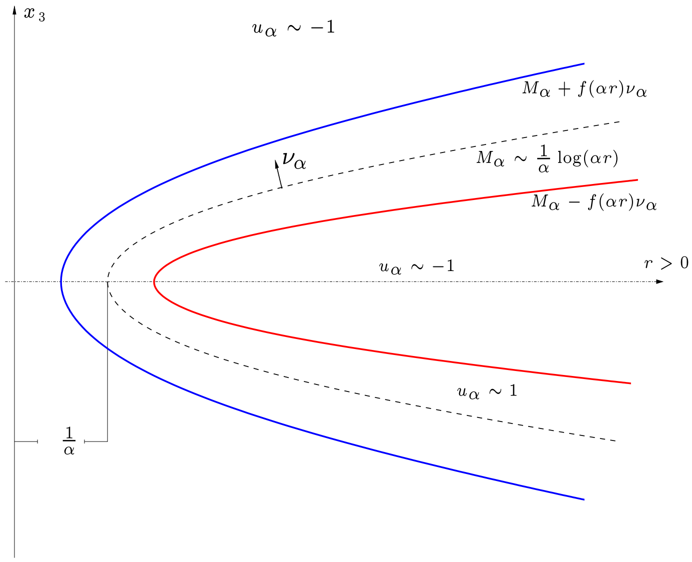

Fig. 2. Solutions for Theorem 2. The surfaces $M_\alpha \pm h(\alpha r)\nu_\alpha$ correspond to the nodal set of these solutions in the case $m = 2$.

2. Geometric setting near the dilated catenoid

We begin with the description of the geometrical background needed for the proof of Theorem 2, since the developments from this section will useful throughout the paper. In this section we compute the Euclidean Laplacian in $\mathbb{R}^3$, in a neighborhood of the dilated catenoid M_α.

Let us consider the curve

$$\gamma(s) = \big(\cosh(s), s\big), \quad s \in \mathbb{R}.$$

The set $\gamma(\mathbb{R})$ corresponds to the catenary curve in $\mathbb{R}^2$, for which we can compute the corresponding signed arch-length as

$$y(s) = \int_0^s \big\|\gamma'(\zeta)\big\| \, d\zeta = \sinh(s), \quad s \in \mathbb{R}.$$

Setting

$$s(y) = \log\big(y + \sqrt{1+y^2}\,\big), \quad y \in \mathbb{R}$$

we can parameterize $\gamma(\mathbb{R})$ using the mapping

$$\gamma\big(s(y)\big) = \big(\sqrt{1+y^2}, \log\big(y + \sqrt{1+y^2}\,\big)\big), \quad y \in \mathbb{R}.$$

Let us now consider the catenoid M in $\mathbb{R}^3$, with $\gamma(\mathbb{R})$ as profile curve. The mapping $Y : \mathbb{R} \times (0, 2\pi) \to \mathbb{R}^3$, defined by

$$Y(y, \theta) := \big(\sqrt{1+y^2}\cos\theta, \sqrt{1+y^2}\sin\theta, \log\big(y + \sqrt{1+y^2}\,\big)\big)$$

gives local coordinates on M in terms of the signed arch-length variable of $\gamma(\mathbb{R})$ and the angle of rotations around the x_3-axis which, in our setting, corresponds to the axis of symmetry of M. Observe also that for $y = (y_1, y_2, y_3) \in M$

$$r(y) := \left|(y_1, y_2)\right| = \sqrt{1+y^2}, \quad y = Y(y,\theta) \in M.$$

We consider local Fermi coordinates

$$X(y,\theta,z) = Y(y,\theta) + z\nu(y,\theta), \quad y \in \mathbb{R},\ \theta \in (0,2\pi),\ z \in \mathbb{R}.$$

This map defines a smooth local change of variables onto the open neighborhood of M, given by

$$\mathcal{N} := \left\{ Y(y,\theta) + z\nu(y,\theta) \colon\ |z| < \eta + \frac{1}{2}\log\left(1+y^2\right) \right\}$$

for some small, but fixed $\eta > 0$. Observe that $|z| = \mathit{dist}(x, M)$, for every $x \in \mathcal{N}$ with $x = X(y,\theta,z)$.

Let us compute the Euclidean Laplacian in $\mathcal{N}$, in terms of these local coordinates, from the formula

$$\Delta_X = \frac{1}{\sqrt{\det(g)}} \partial_i\left(\sqrt{\det(g)} g^{ij} \partial_j\right), \quad i, j = y, \theta, z,$$

where $g_{ij} = \partial_i X \cdot \partial_j X$ corresponds to the ij-th entry of the metric g on $\mathcal{N}$ and $g^{ij} = (g^{-1})_{ij}$.

Computing the metric g, we find that

$$g = \begin{bmatrix} g_{yy} & 0 & 0 \\ 0 & g_{\theta\theta} & 0 \\ 0 & 0 & g_{zz} \end{bmatrix} = \begin{bmatrix} (1+\frac{z}{1+y^2})^2 & 0 & 0 \\ 0 & (1+y^2)(1-\frac{z}{1+y^2})^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so that

$$\sqrt{\det(g)} = \sqrt{1+y^2}\left(1 - \frac{z^2}{(1+y^2)^2}\right).$$

Since

$$\Delta_X = \frac{1}{\sqrt{\det(g)}}\left[\partial_y\left(\sqrt{\det(g)} g_{yy}^{-1} \partial_y\right) + \partial_\theta\left(\sqrt{\det(g)} g_{\theta\theta}^{-1} \partial_\theta\right) + \partial_z\left(\sqrt{\det(g)} \partial_z\right)\right],$$

we find by a direct computation that

$$\Delta_X = \partial_{zz} + \partial_{yy} + \frac{y}{1+y^2}\partial_y + \frac{1}{1+y^2}\partial_{\theta\theta} - \frac{2z}{(1+y^2)^2}\partial_z + D, \tag{2.1}$$

where

$$D = z a_1(y,z)\partial_{yy} + z a_2(y,z)\partial_{\theta\theta} + z b_1(y,z)\partial_y + z^3 b_2(y,z)\partial_z,$$

and the smooth functions $a_i(y,z)$, $b_i(y,z)$ satisfy

$$\begin{gathered} |a_i| + |y D_y a_i| = O\left(|y|^{-2}\right), \qquad |b_1| + |y D_y b_1| = O\left(|y|^{-3}\right), \\ |b_2| + |y D_y b_2| = O\left(|y|^{-8}\right) \end{gathered} \tag{2.2}$$

as $|y| \to \infty$, uniformly on z in the neighborhood $\mathcal{N}$ of M. Actually, it is not hard to check that, inside $\mathcal{N}$ and for $i = 1, 2$

$$a_i(y,z) = a_{i,0}(y) + z a_{i,1}(y,z), \qquad b_1(y,z) = b_{1,0}(y) + z b_{1,1}(y,z),$$
$$b_2(y,z) = b_{2,0}(y) + z^2 b_{2,1}(y,z), \tag{2.3}$$

where

$$a_{i,0}(y) = \frac{(-2)^{i-1}}{(1+y^2)^i}, \qquad b_{1,0}(y) = -\frac{2y}{(1+y^2)^2}, \qquad b_{2,0}(y) = -\frac{2}{(1+y^2)^4},$$

and

$$|a_{i,1}| + |yD_y a_{i,1}| = O\big(|y|^{-(4+2i)}\big), \qquad |b_{1,1}| + |yD_y b_{1,1}| = O\big(|y|^{-5}\big),$$
$$|b_{2,1}| + |yD_y b_{2,1}| = O\big(|y|^{-12}\big).$$

At this point, we remark that since the catenoid is an axially symmetric minimal surface, the functions a_i, b_i, $i = 1, 2$, also share this symmetry and actually they enjoy the additional properties

$$a_i(y,z) = a_i(-y,z), \qquad b_1(y,z) = -b_1(-y,z), \qquad b_2(y,z) = b_2(y,z), \qquad x = X(y,\theta,z) \in \mathcal{N}.$$

Let us now consider a large dilation of the catenoid M, given by

$$M_\alpha = \alpha^{-1} M$$

for a small positive number α.

We parameterize M_α by $Y_\alpha : (y,\theta) \mapsto \alpha^{-1} Y(\alpha y, \theta)$ and define associated local Fermi coordinates

$$X_\alpha(y,\theta,z) = \alpha^{-1} Y(\alpha y,\theta) + z\nu(\alpha y,\theta)$$

on the neighborhood $\mathcal{N}_\alpha = \alpha^{-1}\mathcal{N}$ of M_α. Observe that

$$\mathcal{N}_\alpha = \left\{ Y_\alpha(y,\theta) + z\nu(\alpha y,\theta) \colon\ |z| < \frac{\eta}{\alpha} + \frac{1}{2\alpha}\log\big(1 + (\alpha y)^2\big) \right\}.$$

Scaling formula (2.1) we find that

$$\Delta_{X_\alpha} = \partial_{zz} + \partial_{yy} + \frac{\alpha^2 y}{1+(\alpha y)^2}\partial_y + \frac{\alpha^2}{1+(\alpha y)^2}\partial_{\theta\theta} - \frac{2\alpha^2 z}{(1+(\alpha y)^2)^2}\partial_z + D_\alpha, \tag{2.4}$$

where

$$D_\alpha = \alpha z a_1(\alpha y, \alpha z)\partial_{yy} + \alpha^3 z a_2(\alpha y,\alpha z)\partial_{\theta\theta} + \alpha^2 z b_1(\alpha y,\alpha z)\partial_y + \alpha^4 z^3 b_2(\alpha y,\alpha z)\partial_z$$

and the smooth functions a_i, b_i, $i = 1, 2$, satisfy (2.2) and (2.3).

Let us consider next an arbitrary smooth function $h : \mathbb{R} \to \mathbb{R}$ and local coordinates near M_α, defined by

$$X_{\alpha,h}(y,\theta,t) = \alpha^{-1} Y(\alpha y,\theta) + \big(t + h(\alpha y)\big)\nu(\alpha y,\theta)$$

onto the region $\mathcal{N}_\alpha$, which can be described as

$$\mathcal{N}_\alpha = \left\{ X_{\alpha,h}(y,\theta,t) \colon\ \big|t + h(\alpha y)\big| \leqslant \frac{\eta}{\alpha} + \frac{1}{\alpha}\log\big(\sqrt{1+(\alpha y)^2}\,\big) \right\}.$$

Observe that for $x \in \mathcal{N}_\alpha$ we have $x = X_\alpha(y,\theta,z) = X_{\alpha,h}(y,\theta,t)$, which means $t = z - h(\alpha y)$. We will often emphasize the description of the region $\mathcal{N}_\alpha$ in terms of the local coordinates $X_{\alpha,h}$ by writing $\mathcal{N}_{\alpha,h}$.

We compute directly, from expression (2.4), the Euclidean Laplacian in these new coordinates.

Lemma 2.1. *On the open neighborhood $\mathcal{N}_{\alpha,h}$ of M_α in $\mathbb{R}^3$, in the coordinates $x = X_{\alpha,h}(y,\theta,t)$, the Euclidean Laplacian has the following expression:*

$$\begin{aligned}\Delta_{X_{\alpha,h}} &= \partial_{tt} + \partial_{yy} + \frac{\alpha^2 y}{1+(\alpha y)^2}\partial_y + \frac{\alpha^2}{1+(\alpha y)^2}\partial_{\theta\theta} \\ &\quad - \alpha^2\left\{h''(\alpha y) + \frac{\alpha y}{1+(\alpha y)^2}h'(\alpha y) + \frac{2(t+h)}{(1+(\alpha y)^2)^2}\right\}\partial_t \\ &\quad - 2\alpha h'(\alpha y)\partial_{ty} + \alpha^2\big[h'(\alpha y)\big]^2\partial_{tt} + D_{\alpha,h}, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} D_{\alpha,h} &= \alpha(t+h)a_1\big(\alpha y, \alpha(t+h)\big)\big(\partial_{yy} - 2\alpha h'(\alpha y)\partial_{yt} - \alpha^2 h''(\alpha y)\partial_t + \alpha^2\big[h'(\alpha y)\big]^2\partial_{tt}\big) \\ &\quad + \alpha^3(t+h)a_2\big(\alpha y,\alpha(t+h)\big)\partial_{\theta\theta} + \alpha^2(t+h)b_1\big(\alpha y,\alpha(t+h)\big)\big(\partial_y - \alpha h'(\alpha y)\partial_t\big) \\ &\quad + \alpha^4(t+h)^3 b_2\big(\alpha y,\alpha(t+h)\big)\partial_t \end{aligned} \tag{2.6}$$

and the smooth functions a_i, b_i are those described in (2.2)–(2.3).

Proof. Set $z = t + h(\alpha y)$ and consider a function $U \in C^2(\mathcal{N}_{\alpha,h})$. In the coordinates $X_{\alpha,h}$ as well as in the coordinates X_α, we can write

$$U\big(X_\alpha(y,\theta,z)\big) = u(y,\theta,z) \quad \text{and} \quad U\big(X_{\alpha,h}(y,\theta,t)\big) = v(y,\theta,t)$$

which means that $u(y,\theta,z) = v(y,\theta,z-h(\alpha y))$.

It remains to notice that in the local coordinates $X_{\alpha,h}$

$$\begin{gathered} \partial_z u = \partial_t v, \qquad \partial_{zz} u = \partial_{tt} v, \\ \partial_\theta u = \partial_\theta v, \qquad \partial_{\theta\theta} w = \partial_{\theta\theta} v, \\ \partial_y u = \partial_y v - \alpha h'(\alpha y)\partial_t v, \\ \partial_{yy} u = \partial_{yy} v - 2\alpha h'(\alpha y)\partial_{ty} v - \alpha^2 h''(\alpha y)\partial_t v + \alpha^2\big[h'(\alpha y)\big]^2\partial_{tt} v. \end{gathered}$$

Substituting these partial derivatives into formula (2.4) and using that $z = t + h$, we get the expression (2.5). $\square$

Remark 2.1. The Laplace–Beltrami operator of the dilated catenoid M_α, in the coordinates $Y_\alpha(y,\theta)$, corresponds to the differential operator

$$\Delta_{M_\alpha} = \partial_{yy} + \frac{\alpha^2 y}{1+(\alpha y)^2}\partial_y + \frac{\alpha^2}{1+(\alpha y)^2}\partial_{\theta\theta}$$

with the convention that $M = M_1$. On the other hand, since each of these dilated catenoids is a minimal surface, we have that the Gaussian curvature, K_{M_α} of M_α, is given by the relation

$$2K_{M_\alpha}(y) = -\alpha^2\big|A_M(\alpha y)\big|^2 = -\frac{2\alpha^2}{(1+(\alpha y)^2)^2}, \quad y \in \mathbb{R},$$

where $|A_M(y)|$ is the norm of the second fundamental form of the catenoid M.

Hence, we can write the Euclidean Laplacian in (2.5), as follows

$$\Delta_{X_{\alpha,h}} = \partial_{tt} + \Delta_{M_\alpha} - \alpha^2\{\Delta_M h + (t+h)|A_M|^2\}\partial_t - 2\alpha h'(\alpha y)\partial_{ty} + \alpha^2\big[h'(\alpha y)\big]^2\partial_{tt} + D_{\alpha,h}, \tag{2.7}$$

where the functions h, $\Delta_M h$, $|A_M|^2$ are evaluated at αy.

3. Jacobi–Toda system on the catenoid

In the previous section, we discussed the system of coordinates and differential operators that come into play in the proof of Theorem 2. We continue our discussion providing a detailed description of the approximate nodal set of the solutions predicted by this theorem. As mentioned in the introduction, the location of this nodal set is governed by the nonlinear system of PDEs for $\mathrm{h} = (h_1, h_2, \dots, h_m)$

$$\alpha^2\big(\Delta_M h_l + |A_M|^2 h_l\big) - a_0\big[e^{-\sqrt{2}(h_l - h_{l-1})} - e^{-\sqrt{2}(h_{l+1}-h_l)}\big] = 0, \quad \text{in } M, \tag{3.1}$$

where $a_0 > 0$ is a constant, $\alpha > 0$ is a small parameter and with the convention that

$$-\infty = h_0 < h_1 < \dots < h_m < h_{m+1} = \infty.$$

In this section we provide a complete proof of the following proposition:

Proposition 3.1. *For every $\alpha > 0$ small enough there exists an axially symmetric and smooth vector function $\mathrm{h} = (h_1, \dots, h_m)$ solving system (3.1) and satisfying*

$$h_l = \left(l - \frac{m+1}{2}\right)\left[\sigma_\alpha + \left(1 - \frac{1}{\sqrt{2}\sigma_\alpha}\right)\log\big(|A_M(y)|^{-2}\big)\right] + \tilde{h}_l, \quad l = 1, \dots, m, \tag{3.2}$$

where $\sigma_\alpha \sim \log(\alpha^{-1})$ and the functions $\tilde{h}_l$ satisfy the estimates

$$\big|\tilde{h}_l(y)\big| \leqslant K\sigma^{-\frac{5}{4}}\log\big(2 + r(y)\big), \quad y \in M,$$
$$\big\|\big(1 + r(y)\big)^j D^{(j)}\tilde{h}_l\big\|_{L^\infty(M)} \leqslant K_j\sigma^{-\frac{5}{4}+\frac{j}{2}}, \quad l = 1, \dots, m, \ j = 1, 2, \dots$$

for some large constant $K > 0$, independent of $\alpha > 0$. In addition h is even respect to the arch-length variable of the catenoid M

We split the proof of Proposition 3.1 into a series of steps, each of which is presented as a subsection.

3.1. Decoupling procedure and the approximate solution

We look for a solution $\mathrm{h} = (h_1, h_2, \dots, h_m)$ to (3.1) having the form

$$h_l = \left(l - \frac{m+1}{2}\right)\sigma_\alpha + q_l, \quad l = 1, \dots, m \tag{3.3}$$

where the constant $\sigma = \sigma_\alpha$ solves the algebraic equation

$$\alpha^2\sigma = a_0 e^{-\sqrt{2}\sigma}$$

so that σ_α is a smooth function of α, satisfying the asymptotic expansion

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$$\sigma_\alpha = \log\left(\frac{\sqrt{2}a_0}{\alpha^2}\right) - \log\left(\log\left(\frac{\sqrt{2}a_0}{\alpha^2}\right)\right) + \mathcal{O}\left(\frac{\log\log\log\frac{1}{\alpha^2}}{\log\log\frac{1}{\alpha^2}}\right).$$

In what follows, we omit the explicit dependence of σ respect to α and we set $\delta = \sigma^{-1}$.

Plugging (3.3) into (3.1) and dividing by σ, we obtain the system for $(q_1, \dots, q_m)$

$$\begin{aligned} &\delta\big(\Delta_M q_l + |A_M|^2 q_l\big) - \big[e^{-\sqrt{2}(q_l - q_{l-1})} - e^{-\sqrt{2}(q_{l+1}-q_l)}\big] \\ &\quad + \left(l - \frac{m+1}{2}\right)|A_M|^2 = 0, \quad \text{in } M,\ l = 1, \dots, m. \end{aligned} \tag{3.4}$$

Before solving system (3.4), let us introduce some useful notation. Consider the invertible $m \times m$ real matrix

$$\mathbf{B} := \begin{pmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix} \tag{3.5}$$

and the auxiliary functions

$$\begin{pmatrix} \mathrm{v} \\ v_m \end{pmatrix} := \mathbf{B} \cdot \begin{pmatrix} \mathrm{q} \\ q_m \end{pmatrix}, \qquad \mathrm{q} := \begin{pmatrix} q_1 \\ \vdots \\ q_{m-1} \end{pmatrix}.$$

We notice that the l-th entry of the constant vector $\mathbf{B}^{-1} \cdot \mathbb{1}$ corresponds to $l - \frac{m+1}{2}$.

Let us introduce also the notation

$$e^{\mathrm{v}} := \begin{bmatrix} e^{v_1} \\ \vdots \\ e^{v_{m-1}} \end{bmatrix}, \qquad \mathbb{1} := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

and consider the constant invertible $(m-1) \times (m-1)$ matrix

$$\mathbf{C} = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & \cdots & 1 & -2 \end{pmatrix}. \tag{3.6}$$

With this notation, system (3.4) can be written as

$$\delta\big(\Delta_M \mathrm{v} + |A_M|^2 \mathrm{v}\big) + \mathbf{C} \cdot e^{-\sqrt{2}\mathrm{v}} + |A_M|^2 \cdot \mathbb{1} = 0, \quad \text{in } M \tag{3.7}$$

$$\Delta_M v_m + |A_M|^2 v_m = 0, \quad \text{in } M \tag{3.8}$$

Since the matrix $\mathbf{B}$ in (3.5) is invertible, any information about system (3.7)–(3.8) has a direct translation into system (3.4) and vice versa.

Taking $v_m = 0$ in (3.8), we only need to take care of system (3.7). In order to solve this system, let us set

$$E(\mathrm{v}, \delta, y) := \delta\big(\Delta_M \mathrm{v} + |A_M|^2 \mathrm{v}\big) + \mathbf{C} \cdot e^{-\sqrt{2}\mathrm{v}} + |A_M|^2 \mathbb{1}. \tag{3.9}$$

We want to find an approximate solution v_0 to (3.7) such that $E(\mathrm{v}_0, \delta, y)$ is as close to zero as possible. Writing

$$\mathrm{v}_0(y, \delta) = \omega_0(y) + \delta\omega_1(y)$$

expression (3.9) becomes

$$\begin{aligned} E(\mathrm{v}_0, \delta, y) = {} & \mathbf{C} \cdot e^{-\sqrt{2}\omega_0} + |A_M|^2 \mathbb{1} + \delta\big(\Delta_M \omega_0 + |A_M|^2 \omega_0\big) + \delta D_{\mathrm{v}}\big(\mathbf{C} \cdot e^{-\sqrt{2}\mathrm{v}}\big)_{\mathrm{v}=\omega_0} \cdot \omega_1 \\ & + \delta^2\big(\Delta_M \omega_1 + |A_M|^2 \omega_1\big) + \mathbf{C} \cdot \big[e^{-\sqrt{2}(\omega_0 + \delta\omega_1)} - e^{-\sqrt{2}\omega_0} - \delta D_{\mathrm{v}}\big(e^{-\sqrt{2}\mathrm{v}}\big)_{\mathrm{v}=\omega_0} \omega_1\big]. \end{aligned} \tag{3.10}$$

Proceeding formally by taking $\delta \to 0$, we find that ω_0 must solve the algebraic equation

$$\mathbf{C} \cdot e^{-\sqrt{2}\omega_0} + |A_M|^2 \mathbb{1} = 0, \tag{3.11}$$

where we recall that in local coordinates

$$\big|A_M(y)\big|^2 = \frac{2}{(1 + y^2)^2}, \quad y = Y(y, \theta).$$

From this we write $\omega_0 = (\omega_{0,1}, \ldots, \omega_{0,m-1})$ where

$$\omega_{0,l}(y) = -\frac{1}{\sqrt{2}} \log\left(\frac{1}{2}\big|A_M(y)\big|^2 (m - l) l\right), \quad 1 \leqslant l \leqslant m - 1$$

so that

$$\omega_0 = \frac{1}{\sqrt{2}} \log\big(|A_M|^{-2}\big) \mathbb{1} + c_0 \tag{3.12}$$

for some constant vector c_0. A direct computation yields that

$$\Delta_M \omega_0 + |A_M|^2 \omega_0 = |A_M|^2 (2 \cdot \mathbb{1} + \omega_0). \tag{3.13}$$

With this choice of ω_0, dividing expression (3.10) by δ and taking $\delta \to 0$, we find that ω_1 must solve the algebraic equation

$$D_{\mathrm{v}}\big(\mathbf{C} \cdot e^{-\sqrt{2}\mathrm{v}}\big)_{\mathrm{v}=\omega_0} \cdot \omega_1 + \big(\Delta_M \omega_0 + |A_M|^2 \omega_0\big) = 0. \tag{3.14}$$

Observe that

$$\begin{aligned} D_{\mathrm{v}}\big(\mathbf{C} \cdot e^{-\sqrt{2}\mathrm{v}}\big)_{\mathrm{v}=\omega_0} &= -\sqrt{2}|A_M|^2 \mathbf{C} \cdot diag\left(\frac{(m - j)j}{2}\right)_{(m-1)\times(m-1)} \\ &= \sqrt{2}|A_M|^2 \begin{pmatrix} -2a_1 & a_2 & \ldots & \ldots & 0 & 0 \\ a_1 & -2a_2 & \ldots & \ldots & 0 & 0 \\ 0 & a_2 & -2a_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & 0 \\ 0 & 0 & \ldots & a_{m-3} & -2a_{m-2} & a_{m-1} \\ 0 & 0 & \ldots & 0 & a_{m-2} & -2a_{m-1} \end{pmatrix}, \end{aligned} \tag{3.15}$$

where

$$a_l = \frac{(m-l)l}{2}, \quad l = 1, \dots, m-1.$$

Directly from (3.15) we find that

$$-\mathbf{C} \cdot diag\left(\frac{(m-j)j}{2}\right)_{(m-1)\times(m-1)} \cdot \mathbb{1} = \mathbb{1}.$$

and consequently (3.14) becomes

$$\sqrt{2}\mathbf{C} \cdot diag\left(\frac{(m-j)j}{2}\right)_{m-1} \omega_1 = (2 \cdot \mathbb{1} + \omega_0).$$

It follows that

$$\omega_1 = -\sqrt{2} \cdot \mathbb{1} - \frac{1}{2}\log\left(|A_M|^{-2}\right) \cdot \mathbb{1} + c_1 \tag{3.16}$$

for some constant vector c_1. Therefore, our choice of the approximation to (3.7) is

$$\mathrm{v}_0(y,\delta) = \frac{1}{\sqrt{2}}\left(1 - \frac{\delta}{\sqrt{2}}\right)\log\left(\left|A_M(y)\right|^{-2}\right)\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} + c_0 + \delta c_1$$

and observe that

$$E(\mathrm{v}_0,\delta,y) = \delta^2\left(\Delta_M\omega_1 + |A_M|^2\omega_1\right) + \mathbf{C}\cdot\left[e^{-\sqrt{2}(\omega_0+\delta\omega_1)} - e^{-\sqrt{2}\omega_0} - \delta D_{\mathrm{v}}\left(e^{-\sqrt{2}\mathrm{v}}\right)_{\mathrm{v}=\omega_0}\delta\omega_1\right]. \tag{3.17}$$

From (3.12), (3.16), (3.17) and a direct computation we get the pointwise estimate in M

$$\left|E(\mathrm{v}_0,\delta,y)\right| \leqslant C\delta^2|A_M|^{2(1-\delta)-\varepsilon}\left[1 + \left|\log\left(|A_M|^2\right)\right| + \mathcal{O}\left(\left|\log\left(|A_M|^2\right)\right|^2\right)\right]. \tag{3.18}$$

for some $\varepsilon > 0$ small. To verify estimate (3.18), first recall that

$$r(y) = |y'|, \qquad y = \left(y', y_3\right) \in M$$

which in the local coordinates $y = Y(y,\theta)$ reads as $r(y) = \sqrt{1+y^2}$.

Next, using a Taylor's expansion up to second derivatives in the region of M, where

$$\delta\log\left(|A_M|^2\right) \leqslant K_1$$

we get that

$$\left|e^{-\sqrt{2}(\omega_0+\delta\omega_1)} - e^{-\sqrt{2}\omega_0} - \delta D_{\mathrm{v}}\left(e^{-\sqrt{2}\mathrm{v}}\right)_{\mathrm{v}=\omega_0}\delta\omega_1\right| \leqslant C\delta^2|A_M|^2|\omega_1|^2,$$

where K_1 is independent of δ and y. Since $|A_M|^2 \sim \mathcal{O}(r(y)^{-4})$, this actually occurs in the large region determined by

$$r(y) \leqslant e^{\frac{K_1}{4\delta}}, \quad y \in M$$

while in the remaining part of M, we use the fast decay of $|A_M|^2$ to get that

$$\left|e^{-\sqrt{2}(\omega_0+\delta\omega_1)} - e^{-\sqrt{2}\omega_0} - \delta D_{\mathrm{v}}\big(e^{-\sqrt{2}\mathrm{v}}\big)_{\mathrm{v}=\omega_0}\delta\omega_1\right| \leqslant C|A_M|^2 e^{\delta\log(r^4(y))} \leqslant r(y)^{-\beta}e^{-\frac{c_1}{\delta}}$$

which is exponentially small in δ, provided that we choose β so that $0<\beta<4-4\delta$. Clearly, (3.18) follows at once from these remarks.

3.2. Solving the Jacobi–Toda system

Next, we linearize system (3.7) around the approximate solution $\mathrm{v}_0(y,\delta)$ we have described in the previous subsection.

Let us first introduce the topologies that will be used to set up our functional analytical scheme. For functions g and ζ defined in M, $1<p\leqslant\infty$ and $\beta>\frac{5}{2}$ we consider the norms

$$\|g\|_{p,\beta} := \big\|\big(1+r(y)^{\beta}\big)g\big\|_{L^p(M)}, \tag{3.19}$$

$$\|\zeta\|_{\delta,\infty} := \delta\big\|D^2\zeta\big\|_{\infty,2} + \delta^{\frac{1}{2}}\big\|\big(1+r(y)\big)D\zeta\big\|_{L^\infty(M)} + \big\|\log\big(r(y)+2\big)^{-1}\zeta\big\|_{L^\infty(M)}, \tag{3.20}$$

$$\|\zeta\|_{\delta,p,\beta} := \delta\big\|D^2\zeta\big\|_{p,\beta} + \delta^{\frac{1}{2}}\big\|\big(1+r(y)\big)D\zeta\big\|_{L^\infty(M)} + \big\|\log\big(r(y)+2\big)^{-1}\zeta\big\|_{L^\infty(M)}. \tag{3.21}$$

Next, we study the linearization of system (3.7) around $\mathrm{v}_0(y,\delta)$. Recall that

$$\mathrm{v}_0(y,\delta) = \frac{1}{\sqrt{2}}\left(1-\frac{\delta}{\sqrt{2}}\right)\log\big(\big|A_M(y)\big|^{-2}\big)\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} + c_0 + \delta c_1 \tag{3.22}$$

and we look for a solution to (3.7) of the form

$$\mathrm{v} = \mathrm{v}_0 + \zeta.$$

Thus, we are led to study the system

$$\begin{aligned}&\delta\big(\Delta_M\zeta + |A_M|^2\zeta\big) + D_{\mathrm{v}}\big[\mathbf{C}\cdot e^{-\sqrt{2}\mathrm{v}}\big]_{\mathrm{v}=\mathrm{v}_0}\zeta\\ &\quad = -E(\mathrm{v}_0,\delta) - \big(\mathbf{C}\cdot e^{-\sqrt{2}(\mathrm{v}_0+\zeta)} - \mathbf{C}\cdot e^{-\sqrt{2}\mathrm{v}_0} - D_{\mathrm{v}}\big[\mathbf{C}\cdot e^{-\sqrt{2}\mathrm{v}}\big]_{\mathrm{v}=\mathrm{v}_0}\zeta\big), \quad \text{in } M.\end{aligned} \tag{3.23}$$

Let us observe that

$$D_{\mathrm{v}}\big[\mathbf{C}\cdot e^{-\sqrt{2}\mathrm{v}}\big]_{\mathrm{v}=\omega_0+\delta\omega_1} = D_{\mathrm{v}}\big[\mathbf{C}\cdot e^{-\sqrt{2}\mathrm{v}}\big]_{\mathrm{v}=\omega_0} + \mathbf{C}\cdot\big(\big[D_{\mathrm{v}}e^{-\sqrt{2}\mathrm{v}}\big]_{\mathrm{v}=\omega_0+\delta\omega_1} - D_{\mathrm{v}}\big[e^{-\sqrt{2}\mathrm{v}}\big]_{\mathrm{v}=\omega_0}\big). \tag{3.24}$$

Proceeding as in (3.18), it can be checked that

$$\big\|\mathbf{C}\cdot\big(\big[D_{\mathrm{v}}e^{-\sqrt{2}\mathrm{v}}\big]_{\mathrm{v}=\omega_0+\delta\omega_1} - D_{\mathrm{v}}\big[e^{-\sqrt{2}\mathrm{v}}\big]_{\mathrm{v}=\omega_0}\big)\big\|_{\infty,\beta} \leqslant C\delta \tag{3.25}$$

for any $0<\beta<4-4\delta$. Consequently, we can write system (3.23) as

$$\mathcal{L}_\delta(\zeta) = -E(\mathrm{v}_0,\delta) - \mathrm{Q}(\mathrm{v}_0,\zeta), \quad \text{in } M, \tag{3.26}$$

where

$$\mathcal{L}_\delta(\zeta) := \delta\big(\Delta_M\zeta + |A_M|^2\zeta\big) - \sqrt{2}|A_M|^2\mathbf{C}\cdot A(y,\delta)\zeta,$$
$$A(y,0) := diag\bigg(\frac{(m-j)j}{2}\bigg)_{(m-1)\times(m-1)}$$
$$\big\||A_M|^2\big(A(\cdot,\delta) - A(\cdot,0)\big)\big\|_{\infty,\beta} \leqslant C\delta, \quad 0<\beta<4-4\delta$$

and

$$\mathrm{Q}(\mathrm{v}_0,\zeta) := \mathbf{C}\cdot e^{-\sqrt{2}(\mathrm{v}_0+\zeta)} - \mathbf{C}\cdot e^{-\sqrt{2}\mathrm{v}_0} - D_{\mathrm{v}}\big[\mathbf{C}\cdot e^{-\sqrt{2}\mathrm{v}}\big]_{\mathrm{v}=\mathrm{v}_0}\zeta.$$

The following proposition provides a suitable linear theory needed to solve the linear equation

$$\mathcal{L}_\delta(\zeta) = \tilde{g}, \quad \text{on } M \tag{3.27}$$

in the class of axially symmetric even functions.

Proposition 3.2. *For every $\delta > 0$ small enough and any given axially symmetric even vector function $\tilde{g}$ defined on M, with*

$$\|\tilde{g}\|_{p,\beta} < \infty$$

for $\frac{4}{3} < p \leqslant \infty$ and $\frac{5}{2} < \beta < 4 - \frac{2}{p}$, there exists a unique axially symmetric even solution ζ to system (3.27) satisfying the estimates

$$\|\zeta\|_{\delta,p,\beta} \leqslant C\delta^{-\frac{3}{4}}\|\tilde{g}\|_{p,\beta}, \tag{3.28}$$
$$\|\zeta\|_{\delta,\infty} \leqslant C\delta^{-\frac{3}{4}}\|\tilde{g}\|_{\infty,\beta}. \tag{3.29}$$

We remark that the constant $C > 0$ in Proposition 3.2 does not depend on δ but rather on β and p. We provide the proof of this result in next section.

We finish this section solving system (3.26). Let $\zeta = T_\delta(\tilde{g})$ denote the linear operator provided by Proposition 3.2. We recast system (3.26) as the fixed point problem for the vector function ζ

$$\zeta = R(\zeta), \qquad R(\zeta) := T_\delta\big[-E(\mathrm{v}_0,\delta) - \mathrm{Q}(\mathrm{v}_0,\zeta)\big]$$

in the Banach space X of smooth vector functions ζ with the norm

$$\|\zeta\|_X := \|\zeta\|_{\delta,\infty} < \infty.$$

From (3.18) and for any β such that $2 < \beta < 4 - 4\delta$, we get that

$$\big\|E(\mathrm{v}_0,\delta)\big\|_{\infty,\beta} \leqslant C\delta^2 \tag{3.30}$$

and consequently, from (3.29) we obtain that

$$\big\|R(0)\big\|_X = \big\|T_\delta\big[E(\mathrm{v}_0,\delta)\big]\big\|_X \leqslant C\delta^{\frac{5}{4}}.$$

On the other hand, proceeding as we did to verify (3.18), for any $\frac{5}{2} < \beta < 4 - 4\delta$ and any ζ such that

$$\|\zeta\|_X \leqslant C\delta^{\frac{5}{4}} \tag{3.31}$$

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it follows that

$$\left\|T_\delta\left[\mathrm{Q}(\mathrm{v}_0,\zeta)\right]\right\|_X \leqslant C\delta^{-\frac{3}{4}}\left\|\mathrm{Q}(\mathrm{v}_0,\zeta)\right\|_{\infty,\beta} \leqslant C\delta^{-\frac{3}{4}}\|\zeta\|_X^2 = \mathcal{O}\left(\delta^{\frac{7}{4}}\right).$$

Finally, to check the Lipschitz character of $\mathrm{Q}(\mathrm{v}_0,\zeta)$ respect to ζ, we simply observe that for ζ_1,ζ_2 satisfying (3.31), we have

$$\mathrm{Q}(\mathrm{v}_0,\zeta_1)-\mathrm{Q}(\mathrm{v}_0,\zeta_2) = \mathbf{C}\cdot\left[e^{-\sqrt{2}(\mathrm{v}_0+\zeta_1)} - e^{-\sqrt{2}(\mathrm{v}_0+\zeta_2)} - D_{\mathrm{v}}\left(e^{-\sqrt{2}\mathrm{v}}\right)_{\mathrm{v}=\mathrm{v}_0}(\zeta_1-\zeta_2)\right].$$

From this and proceeding again as we did to obtain (3.30), the inequality

$$\left\|\mathrm{Q}(\mathrm{v}_0,\zeta_1)-\mathrm{Q}(\mathrm{v}_0,\zeta_2)\right\|_{\infty,\beta} \leqslant C\delta^{\frac{5}{4}}\|\zeta_1-\zeta_2\|_X \tag{3.32}$$

follows. This implies that

$$\left\|R(\zeta_1)-R(\zeta_2)\right\|_X \leqslant C\delta^{-\frac{3}{4}}\left\|\mathrm{Q}(\mathrm{v}_0,\zeta_1)-\mathrm{Q}(\mathrm{v}_0,\zeta_2)\right\|_{\infty,\beta} \leqslant C\delta^{\frac{1}{2}}\|\zeta_1-\zeta_2\|_X.$$

Hence, the function R maps the ball in X of radius $K\delta^{\frac{5}{4}}$ onto itself, provided the constant $K>0$ is chosen large enough, but independent of $\delta>0$ small. A direct application of Banach fixed point theorem allows us to solve system (3.26). We have thus proven the following proposition:

Proposition 3.3. *For every $\delta>0$ small and β such that $\frac{5}{2}<\beta<4(1-\delta)$ there exists a unique axially symmetric even and smooth solution ζ to the system*

$$\mathcal{L}_\delta(\zeta) = -E(\mathrm{v}_0,\delta)-\mathrm{Q}(\mathrm{v}_0,\zeta), \quad \text{in } M$$

satisfying

$$\|\zeta\|_{\delta,\infty} \leqslant K\delta^{\frac{5}{4}}, \qquad \left\|\left(1+r(y)\right)^j D^{(j)}\zeta\right\|_\infty \leqslant K\delta^{\frac{5}{4}-\frac{j}{2}}, \quad j=1,2,\ldots$$

To conclude the proof of Proposition 3.1 simply notice that, from the previous proposition and a direct computation, the solution $\mathrm{h} = \mathbf{B}^{-1}[\mathrm{v}_0+\zeta]$ is such that

$$h_l = \left(l-\frac{m+1}{2}\right)\left[\sigma_\alpha + \left(1-\frac{1}{\sqrt{2}\sigma_\alpha}\right)\log\left(\left|A_M(y)\right|^{-2}\right)\right] + \tilde{h}_l, \quad l=1,\ldots,m$$

with the $\tilde{h}_l$ as predicted in Proposition 3.1.

4. Jacobi and linearized Jacobi–Toda operators on the catenoid

This section is devoted to prove Proposition 3.2 and the study of the linearization of the decoupled Jacobi–Toda system around the exact solution we found in the previous section.

4.1. Linearized Jacobi–Toda operator

We first prove Proposition 3.2. In order to do so, we study the linear system

$$\delta\Delta_M\zeta + |A_M|^2\left(-\sqrt{2}\mathbf{C}\cdot A(y,0)+\delta\mathbf{I}\right)\zeta = \tilde{g}, \quad \text{in } M, \tag{4.1}$$

where we recall that

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$$A(y,0) = diag\left(\frac{(m-j)j}{2}\right)_{(m-1)\times(m-1)}$$

and the matrix $\mathbf{C}$ is given in (3.6). Actually, a direct computation shows that the numbers

$$1, \frac{1}{2}, \ldots, \frac{m-1}{m}$$

are the $m-1$ eigenvalues of the matrix $-\mathbf{C}$, so that $-\mathbf{C}$ is symmetric and positive definite. Let us write

$$\zeta = [-\mathbf{C}]^{\frac{1}{2}}\psi, \qquad \tilde{g} = [-\mathbf{C}]^{\frac{1}{2}}\bar{g}.$$

System (4.1) becomes

$$\delta\Delta_M\psi + |A_M|^2(\delta\mathbf{I} + \mathcal{B})\psi = \bar{g}, \quad \text{in } M, \tag{4.2}$$

where the matrix $\mathcal{B}$ is given by

$$\mathcal{B} = \frac{1}{\sqrt{2}}[-\mathbf{C}]^{\frac{1}{2}}\, diag\big((m-j)j\big)_{(m-1)\times(m-1)}[-\mathbf{C}]^{\frac{1}{2}}.$$

Next, we consider the eigenvectors $\hat{e}_1, \ldots, \hat{e}_{m-1}$ of the matrix $\mathcal{B}$, i.e.

$$\mathcal{B}\cdot\hat{e}_i = \lambda_i\hat{e}_i, \quad i = 1, \ldots, m-1$$

and we write

$$\psi = \sum_{i=1}^{m-1}\psi_i\hat{e}_i, \qquad \bar{g} = \sum_{i=1}^{m-1}\bar{g}_i\hat{e}_i.$$

Hence, system (4.2) decouples into $m-1$ scalar equations, namely

$$\delta\Delta_M\psi_i + |A_M|^2(\lambda_i + \delta)\psi_i = \bar{g}_i, \quad \text{in } M,\ i = 1, \ldots, m-1. \tag{4.3}$$

The eigenvalues $\lambda_1, \ldots, \lambda_{m-1}$ are positive, a fact that makes invertibility of each equation in (4.3) a very delicate matter.

Without any loss of generality, we study solvability theory for the model linear equation

$$L_\delta(\psi) = \delta\Delta_M\psi + |A_M|^2\psi = \tilde{g}, \quad \text{in } M. \tag{4.4}$$

Since we are working in the class of axially symmetric and even functions, we only need to study solutions to

$$L_\delta\psi = 0, \quad \text{in } M \cap \{x_3 \geqslant 0\}$$

which in the arch-length variable of M reads as the ODE

$$\delta\left(\psi''(y) + \frac{y}{1+y^2}\psi'(y)\right) + \frac{2}{(1+y^2)^2}\psi(y) = 0, \quad y \geqslant 0. \tag{4.5}$$

Let us denote $y_\delta > 0$, the real number such that $\sqrt{1+y_\delta^2} = \frac{1}{\sqrt{\delta}}$.

Consider the change of variables $y = \sinh(t)$ and consider the outer region $y > y_\delta$. Let us choose $T_\delta > 0$ so that $\delta \cosh^2(T_\delta) = 2$. Hence writing solutions to (4.5) in the form $\psi(y) = \phi(t)$, we see that the function ϕ must satisfy

$$\partial_{tt}\phi + p_\delta(t)\phi = 0, \qquad p_\delta(t) := 2\delta^{-1}\operatorname{sech}^2(t) \quad t > T_\delta \tag{4.6}$$

The following lemma gives us a precise description of the solutions to (4.6) in the outer region $t > T_\delta$.

Lemma 4.1. *The linear ODE (4.6) has two linearly independent solutions, $\phi_1(t)$, $\phi_2(t)$, satisfying that*

$$\phi_1(t) = t + \mathcal{O}(1), \qquad \partial_t\phi_1(t) = 1 + \mathcal{O}\big(t^{-1}\big), \quad \text{for } t > T_\delta, \tag{4.7}$$

$$\phi_2(t) = 1 + \mathcal{O}\big(t^{-1}\big), \qquad \partial_t\phi_2(t) = \mathcal{O}\big(t^{-1}\big), \quad \text{for } t > T_\delta, \tag{4.8}$$

provided δ is small enough, which amounts to the fact that T_δ is large enough. Even more, $\phi_2(t)$ satisfies the estimate

$$\big|\partial_t\phi_2(t)\big| \leqslant C\|\phi_2\|_{L^\infty(T_\delta,\infty)} p_\delta(t), \quad t > T_\delta. \tag{4.9}$$

Proof. First let us look for a solution $\phi_1(t)$ to (4.6) of the form $\phi_1(t) = tv(t)$. We find that $v(t)$ must solve

$$\partial_t\big(t^2\partial_t v(t)\big) + p_\delta(t)t^2 v(t) = 0.$$

Setting $z(t) = t^2\partial_t v(t)$, we obtain the first order IVP for $z(t)$ and $v(t)$

$$\partial_t z(t) = -p_\delta(t)t^2 v(t), \qquad \partial_t v(t) = \frac{1}{t^2}z(t), \qquad z(T_\delta) = z_0, \qquad v(T_\delta) = v_0.$$

Integrating each equation on the system, we find that

$$z(t) = z_0 - \int_{T_\delta}^{t} p_\delta(\tau)\tau^2 v(\tau)\,d\tau, \qquad v(t) = v_0 + \int_{T_\delta}^{t} \frac{1}{\tau^2}z(\tau)\,d\tau.$$

Hence, using this integral formulas and Fubini's theorem, we obtain the integral representation for $z(t)$

$$z(t) = z_0 - v_0\int_{T_\delta}^{t} p_\delta(\tau)\tau^2\,d\tau - \int_{T_\delta}^{t} \frac{1}{\tau^2}z(\tau)\int_{\tau}^{t} p_\delta(s)s^2\,ds\,d\tau.$$

Next, we prove that $z(t)$ is bounded. First observe that

$$0 \leqslant \int_{T_\delta}^{t} p_\delta(\tau)\tau^2\,d\tau \leqslant \int_{T_\delta}^{\infty} p_\delta(\tau)\tau^2\,d\tau \leqslant C\delta^{-1}T_\delta^2 e^{-2T_\delta} \leqslant CT_\delta^2,$$

where $C > 0$ is independent of δ, provided $\delta > 0$ is small enough. On the other hand,

$$\big|z(t)\big| \leqslant C\big(|z_0| + \delta^{-1}|v_0|\big) + \int_{T_\delta}^{t} p_\delta(\tau)\big|z(\tau)\big|\,d\tau$$

and directly from Gronwall's inequality we obtain that

$$\left|z(t)\right| \leqslant C\left(|z_0| + \delta^{-1}|v_0|\right) \exp\left(\int_{T_\delta}^{t} p_\delta(\tau)\,d\tau\right).$$

Since

$$\int_{T_\delta}^{\infty} p_\delta(\tau)\,d\tau \leqslant \frac{C}{\delta} e^{-2T_\delta}$$

then for δ small enough, and taking $v_0 = 0$, we find that $|z(t)| \leqslant C|z_0|$, for $t > T_\delta$.

Plugging this into the integral formula for $z(t)$ we observe that

$$z(t) = z_0 + \int_{T_\delta}^{t} z(\tau)\frac{1}{\tau^2}\int_{\tau}^{t} p_\delta(s)s^2\,ds\,d\tau.$$

Since $z(t)$ is bounded, we obtain that

$$z(\infty) = \lim_{t\to\infty} z(t) = z_0 + \int_{T_\delta}^{\infty} z(\tau)\frac{1}{\tau^2}\int_{\tau}^{\infty} p_\delta(s)s^2\,ds\,d\tau$$

and without any loss of generality we write

$$z(t) = 1 + \int_{t}^{\infty} z(\tau)\frac{1}{\tau^2}\int_{\tau}^{t} p_\delta(s)s^2\,ds\,d\tau, \quad t > T_\delta.$$

Observe that

$$\left|z(t) - 1\right| \leqslant Cp_\delta(t) \leqslant Ce^{-2(t-T_\delta)}, \quad t > T_\delta.$$

From the integral formula for $v(t)$, we obtain that

$$v(t) = v(\infty) + \int_{t}^{\infty} z(\tau)\frac{1}{\tau^2}\,d\tau = v(\infty) + \mathcal{O}\left(\frac{1}{t}\right)$$

so that, we may choose

$$\phi_1(t) = t + \mathcal{O}(1), \quad t > T_\delta, \qquad \partial_t\phi_1(t) = v(t) + t\partial_t v(t) = 1 + \mathcal{O}\left(\frac{1}{t}\right).$$

Using the reduction of order formula, we find the second solution $\phi_2(t)$, satisfying

$$\phi_2(t) = 1 + \mathcal{O}\left(\frac{1}{t}\right), \qquad \partial_t\phi_2(t) = \mathcal{O}\left(\frac{1}{t}\right).$$

To find estimate (4.9), observe that $\partial_t\phi_2(\infty) = 0$. So we obtain from (4.6) that

$$\partial_t\phi_2(t) = -\int_{t}^{\infty} p_\delta(\tau)\phi_2(\tau)\,d\tau, \quad t > T_\delta$$

from which

$$\left|\partial_t \phi_2(t)\right| \leqslant C\|\phi_2\|_{L^\infty(T_\delta,\infty)} p_\delta(t), \quad \text{for } t > T_\delta.$$

This concludes the proof of the lemma. □

Next, we describe solutions to (4.5) in the whole line and in the arch-length variable y. Let $\psi_1(y), \psi_2(y)$ be two linearly independent solutions of (4.5) satisfying

$$\psi_i(0) = c_{i,1}, \qquad \partial_y \psi_i(0) = c_{i,2}\delta^{-\frac{1}{2}}, \quad i = 1, 2, \tag{4.10}$$

where $c_{1,1}c_{2,2} - c_{1,2}c_{2,1} = 1$, so that the Wronskian is given by

$$W(\psi_1, \psi_2) = \frac{\delta^{-\frac{1}{2}}}{\sqrt{1+y^2}}, \quad \forall y \in \mathbb{R}.$$

The following proposition completes the description of the kernel.

Proposition 4.1. *The fundamental set* $\{\psi_1, \psi_2\}$ *of* (4.5) *satisfies the following estimates*

$$\left|\psi_i(y)\right| \leqslant C\left(1+y^2\right)^{\frac{1}{4}}, \qquad \left|\partial_y \psi_i(y)\right| \leqslant C\delta^{-\frac{1}{2}}\left(1+y^2\right)^{-\frac{1}{4}}, \quad 0 \leqslant y \leqslant y_\delta, \tag{4.11}$$

$$\left|\psi_i(y)\right| \leqslant C\delta^{-\frac{1}{4}}\left|\log(\delta)\right| \ln\left(1+|y|\right), \qquad \left(1+|y|\right)\left|\partial_y \psi_i(y)\right| \leqslant C\delta^{-\frac{1}{4}}, \quad y \geqslant y_\delta. \tag{4.12}$$

Proof. We pass to the sphere S^2 using the stereographic projection

$$y = \tan(\theta), \quad \text{for } 0 < \theta < \theta_\delta,$$

where the number $\theta < \theta_\delta$ is such that $y_\delta = \tan(\theta_\delta)$, $0 < \theta_\delta < \frac{\pi}{2}$. Writing

$$\psi(y) = \varphi(\theta), \quad \text{for } 0 < \theta < \theta_\delta$$

we find that φ solves the equation

$$\partial_{\theta\theta}\varphi(\theta) - \tan(\theta)\partial_\theta \varphi(\theta) + \frac{2}{\delta}\varphi(\theta) = 0. \tag{4.13}$$

Assume further that

$$\psi(y) = \frac{1}{\sqrt{\cos(\theta)}}\gamma\left(\frac{\theta}{\sqrt{\delta}}\right), \quad \text{for } 0 < \theta < \theta_\delta. \tag{4.14}$$

so that

$$\partial_{ss}\gamma(s) + \left(\left[1 + \frac{\delta}{4}\right] + \frac{\delta}{4}\sec^2(\sqrt{\delta}s)\right)\gamma(s) = 0, \quad \text{for } 0 < s < s_\delta := \frac{\theta_\delta}{\sqrt{\delta}}.$$

We claim that $\gamma(s)$ and $\partial_s\gamma(s)$ are uniformly bounded in $(0, s_\delta)$. To prove this claim, we consider the pointwise energy

$$J(s) := \left|\partial_s \gamma(s)\right|^2 + \left[1 + \frac{\delta}{4}\right]\left|\gamma(s)\right|^2$$

for which

$$\partial_s J(s) = -2\partial_s\gamma(s)\gamma(s)\frac{\delta}{4}\sec^2(\sqrt{\delta}s).$$

Hence, for a constant $C > 0$ independent of $\delta > 0$, it follows that

$$\left|\partial_s J(s)\right| \leqslant CJ(s)\frac{\delta}{4}\sec^2(\sqrt{\delta}s)$$

and consequently

$$0 \leqslant J(s) \leqslant J(0) + C\frac{\delta}{4}\int_0^s J(\xi)\sec^2(\sqrt{\delta}\xi)\,d\xi, \quad \text{for } 0 < s < s_\delta.$$

Using Gronwall's inequality, we find that

$$J(s) \leqslant J(0)\exp\left(C\frac{\delta}{4}\int_0^{s_\delta}\sec^2(\sqrt{\delta}\xi)\,d\xi\right). \tag{4.15}$$

We compute explicitly the integral in (4.15) to find that

$$\frac{\delta}{4}\int_0^{s_\delta}\sec^2(\sqrt{\delta}\xi)\,d\xi = \frac{\sqrt{\delta}}{4}\tan(\sqrt{\delta}s_\delta) = \frac{\sqrt{\delta}}{4}\tan(\theta_\delta) \leqslant c_0,$$

where c_0 does not depend on $\delta > 0$. Hence, we find that

$$J(s) := \left|\partial_s\gamma(s)\right|^2 + \left[1 + \frac{\delta}{4}\right]\left|\gamma(s)\right|^2 \leqslant CJ(0), \quad 0 < s < \frac{\theta_\delta}{\sqrt{\delta}}$$

and so the claim is proven. Pulling back the change of variables given in (4.14) and since

$$\left(1 + y^2\right)\partial_y\psi(y) = \frac{1}{\sqrt{\delta}}\frac{\partial_s\gamma(\frac{\theta}{\sqrt{\delta}})}{\sqrt{\cos(\theta)}} + \frac{\sin(\theta)\gamma(\frac{\theta}{\sqrt{\delta}})}{2\cos^{\frac{3}{2}}(\theta)}. \tag{4.16}$$

we find that

$$\psi(0) = \gamma(0), \qquad \partial_y\psi(0) = \delta^{-\frac{1}{2}}\partial_s\gamma(0).$$

and consequently we obtain (4.11).

On the other hand, using Lemma 4.1 we may find another fundamental set for (4.5), say $\{\tilde{\psi}_1(y), \tilde{\psi}_2(y)\}$, such that

$$\begin{aligned}
&\tilde{\psi}_1(y) = \ln\left(1 + |y|\right) + \mathcal{O}(1), \qquad \left(1 + |y|\right)\partial_y\tilde{\psi}_1(y) = 1 + \mathcal{O}\left(\ln\left(1 + |y|\right)^{-1}\right), \quad y \geqslant y_\delta,\\
&\tilde{\psi}_2(y) = 1 + \mathcal{O}\left(\ln\left(1 + |y|\right)^{-1}\right), \qquad \left(1 + |y|\right)\partial_y\tilde{\psi}_2(y) = \mathcal{O}\left(\log\left(1 + |y|\right)|^{-1}\right), \quad y \geqslant y_\delta,
\end{aligned}$$

and with Wronski determinant

$$0 < W(\tilde{\psi}_1, \tilde{\psi}_2) = c\left(1 + y^2\right)^{-\frac{1}{2}}.$$

Let us consider next equation (4.5) for ψ_i in the region $y > y_\delta$. Since $\gamma(s)$, $|\partial_s\gamma(s)|$ are uniformly bounded, we find from (4.14) and (4.16) the conditions

$$\psi_i(y_\delta) = \mathcal{O}\big(\delta^{-\frac{1}{4}}\big), \qquad \partial_y\psi_i(y_\delta) = \mathcal{O}\big(\delta^{\frac{1}{4}}\big) \tag{4.17}$$

and we write

$$\psi_i(y) = c_{i,1}\tilde{\psi}_1 + c_{i,2}\tilde{\psi}_2, \quad y \geqslant y_\delta, \; i = 1,2.$$

A direct computation shows that

$$\begin{bmatrix} c_{i,1} \\ c_{i,2} \end{bmatrix} = c\delta^{-\frac{1}{2}} \begin{bmatrix} \partial_y\tilde{\psi}_2(y_\delta) & -\tilde{\psi}_2(y_\delta) \\ -\partial_y\tilde{\psi}_1(y_\delta) & \tilde{\psi}_1(y_\delta) \end{bmatrix} \cdot \begin{bmatrix} \psi_i(y_\delta) \\ \partial_y\psi_i(y_\delta) \end{bmatrix}.$$

From this we obtain that

$$c_{i,1} = \mathcal{O}\big(\delta^{-\frac{1}{4}}\big), \qquad c_{i,2} = \mathcal{O}\big(\delta^{-\frac{1}{4}}\big|\log(\delta)\big|\big)$$

and clearly (4.12) follows at once from these remarks. □

Proof of Proposition 3.2. Using Proposition 4.1 we choose a solution to (4.4) defined by the variations of parameters formula

$$\psi(y) = -\delta^{-\frac{1}{2}}\psi_1(y)\int_0^y \sqrt{1+\xi^2}\psi_2(\xi)\tilde{g}(\xi)\,d\xi + \delta^{-\frac{1}{2}}\psi_2(y)\int_0^y \sqrt{1+\xi^2}\psi_1(\xi)\tilde{g}(\xi)\,d\xi. \tag{4.18}$$

In order to estimate the size of ψ, we observe that for $2 \leqslant p < \infty$, $\beta > \frac{5}{2}$ and $0 < y < y_\delta$, it holds that

$$\int_0^y \sqrt{1+\xi^2}\big|\psi_i(\xi)\big|\big|\tilde{g}(\xi)\big|\,d\xi \leqslant C\|\tilde{g}\|_{p,\beta}\left(\int_0^y (1+|\xi|)^{(1+\frac{p'}{2}-\beta p')}\,d\xi\right)^{\frac{1}{p'}}.$$

Directly from this inequality and using (4.11), we find that

$$\left|\psi_i(y)\int_0^y \sqrt{1+\xi^2}\psi_j(\xi)\tilde{g}(\xi)\,d\xi\right| \leqslant C\delta^{-\frac{1}{4}}\|\tilde{g}\|_{p,\beta}, \quad i,j = 1,2, \; i \neq j$$

and since we are taking $\beta > \frac{5}{2}$ and using again (4.11), we get that

$$\delta^{\frac{1}{2}}\sqrt{1+y^2}\big|\psi'(y)\big| + \big|\psi(y)\big| \leqslant C\delta^{-\frac{3}{4}}\|\tilde{g}\|_{p,\beta}, \quad 0 < y \leqslant y_\delta. \tag{4.19}$$

Proceeding as above, we observe that for $y > y_\delta$

$$\int_0^y \sqrt{1+\xi^2}\big|\psi_i(\xi)\big|\big|\tilde{g}(\xi)\big|\,d\xi \leqslant C\|\tilde{g}\|_{p,\beta} + \int_{y_\delta}^y \sqrt{1+\xi^2}\big|\psi_i(\xi)\big|\big|\tilde{g}(\xi)\big|\,d\xi$$

and using (4.12) and since $\beta > \frac{5}{2}$, we find that for some $\varepsilon > 0$ small

$$\int_{y_\delta}^{y} \sqrt{1+\xi^2}\big|\psi_i(\xi)\big|\big|\tilde g(\xi)\big|\,d\xi \leqslant C\big|\log(\delta)\big|\delta^{-\frac14}\|\tilde g\|_{p,\beta}\Bigg(\int_{y_\delta}^{y}\big(1+|\xi|\big)^{(1-\beta p')}\log\big(1+|\xi|\big)^{p'}\Bigg)^{\frac{1}{p'}}$$
$$\leqslant C\delta^{\varepsilon}\|\tilde g\|_{p,\beta}.$$

Hence, using again (4.12), it holds that

$$\delta^{\frac12}\sqrt{1+y^2}\big|\psi'(y)\big| + \log\big(2+|y|\big)^{-1}\big|\psi(y)\big| \leqslant C\delta^{-\frac34}\|\tilde g\|_{p,\beta}, \quad y \geqslant y_\delta. \tag{4.20}$$

Putting together, estimate (4.19) and (4.20) we obtain that

$$\sqrt{\delta}\big\|\big(1+r(y)\big)D\psi\big\|_{L^\infty(M)} + \big\|\log\big(2+r(y)\big)^{-1}\psi\big\|_{L^\infty(M)} \leqslant C\delta^{-\frac34}\|\tilde g\|_{p,\beta}. \tag{4.21}$$

Finally, observe that for $2 \leqslant p < \infty$, $\beta < 3$ and some $\varepsilon > 0$ arbitrarily small, we have that

$$\int_M \big(1+r(y)^\beta\big)^p\big|A_M(y)\big|^{2p}\big|\psi(y)\big|^p\,dV_M \leqslant C\big\|(\log\big(r(y)+2\big)^{-1}\psi\big\|_{L^\infty(M)}\int_M\big(1+r(y)\big)^{(\beta-4-\varepsilon)p}\,dV_M.$$

Since $(\beta-4)p < -2$, we obtain that

$$\big\||A_M|^2\psi\big\|_{p,\beta} \leqslant C\big\|\big(\log\big(r(y)+2\big)\big)^{-1}\psi\big\|_{L^\infty(M)} \leqslant C\delta^{-\frac34}\|\tilde g\|_{p,\beta}.$$

and so, from (4.4)

$$\|\psi\|_{\delta,p,\beta} \leqslant C\delta^{-\frac34}\|\tilde g\|_{p,\beta}$$

where

$$\|\psi\|_{\delta,p,\beta} = \delta\big\|D^2\psi\big\|_{p,\beta} + \delta^{\frac12}\big\|\big(1+r(y)\big)D\psi\big\|_{L^\infty(M)} + \big\|\log\big(2+r(y)\big)^{-1}\psi\big\|_{L^\infty(M)}.$$

The case $p=\infty$ is treated in an analogous fashion.

To finish the proof of Proposition 3.2, we simply notice that linear system (4.1) can be written as the fixed point problem

$$\psi = L_\delta^{-1}[\tilde g] - L_\delta^{-1}\big[-|A_M|^2\big(A(y,\delta) - A(y,0)\psi\big)\big]$$

and as we observed before, it holds that

$$\big\||A_M|^2\big(A(\cdot,\delta) - A(\cdot,0)\big)\big\|_{p,\beta} \leqslant C\delta,$$

then a direct application of the contraction mapping principle, in both of the norms (3.20)–(3.21) for ψ, completes the proof of Proposition 3.2. □

4.2. The Jacobi operator in M

To study the linearization of the system (3.1), we also need to develop solvability theory for the equation

$$\mathcal{J}_M(v) = \Delta_M v + |A_M|^2 v = g, \quad \text{in } M. \tag{4.22}$$

Operator $\mathcal{J}_M$ in Eq. (4.22) corresponds to the linearization around the catenoid M of the mean curvature operator.

It is well known that the catenoid M is L^∞-nondegenerate, in the sense that the functions $z_i = \nu \cdot e_i$, for $i = 1, 2, 3$ are the only bounded solutions to the equation

$$\mathcal{J}_M(v) = \Delta_M v + |A_M|^2 v = 0, \quad \text{in } M,$$

where e_1, e_2, e_3 corresponds to the canonical basis in $\mathbb{R}^3$. One can check directly that $z_3(y)$, which has the explicit expression

$$z_3(y) = \frac{y}{\sqrt{1+y^2}}, \quad y = Y(y,\theta) \in M$$

is the only bounded axially symmetric Jacobi field. Hence, using the reduction of order formula with the ansatz

$$z_4(y) = 1 + s(y) z_3(y), \quad y \neq 0$$

one can also deduce the existence of another axially symmetric element of the kernel of $\mathcal{J}_M$, with logarithmic growth, associated to the dilations of the catenoid M, namely

$$z_4(y) := Y(y,\theta) \cdot \nu(y,\theta) = 1 - \ln\big(y + \sqrt{1+y^2}\,\big) \frac{y}{\sqrt{1+y^2}}, \quad y = Y(y,\theta) \in M.$$

We compute the derivatives of z_3 and z_4, with respect to y, so we get

$$\partial_y z_3'(y) = -\frac{1}{(1+y^2)^{\frac{3}{2}}} = \mathcal{O}\big(|y|^{-3}\big), \tag{4.23}$$

$$\partial_y z_4'(y) = -\ln\big(y + \sqrt{1+y^2}\,\big)\big(1+y^2\big)^{-\frac{3}{2}} - \frac{y}{1+y^2} = \mathcal{O}\big(|y|^{-1}\big). \tag{4.24}$$

Since we are working in the class of axially symmetric functions, we use the variations of parameters formula to define $\mathcal{J}^{-1}(g) := v$, where

$$v(y) := -z_3(y) \int_0^y \sqrt{1+\xi^2}\, g(\xi) z_4(\xi)\, d\xi + z_4(y) \int_{-\infty}^y \sqrt{1+\xi^2}\, g(\xi) z_3(\xi)\, d\xi \tag{4.25}$$

for any function g satisfying that

$$\|g\|_{p,\beta} := \big\|\big(1 + r(y)^\beta\big) g\big\|_{L^p(M)} < \infty.$$

Formula (4.25) defines a function v that solves equation (4.22). We remark that, under the orthogonality condition

$$\int_{-\infty}^{\infty} \sqrt{1+\xi^2}\, g(\xi) z_3(\xi)\, d\xi = 0 \tag{4.26}$$

this solution is unique in the class of bounded functions with $v'(0) = 0$ and the following lemma gives us an estimate on the size of $\mathcal{J}^{-1}$.

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Lemma 4.2. *Let g be an axially symmetric function satisfying condition (4.26), and such that $\|g\|_{p,\beta} < \infty$, for $1 < p \leqslant \infty$ and $2 < \beta < 4 - \frac{2}{p}$. Then, the function v, given by formula (4.25), defines an axially symmetric solution to*

$$\Delta_M v + |A_M|^2 v = g, \quad \text{in } M,$$

such that $v'(0) = 0$ and the following estimate holds true

$$\|v\|_{2,p,\beta} \leqslant C\|g\|_{p,\beta}, \tag{4.27}$$

where

$$\|v\|_{2,p,\beta} := \|v\|_{L^\infty(M)} + \left\|r^{\beta-1}(y)\nabla v\right\|_{L^\infty(M)} + \left\|D^2 v\right\|_{p,\beta}.$$

The proof of this lemma follows calculations similar to those in the proof of Proposition 3.2, so we leave details to the reader.

Remark 4.1. To prove Lemma 4.2, we simply notice that an even axially symmetric function g in $L^1(M)$, automatically satisfies the orthogonality condition (4.26). In such a case, formula (4.25) defines an even function.

5. Approximation of the solution in Theorem 2

To define our approximate solution to problem (1.1), let us first observe that the heteroclinic solution to

$$w''(s) + w\left(1 - w^2\right) = 0, \quad s \in \mathbb{R}, \qquad w(\pm\infty) = \pm 1$$

is given explicitly by

$$w(s) = \tanh\left(\frac{s}{\sqrt{2}}\right), \quad s \in \mathbb{R}$$

and has the asymptotic properties

$$\begin{aligned} w(s) &= 1 - 2e^{-\sqrt{2}s} + \mathcal{O}\left(e^{-2\sqrt{2}|s|}\right), \quad s > 1, \\ w(s) &= -1 + 2e^{\sqrt{2}s} + \mathcal{O}\left(e^{-2\sqrt{2}|s|}\right), \quad s < -1, \\ w'(s) &= 2\sqrt{2}e^{-\sqrt{2}|s|} + \mathcal{O}\left(e^{-2\sqrt{2}|s|}\right), \quad |s| > 1, \end{aligned} \tag{5.1}$$

where $w' = \frac{dw}{ds}$.

5.1. *The first local approximation*

Let us consider the vector function $\mathrm{h} = (h_1, \ldots, h_m)$ given in Proposition 3.1 and solving the Jacobi–Toda system. Recall that every h_l has the form

$$h_l(y) = \left(l - \frac{m+1}{2}\right)\left[\sigma + \sqrt{2}\left(1 - \frac{1}{\sigma}\right)\log\left(1 + y^2\right)\right] + \tilde{h}_l(y), \quad y \in \mathbb{R}, \tag{5.2}$$

where

$$\left|\tilde{h}_l(y)\right| \leqslant K\sigma^{-\frac{5}{4}} \log\big(2+r(y)\big), \quad y \in M,$$
$$\left\|\big(1+r(y)\big)^j D^{(j)}\tilde{h}_l\right\|_{L^\infty(M)} \leqslant K\sigma^{-\frac{5}{4}+\frac{j}{2}}, \quad l=1,\dots,m,\ j\in\mathbb{N},$$

where σ is the unique positive real number that solves the algebraic equation

$$\alpha^2\sigma = a_0 e^{-\sqrt{2}\sigma}. \tag{5.3}$$

Let us also consider a parameter vector function $\mathrm{v} = (\mathrm{v}_1,\dots,\mathrm{v}_m)$ satisfying the a priori estimate that

$$\sigma^{-\frac{1}{4}}\left\|\big(1+r(y)\big)D\mathrm{v}_l\right\|_{L^\infty(M)} + \left\|\big(\log\big(2+r(y)\big)\big)^{-1}\mathrm{v}_l\right\|_{L^\infty(M)} \leqslant K_1\alpha^{\tau_0}\sigma^{\frac{3}{4}} \tag{5.4}$$

for some $\tau_0 > 0$ small and $K_1 > 0$ a universal constant to be chosen large but independent of $\alpha > 0$.

Let us consider m normal graphs over M of the axially symmetric functions $f_l = h_l + \mathrm{v}_l \in C^2(M)$, $l = 1,\dots,m$. With a slight abuse of notation we write

$$f_l\big(Y(y,\theta)\big) = f_l(y), \quad (y,\theta)\in\mathbb{R}\times(0,2\pi),\ l=1,\dots,m.$$

From (5.2)–(5.4), we observe that

$$f_{l+1}(y) - f_l(y) \geqslant \sigma + \sqrt{2}\left(1 - \frac{1}{\sigma} - M\sigma^{-\frac{5}{4}}\right)\log\big(1+y^2\big), \quad y\in\mathbb{R}, \tag{5.5}$$

for some positive universal constant $M > 0$ and for every fixed $j = 1,\dots,m-1$.

In the region $\mathcal{N}_\alpha$ we consider as a local approximation the function

$$U_0(x) = \sum_{j=1}^m w_j\big(z - f_j(\alpha y)\big) + \frac{(-1)^{m-1}-1}{2}, \quad w_j(s) = (-1)^{j-1}w(s), \tag{5.6}$$

where $x = X_\alpha(y,\theta,z) \in \mathcal{N}_\alpha$. Observe that for points $x \in \mathcal{N}_\alpha$, for which z is close enough to $h_j(\alpha y)$, we have that

$$U_0(x) \approx w_j\big(z - f_j(\alpha y)\big).$$

For $l = 1,\dots,m$ fixed, we consider the set

$$A_l = \left\{X_\alpha(y,\theta,z)\colon\ \left|z - f_l(\alpha y)\right| \leqslant \frac{1}{2}\left[\sigma + \sqrt{2}\left(1 - \frac{1}{\sigma} - M\sigma^{-\frac{5}{4}}\right)\log\big(1+(\alpha y)^2\big)\right]\right\}.$$

From (5.2) it is direct to check that $A_l \subset \mathcal{N}_\alpha$, for every $\alpha > 0$ small enough. Setting $t = z - f_l(\alpha y)$, the set A_l can also be describe in terms of the local coordinates $X_{\alpha,f_l}(y,\theta,t)$ as

$$A_l = \left\{X_{\alpha,f_l}(y,\theta,t)\colon\ |t| \leqslant \frac{1}{2}\left[\sigma + \sqrt{2}\left(1 - \frac{1}{\sigma} - M\sigma^{-\frac{5}{4}}\right)\log\big(1+(\alpha y)^2\big)\right]\right\}.$$

Next, with the aid of Lemma 2.1, we compute the error

$$S(U_0) = \Delta U_0 + F(U_0), \quad \text{in } A_l,\ l=1,\dots,m$$

of the approximation U_0 defined in (5.6) and where $F(u) = u(1-u^2)$.

We proceed as in Lemma 2.4 in [13], collecting all the computations in the following lemma:

Lemma 5.1. *For $l = 1, \dots, m$ and $x = X_{\alpha,f_l}(y,\theta,t) \in A_l$, it holds that*

$$\begin{aligned}(-1)^{l-1}S(U_0) &= -\alpha^2\big(\Delta_M f_l + |A_M|^2 f_l\big)w'(t)\\ &\quad + 6\big(1-w^2(t)\big)\big[e^{-\sqrt{2}t}e^{-\sqrt{2}(f_l-f_{l-1})} - e^{\sqrt{2}t}e^{-\sqrt{2}(f_{l+1}-f_l)}\big]\\ &\quad - \alpha^2|A_M|^2 t w'(t) + \alpha^2\big[f_l'\big]^2 w''(t) - \alpha^3(t+f_l)a_1\big(\alpha y, \alpha(t+f_l)\big)f_l'' w'(t)\\ &\quad - \alpha^2 \sum_{|j-l|\geqslant 1}\big(\Delta_M f_j - \alpha(t+f_l)a_1\big(\alpha y,\alpha(t+f_l)\big)f_j''\big)w_j'(t+f_l-f_j)\\ &\quad + \mathrm{R}_l(\alpha y, t, \mathrm{v}_1, \dots, \mathrm{v}_m, D\mathrm{v}_1, \dots, D\mathrm{v}_m),\end{aligned} \tag{5.7}$$

where $\mathrm{R}_l = \mathrm{R}_l(\alpha y, t, p, q)$ is smooth on its arguments and

$$\big|D_p\mathrm{R}_l(\alpha y,t,p,q)\big| + \big|D_q\mathrm{R}_l(\alpha y,t,p,q)\big| + \big|\mathrm{R}_l(\alpha y,t,p,q)\big| \leqslant C\alpha^{2+\tau}\big(1+|\alpha y|\big)^{-4}e^{-\varrho|t|} \tag{5.8}$$

for $0<\varrho<\sqrt{2}$, some $0<\tau<1$ and where

$$p = (\mathrm{v}_1, \dots, \mathrm{v}_m), \qquad q = (D\mathrm{v}_1, \dots, D\mathrm{v}_m).$$

Proof. Denote

$$E_1 = F\big((-1)^{l-1}U_0\big), \qquad E_2 = \Delta_{X_{\alpha,h_l}}\big[(-1)^{l-1}U_0(x)\big].$$

We first compute E_1. We begin noticing that

$$F(U_0) = \sum_{j=1}^m F\big(w_j(t+f_l-f_j)\big) + \Bigg[F\big(U_0(x)\big) - \sum_{j=1}^m F\big(w_j(t+f_l-f_j)\big)\Bigg].$$

Since $F(u) = u(1-u^2)$, for $u \in \mathbb{R}$, we find that

$$0 \leqslant F(u) \leqslant |1-u||1+u|, \quad \forall u \in [-1,1]. \tag{5.9}$$

On the other hand, for $|j-l| \geqslant 1$, we have that

$$|f_l - f_j| = |l-j|\left[\sigma + \sqrt{2}\left(1 - \frac{1}{\sigma} + \mathcal{O}\big(\sigma^{-\frac{5}{4}}\big)\right)\log\big(1+(\alpha y)^2\big)\right]$$

and recall that

$$\alpha^2\sigma = a_0 e^{-\sqrt{2}\sigma}.$$

Hence, we obtain for $|j-l| \geqslant 1$ and for $\varepsilon \in [0,1)$ that

$$\begin{aligned}|t+f_l-f_j| &\geqslant |l-j|\left[\sigma + \sqrt{2}\left(1-\frac{1}{\sigma}+\mathcal{O}\big(\sigma^{-\frac{5}{4}}\big)\right)\log\big(1+(\alpha y)^2\big)\right] - |t|\\ &\geqslant \left(|j-l| - \frac{1+\varepsilon}{2}\right)\left[\sigma + \sqrt{2}\left(1-\frac{1}{\sigma}\right)\log\big(1+(\alpha y)^2\big)\right] + \varepsilon|t|.\end{aligned}$$

Assume for the moment that $2 \leqslant l \leqslant m-1$. For $x = X_{\alpha,f_l}(y,\theta,t) \in A_l$ and $1 \leqslant j < l$, it holds that

$$t + f_l(\alpha y) - f_j(\alpha y) \geqslant \frac{1}{2}\left[\sigma + \sqrt{2}\left(1 - \frac{1}{\sigma} - M\sigma^{-\frac{5}{4}}\right)\log\left(1 + (\alpha y)^2\right)\right]$$

while for $l < j \leqslant m$, it holds that

$$t + f_l(\alpha y) - f_j(\alpha y) \leqslant -\frac{1}{2}\left[\sigma + \sqrt{2}\left(1 - \frac{1}{\sigma} - M\sigma^{-\frac{5}{4}}\right)\log\left(1 + (\alpha y)^2\right)\right].$$

Using the asymptotic behavior of $w(s)$ from (5.1), we find that

$$w(t + f_l - f_j) = 1 - 2e^{-\sqrt{2}t}e^{-\sqrt{2}(f_l - f_j)} + \mathcal{O}\left(e^{-2\sqrt{2}|t + f_l - f_j|}\right), \quad 1 \leqslant j < l,$$
$$w(t + f_l - f_j) = -1 + 2e^{\sqrt{2}t}e^{\sqrt{2}(f_l - f_j)} + \mathcal{O}\left(e^{-2\sqrt{2}|t + f_l - f_j|}\right), \quad l < j \leqslant m.$$

From (5.9) and the remarks made above, we conclude that

$$\left|\sum_{|j-l|\geqslant 2} F\left(w_j(t + f_l - f_j)\right)\right| \leqslant C \max_{|j-l|\geqslant 2} e^{-\sqrt{2}|t + f_l - f_j|} \leqslant C\alpha^{2+\tau}\left(1 + |\alpha y|\right)^{-4}e^{-\varrho|t|}$$

for some $0 < \varrho < \sqrt{2}$ independent of $\alpha > 0$ and $0 < \tau < 1$ depending only on $\varrho > 0$.

From the previous estimate we also observe that

$$\begin{aligned}
&(-1)^{l-1}\left[F\left(U_0(x)\right) - \sum_{j=1}^{m} F\left(w_j(t + f_l - f_j)\right)\right] \\
&\quad = (-1)^{l-1}F\left(U_0(x)\right) + F\left(w(t + f_l - f_{l-1})\right) - F\left(w(t)\right) + F\left(w(t + f_l - f_{l+1})\right) \\
&\qquad + \bar{R}_l(\alpha y, t, \mathrm{v}_1, \ldots, \mathrm{v}_m),
\end{aligned} \tag{5.10}$$

where

$$\left|D_p\bar{R}_l(\alpha y, t, p)\right| + \left|\bar{R}_l(\alpha y, t, p)\right| \leqslant C\alpha^{2+\tau}\left(1 + |\alpha y|\right)^{-4}e^{-\varrho|t|}.$$

for $p = (\mathrm{v}_1, \ldots, \mathrm{v}_m)$.

Let us now denote

$$a_1 = w(t + f_l - f_{l-1}) - 1, \qquad a_2 = w(t + f_l - f_{l+1}) + 1.$$

From the mean value theorem, we can choose numbers $s_i \in (0, 1)$, for $i = 1, 2, 3$, such that

$$F\left(w(t + f_l - f_{l-1})\right) = F(1) + F'(1)a_1 + \frac{1}{2}F''(1 + s_1 a_1)a_1^2,$$
$$F\left(w(t + f_l - f_{l+1})\right) = F(-1) + F'(-1)a_2 + \frac{1}{2}F''(-1 + s_2 a_2)a_2^2,$$

$$\begin{aligned}
(-1)^{l-1}F\left(U_0(x)\right) &= F(w) - F'(w)(a_1 + a_2) + F'(w)\sum_{|j-l|\geqslant 2}(-1)^{j-l}\left[w(t + f_l - f_j) - sign(l - j)\right] \\
&\quad + \frac{1}{2}F''\left[w + s_3\left((-1)^{l-1}U_0 - w\right)\right]\left(\sum_{|j-l|\geqslant 1}(-1)^{j+l}w(t + f_l - f_j) - sign(l - j)\right)^2.
\end{aligned}$$

Hence, using that $F'(1) = F'(-1)$, we obtain that

$$\begin{aligned}(-1)^{l-1}F(U_0) = {} & \sum_{j=1}^{m}(-1)^{l-1}F\big(w_j(t+f_l-f_j)\big) \\ & + 6\big(1-w^2(t)\big)\big[e^{-\sqrt{2}t}e^{-\sqrt{2}(f_l-f_{l-1})} - e^{\sqrt{2}t}e^{-\sqrt{2}(f_{l+1}-f_l)}\big] \\ & + R_l(\alpha y, t, \mathrm{v}_1, \dots, \mathrm{v}_m), \end{aligned} \tag{5.11}$$

where for $p = (\mathrm{v}_1, \dots, \mathrm{v}_m)$

$$\big|D_p R_l(\alpha y, p)\big| + \big|R_l(\alpha y, t, p)\big| \leqslant C\alpha^{2+\tau}\big(1+|\alpha y|\big)^{-4}e^{-\varrho|t|}. \tag{5.12}$$

The remaining cases, namely $l = 1$ and $l = m$, are treated in an similar fashion, replacing the term

$$e^{-\sqrt{2}t}e^{-\sqrt{2}(f_l-f_{l-1})} - e^{\sqrt{2}t}e^{-\sqrt{2}(f_{l+1}-f_l)}$$

by the respective terms

$$-e^{\sqrt{2}t}e^{-\sqrt{2}(f_2-f_1)}, \qquad e^{-\sqrt{2}t}e^{-\sqrt{2}(f_m-f_{m-1})}.$$

So far, we have only written the term E_1 in a convenient way. We still have to compute E_2. In order to do so, we write

$$E_2 = \Delta_{X_{\alpha,f_l}} w(t) + \sum_{|j-l|\geqslant 1} \Delta_{X_{\alpha,f_l}}\big[(-1)^{l-1}w_j(t+f_l-f_j)\big] = E_{21} + E_{22}.$$

Directly from Lemma 2.1, we obtain that

$$\begin{aligned} E_{21} = {} & w''(t) - \alpha^2\big(\Delta_M f_l + |A_M|^2 f_l\big)w'(t) - \alpha^2|A_M|^2 t w'(t) + \alpha^2\big[f_l'\big]^2 w''(t) \\ & - \alpha^3(t+f_l)a_1\big(\alpha y, \alpha(t+f_l)\big)\big\{f_l'' w'(t) - \big[f_l'\big]^2 w''(t)\big\} \\ & - \alpha^3(t+f_l)b_1\big(\alpha y, \alpha(t+f_l)\big)h_l' w'(t) - \alpha^4(t+f_l)^3 b_2\big(\alpha y, \alpha(t+f_l)\big)w'(t). \end{aligned}$$

Using assumptions (5.2)–(5.4), we can write E_{21} as follows:

$$\begin{aligned} E_{21} = {} & w''(t) - \alpha^2\big(\Delta_M f_l + |A_M|^2 f_l\big)w'(t) - \alpha^2|A_M|^2 t w'(t) + \alpha^2\big[f_l'\big]^2 w''(t) \\ & - \alpha^3(t+f_l)a_1\big(\alpha y, \alpha(t+f_l)\big)h_l'' w'(t) + \mathrm{Q}_{21}(\alpha y, t, \mathrm{v}_l, D\mathrm{v}_l), \end{aligned} \tag{5.13}$$

where

$$\mathrm{Q}_{21} = \mathrm{Q}_{21}(\alpha y, t, p, q)$$

and

$$\big|D_p \mathrm{Q}_{21}(\alpha y, t, p, q)\big| + \big|D_q \mathrm{Q}_{21}(\alpha y, t, p, q)\big| + \big|\mathrm{Q}_{21}(\alpha y, t, p, q)\big| \leqslant C\alpha^3\big(1+|\alpha y|\big)^{-4}e^{-\varrho|t|} \tag{5.14}$$

for some $0 < \varrho < \sqrt{2}$.

Next, we compute E_{22}. A direct computation yields that

$$
\begin{aligned}
(-1)^{l-1}E_{22} = \sum_{|j-l|\geqslant 1} & w_j''(t+f_l-f_j) \\
& - \alpha^2 \sum_{|j-l|\geqslant 1} \left(\left[\Delta_M f_j + |A_M|^2(f_l+t)\right] w_j'(t+f_l-f_j) - \left[f_j'\right]^2 w_j''(t+f_l-h_j)\right) \\
& - \alpha^3(t+f_l)a_1\left(\alpha y, \alpha(t+f_l)\right) \sum_{|j-l|\geqslant 1} \left(f_j'' w_j'(t+f_l-f_j) - \left[f_j'\right]^2 w_j''(t+f_l-f_j)\right) \\
& - \alpha^3(t+f_l)b_1\left(\alpha y, \alpha(t+f_l)\right) \sum_{|j-l|\geqslant 1} \left(f_j' w_j' - \alpha(t+f_l)^3 b_2\left(\alpha y, \alpha(t+f_l)\right) w_j'(t+f_l-f_j)\right).
\end{aligned}
$$

Using the fact that for $\varepsilon \in (0,1)$ and $|j-l| \geqslant 1$

$$
|t+f_l-f_j| \geqslant \left(1 - \frac{1+\varepsilon}{2}\right)\left[\sigma + \sqrt{2}\left(1-\frac{1}{\sigma}\right)\log\left(1+(\alpha y)^2\right)\right] + \varepsilon|t|
$$

and proceeding as above, we can write E_{22} as follows

$$
\begin{aligned}
(-1)^{l-1}E_{22} = {} & w_j''(t+f_l-f_j) \\
& - \alpha^2 \sum_{|j-l|\geqslant 1} \left(\Delta_M f_j - \alpha(t+f_l)a_1\left(\alpha y, \alpha(t+f_l)\right) f_j''\right) w_j'(t+f_l-f_j) \\
& + \mathrm{Q}_{22}(\alpha y, t, \mathrm{v}_1, \ldots, \mathrm{v}_m, D\mathrm{v}_1, \ldots, D\mathrm{v}_m),
\end{aligned} \tag{5.15}
$$

where

$$
\mathrm{Q}_{22} = \mathrm{Q}_{22}(\alpha y, t, p, q)
$$

and

$$
\left|D_p Q_{22}(\alpha y, t, v, q)\right| + \left|D_q Q_{22}(\alpha y, t, p, q)\right| + |Q_{22}| \leqslant C\alpha^{2+\tau}\left(1+|\alpha y|\right)^{-4} e^{-\varrho|t|} \tag{5.16}
$$

for some $0 < \varrho < \sqrt{2}$ and some $0 < \tau < 1$.

Setting $\mathrm{R}_l = R_l + \mathrm{Q}_{21} + \mathrm{Q}_{22}$, we have that $\mathrm{R}_l = \mathrm{R}_l(\alpha y, t, p, q)$ is smooth on its arguments and

$$
\left|D_p \mathrm{R}_l(\alpha y, t, p, q)\right| + \left|D_q \mathrm{R}_l(\alpha y, t, p, q)\right| + \left|\mathrm{R}_l(\alpha y, t, p, q)\right| \leqslant C\alpha^{2+\tau}\left(1+|\alpha y|\right)^{-4} e^{-\varrho|t|}
$$

for $0 < \varrho < \sqrt{2}$ and $0 < \tau < 1$. Putting together (5.11)–(5.15) and using that $w_j'' + F(w_j) = 0$, we obtain expressions (5.7) and (5.8) and the proof of the lemma is complete. □

5.2. Improvement of the local approximation

For subsequents developments, it will be useful to have more precise information about the asymptotics of the solution we are looking for, so we improve our first approximation U_0. In order to do so, we write

$$
6\left(1-w^2(t)\right)e^{-\sqrt{2}t} = a_0 w'(t) + g_0(t), \qquad \int_{\mathbb{R}} g_0(t) w'(t)\, dt = 0. \tag{5.17}
$$

Using (5.17), the fact that the vector function h is an exact solution of the Jacobi–Toda system in M and Lemma 5.1, we observe that

$$
\begin{aligned}
(-1)^{l-1}S(U_0) :=\ & -\alpha^2\big(\Delta_M \mathrm{v}_l + |A_M|^2\mathrm{v}_l\big)w'(t) \\
& + g_0(-t)e^{-\sqrt{2}(h_l-h_{l-1})} - g_0(t)e^{-\sqrt{2}(h_{l+1}-h_l)} + \alpha^2\big[h_l'\big]^2 w''(t) - \alpha^2|A_M|^2 t w'(t) \\
& + 6\big(1-w^2(t)\big)e^{-\sqrt{2}t}e^{-\sqrt{2}(h_l-h_{l-1})}\big[e^{-\sqrt{2}(\mathrm{v}_l-\mathrm{v}_{l-1})}-1\big] \\
& - 6\big(1-w^2(t)\big)e^{\sqrt{2}t}e^{-\sqrt{2}(h_{l+1}-h_l)}\big[e^{-\sqrt{2}(\mathrm{v}_{l+1}-\mathrm{v}_l)}-1\big] \\
& + \alpha^2\mathrm{v}_l'\big(2h_l'+\mathrm{v}_l'\big)w''(t) - \alpha^3(t+f_l)a_1\big(\alpha y,\alpha(t+f_l)\big)f_l''w'(t) \\
& - \alpha^2\sum_{|j-l|\geqslant 1}\big(\Delta_M f_j - \alpha(t+f_l)a_1\big(\alpha y,\alpha(t+f_l)\big)f_j''\big)w_j'(t+f_l-f_j) \\
& + \mathrm{R}_l(\alpha y, t, \mathrm{v}_1,\dots,\mathrm{v}_m, D\mathrm{v}_1,\dots,D\mathrm{v}_m),
\end{aligned}
\tag{5.18}
$$

where $\mathrm{R}_l = \mathrm{R}_l(\alpha y, t, p, q)$ is smooth on its arguments and satisfies (5.8) for $0<\varrho<\sqrt{2}$ and some $0<\tau<1$.

Let us consider $\psi_0(t)$ to be the bounded solution to the equation

$$\partial_{tt}\psi_0(t) + F'\big(w(t)\big)\psi_0(t) = g_0(t), \quad t\in\mathbb{R}$$

given explicitly by the variations of parameters formula

$$\psi_0(t) = w'(t)\int_0^t w'(s)^{-2}\int_s^\infty w'(\xi)g_0(\xi)\,d\xi\,ds. \tag{5.19}$$

From (5.19), we obtain the estimate

$$\big\|\big(1+e^{2\sqrt{2}t}\chi_{\{t>0\}}\big)\partial_t^{(j)}\psi_0\big\|_{L^\infty(\mathbb{R})} \leqslant C_j, \quad j\in\mathbb{N}.$$

Let us also consider functions $\psi_1(t)$ and $\psi_2(t)$ so that

$$\partial_{tt}\psi_1(t) + F'\big(w(t)\big)\psi_1(t) = -w''(t), \quad t\in\mathbb{R}, \tag{5.20}$$

$$\partial_{tt}\psi_2(t) + F'\big(w(t)\big)\psi_2(t) = tw'(t), \quad t\in\mathbb{R}. \tag{5.21}$$

Proceeding as before, we see that

$$\psi_2(t) = -w(t)\int_0^t w'(s)^{-2}\int_s^\infty \xi w'(\xi)^2\,d\xi\,ds$$

and $\psi_1(t) = -\frac{1}{2}tw'(t)$, from where the following estimate follows at once

$$\big\|e^{\varrho|t|}\partial_t^{(j)}\psi_i\big\|_{L^\infty(\mathbb{R})} \leqslant C_j, \quad i=1,2,\ j\in\mathbb{N},\ 0<\varrho<\sqrt{2}.$$

So, we consider as a second approximation in the region $\mathcal{N}_\alpha$, the function

$$U_1(x) = U_0 + \sum_{j=1}^m \phi_{j,0} \tag{5.22}$$

where for every $l=1,\dots,m$ and in the coordinates X_{α,f_l}

$$(-1)^{l-1}\phi_{l,0}(y,t) = -e^{-\sqrt{2}(h_l-h_{l-1})}\psi_0(-t) + e^{-\sqrt{2}(h_{l+1}-h_l)}\psi_0(t) + \alpha^2\big[h_l'(\alpha y)\big]^2\psi_1(t) + \alpha^2\big|A_M(\alpha y)\big|^2\psi_2(t).$$

The new error created reads as

$$S(U_1) := S(U_0) + \sum_{j=1}^m \partial_{tt}\phi_{j,0} + F'\big(w_j(t)\big)\phi_{j,0} + \sum_{j=1}^m \Delta_{M_\alpha}\phi_{j,0} + B_j(\phi_{j,0}) + \big[F'(U_1) - F'\big(w_j(t)\big)\big]\phi_{j,0}.$$

Directly from (5.18) in each one of the sets A_l, the error reads at main order as follows:

$$\begin{aligned}(-1)^{l-1}S(U_1) = {} & -\alpha^2\big(\Delta_M \mathrm{v}_l + |A_M|^2\mathrm{v}_l\big)w'(t) \\ & + 6\big(1-w^2(t)\big)e^{-\sqrt{2}t}e^{-\sqrt{2}(h_l-h_{l-1})}\big[e^{-\sqrt{2}(\mathrm{v}_l-\mathrm{v}_{l-1})} - 1\big] \\ & - 6\big(1-w^2(t)\big)e^{\sqrt{2}t}e^{-\sqrt{2}(h_{l+1}-h_l)}\big[e^{-\sqrt{2}(\mathrm{v}_{l+1}-\mathrm{v}_l)} - 1\big] \\ & + \alpha^2\mathrm{v}_l'\big(2h_l' + \mathrm{v}_l'\big)w''(t) - \alpha^3(t+f_l)a_1\big(\alpha y, \alpha(t+f_l)\big)f_l''w'(t) \\ & - \alpha^2\big(\Delta_M f_j - \alpha(t+f_l)a_1\big(\alpha y, \alpha(t+f_l)\big)f_j''\big)w_j'(t+f_l-f_j) + \tilde{\mathrm{R}}_l,\end{aligned} \tag{5.23}$$

where

$$\tilde{\mathrm{R}}_l = \tilde{\mathrm{R}}_l(\alpha y, t, \mathrm{v}_1, \ldots, \mathrm{v}_m, D\mathrm{v}_1, \ldots, D\mathrm{v}_m)$$

and

$$\big|D_p\tilde{\mathrm{R}}(\alpha y,t,p,q)\big| + \big|D_q\tilde{\mathrm{R}}(\alpha y,t,p,q)\big| + \big|\tilde{\mathrm{R}}(y,t,p,q)\big| \leqslant C\alpha^{2+\tau}r_\alpha(y)^{-4}e^{-\varrho|t|} \tag{5.24}$$

for some $0<\varrho<\sqrt{2}$ and some $0<\tau<1$.

5.3. Global approximation

The approximation U_1 is so far defined only on the neighborhood $\mathcal{N}_\alpha$ of M_α. To define our global approximation, we use the non-negative function $\beta \in C^\infty(\mathbb{R})$ from the previous sections to define the cut-off function

$$\beta_\alpha(x) = \beta\left(|z| - \frac{\eta}{\alpha} - 2\sqrt{2}(m+1)\log\big(r(\alpha y)\big) + 3\right), \quad x = X_\alpha(y,\theta,z) \in \mathcal{N}_\alpha$$

for which we observe that is supported in a region that expands logarithmically in $r_\alpha(y)$. With the aid of this function, we set up as approximation in $\mathbb{R}^3$, the function

$$\mathrm{w}(x) = \beta_\alpha(x)U_1 + \big(1-\beta_\alpha(x)\big)\mathbb{H}, \quad x \in \mathbb{R}^3 \tag{5.25}$$

where $\mathbb{H}$ is the function

$$\mathbb{H}(x) = \begin{cases} 1, & x \in S_\alpha^+ \\ (-1)^m, & x \in S_\alpha^- \end{cases}$$

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and $S^{\pm}_{\alpha} = \alpha^{-1}S^{\pm}$, $S^{\pm}$ being the two connected components of $\mathbb{R}^3 - M$ for which S^+ is the component containing the x_3-axis.

We compute the new error as follows

$$S(\mathrm{w}) = \Delta\mathrm{w} + F(\mathrm{w}) = \beta_\alpha(x)S(U_1) + E$$

where

$$E = 2\nabla\beta_\alpha\nabla U_1 + \Delta\beta_\alpha(U_1 - \mathbb{H}) + F\big(\beta_\alpha U_1 + (1-\beta_\alpha)\mathbb{H}\big) - \beta_\alpha F(U_1).$$

Due to the choice of $\beta_\alpha(x)$ and the explicit form of the error the term E, the error created only takes into account values of β_α for $x \in \mathbb{R}^3$ in the region

$$x = X_\alpha(y,\theta,z), \qquad |z| \geqslant \frac{\eta}{\alpha} + 4\ln\big(r_\alpha(y)\big) - 2,$$

and so, we get the following estimate for the term E

$$|D_y E| + |E| \leqslant Ce^{-\frac{\eta}{\alpha}} r_\alpha^{-4}(y).$$

We observe that the error E decays rapidly and is exponentially small in $\alpha > 0$, so that its contribution is basically negligible.

6. Proof of Theorem 2

Since the proof of Theorem 2 is fairly technical, first we sketch the steps of the proof and then leave the detailed proofs of the propositions and lemmas mentioned here to subsequent sections.

First, we introduce the norms we will use to set up an appropriate functional analytic scheme for the proof of Theorem 1. Let us recall the notation

$$r(x) = \sqrt{x_1^2 + x_2^2}, \quad x = (x_1,x_2,x_3) \in \mathbb{R}^3$$

and let us define for $\alpha > 0$, $\mu > 0$ and $f(x)$, defined in $\mathbb{R}^3$, the norm

$$\|f\|_{p,\mu,\sim} := \sup_{x\in\mathbb{R}^3} \big(1 + r(\alpha x)\big)^{\mu} \|f\|_{L^p(B_1(x))}, \quad p > 1. \tag{6.1}$$

We also consider $0 < \varrho < \sqrt{2}$, $\mu > 0$, $\alpha > 0$ and functions $g = g(y,t)$ and $\phi = \phi(y,t)$, defined for every $(y,t) \in M_\alpha \times \mathbb{R}$. Let us define the norms

$$\|g\|_{p,\mu,\varrho} := \sup_{(y,t)\in M_\alpha\times\mathbb{R}} \big(1 + r(\alpha y)\big)^{\mu} e^{\varrho|t|} \|g\|_{L^p(B_1(y,t))} \tag{6.2}$$

$$\|\phi\|_{\infty,\mu,\varrho} := \big\|\big(1 + r(\alpha y)^{\mu}\big) e^{\varrho|t|}\phi\big\|_{L^\infty(M_\alpha\times\mathbb{R})} \tag{6.3}$$

$$\|\phi\|_{2,p,\mu,\varrho} := \big\|D^2\phi\big\|_{p,\mu,\varrho} + \|D\phi\|_{\infty,\mu,\varrho} + \|\phi\|_{\infty,\mu,\varrho}. \tag{6.4}$$

Finally, for functions v and $\tilde{g}$ defined in M, we recall the norms

$$\|\tilde{g}\|_{p,\beta} := \big\|\big(1 + r(y)^{\beta}\big)\tilde{g}\big\|_{L^p(M)} \tag{6.5}$$

$$\|\mathrm{v}\|_{\delta,p,\beta} := \delta\big\|D^2\mathrm{v}\big\|_{p,\beta} + \delta^{\frac{1}{2}}\big\|\big(1 + r(y)\big)D\mathrm{v}\big\|_{L^\infty(M)} + \big\|\log\big(r(y)+2\big)^{-1}\mathrm{v}\big\|_{L^\infty(M)}. \tag{6.6}$$

Now, in order to prove Theorem 1, let us look for a solution to Eq. (1.1) of the form

$$U(x) = \mathrm{w}(x) + \varphi(x),$$

where $\mathrm{w}(x)$ is the global approximation defined in (5.25) and φ is going to be chosen small. Hence, since $F(u) = u(1-u^2)$, for $U(x)$ being a genuine solution to (1.1), we see that φ must solve the equation

$$\Delta\varphi + F'(\mathrm{w})\varphi + S(\mathrm{w}) + N(\varphi) = 0, \quad \text{in } \mathbb{R}^3$$

or equivalently

$$\Delta\varphi + F'(\mathrm{w})\varphi = -S(\mathrm{w}) - N(\varphi) = -\beta_\alpha S(U_1) - E - N(\varphi), \tag{6.7}$$

where

$$N(\varphi) = F(\mathrm{w} + \varphi) - F(\mathrm{w}) - F'(\mathrm{w})\varphi.$$

6.1. Gluing procedure

In order to solve equation (6.7), we consider a non-negative function $\beta \in C^\infty(\mathbb{R})$ such that

$$\beta(s) = \begin{cases} 1, & |s| \leqslant 1 \\ 0, & |s| \geqslant 2 \end{cases}$$

and define for $l = 1, \dots, m$ and $n \in \mathbb{N}$, the cut off function for $x = X_{\alpha,f_l}(y,\theta,t) \in \mathcal{N}_{\alpha,f_l}$

$$\zeta_{l,n}(x) = \beta\left(|t| - \frac{1}{2}\left[\sigma + \sqrt{2}\left(1 - \frac{1}{\sigma} - M\sigma^{-\frac{5}{4}}\right)\log\left(1 + (\alpha y)^2\right)\right] + n\right). \tag{6.8}$$

Observe that for $k \neq l$ and $n \in \mathbb{N}$, $\zeta_{l,n} \cdot \zeta_{k,n} = 0$. Observe that for $k \neq l$, $\zeta_{l,n} \cdot \zeta_{k,n} = 0$.
Now we look for a solution $\varphi(x)$ in the particular form

$$\varphi(x) = \sum_{j=1}^m \zeta_{j,3}(x)\varphi_j(y,z) + \psi(x),$$

where the functions $\varphi_j(y,z)$ are defined in $M_\alpha \times \mathbb{R}$ and the function $\psi(x)$ is defined in the whole $\mathbb{R}^3$. So, from Eq. (6.7) and noticing that $\zeta_{j,2} \cdot \zeta_{j,3} = \zeta_{j,3}$, we find that

$$\begin{aligned}
&\sum_{j=1}^m \zeta_{j,3}\left[\Delta_{\mathcal{N}_\alpha}\varphi_j + F'(\zeta_{j,2}\mathrm{w})\varphi_j + \zeta_{j,2}S(\mathrm{w}) + \zeta_{j,2}N(\varphi_j + \psi) + \zeta_{j,2}\big(F'(\mathrm{w}) + 2\big)\psi\right] \\
&\quad + \Delta\psi - \left[2 - \left(1 - \sum_{j=1}^m \zeta_{j,3}\right)\big(F'(\mathrm{w}) + 2\big)\right]\psi + \left(1 - \sum_{j=1}^m \zeta_{j,3}\right)S(\mathrm{w}) \\
&\quad + \sum_{j=1}^m 2\nabla\zeta_{j,3} \cdot \nabla_{\mathcal{N}_\alpha}\varphi_j + \varphi_j\Delta\zeta_{j,3} + (1 - \zeta_{j,3})N\left[\psi + \sum_{i=1}^m \zeta_{i,2}\varphi_i\right] = 0.
\end{aligned}$$

Hence, to construct a solution to (6.7), it suffices to solve the system of PDEs

$$\Delta\psi - \left[2 - \left(1 - \sum_{j=1}^{m} \zeta_{j,2}\right)\left(F'(\mathrm{w}) + 2\right)\right]\psi = -\left(1 - \sum_{j=1}^{m} \zeta_{j,2}\right) S(\mathrm{w}) - \sum_{j=1}^{m} 2\nabla\zeta_{j,2} \cdot \nabla_{\mathcal{N}_\alpha}\varphi_j - \varphi_j \Delta\zeta_{j,2}$$
$$- \left(1 - \sum_{j=1}^{m} \zeta_{j,3}\right) N\left[\sum_{i=1}^{m} \zeta_{i,2}\varphi_i + \psi\right], \quad \text{in } \mathbb{R}^3 \tag{6.9}$$

$$\Delta_{\mathcal{N}_\alpha}\varphi_l + F'(\zeta_{l,2}\mathrm{w})\varphi_l = -\zeta_{l,2} S(\mathrm{w}) - \zeta_{l,2} N(\varphi_l + \psi)$$
$$- \zeta_{l,2}\left(F'(\mathrm{w}) + 2\right)\psi, \quad \text{for } \left|z - f_l(\alpha y)\right| \leqslant \rho_\alpha(y), \quad l = 1, \dots, m, \tag{6.10}$$

where

$$\rho_\alpha(y) := \frac{1}{2}\left[\sigma_\alpha + \sqrt{2}\left(1 - \frac{1}{\sigma_\alpha}\right)\log\left(1 + (\alpha y)^2\right)\right], \quad y = Y_\alpha(y, \theta) \in M_\alpha.$$

Now, we extend Eq. (6.10) to the whole $M_\alpha \times \mathbb{R}$. First, let us introduce the differential operator

$$B_l := \zeta_{l,2}[\Delta_{\mathcal{N}_{\alpha,f_l}} - \partial_{tt} - \Delta_{M_\alpha}]$$

for $l = 1, \dots, m$. Recall that Δ_{M_α} is nothing but the Laplace–Beltrami and which in the local coordinates $Y_\alpha(y, \theta)$, has the expression

$$\Delta_{M_\alpha} = \partial_{yy} + \frac{\alpha^2 y}{1 + (\alpha y)^2}\partial_y + \frac{\alpha^2}{1 + (\alpha y)^2}\partial_{\theta\theta}.$$

Clearly, B_l vanishes in the domain

$$|t| \geqslant \frac{1}{2}\left[\sigma_\alpha + 2\left(1 - \frac{1}{\sigma_\alpha}\right)\ln\left(1 + (\alpha y)^2\right)\right] - 1.$$

We look for a solution to (6.10) having the form

$$\phi_l(y, t) = \varphi_l\left(y, t + f_l(\alpha y)\right), \qquad x = X_{\alpha, f_l}(y, \theta, t)$$

and so, instead of Eq. (6.10), we consider the equation

$$\partial_{tt}\phi_l + \Delta_{M_\alpha}\phi_l + F'\left(w_l(t)\right)\phi_l = -S_l(\mathrm{w}) - B_l(\phi_l) - \left[F'(\zeta_{l,2}\mathrm{w}) - F'\left(w_l(t)\right)\right]\phi_l$$
$$- \zeta_{l,2}\left(F'(\mathrm{w}) + 2\right)\psi - \zeta_{l,2} N(\phi_l + \psi), \quad \text{in } M_\alpha \times \mathbb{R}, \tag{6.11}$$

where we have denoted

$$(-1)^{l-1} S_l(\mathrm{w}) = -\alpha^2\left(\Delta_M \mathrm{v}_l + |A_M|^2 \mathrm{v}_l\right) w'(t)$$
$$+ 6\left(1 - w^2(t)\right) e^{-\sqrt{2}t} \zeta_{l,2} e^{-\sqrt{2}(h_l - h_{l-1})}\left[e^{-\sqrt{2}(\mathrm{v}_l - \mathrm{v}_{l-1})} - 1\right]$$
$$- 6\left(1 - w^2(t)\right) e^{\sqrt{2}t} \zeta_{l,2} e^{-\sqrt{2}(h_{l+1} - h_l)}\left[e^{-\sqrt{2}(\mathrm{v}_{l+1} - \mathrm{v}_l)} - 1\right]$$
$$+ \alpha^2 \mathrm{v}_l'\left(2h_l' + \mathrm{v}_l'\right) w''(t) + \zeta_{l,2}\left[-\alpha^3 (t + f_l) a_1\left(\alpha y, \alpha(t + f_l)\right) f_l'' w'(t)\right.$$
$$\left. - \alpha^2\left(\Delta_M f_j - \alpha(t + f_l) a_1\left(\alpha y, \alpha(t + f_l)\right) f_j''\right) w_j'(t + f_l - f_j) + \tilde{\mathrm{R}}_l\right], \tag{6.12}$$

where we recall that

$$\tilde{\mathrm{R}}_l = \mathrm{R}_l(\alpha y, t, \mathrm{v}_1, \ldots, \mathrm{v}_m, D\mathrm{v}_1, \ldots, D\mathrm{v}_m)$$

and

$$\left|D_p \tilde{\mathrm{R}}(\alpha y, t, p, q)\right| + \left|D_q \tilde{\mathrm{R}}(\alpha y, t, p, q)\right| + \left|\tilde{\mathrm{R}}(y, t, p, q)\right| \leqslant C\alpha^{2+\tau} r_\alpha(y)^{-4} e^{-\varrho|t|} \tag{6.13}$$

for $0 < \varrho < \sqrt{2}$ and $0 < \tau < 1$. Observe that $S_l(\mathrm{w})$ coincides with $S(U_1)$ where $\zeta_{l,2} = 1$, but we have basically cut-off the parts in $S(U_1)$ that, in the local coordinates X_{α, f_l}, are not defined for all $t \in \mathbb{R}$.

Using (6.12) and (6.13) and since the support of $\zeta_{l,2}$ is contained in a region of the form

$$|t| \leqslant \frac{1}{2}\left[\sigma_\alpha - 2\left(1 - \frac{1}{\sigma_\alpha}\right) \ln\left(1 + (\alpha y)^2\right)\right]$$

we compute directly the size of this error to obtain that

$$\left\|S_l(\mathrm{w})\right\|_{p,2,\varrho} \leqslant C\alpha^{2+\tau_1} \tag{6.14}$$

for some $0 < \varrho < \sqrt{2}$, some constant $C > 0$ and some $0 < \tau_1 < \tau_0$ small, independent of $\alpha > 0$.

Hence we solve system (6.9)–(6.11). We first solve equation (6.9), using the fact that the potential $2 - (1 - \sum_{j=1}^m \zeta_{j3})(F'(\mathrm{w}) + 2)$ is uniformly positive, so that the linear operator there behaves like $\Delta_{\mathbb{R}^3} - 2$. A solution $\psi = \Psi(\phi_1, \ldots, \phi_m)$ is then found using contraction mapping principle. We collect this discussion in the following proposition, that will be proven in detail in Section 7.

Proposition 6.1. *Assume $0 < \varrho < \sqrt{2}$, $\mu > 0$, $p > 2$ and let the functions f_l's be as in (5.2)–(5.4). Then, for every $\alpha > 0$ sufficiently small and for m fixed functions $\phi_1, \ldots, \phi_m$, satisfying that*

$$\|\phi_l\|_{2,p,\mu,\varrho} \leqslant 1, \quad l = 1, \ldots, m$$

Eq. (6.9) has a unique solution $\psi = \Psi(\phi_1, \ldots, \phi_m)$. Even more, the operator $\psi = \Psi(\phi_1, \ldots, \phi_m)$ turns out to be Lipschitz in every ϕ_j. More precisely, $\psi = \Psi(\phi_1, \ldots, \phi_m)$ satisfies that

$$\begin{aligned}\|\psi\|_X &:= \left\|D^2\psi\right\|_{p,\hat{\mu},\sim} + \left\|\left(1 + r^{\hat{\mu}}(\alpha x)\right) D\psi\right\|_{L^\infty(\mathbb{R}^3)} + \left\|\left(1 + r^{\hat{\mu}}(\alpha x)\right)\psi\right\|_{L^\infty(\mathbb{R}^3)} \\ &\leqslant C\left(\alpha^{2+\frac{\varrho}{\sqrt{2}}-\varepsilon} + \alpha^{\frac{\varrho}{\sqrt{2}}-\varepsilon}\sum_{j=1}^m \|\phi_j\|_{2,p,\mu,\varrho}\right),\end{aligned} \tag{6.15}$$

where $0 < \hat{\mu} < \min(2\mu, \mu + \varrho\sqrt{2}, 2 + \varrho\sqrt{2})$ and

$$\left\|\Psi(\phi_j) - \Psi(\hat{\phi}_j)\right\|_X \leqslant C\alpha^{\frac{\varrho}{\sqrt{2}}-\varepsilon}\|\phi_j - \hat{\phi}_j\|_{2,p,\mu,\varrho}. \tag{6.16}$$

Hence, using Proposition 6.1, we solve Eq. (6.11) with $\psi = \Psi(\phi_1, \ldots, \phi_m)$. Let us set

$$\begin{aligned}\mathbf{N}_l(\phi_1, \ldots, \phi_l, \ldots, \phi_m) := {} & B_l(\phi_l) + \left[F'(\zeta_{l,2}\mathrm{w}) - F'\left(w(t)\right)\right]\phi_l \\ & + \zeta_{l,2}\left(F'(\mathrm{w}) + 2\right)\Psi(\phi_1, \ldots, \phi_m) + \zeta_{l,2} N\left[\phi_l + \Psi(\phi_1, \ldots, \phi_m)\right].\end{aligned}$$

So, setting $\Phi = (\phi_1, \ldots, \phi_m)$, we only need to solve

$$\partial_{tt}\phi_l + \Delta_{M_\alpha}\phi_l + F'\left(w_l(t)\right)\phi_l = -S_l(\mathrm{w}) - \mathbf{N}_l(\Phi), \quad \text{in } M_\alpha \times \mathbb{R} \tag{6.17}$$

for every $l = 1, \ldots, m$.

To treat system (6.17), we solve a nonlinear and nonlocal problem for ϕ_l, in such a way that we eliminate the parts of the error that do not contribute to the projections onto $w'(t)$. This step can be though as an improvement of the approximation w. We use the fact that the error has the size

$$\left\|S_l(\mathrm{w})\right\|_{p,2,\varrho} \leqslant \alpha^{2+\tau_1} \tag{6.18}$$

and as we will see in Section 7 for $0 < \tau_1 < \tau_0$, $\mathbf{N}_l(\phi)$ satisfies

$$\left\|\mathbf{N}_l(\varPhi)\right\|_{p,4,\varrho} \leqslant C\alpha^{3+\tau_1}, \tag{6.19}$$

$$\left\|\mathbf{N}_l(\varPhi_1) - \mathbf{N}_l(\varPhi_2)\right\|_{p,4,\varrho} \leqslant C\alpha\|\varPhi_1 - \varPhi_2\|_{2,p,2,\varrho}, \tag{6.20}$$

for $\varPhi_1, \varPhi_2 \in B_\alpha$ a ball of radius $\mathcal{O}(\alpha^{2+\tau_1})$ in the product norm $\|\varPhi\|_{2,p,2,\varrho}$. A direct application of the contraction mapping principle allows us to solve the projected system

$$\partial_{tt}\phi_l + \Delta_{M_\alpha}\phi_l + F'\big(w_l(t)\big)\phi_l = -S_l(\mathrm{w}) - \mathbf{N}_l(\varPhi) + c_l(y)w'(t), \quad \text{in } M_\alpha \times \mathbb{R}, \tag{6.21}$$

$$\int_{\mathbb{R}} \phi_l(y,t)w'(t)\,dt = 0, \quad l = 1,\ldots,m, \tag{6.22}$$

where

$$c_l(y) = \int_{\mathbb{R}} \big[S_l(\mathrm{w}) + \mathbf{N}_l(\varPhi)\big]w'(t)\,dt, \quad \forall l = 1,\ldots,m.$$

This solution ϕ_l, defines a Lipschitz operator $\phi_l = \varPhi_l(\mathrm{v}_1,\ldots,\mathrm{v}_m)$ for the product norm

$$\left\|(\mathrm{v}_1,\ldots,\mathrm{v}_m)\right\|_{\delta,p,\beta} := \sum_{j=1}^m \|\mathrm{v}_j\|_{\delta,p,\beta}.$$

This information is collected in the following proposition:

Proposition 6.2. *Assume $0 < \mu \leqslant 2$, $0 < \varrho < \sqrt{2}$ and $p > 2$. For every $\alpha > 0$ small enough, there exists a universal constant $C > 0$, such that system (6.21)–(6.22) has a unique solution $(\phi_1,\ldots,\phi_m) = \varPhi(\mathrm{v}_1,\ldots,\mathrm{v}_m)$, satisfying*

$$\|\varPhi\|_{2,p,2,\varrho} \leqslant C\alpha^{2+\tau_1}$$

and

$$\left\|\varPhi(\mathrm{v}_1,\ldots,\mathrm{v}_m) - \varPhi(\hat{\mathrm{v}}_1,\ldots,\hat{\mathrm{v}}_m)\right\|_{2,p,2,\varrho} \leqslant C\alpha^{2+\tau_1}\left\|(\mathrm{v}_1,\ldots,\mathrm{v}_m) - (\hat{\mathrm{v}}_1,\ldots,\hat{\mathrm{v}}_m)\right\|_{\delta,p,\beta}$$

for some fixed $\beta \in (\frac{5}{2}, 4 - 4\delta)$.

6.2. Solving the Jacobi–Toda system to adjust the nodal sets

First, to estimate the size of the error of the projected problem, we borrow a result from Section 8 in [12].

Lemma 6.1. *Assume $g(y,t)$ is a function defined in $M_\alpha \times \mathbb{R}$ and for which*

$$\sup_{(y,t)\in M_\alpha\times\mathbb{R}} \big(1+r(\alpha y)^\mu\big)e^{\varrho|t|}\|g\|_{L^p(B_1(y,t))}<\infty$$

for some $\varrho,\mu>0$ *and* $p>2$. *The function defined in* M *as*

$$q(y):=\int_{\mathbb{R}} g\left(\frac{y}{\alpha},t\right)w'(t)\,dt$$

satisfies

$$\|q\|_{p,\beta}\leqslant C\sup_{(y,t)\in M_\alpha\times\mathbb{R}}\big(1+r(y)^\mu\big)e^{\varrho|t|}\|g\|_{L^p(B_1(y,t))}$$

provided

$$\mu>\beta+\frac{2}{p}.$$

To conclude the proof of Theorem 2, we choose the vector function $\mathrm{v}=(\mathrm{v}_1,\dots,\mathrm{v}_m)$ in such a way that

$$c_l(y)=\int_{\mathbb{R}}\big[S_l(\mathrm{w})+\mathbf{N}_l(\varPhi)\big]w'(t)\,dt=0,\quad \forall l=1,\dots,m.$$

Using (6.12), we find that making these projections zero is equivalent to solve the nonlinear and nonlocal system of equations

$$\alpha^2\big(\Delta_M\mathrm{v}_l+|A_M|^2\mathrm{v}_l\big)-\sqrt{2}a_0\big[e^{-\sqrt{2}(h_l-h_{l-1})}(\mathrm{v}_l-\mathrm{v}_{l+1})-e^{-\sqrt{2}(h_{l+1}-h_l)}(\mathrm{v}_{l+1}-\mathrm{v}_l)\big]=\alpha^2\mathrm{Q}_l(\mathrm{v}),\tag{6.23}$$

where

$$\mathrm{Q}_l(\mathrm{v}):=G_{l,1}(\mathrm{v})+G_{l,2}(\mathrm{v})$$

$$\begin{aligned}
\alpha^2G_{l,1}(\mathrm{v}):=\int_{\mathbb{R}}&\zeta_{j2}\big[-\alpha^3(t+f_j)a_1\big(\alpha y,\alpha(t+f_j)\big)f_j''w'(t)\\
&-\alpha^2\big(\Delta_Mf_j-\alpha(t+f_l)a_1\big(\alpha y,\alpha(t+f_l)\big)f_j''\big)w_j'(t+f_l-f_j)+\tilde{\mathrm{R}}_l\big]w'(t)\,dt\\
&-a_0e^{-\sqrt{2}(h_l-h_{l-1})}\big(e^{-\sqrt{2}(\mathrm{v}_l-\mathrm{v}_{l-1})}-1+\sqrt{2}(\mathrm{v}_l-\mathrm{v}_{l-1})\big)\\
&+a_0e^{-\sqrt{2}(h_{l+1}-h_l)}\big(e^{-\sqrt{2}(\mathrm{v}_{l+1}-\mathrm{v}_l)}-1+\sqrt{2}(\mathrm{v}_{l+1}-\mathrm{v}_l)\big)\\
&-\int_{\mathbb{R}}6\big(1-w^2(t)\big)e^{-\sqrt{2}t}(1-\zeta_{l,2})w'(t)\,dt\,e^{-\sqrt{2}(h_l-h_{l-1})}\big[e^{-\sqrt{2}(\mathrm{v}_l-\mathrm{v}_{l-1})}-1\big]\\
&+\int_{\mathbb{R}}6\big(1-w^2(t)\big)e^{\sqrt{2}t}(1-\zeta_{l,2})w'(t)\,dt\,e^{-\sqrt{2}(h_{l+1}-h_l)}\big[e^{-\sqrt{2}(\mathrm{v}_{l+1}-\mathrm{v}_l)}-1\big],
\end{aligned}$$

$$\alpha^2G_{l,2}(\mathrm{v}):=\int_{\mathbb{R}}\mathbf{N}_l(\varPhi)w'(t)\,dt,$$

where we set $\varPhi=(\varPhi_1,\dots,\varPhi_m)$ and

$$a_0=\|w'\|^{-2}_{L^2(\mathbb{R})}\int_{\mathbb{R}}6\big(1-w^2(t)\big)e^{-\sqrt{2}t}w'(t)\,dt.$$

Direct computations using (6.12) and Lemma 6.1 yield the estimates

$$\|G_{l,1}(\mathrm{v})\|_{p,\beta} \leqslant C\alpha^{\tau_0}$$
$$\|G_{l,1}(\mathrm{v}) - G_{l,1}(\hat{\mathrm{v}})\|_{p,\beta} \leqslant C\alpha^{\tau_0}\|\mathrm{v} - \hat{\mathrm{v}}\|_{\delta,p,\beta}$$

for some $0 < \tau_0 < 1$ fixed independent of $\alpha > 0$.

From (6.19) and Lemma 6.1 we also have that for any $p > 2$ and $0 < \beta < 4 - \frac{2}{p}$

$$\|G_{l,2}(\mathrm{v})\|_{p,\beta} \leqslant \alpha^{-2}\left\| \int_{\mathbb{R}} \mathbf{N}_l(\Phi) w'(t)\, dt \right\|_{p,\beta} \leqslant C\alpha^{1+\tau_1}.$$

On the other hand, it is direct to check from (6.20) and Proposition 6.2 that

$$\|G_{l,2}(\mathrm{v}) - G_{l,2}(\hat{\mathrm{v}})\|_{p,\beta} \leqslant C\alpha^{1+\tau_1}\|\mathrm{v} - \hat{\mathrm{v}}\|_{\delta,p,\beta}$$

Hence we find that

$$\mathrm{Q}(\mathrm{v}) := \big(\mathrm{Q}_1(\mathrm{v},), \ldots, \mathrm{Q}_m(\mathrm{v},)\big)$$

satisfies

$$\|\mathrm{Q}(\mathrm{v})\|_{p,\beta} \leqslant C\alpha^{\tau_0}$$
$$\|\mathrm{Q}(\mathrm{v}) - \mathrm{Q}(\hat{\mathrm{v}})\|_{p,\beta} \leqslant C\alpha^{\tau_0}\|\mathrm{v} - \hat{\mathrm{v}}\|_{\delta,p,\beta}.$$

Since we are linearizing the Jacobi–Toda system (6.23) around the exact solution h, we can proceed as in the proof Proposition 3.1 to solve this system. We see that using Propositions 3.2 and 3.3 and a direct application of contraction mapping principle in a ball of radius $\mathcal{O}(\alpha^{\tau_0}\sigma^{\frac{3}{4}})$ in the product topology $\|\mathrm{v}\|_{\delta,p,\beta}$ yields the existence of functions $\mathrm{v}_1, \ldots, \mathrm{v}_m$ satisfying (5.4), so that

$$c_l(y) = \int_{\mathbb{R}} \big[S_l(\mathrm{w}) + \mathbf{N}_l(\Phi)\big] w'(t)\, dt = 0, \quad \forall l = 1, \ldots, m$$

and this completes the proof of the theorem. We omit the details since the procedure is similar to the decoupling developed in Section 3.2.

In Section 7 we will carry out the proofs of the auxiliary results mentioned in this section.

7. Gluing reduction and solution to the projected problem

In this section, we prove Propositions 6.1 and 6.2. The notations we use in this section have been set up in Sections 4 and 5.

7.1. Solving the gluing system

Given fixed functions $\phi_1, \ldots, \phi_m$ such that $\|\phi_l\|_{2,p,\mu,\varrho} \leqslant 1$ for $l = 1, \ldots, m$, we solve problem (6.9). To begin with, we observe that there exist constants $a < b$, independent of α, such that

$$0 < a \leqslant Q_\alpha(x) \leqslant b, \quad \text{for every } x \in \mathbb{R}^3,$$

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where we set

$$Q_\alpha(x) = 2 - \left(1 - \sum_{j=1}^{m} \zeta_{j2}\right)\left[F'(\mathrm{w}) + 2\right].$$

Using this remark, we study the problem

$$\Delta\psi - Q_\alpha(x)\psi = g(x), \quad x \in \mathbb{R}^3 \tag{7.1}$$

for a given $g = g(x)$ such that

$$\|g\|_{p,\hat{\mu},\sim} := \sup_{x\in\mathbb{R}^3} \left(1 + R^{\hat{\mu}}(\alpha x)\right)\|g\|_{L^p(B_1(x))}.$$

Solvability theory for Eq. (7.1) is collected in the following lemma whose proof follows the same lines as in Lemma 7.1 in [12] and [14].

Lemma 7.1. *Assume $p > 2$ and $\hat{\mu} \geqslant 0$. There exists a constant $C > 0$ and $\alpha_0 > 0$ small enough such that for $0 < \alpha < \alpha_0$ and any given $g = g(x)$ with $\|g\|_{p,\hat{\mu},\sim} < \infty$, Eq. (7.1) has a unique solution $\psi = \psi(g)$, satisfying the a-priori estimate*

$$\|\psi\|_X \leqslant C\|g\|_{p,\hat{\mu},\sim},$$

where

$$\|\psi\|_X := \left\|D^2\psi\right\|_{p,\hat{\mu},\sim} + \left\|\left(1 + r(\alpha x)^{\hat{\mu}}(x)\right)D\psi\right\|_{L^\infty(\mathbb{R}^3)} + \left\|\left(1 + r^{\hat{\mu}}(\alpha x)\right)\psi\right\|_{L^\infty(\mathbb{R}^3)}.$$

Now we prove Proposition 6.1. Denote by X, the space of functions $\psi \in W^{2,p}_{loc}(\mathbb{R}^3)$ such that $\|\psi\|_X < \infty$ and let us denote by $\Gamma(g) = \psi$ the solution to Eq. (7.1) from the previous lemma. We see that the linear map Γ is continuous, i.e.

$$\left\|\Gamma(g)\right\|_X \leqslant C\|g\|_{p,\hat{\mu},\sim}$$

with $0 < \hat{\mu} < \min(2\mu, \mu + \varrho\sqrt{2}, 2 + \varrho\sqrt{2})$. Using this we can recast (6.9) as a fixed point problem, in the following manner

$$\psi = -\Gamma\left(\left(1 - \sum_{j=1}^{m} \zeta_{j2}\right)S(\mathrm{w}) + g_1 + \left(1 - \sum_{j=1}^{m} \zeta_{j2}\right)N\left[\sum_{i=1}^{m} \zeta_{i3}\phi_i + \psi\right]\right), \tag{7.2}$$

where

$$g_1 = \sum_{j=1}^{m} 2\nabla\zeta_{j2} \cdot \nabla\phi_j + \phi_j\Delta\zeta_{j2}.$$

Under conditions (5.2)–(5.4) and $\max_{1\leqslant l\leqslant m} \|\phi_l\|_{2,p,\mu,\varrho} \leqslant 1$, we estimate the size of the right-hand side in (7.2).

Recall that $S(\mathrm{w}) = \beta_\alpha(x)S(U_1) + E$, where

$$|D_y E| + |E| \leqslant Ce^{-\frac{\eta}{\alpha}} r_\alpha^{-4}(y).$$

So, we estimate directly using (8.32), to get

$$\left|\left(1-\sum_{j=1}^{m}\zeta_{j2}\right)S(\mathrm{w})\right| \leqslant C\sum_{j=1}^{m}\alpha^2\big(1+r_\alpha(y)\big)^{-2}e^{-\varrho|t|}(1-\zeta_{j2})$$
$$\leqslant C\alpha^{2+\frac{\varrho}{\sqrt{2}}}\sigma^{\frac{\varrho}{2\sqrt{2}}}\big(1+r_\alpha(y)\big)^{-2(1+\frac{\varrho}{\sqrt{2}})}$$

this means that

$$\left|\left(1-\sum_{j=1}^{m}\zeta_{j2}(x)\right)S(\mathrm{w})\right| \leqslant C\alpha^{2+\frac{\varrho}{\sqrt{2}}}\sigma^{\frac{\varrho}{2\sqrt{2}}}\big(1+R_\alpha(x)\big)^{-2(1+\frac{\varrho}{\sqrt{2}})}.$$

Consequently we get, for $0<\hat{\mu}<2(1+\frac{\varrho}{\sqrt{2}})$ that

$$\left\|\left(1-\sum_{j=1}^{m}\zeta_{j,2}\right)S(\mathrm{w})\right\|_{p,\hat{\mu},\sim} \leqslant C\alpha^{2+\frac{\varrho}{\sqrt{2}}-\varepsilon}$$

for some $\varepsilon>0$ sufficiently small.

As for the second term in the right-hand side of (7.2), the following holds true

$$|2\nabla\zeta_{j,2}\cdot\nabla\phi_j+\phi_j\Delta\zeta_{j,2}| \leqslant C(1-\zeta_{j2})\big(1+r^{\mu}(\alpha y)\big)^{-1}e^{-\varrho|t|}\|\phi_j\|_{2,p,\mu,\varrho}$$
$$\leqslant C\alpha^{\frac{\varrho}{\sqrt{2}}}\sigma^{\frac{\varrho}{2\sqrt{2}}}\big(1+r^{\mu+\frac{\varrho}{\sqrt{2}}}(\alpha y)\big)^{-1}\|\phi_j\|_{2,p,\mu,\varrho}.$$

This implies that

$$\|2\nabla\zeta_{j,2}\cdot\nabla\phi_j+\phi_j\Delta\zeta_{j,2}\|_{p,\mu+\varrho\sqrt{2}-\varepsilon,\sim} \leqslant C\alpha^{\frac{\varrho}{\sqrt{2}}-\varepsilon}\sum_{j=1}^{m}\|\phi_j\|_{2,p,\mu,\varrho}.$$

Finally we must check the Lipschitz character of $(1-\sum_{j=1}^{m}\zeta_{j2})N[\sum_{i=1}^{m}\zeta_{i2}\phi_i+\psi]$. Take $\psi_1,\psi_2\in X$. Then

$$\left(1-\sum_{j=1}^{m}\zeta_{j2}\right)\left|N\left[\sum_{i=1}^{m}\zeta_{i2}\phi_i+\psi_1\right]-N\left[\sum_{i=1}^{m}\zeta_{i2}\phi_i+\psi_2\right]\right|$$
$$\leqslant\left(1-\sum_{j=1}^{m}\zeta_{j2}\right)\left|F\left(\mathrm{w}+\sum_{i=1}^{m}\zeta_{j1}\phi_i+\psi_1\right)-F\left(\mathrm{w}+\sum_{i=1}^{m}\zeta_{i1}\phi_i+\psi_2\right)-F'(\mathrm{w})(\psi_1-\psi_2)\right|$$
$$\leqslant C\left(1-\sum_{j=1}^{m}\zeta_{j2}\right)\sup_{s\in[0,1]}\left|\sum_{i=1}^{m}\zeta_{i1}\phi_i+s\psi_1+(1-s)\psi_2\right||\psi_1-\psi_2|$$
$$\leqslant C\alpha^{\varrho-\varepsilon}\left(\sum_{i=1}^{m}\|\phi_i\|_{\infty,\mu,\varrho}+\|\psi_1\|_X+\|\psi_2\|_X\right)|\psi_1-\psi_2|$$

So, we see that

$$\left\|\left(1-\sum_{j=1}^{m}\zeta_{j2}\right)N\left[\sum_{i=1}^{m}\zeta_{i2}\phi_i+\psi_1\right]-\left(1-\sum_{j=1}^{m}\zeta_{j2}\right)N\left[\sum_{i=1}^{m}\zeta_{i2}\phi_i+\psi_2\right]\right\|_{p,2\hat{\mu},\sim}$$
$$\leqslant C\alpha^{\frac{\varrho}{\sqrt{2}}-\varepsilon}\|\psi_1-\psi_2\|_{\infty,\hat{\mu},\sim}.$$

In particular, we take advantage of the fact that $N(\varphi) \sim \varphi^2$, to find that

$$\left\| \left(1 - \sum_{j=1}^{m} \zeta_{j2}\right) N\left(\sum_{i=1}^{m} \zeta_{i2}\phi_i\right) \right\|_{p,2\mu,\sim} \leqslant C\alpha^{2\varrho-\varepsilon} \sum_{j=1}^{m} \|\phi_j\|^2_{2,p,\mu,\varrho}.$$

Consider $\tilde{\Gamma} : X \to X$, $\tilde{\Gamma} = \tilde{\Gamma}(\psi)$ the operator given by the right-hand side of (7.2). From the previous remarks we have that $\tilde{\Gamma}$ is a contraction provided α is small enough and so we have found $\psi = \tilde{\Gamma}(\psi)$ the solution to (6.9) with

$$\|\psi\|_X \leqslant C\left(\alpha^{2+\frac{\varrho}{\sqrt{2}}-\varepsilon} + \alpha^{\frac{\varrho}{\sqrt{2}}-\varepsilon} \sum_{j=1}^{m} \|\phi_j\|_{2,p,\mu,\rho}\right).$$

We can check directly that $\Psi(\Phi) = \psi$ is Lipschitz in $\Phi = (\phi_1, \ldots, \phi_m)$, i.e.

$$\begin{aligned}\big\|\Psi(\Phi_1) - \Psi(\Phi_2)\big\|_X &\leqslant C\left\|\left(1 - \sum_{j=1}^{m} \zeta_{j2}\right)\left[N\left(\sum_{i=1}^{m} \zeta_{i2}\phi_{i1} + \Psi(\Phi_1)\right) - N\left(\sum_{i=1}^{m} \zeta_{i2}\phi_{i2} + \Psi(\Phi_2)\right)\right]\right\|_{p,2\mu,\sim} \\ &\leqslant C\alpha^{\varrho-\varepsilon}\big(\big\|\Psi(\Phi_1) - \Psi(\Phi_2)\big\|_X + \|\Phi_1 - \Phi_2\|_{2,p,\mu,\varrho}\big).\end{aligned}$$

Hence for α small, we conclude

$$\big\|\Psi(\Phi_1) - \Psi(\Phi_2)\big\|_X \leqslant C\alpha^{\tau}\|\Phi_1 - \Phi_2\|_{2,p,\mu,\varrho}.$$

7.2. Solving the projected system (6.21)–(6.22)

Now we solve system

$$\partial_{tt}\phi_l + \Delta_{M_\alpha}\phi_l + F'\big(w_l(t)\big)\phi_l = -S_l(\mathrm{w}) - \mathbf{N}_l(\phi_l) + c_l(y)w'(t), \quad \text{in } M_\alpha \times \mathbb{R}.$$
$$\int_{\mathbb{R}} \phi_l(y,t)w'(t)\,dt = 0.$$

To do so, we need to study solvability for the linear equation

$$\partial_{tt}\phi + \Delta_{M_\alpha}\phi + F'\big(w(t)\big)\phi = g(y,t) + c(y)w'(t), \quad \text{in } M_\alpha \times \mathbb{R}, \tag{7.3}$$
$$\int_{\mathbb{R}} \phi(y,t)w'(t)\,dt = 0. \tag{7.4}$$

Solvability of (7.3)–(7.4) is based upon the fact that the heteroclinic solution $w(t)$ is nondegenerate in the sense, that the following property holds true.

Lemma 7.2. *Assume that $\phi \in L^\infty(\mathbb{R}^3)$ and assume $\phi = \phi(x_1, x_2, t)$ satisfies*

$$L(\phi) := \partial_{tt}\phi + \Delta_{\mathbb{R}^2}\phi + F'\big(w(t)\big)\phi = 0, \qquad \text{in } \mathbb{R}^2 \times \mathbb{R}. \tag{7.5}$$

Then $\phi(x_1, x_2, t) = Cw'(t)$, for some constant $C \in \mathbb{R}$.

For the detailed proof of this lemma we refer the reader to [12,14] and references therein.

The linear theory we need to solve system (6.22), is collected in the following proposition, whose proof is again contained in essence in Proposition 4.1 in [12] and [14].

Proposition 7.1. *Assume $p>2$, $0<\varrho<\sqrt{2}$ and $\mu\geqslant 0$. There exist $C>0$, a universal constant and $\alpha_0>0$ small such that, for every $\alpha\in(0,\alpha_0)$ and any given g with $\|g\|_{p,\mu,\varrho}<\infty$, problem (7.3)–(7.4) has a unique solution (ϕ,c) with $\|\phi\|_{p,\mu,\varrho}<\infty$, satisfying the a priori estimate*

$$\left\|D^2\phi\right\|_{p,\mu,\varrho}+\|D\phi\|_{\infty,\mu,\varrho}+\|\phi\|_{\infty,\mu,\varrho}\leqslant C\|g\|_{p,\mu,\varrho}.$$

Using Proposition 7.1, we are ready to solve system (6.21)–(6.22). First, recall that as stated in (6.14)

$$\left\|S_l(\mathrm{w})\right\|_{p,2,\varrho}\leqslant C\alpha^{2+\tau_1} \tag{7.6}$$

for some $0<\tau_1<\tau_0$ small enough.

From Proposition 6.1 we have a nonlocal operator $\psi=\Psi(\phi_1,\dots,\phi_m)$. We want to solve the following problem:

Recall that for $\Phi=(\phi_1,\dots,\phi_m)$,

$$\mathbf{N}_l(\Phi):=B_l(\phi_l)+\left[F'(\zeta_{l2}\mathrm{w})-F'\big(w_l(t)\big)\right]\phi_l+\zeta_{l2}\left[F'(\mathrm{w})+2\right]\Psi(\Phi)+\zeta_{l2}N\big(\phi_l+\Psi(\Phi)\big).$$

Let us denote

$$\begin{gathered}N_1(\Phi):=B_l(\phi_l)+\left[F'(\zeta_{l2}\mathrm{w})-F'\big(w_l(t)\big)\right]\phi_l,\\ N_2(\Phi):=\zeta_{l2}\left[F'(\mathrm{w})+2\right]\Psi(\Phi),\\ N_3(\Phi):=\zeta_{l2}N\big(\phi_l+\Psi(\Phi)\big).\end{gathered}$$

We need to investigate the Lipschitz character of N_i, $i=1,2,3$. We begin with N_3. Observe that

$$\begin{aligned}\left|N_3(\Phi_1)-N_3(\Phi_2)\right|&=\zeta_{l2}\left|N\big(\phi_{l1}+\Psi(\Phi_1)\big)-N\big(\phi_{l2}+\Psi(\Phi_2)\big)\right|\\&\leqslant C\zeta_{l2}\sup_{\tau\in[0,1]}\left|\tau\big(\phi_{l1}+\Psi(\Phi_1)\big)+(1-\tau)\big(\phi_{l2}+\Psi(\phi_{l2})\big)\right|\cdot\left|\phi_{l1}-\phi_{l2}+\Psi(\Phi_1)-\Psi(\Phi_2)\right|\\&\leqslant C\left[\left|\Psi(\Phi_2)\right|+|\phi_{l1}-\phi_{l2}|+\left|\Psi(\Phi_1)-\Psi(\Phi_2)\right|+|\phi_{l2}|\right]\cdot\left[|\phi_{l1}-\phi_{l2}|+\left|\Psi(\Phi_1)-\Psi(\Phi_2)\right|\right].\end{aligned}$$

This implies that

$$\begin{aligned}&\left\|N_3(\Phi_1)-N_3(\Phi_2)\right\|_{p,2\mu,\varrho}\\&\quad\leqslant C\left[\alpha^{2+\frac{\varrho}{\sqrt{2}}-\varepsilon}+\sum_{j=1}^m\|\phi_{j1}\|_{\infty,\mu,\varrho}+\sum_{j=1}^m\|\phi_{j2}\|_{\infty,\mu,\varrho}\right]\cdot\sum_{j=1}^m\|\phi_{j1}-\phi_{j2}\|_{\infty,\mu,\varrho}.\end{aligned}$$

Now we check on $N_1(\Phi)$. Clearly, we just have to pay attention to $B_l(\phi_l)$. But notice that $B_l(\phi_l)$ is linear in ϕ_l and

$$\begin{aligned}B_l(\phi_l)=-\alpha^2\left\{f_l''(\alpha y)+\frac{\alpha y}{1+(\alpha y)^2}f_l'(\alpha y)+\frac{2(t+f_l)}{(1+(\alpha y)^2)^2}\right\}\partial_t\phi_l\\-2\alpha f_l'(\alpha y)\partial_{ty}\phi_l+\alpha^2\left[f_l'(\alpha y)\right]^2\partial_{tt}\phi_l+D_{\alpha,f_l}(\phi_l),\end{aligned}$$

where the differential operator D_{α,f_l} is given in (2.6). From assumptions (5.2)–(5.5) made on the functions f_l's, we have that

$$\left\|N_1(\Phi_1)-N_1(\Phi_2)\right\|_{p,2+\mu,\varrho}\leqslant C\alpha\|\Phi_1-\Phi_2\|_{2,p,\mu,\varrho}.$$

Then, assuming that $\max_{1\leqslant j\leqslant m}\|\phi_j\|_{2,p,\mu,\varrho}\leqslant A\alpha^{2+\tau_1}$, we have that

$$\big\|\mathbf{N}_l(\Phi)\big\|_{p,2+\mu,\varrho}\leqslant C\alpha^{3+\tau_1}.$$

Letting $T(g)=\phi$ be the linear operator given by Lemma 7.1, we recast problem (6.21) as the fixed point problem

$$\phi_l=T\big(-S_l(\mathrm{w})-\mathbf{N}_l(\Phi)\big)=:\mathcal{T}_l(\Phi),\quad l=1,\dots,m$$

in the ball

$$B_\alpha:=\big\{\Phi=(\phi_1,\dots,\phi_m)\colon\ \|\Phi\|_{**}\leqslant A\alpha^{2+\tau_1},\ j=1,\dots,m\big\},$$

where clearly we are working in the space of function $\Phi\in W^{2,p}_{loc}(M_\alpha\times\mathbb{R})$ endowed with the norm

$$\|\Phi\|_{**}:=\sum_{j=1}^m\|\phi_j\|_{2,p,2,\varrho}.$$

Observe that

$$\big\|\mathcal{T}_l(\Phi_1)-\mathcal{T}_l(\Phi_2)\big\|_{**}\leqslant C\big\|\mathbf{N}_l(\Phi_1)-\mathbf{N}_l(\Phi_2)\big\|_{p,4,\varrho}\leqslant C\alpha\|\Phi_1-\Phi_2\|_{**},\quad \Phi_1,\Phi_2\in B_\alpha.$$

On the other hand, because C and K_1 are universal constants and taking A large enough independent of $\alpha>0$, we have that

$$\big\|\mathcal{T}_l(\Phi)\big\|_{**}\leqslant C\big(\big\|S_l(\mathrm{w})\big\|_{p,2,\varrho}+\big\|N_l(\Phi)\big\|_{p,4,\varrho}\big)\leqslant A\alpha^{2+\tau_1},\quad \phi\in B_\alpha.$$

Hence, the mapping $\mathcal{T}=(\mathcal{T}_1,\dots,\mathcal{T}_m)$ is a contraction from the ball B_α onto itself. From the contraction mapping principle we get a unique solution

$$\Phi=\Phi(\mathrm{v}_1,\dots,\mathrm{v}_m)$$

as required. As for the Lipschitz character of $\Phi(\mathrm{v}_1,\dots,\mathrm{v}_m)$ it comes from a lengthy by direct computation from the fact that

$$\begin{aligned}\big\|\Phi(\mathrm{v}_1,\dots,\mathrm{v}_m)-\Phi(\tilde{\mathrm{v}}_1,\dots,\tilde{\mathrm{v}}_m)\big\|_{2,p,2,\varrho}\leqslant{}& C\sum_{j=1}^m\big\|S_j(\mathrm{w},\mathrm{v}_1,\dots,\mathrm{v}_m)-S_j(\mathrm{w},\tilde{\mathrm{v}}_1,\dots,\tilde{\mathrm{v}}_m)\big\|_{p,2,\varrho}\\&+\sum_{j=1}^m\big\|N_j\big(\Phi(\mathrm{v}_1,\dots,\mathrm{v}_m)\big)-Nj\big(\Phi(\tilde{\mathrm{v}}_1,\dots,\tilde{\mathrm{v}}_m)\big)\big\|_{p,4,\varrho}.\end{aligned}$$

We left to the reader to check on the details of the proof of the following estimate

$$\big\|\Phi(\mathrm{v}_1,\dots,\mathrm{v}_m)-\Phi(\tilde{\mathrm{v}}_1,\dots,\tilde{\mathrm{v}}_m)\big\|_{2,p,2,\varrho}\leqslant C\alpha^{2+\tau_1}\sum_{j=1}^m\|\mathrm{v}_j-\tilde{\mathrm{v}}_j\|_{\delta,p,\beta}$$

for $(\mathrm{v}_1,\dots,\mathrm{v}_m)$ and $(\tilde{\mathrm{v}}_1,\dots,\tilde{\mathrm{v}}_m)$ satisfying (5.2) and (5.4). This completes the proof of Proposition 6.2 and consequently the proof of Theorem 2.

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8. Proof of Theorem 1

This section is devoted to the construction of the solutions predicted in Theorem 1. We skip details that are similar to the proof of Theorem 2. We begin by describing the location of the nodal set of the solutions predicted by this theorem.

8.1. *Toda system in* $\mathbb{R}^2$ *and its linearization*

In this part we describe the way we solve the Toda System of PDEs

$$\Delta f_1 + a_0 e^{-\sqrt{2}(f_2 - f_1)} = g_1, \quad \text{in } \mathbb{R}^2 \tag{8.1}$$

$$\Delta f_2 - a_0 e^{-\sqrt{2}(f_2 - f_1)} = g_2, \quad \text{in } \mathbb{R}^2 \tag{8.2}$$

where

$$a_0 = \|w'\|^{-2}_{L^2(\mathbb{R})} \int_{\mathbb{R}} 6\big(1 - w^2(t)\big) e^{\sqrt{2}t} w'(t)\, dt > 0.$$

A decoupling procedure similar to the one performed in Section 3, implies that system (8.1)–(8.2) becomes

$$\Delta(f_2 - f_1) - 2a_0 e^{-\sqrt{2}(f_2 - f_1)} = g_2 - g_1, \quad \text{in } \mathbb{R}^2, \tag{8.3}$$

$$\Delta(f_1 + f_2) = g_1 + g_2, \quad \text{in } \mathbb{R}^2. \tag{8.4}$$

Let us look for a radially symmetric smooth solution to (8.1)–(8.2) having the form

$$f_1(x') = q_1(x') + v_1(x'), \qquad f_2(x') = q_2(x') + v_2(x'), \quad x' \in \mathbb{R}^2, \tag{8.5}$$

where the vector function (q_1, q_2) solves the system of PDEs

$$\Delta q_1 + a_0 e^{-\sqrt{2}(q_2 - q_1)} = 0, \quad \text{in } \mathbb{R}^2, \tag{8.6}$$

$$\Delta q_2 - a_0 e^{-\sqrt{2}(q_2 - q_1)} = 0, \quad \text{in } \mathbb{R}^2. \tag{8.7}$$

Since we are looking for an axially symmetric nodal sets that are also symmetric respect to the x_3-axis, we assume that $q_2 = -q_1 = q$, so that the system (8.6)–(8.7) reduces to a Liouville equation, namely

$$\Delta q - a_0 e^{-2\sqrt{2}q} = 0, \quad \text{in } \mathbb{R}^2. \tag{8.8}$$

It is known that every radially symmetric solution to (8.8) is given by

$$q(x', \rho, \gamma) = \frac{1}{2\sqrt{2}} \log\left(\frac{\sqrt{2}a_0}{4\rho^2\gamma^2}\big(1 + \rho^2|x'|^{2\gamma}\big)^2\right) - \frac{(\gamma - 1)}{\sqrt{2}} \log\big(|x'|\big), \quad r > 0. \tag{8.9}$$

Since we are looking for smooth solutions to (8.8) with the initial conditions

$$q(0) = a > 0, \qquad \nabla_{x'} q(0) = 0$$

this forces $\gamma = 1$, so that

$$q(x',\rho)=\frac{1}{2\sqrt{2}}\log\left(\frac{\sqrt{2}a_0}{4\rho^2}\left(1+\rho^2|x'|^2\right)^2\right),\quad \rho>0. \tag{8.10}$$

From the fact that $q(0)=a>0$, we obtain

$$\log\left(\frac{\sqrt{2}a_0}{4\rho^2}\right)=\frac{2a}{\sqrt{2}}.$$

Remark 8.1. Observe that ρ is a free parameter that determines the conditions at the origin in (8.8). Without any loss of generality we assume that $\rho=1$, but it is important to keep in mind that the function q is smooth respect to this parameter $\rho>0$. We also remark that in the case when ρ lies in a fixed and compact interval of $\mathbb{R}_+$, the topologies considered and the procedure we carry out below, can be done independent of ρ.

Decoupling and linearizing (8.1)–(8.2) around the exact solution (q_1,q_2) as we did in Section 3.2, we obtain the nonlinear system

$$\Delta v_1+2\sqrt{2}a_0e^{-2\sqrt{2}q}v_1+\mathrm{N}(v_1)=\tilde{g}_1,\quad \text{in } \mathbb{R}^2, \tag{8.11}$$

$$\Delta v_2=\tilde{g}_2,\quad \text{in } \mathbb{R}^2, \tag{8.12}$$

where we consider right-hand side functions $\tilde{g}_j$ such that

$$\|\tilde{g}_j\|_{p,\beta}:=\left\|\left(1+|x'|^\beta\right)\tilde{g}_j\right\|_{L^p(\mathbb{R}^2)}<\infty,\quad j=1,2 \tag{8.13}$$

for some $p>1$ and $\beta\geqslant 0$ and where we have denoted

$$\mathrm{N}(v_1)=-e^{-2\sqrt{2}q}\left[e^{-\sqrt{2}v_1}-1+\sqrt{2}v_1\right]. \tag{8.14}$$

Let us consider first the linear system associated to (8.11)–(8.12), namely

$$\Delta v_1+2\sqrt{2}a_0e^{-2\sqrt{2}q}v_1=\tilde{g}_1,\quad \text{in } \mathbb{R}^2, \tag{8.15}$$

$$\Delta v_2=\tilde{g}_2,\quad \text{in } \mathbb{R}^2. \tag{8.16}$$

Since our setting is radially symmetric, we deal with this system using variations of parameters formula. We solve first Eq. (8.15). Taking derivatives in (8.9) respect to γ and ρ, for $\gamma=1$ and $\rho=1$, we find that the functions $\psi_1(r)=\partial_\gamma q(r,1,1)$ and $\psi_2(r)=\partial_\rho q(r,1,1)$ span the set of radially symmetric solutions to

$$\Delta\psi+2\sqrt{2}a_0e^{-2\sqrt{2}q}\psi=0,\quad \text{in } \mathbb{R}^2,$$

where

$$\sqrt{2}\psi_1(r)=\frac{\log(r)(r^2-1)}{r^2+1}-1,\qquad \sqrt{2}\psi_2(r)=\frac{r^2-1}{r^2+1}. \tag{8.17}$$

Observe that ψ_1 is clearly singular at the origin. Observe also that

$$\partial_r\psi_1(r)=\frac{-1+r^4+4r^2\log(r)}{\sqrt{2}r(1+r^2)^2},\qquad \partial_r\psi_2(r)=\frac{2\sqrt{2}r}{(1+r^2)^2} \tag{8.18}$$

so that from (8.18) we find that

$$\frac{c}{r} \leqslant \left|\partial_r \psi_1(r)\right| \leqslant \frac{C}{r}, \qquad \left|\partial_r \psi_2(r)\right| \leqslant \frac{Cr}{1+r^4}, \quad r > 0. \tag{8.19}$$

We compute the Wronskian

$$\mathrm{W}(\psi_1, \psi_2) := \psi_1 \partial_r \psi_2 - \psi_2 \partial_r \psi_1 = -\frac{1}{2r}$$

and we observe that the function

$$v_1(r) = 2\psi_1(r) \int_0^r \xi \psi_2(\xi) \tilde{g}_1(\xi)\, d\xi + 2\psi_2(r) \int_r^\infty \xi \psi_1(\xi) \tilde{g}_1(\xi)\, d\xi \tag{8.20}$$

defines a smooth solution to Eq. (8.15). From (8.17) and (8.18), we directly check that $\partial_r v_1(0) = 0$ and that

$$\|v_1\|_{2,p,\beta} \leqslant C \|\tilde{g}_1\|_{p,\beta}, \quad p, \beta > 2,$$

where

$$\|v_1\|_{2,p,\beta} := \left\|D^2 v_1\right\|_{p,\beta} + \left\|\left(1+|x'|\right) D v_1\right\|_{L^\infty(\mathbb{R}^2)} + \left\|\log\left(2+|x'|\right)^{-1} v_1\right\|_{L^\infty(\mathbb{R}^2)}. \tag{8.21}$$

Next, we observe that (8.16) has a radially symmetric smooth solution given by

$$v_2(r) := \int_r^\infty \xi \log(\xi) \tilde{g}_2(\xi)\, d\xi + \log(r) \int_0^r \xi \tilde{g}_2(\xi)\, d\xi. \tag{8.22}$$

Taking $p, \beta > 2$, we see directly from this formula that

$$\|v_2\|_{2,p,\beta} \leqslant C \|\tilde{g}_2\|_{p,\beta}.$$

We are now in position to invert the linear system (8.11)–(8.12). We collect this information in the following lemma:

Lemma 8.1. *Assume $p > 2$, $0 < \beta < 4 - \frac{2}{p}$ and consider a vector function $(\tilde{g}_1, \tilde{g}_2)$ satisfying*

$$\|\tilde{g}_j\|_{p,\beta} \leqslant C\alpha^{\kappa_1}, \quad j = 1, 2$$

for some small parameter $\alpha > 0$ and some $\kappa_1 > 0$. Then, the vector function (v_1, v_2) defined (8.20)–(8.22) is the solution to the system (8.11)–(8.12) and satisfies that

$$\|v_j\|_{2,p,\beta} \leqslant C \max_{k=1,2} \|\tilde{g}_k\|_{p,\beta}, \quad j = 1, 2.$$

Even more this solution turns out to be Lipschitz in the vector function $(\tilde{g}_1, \tilde{g}_2)$, namely

$$\|v_j - \hat{v}_j\|_{2,p,\beta} \leqslant C \max_{k=1,2} \|\tilde{g}_k - \hat{g}_k\|_{p,\beta}, \quad j = 1, 2.$$

The proof of this lemma is straightforward from the previous comment, proceeding as in Section 4. Let us remark that in the case where $\tilde{g}_j$, $j = 1, 2$, are nonlocal operators in (v_1, v_2) having small Lipschitz constant a direct application of Banach fixed point theorem will also lead to the existence of a unique solution to (8.11)–(8.12).

Remark 8.2. When looking for solutions to (8.4)–(8.3) that are symmetric respect to the x_3-axis, i.e. $f_2 = -f_1$ then $\tilde{g}_2 = 0$ and consequently the function v_2 defined in (8.22) is zero. Hence, we deal only with the single linear equation (8.15).

8.2. Approximate solution to the projected problem

Now that we have described the location of the nodal set of our solution, we proceed to set up our approximation. Consider a radially symmetric solution (q_1, q_2) to the system

$$\Delta q_1 + a_0 e^{-\sqrt{2}(q_2 - q_1)} = 0, \qquad \Delta q_2 - a_0 e^{-\sqrt{2}(q_2 - q_1)} = 0, \quad \text{in } \mathbb{R}^2, \tag{8.23}$$

where

$$a_0 := \|w'\|^{-2}_{L^2(\mathbb{R})} \int_{\mathbb{R}} 6\big(1 - w(t)^2\big) e^{\sqrt{2}t} w'(t)\, dt.$$

Recall from the previous section that we have chosen $-q_1 = q_2 = q$, and the function q is a solution to the Liouville equation

$$\Delta q - a_0 e^{-2\sqrt{2}q} = 0, \quad \text{in } \mathbb{R}^2$$

given explicitly by

$$q(x', \rho) = \frac{1}{2\sqrt{2}} \log\left(\frac{\sqrt{2}a_0}{4\rho^2} \big(1 + \rho^2 |x'|^2\big)^2 \right), \tag{8.24}$$

and observe that, for every $\alpha > 0$ the vector function $(\mathrm{q}_{1\alpha}, \mathrm{q}_{2\alpha})$, defined by

$$\mathrm{q}_{1,\alpha}(x') = -\frac{1}{2\sqrt{2}} \log\left(\frac{1}{\alpha^2}\right) - q(\alpha x'), \qquad \mathrm{q}_{2,\alpha}(x') = \frac{1}{2\sqrt{2}} \log\left(\frac{1}{\alpha^2}\right) + q(\alpha x'), \quad r > 0$$

are also smooth radially symmetric solutions to (8.23).

Now, for $\alpha > 0$ small, consider a parameter function v, satisfying

$$\|v\|_{2,p,\beta} := \big\|D^2 v\big\|_{p,\beta} + \big\|\big(1 + |x'|\big) Dv\big\|_{L^\infty(\mathbb{R}^2)} + \big\|\log\big(2 + |x'|\big)^{-1} v\big\|_{L^\infty(\mathbb{R}^2)} \leqslant K\alpha^2 \big|\log(\alpha)\big| \tag{8.25}$$

for some $K > 0$ that will be chosen later and independent of $\alpha > 0$ and consider the functions

$$\mathrm{f}_{l\alpha}(x') = \mathrm{q}_{l\alpha}(x') + \mathrm{v}_{l\alpha}(x'), \quad l = 1, 2, \tag{8.26}$$

where $\mathrm{v}_{2\alpha} = -\mathrm{v}_{1\alpha} = \mathrm{v}_\alpha$ and $\mathrm{v}_\alpha(x') = v(\alpha x')$.

Proceeding as in the proof of Theorem 2, we consider as local approximation the function

$$U_0(x) = w\big(x_3 - \mathrm{f}_{1\alpha}(x')\big) + w\big(x_3 - \mathrm{f}_{2\alpha}(x')\big) - 1, \quad x \in \mathbb{R}^3. \tag{8.27}$$

As in Section 5.1, let us consider the sets

$$A_l := \left\{ x = (x', x_3) \colon \big|x_3 - \mathrm{f}_{j\alpha}(x')\big| \leqslant \frac{1}{2}\big(\mathrm{f}_{2\alpha}(x') - \mathrm{f}_{1\alpha}(x')\big) \right\}, \quad l = 1, 2.$$

Writing $z = t + \mathrm{f}_{l\alpha}(x')$, we notice that A_l can be described as

$$A_l := \left\{ x = (x', t) \colon \ |t| \leqslant \frac{1}{2}\big(\mathrm{f}_{2\alpha}(x') - \mathrm{f}_{1\alpha}(x')\big) \right\}, \quad l = 1, 2.$$

Hence, we can estate the following lemma regarding the error of this approximation in the set A_l.

Lemma 8.2. *For $l = 1, 2$ and every $x \in A_l$, $x = (x', t)$, we have that*

$$\begin{aligned}
(-1)^{l-1} S(U_0) = {} & -\Delta_{\mathbb{R}^2} \mathrm{f}_{l\alpha} w'(t) + (-1)^l 6\big(1 - w^2(t)\big) e^{(-1)^{l-1}\sqrt{2}t} e^{-\sqrt{2}(\mathrm{f}_{2\alpha} - \mathrm{f}_{1\alpha})} \\
& + |\nabla \mathrm{f}_{l\alpha}|^2 w''(t) - \Delta_{\mathbb{R}^2} \mathrm{f}_{j\alpha} w'(t + \mathrm{f}_{l\alpha} - \mathrm{f}_{j\alpha}) + |\nabla \mathrm{f}_{j\alpha}|^2 w''(t + \mathrm{f}_{l\alpha} - f_{j\alpha}) \\
& + \big[(-1)^{l-1} 6\big(1 - w^2(t)\big) + 12\big(1 + (-1)^l w(t)\big)\big] e^{(-1)^{l-1} 2\sqrt{2}t} e^{-2\sqrt{2}(\mathrm{f}_{2\alpha} - \mathrm{f}_{1\alpha})} \\
& + \mathrm{R}_l\big(\alpha x', t, v, Dv\big),
\end{aligned} \tag{8.28}$$

where $\mathrm{R}_l = \mathrm{R}_l(\alpha y, t, p, q)$ is smooth on its arguments and

$$\big|D_p \mathrm{R}_l\big(\alpha x', t, p, q\big)\big| + \big|D_q \mathrm{R}_l\big(\alpha x', t, p, q\big)\big| + \big|\mathrm{R}_l\big(\alpha x', t, p, q\big)\big| \leqslant C\alpha^{2+\tau}\big(1 + |\alpha x'|\big)^{-4} e^{-\varrho|t|} \tag{8.29}$$

for some $0 < \tau < 1$ small and some $0 < \varrho < \sqrt{2}$ and where $p = v$ and $q = Dv$.

Proof. The proof of this lemma follows the same lines of Lemma 5.1, with no significant changes and actually with easier computations. So, we only remark that in the set A_1

$$U_0\big(x', t\big) = w(t) - w(t + \mathrm{f}_{1\alpha} - \mathrm{f}_{2\alpha}) - 1,$$

where the function $w(s)$ is the heteroclinic solution to

$$w'' + F(w) = 0, \qquad w(\pm\infty) = \pm 1, \quad w' > 0$$

having the asymptotic expansion

$$\begin{aligned}
w(s) &= 1 - 2e^{-\sqrt{2}s} + 2e^{-2\sqrt{2}s} + \mathcal{O}\big(e^{-2\sqrt{2}|s|}\big), \quad s > 0, \\
w(s) &= -1 + 2e^{\sqrt{2}s} - 2e^{-2\sqrt{2}s} + \mathcal{O}\big(e^{-3\sqrt{2}|s|}\big), \quad s < 0,
\end{aligned} \tag{8.30}$$

where these relations can be differentiated. Using that $F(\pm 1) = 0$,

$$\begin{aligned}
F(U_0) = {} & F\big(w(t)\big) - F\big(w(t + \mathrm{f}_{1\alpha} - \mathrm{f}_{2\alpha})\big) - \big(F'\big(w(t)\big) - F'(-1)\big)\big[w(t + \mathrm{f}_{1\alpha} - \mathrm{f}_{2\alpha}) + 1\big] \\
& + \frac{1}{2}\big(F''\big(w(t)\big) + F'(-1)\big)\big[w(t + \mathrm{f}_{1\alpha} - \mathrm{f}_{2\alpha}) + 1\big]^2 + \mathcal{O}\big(\big[w(t + \mathrm{f}_{1\alpha} - \mathrm{f}_{2\alpha}) + 1\big]^3\big).
\end{aligned}$$

From (8.30) we obtain that

$$\begin{aligned}
F(U_0) = {} & F\big(w(t)\big) - F\big(w(t + \mathrm{f}_{1\alpha} - \mathrm{f}_{2\alpha})\big) - 6\big(1 - w^2(t)\big) e^{\sqrt{2}t} e^{-\sqrt{2}(\mathrm{f}_{2\alpha} - \mathrm{f}_{1\alpha})} \\
& + 6\big[\big(1 - w^2(t)\big) + 2\big(1 - w(t)\big)\big] e^{2\sqrt{2}t} e^{-2\sqrt{2}(\mathrm{f}_{2\alpha} - \mathrm{f}_{1\alpha})} + \mathcal{O}\big(e^{-3\sqrt{2}|t + \mathrm{f}_{1\alpha} - \mathrm{f}_{2\alpha}|}\big).
\end{aligned}$$

Similar computations hold true in the set A_2 and this completes the proof of the lemma. □

Using the fact that the vector function $\mathrm{q} = (q_1, q_2)$ is an exact solution to the Toda system in $\mathbb{R}^2$ and using the function for g_0 described in (5.17), we can write expression (8.28) as

$$
\begin{aligned}
(-1)^{l-1} S(U_0) = {} & -\Delta_{\mathbb{R}^2} \mathrm{v}_{l\alpha} w'(t) + (-1)^l 6\big(1 - w^2(t)\big) e^{(-1)^{l-1}\sqrt{2}t} e^{-\sqrt{2}(\mathrm{q}_{2,\alpha} - \mathrm{q}_{1,\alpha})}\big(e^{-\sqrt{2}(\mathrm{v}_{2\alpha} - \mathrm{v}_{1\alpha})} - 1\big) \\
& + (-1)^l g_0\big((-1)^l t\big) e^{-\sqrt{2}(\mathrm{q}_{2,\alpha} - \mathrm{q}_{1,\alpha})} + |\nabla \mathrm{q}_{l\alpha}|^2 w''(t) \\
& + \nabla \mathrm{v}_{l\alpha}(2\nabla \mathrm{q}_{l\alpha} + \nabla \mathrm{v}_{l\alpha}) w''(t) - \Delta_{\mathbb{R}^2} \mathrm{f}_{j\alpha} w'(t + \mathrm{f}_{l\alpha} - \mathrm{f}_{j\alpha}) + |\nabla \mathrm{f}_{j\alpha}|^2 w''(t + \mathrm{f}_{l\alpha} - f_{j\alpha}) \\
& + \big[(-1)^{l-1} 6\big(1 - w^2(t)\big) + 12\big(1 + (-1)^l w(t)\big)\big] e^{(-1)^{l-1} 2\sqrt{2}t} e^{-2\sqrt{2}(\mathrm{f}_{2\alpha} - \mathrm{f}_{1\alpha})} \\
& + \mathrm{R}_l\big(\alpha x', t, v, Dv\big).
\end{aligned}
\tag{8.31}
$$

Next, we improve the approximation by considering the function

$$
U_1\big(x', x_3\big) = U_0\big(x', x_3\big) + \varphi_{1,0}\big(x', x_3 - \mathrm{f}_{1\alpha}\big) - \varphi_{2,0}\big(x', x_3 - \mathrm{f}_{2\alpha}\big)
$$

and

$$
(-1)^{l+1} \varphi_{l,0}\big(x', t\big) = e^{-\sqrt{2}(\mathrm{q}_{2\alpha} - \mathrm{q}_{1\alpha})} \psi_0\big((-1)^l t\big) + |\nabla \mathrm{q}_{l\alpha}|^2 \psi_1(t),
$$

where the functions $\psi_0(t)$ is the one described in (5.19) and $\psi_1(t) = -\frac{1}{2} t w'(t)$.

We recall that

$$
\mathrm{q}_{2\alpha}\big(x'\big) = -\mathrm{q}_{1\alpha}\big(x'\big) = \frac{1}{2\sqrt{2}} \log\left(\frac{1}{\alpha^2}\right) + \frac{1}{2\sqrt{2}} \log\left(\frac{\sqrt{2}a_0}{4}\big(1 + \big|\alpha x'\big|^2\big)^2\right)
$$

so that

$$
e^{-\sqrt{2}(\mathrm{q}_{2\alpha} - \mathrm{q}_{1\alpha})} = \frac{\alpha^2}{a_0\sqrt{2}} \frac{4}{(1 + |\alpha x'|^2)^2}.
$$

Proceeding as in Section 5.2 and setting $z = t + \mathrm{f}_{l\alpha}$, we compute the new error created in the region A_l

$$
\begin{aligned}
(-1)^{l-1} S(U_1) = {} & -\Delta_{\mathbb{R}^2} \mathrm{v}_{l\alpha} w'(t) - (-1)^l 6\big(1 - w^2(t)\big) e^{\sqrt{2}(-1)^{l-1}t} e^{-\sqrt{2}(\mathrm{q}_{2\alpha} - \mathrm{q}_{1\alpha})}\big[e^{-\sqrt{2}(\mathrm{v}_{2\alpha} - \mathrm{v}_{1\alpha})} - 1\big] \\
& + \nabla \mathrm{v}_{l\alpha}(2\nabla \mathrm{q}_{l\alpha} + \nabla \mathrm{v}_{l\alpha}) w''(t) - \Delta_{\mathbb{R}^2} \mathrm{f}_{j\alpha} w'(t + \mathrm{f}_{l\alpha} - \mathrm{f}_{j\alpha}) + |\nabla \mathrm{f}_{j\alpha}|^2 w''(t + \mathrm{f}_{l\alpha} - f_{j\alpha}) \\
& + \big[(-1)^{l-1} 6\big(1 - w^2(t)\big) + 12\big(1 + (-1)^l w(t)\big)\big] e^{(-1)^{l-1} 2\sqrt{2}t} e^{-2\sqrt{2}(\mathrm{f}_{2\alpha} - \mathrm{f}_{1\alpha})} + \tilde{\mathrm{R}}_l,
\end{aligned}
\tag{8.32}
$$

where

$$
\tilde{\mathrm{R}}_l = \tilde{\mathrm{R}}_l\big(\alpha x', t, v, Dv\big)
$$

and

$$
\big|D_p \tilde{\mathrm{R}}_l\big(\alpha x', t, p, q\big)\big| + \big|D_q \tilde{\mathrm{R}}_l\big(\alpha\alpha x', t, p, q\big)\big| + \big|\tilde{\mathrm{R}}_l\big(\alpha x', t, p, q\big)\big| \leqslant C\alpha^{2+\tau}\big(1 + \big|\alpha x'\big|\big)^{-4} e^{-\varrho|t|}
\tag{8.33}
$$

for some $0 < \varrho < \sqrt{2}$ and some $0 < \tau < 1$. Actually, from the proof of Lemma 8.2 we have that

$$
\big|\tilde{\mathrm{R}}_l\big(\alpha x', t, v, Dv\big)\big| \leqslant C e^{-3\sqrt{2}|t + \mathrm{f}_{1\alpha} - \mathrm{f}_{2\alpha}|}, \quad \text{in } A_1.
$$

The next step, consists on defining the global approximation to the solution. We consider again the smooth cut-off function $\beta \in C_c^\infty(\mathbb{R})$, such that $\beta(t) = 1$, for $|t| \leqslant 1/2$ and $\beta(t) = 0$, for $|t| \leqslant 1$. Now, for $\alpha > 0$ small we define the cut-off function

$$
\beta_\alpha(x) := \beta\left(|x_3| - \frac{\eta}{\alpha} - 4\log\big(\big|\alpha x'\big| + 3\big)\right), \quad x = \big(x', x_3\big) \in \mathbb{R}^3.
$$

We see that β_α is supported in a region that expands logarithmically in $|\alpha x'|$ and we consider as global approximation the function

$$\mathrm{w}(x) := \beta_\alpha(x)U_1(x) + \big(1 - \beta_\alpha(x)\big)(-1). \tag{8.34}$$

Recalling that $F(u) = u(1-u^2)$, we compute the new error as follows

$$S(\mathrm{w}) = \Delta\mathrm{w} + F(\mathrm{w}) = \beta_\alpha(x)S(U_1) + E,$$

where

$$E = 2\nabla\beta_\alpha\nabla U_1 + \Delta\beta_\alpha(U_1+1) + F\big(\beta_\alpha U_1 - (1-\beta_\alpha)\big) - \beta_\alpha F(U_1).$$

Due to the choice of $\beta_\alpha(x)$, the error term E only takes into account values of β_α for $x \in \mathbb{R}^3$ in the region

$$|x_3| \geqslant \frac{\eta}{\alpha} + 4\ln\big(\left|\alpha x'\right| + 3\big) - 2, \quad x = \big(x', x_3\big) \in \mathbb{R}^3$$

and so, we get the following estimate for the term E

$$|\nabla E| + |E| \leqslant Ce^{-\frac{\eta}{\alpha}}\big(1 + \left|\alpha x'\right|\big)^{-4}.$$

We observe that the error E decays rapidly and is exponentially small in $\alpha > 0$, so that its contribution is negligible.

Remark 8.3. The local approximation U_1 is clearly axially symmetric and even in the z-axis. This is due to the fact that the graph of the function $\mathrm{f}_{1\alpha}$ is a reflection through the z-axis of the graph of the function $\mathrm{f}_{2\alpha}$. Of course, this is also true for the global approximation w. Observe also that for the moment, we are omitting the role of the parameter $\rho > 0$, but clearly the approximations U_1 and w and the error created depend smoothly on it.

8.3. Outline of the Lyapunov–Schmidt reduction

Let us consider first an appropriate functional setting to work with. Consider the norms

$$\|f\|_{p,\hat{\mu},\sim} := \sup_{x\in\mathbb{R}^3}\big(1 + \left|\alpha x'\right|\big)^{\hat{\mu}}\|f\|_{L^p(B_1(x))}, \quad p > 1. \tag{8.35}$$

and

$$\|\psi\|_{2,p,\hat{\mu},\sim} := \left\|D^2\psi\right\|_{p,\hat{\mu},\sim} + \|D\psi\|_{\infty,\hat{\mu},\sim} + \|\psi\|_{\infty,\hat{\mu},\sim} \tag{8.36}$$

where $0 < \hat{\mu} \leqslant \min(2\mu, \mu + \varrho\sqrt{2}, 2 + \varrho\sqrt{2}\,)$.

We also consider $0 < \varrho < \sqrt{2}$, $\mu > 0$, $\alpha > 0$ and functions $g = g(x', t)$ and $\phi = \phi(y, t)$, defined for every $(y, t) \in M_\alpha \times \mathbb{R}$. Let us set the norms

$$\|g\|_{p,\mu,\varrho} := \sup_{(x',t)\in R^2\times\mathbb{R}}\big(1 + \left|\alpha x'\right|\big)^{\mu}e^{\varrho|t|}\|g\|_{L^p(B_1(x',t))} \tag{8.37}$$

$$\|\phi\|_{\infty,\mu,\varrho} := \left\|\big(1 + \left|\alpha x'\right|^{\mu}\big)e^{\varrho|t|}\phi\right\|_{L^\infty(\mathbb{R}^2\times\mathbb{R})} \tag{8.38}$$

$$\|\phi\|_{2,p,\mu,\varrho} := \left\|D^2\phi\right\|_{p,\mu,\varrho} + \|D\phi\|_{\infty,\mu,\varrho} + \|\phi\|_{\infty,\mu,\varrho}. \tag{8.39}$$

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Finally, for functions v and $\tilde{g}$ defined in $\mathbb{R}^2$, recall the definition of the norms

$$\|\tilde{g}\|_{p,\beta} := \left\|\left(1+|x'|^{\beta}\right)\tilde{g}\right\|_{L^p(\mathbb{R}^2)}, \tag{8.40}$$

$$\|v\|_{2,p,\beta} := \left\|D^2 v\right\|_{p,\beta} + \left\|\left(1+|x'|\right)Dv\right\|_{L^\infty(\mathbb{R}^2)} + \left\|\log\left(|x'|+2\right)^{-1} v\right\|_{L^\infty(\mathbb{R}^2)}. \tag{8.41}$$

Observe that the functional setting we are considering in this part is basically the same one used for the proof of Theorem 1.

Let us recall that our goal is to find an axially symmetric solution to Eq. (1.1) which is close to the function w defined in (8.34).

We proceed as in Section 6, with no significant changes, so we rather prefer to give an outline of the scheme. We consider for $l = 1, 2$ and $n \in \mathbb{N}$, the cut off function

$$\zeta_{l,n}(x) = \beta\left(|t| - \frac{1}{2}\left[\mathrm{f}_{2\alpha}(x') - \mathrm{f}_{1\alpha}(x')\right] + n\right), \quad x = \left(x', t + \mathrm{f}_{l\alpha}\right) \in \mathbb{R}^2. \tag{8.42}$$

A crucial observation we make is that, under assumptions (8.25), directly from Lemma 8.2 and the choice of the functional setting, the error

$$\begin{aligned}(-1)^{l-1}S_l(\mathrm{w}) := {} & -\Delta_{\mathbb{R}^2}\mathrm{v}_{l\alpha}w'(t) - (-1)^l 6\left(1-w^2(t)\right)e^{\sqrt{2}(-1)^{l-1}t}e^{-\sqrt{2}(\mathrm{q}_{2\alpha}-\mathrm{q}_{1\alpha})}\zeta_{l,2}\left[e^{-\sqrt{2}(\mathrm{v}_{2\alpha}-\mathrm{v}_{1\alpha})} - 1\right] \\ & + \nabla\mathrm{v}_{l\alpha}(2\nabla\mathrm{q}_{l\alpha} + \nabla\mathrm{v}_{l\alpha})w''(t) - \zeta_{l,2}\Delta_{\mathbb{R}^2}\mathrm{f}_{j\alpha}w'(t+\mathrm{f}_{l\alpha}-\mathrm{f}_{j\alpha}) + \zeta_{l,2}|\nabla\mathrm{f}_{j\alpha}|^2 w''(t+\mathrm{f}_{l\alpha}-f_{j\alpha}) \\ & + \zeta_{l,2}\left[(-1)^{l-1}6\left(1-w^2(t)\right) + 12\left(1+(-1)^l w(t)\right)\right]e^{(-1)^{l-1}2\sqrt{2}t}e^{-2\sqrt{2}(\mathrm{f}_{2\alpha}-\mathrm{f}_{1\alpha})} + \zeta_{l,2}\tilde{\mathrm{R}}_l\end{aligned} \tag{8.43}$$

has the size

$$\left\|S_l(\mathrm{w})\right\|_{p,2,\varrho} \leqslant C\alpha^{2+\tau_1}. \tag{8.44}$$

where $0 < \varrho < \sqrt{2}$ and $0 < \tau_1 \leqslant 1$ is arbitrarily close or equal to 1, in which case ϱ goes or equals 0, independently of $\alpha > 0$. The following proposition collects estimates regarding (8.44).

Proposition 8.1. *Assume $\varrho \in (0, \sqrt{2}\,)$ and that the functions $\mathrm{f}_{j\alpha}$ satisfy condition (8.25). Then there exist a constant $C > 0$ and a small number $0 < \tau_1 \leqslant 1$, both independent of $\alpha > 0$, such that*

$$\left\|S_l(\mathrm{w})\right\|_{p,2,\varrho} \leqslant C\alpha^{2+\tau_1} \tag{8.45}$$

and

$$\left\|S_l(\mathrm{w}, v) - S_l(\mathrm{w}, \tilde{v})\right\|_{p,2,\varrho} \leqslant C\alpha^{2+\tau_1}\|v-\tilde{v}\|_{2,p,\beta}, \tag{8.46}$$

where

$$\|v\|_{2,p,\beta} := \left\|D^2 v\right\|_{p,\beta} + \left\|\left(1+|x'|\right)Dv\right\|_{L^\infty(\mathbb{R}^2)} + \left\|\log\left(2+|x'|\right)^{-1} v\right\|_{L^\infty(\mathbb{R}^2)}. \tag{8.47}$$

As before, we look for a solution to (1.1) of the form

$$U = \mathrm{w} + \zeta_{1,3}(x)\phi_1\left(x', x_3 - \mathrm{f}_{1\alpha}\right) - \zeta_{2,3}(x)\phi_2\left(x', x_3 - \mathrm{f}_{2\alpha}\right) + \psi \tag{8.48}$$

so that we fall into a system of elliptic PDEs for ϕ_1, ϕ_2 and ψ similar to (6.9)–(6.11).

The linear theory needed to solve this problem is a copy of the one sketched in Section 7, but applied to the system

$$\Delta\psi(x) - 2\psi(x) = h(x), \quad x \in \mathbb{R}^3, \tag{8.49}$$

$$\partial_{tt}\phi_l(x',t) + \Delta_{\mathbb{R}^2}\phi_l(x',t) + F'(w(t))\phi_l(x',t) = g_l(x',t) + c_l(x')w'(t), \quad \text{in } \mathbb{R}^2 \times \mathbb{R} \tag{8.50}$$

in the class of axially symmetric functions and in the topologies induced by the norms set above. In particular, the nonlinear nonlocal system of equations for the functions ϕ_l reads as

$$\begin{aligned}\partial_{tt}\phi_l + \Delta_{\mathbb{R}^2}\phi_l + F'(w(t))\phi_l = {} & -S_l(\mathrm{w}) - B_l(\phi_l) - \left[F'(\zeta_{l,2}\mathrm{w}) - (-1)^{l-1}F'(w(t))\right]\phi_l \\ & - \zeta_{l,2}\left(F'(\mathrm{w}) + 2\right)\psi - \zeta_{l,2}N(\phi_l + \psi), \quad \text{in } \mathbb{R}^2 \times \mathbb{R}\end{aligned} \tag{8.51}$$

with

$$B_l(\phi_l) := -\Delta_{\mathbb{R}^2}\mathrm{f}_{l\alpha}\partial_t\phi_l - 2\nabla\mathrm{f}_{l\alpha}\nabla_{x'}\partial_t\phi_l + |\nabla\mathrm{f}_{l\alpha}|^2\partial_{tt}\phi_l$$

and

$$N(\phi_l + \psi) = F(\mathrm{w} + \phi_l) - F(\mathrm{w}) - F'(\mathrm{w})(\phi_l + \psi),$$

from where we get that

$$\|\phi_l\|_{2,p,2,\varrho} \leqslant C\alpha^{2+\tau_1}, \qquad \int_{\mathbb{R}} \phi_l(x',t)w'(t)\,dt = 0, \quad l = 1,2, \tag{8.52}$$

with τ_1 as above.

As we already saw, the Lyapunov–Schmidt reduction scheme is based upon the fact that we can find functions v_1, v_2 satisfying (8.25) such that the functions $c_l(x')$, $l = 1,2$ in (8.50) are zero.

8.4. Solving the reduced problem

Let us recall that

$$\mathrm{w}(x) := \beta_\alpha(x)U_1(x) + \left(1 - \beta_\alpha(x)\right)(-1) \tag{8.53}$$

where

$$\begin{aligned}U_1(x) = {} & w\left(x_3 - \mathrm{f}_{1\alpha}(x')\right) - w\left(x_3 - \mathrm{f}_{2\alpha}(x')\right) - 1 + \phi_{1,0}\left(x', x_3 - \mathrm{f}_{1\alpha}(x')\right) \\ & - \phi_{2,0}\left(x', x_3 - \mathrm{f}_{2\alpha}(x')\right)\end{aligned} \tag{8.54}$$

where for $l = 1,2$

$$\phi_{l,0}(x',t) = (-1)^{l+1}e^{-\sqrt{2}(\mathrm{q}_{2\alpha} - \mathrm{q}_{1\alpha})}\psi_0\left((-1)^{l+1}t\right) + |\nabla\mathrm{q}_{l\alpha}|^2\psi_1(t) \tag{8.55}$$

the functions ψ_0, ψ_1 are those described in (5.19) and (5.20).

Next we make use of the symmetries we have assumed for the nodal set and the local and global approximations. From the structure of Eq. (1.1) and using the fact that the approximation w is axially symmetric and even respect to the x_3-axis, we find that the functions ϕ_1, ϕ_2 and ψ share also this symmetry.

In this setting the error $S_1(\mathrm{w})$ in the region A_1 and in terms of the parameter function v_α, reads as

$$\begin{aligned} S_1(\mathrm{w}) = {} & \Delta_{\mathbb{R}^2}\mathrm{v}_\alpha w'(t) - 6\big(1-w^2(t)\big)e^{\sqrt{2}t}e^{-2\sqrt{2}\mathrm{q}_\alpha}\zeta_{l,2}\big[e^{-2\sqrt{2}\mathrm{v}_\alpha}-1\big] \\ & + \nabla\mathrm{v}_\alpha(2\nabla\mathrm{q}_\alpha+\nabla\mathrm{v}_\alpha)w''(t) + \zeta_{l,2}\Delta_{\mathbb{R}^2}\mathrm{f}_\alpha w'(t-2\mathrm{f}_\alpha) + \zeta_{l,2}|\nabla\mathrm{f}_\alpha|^2 w''(t-2\mathrm{f}_\alpha) \\ & + \zeta_{l,2}\big[6\big(1-w^2(t)\big)+12\big(1-w(t)\big)\big]e^{2\sqrt{2}t}e^{-4\sqrt{2}\mathrm{f}_\alpha} + \zeta_{l,2}\tilde{\mathrm{R}} \end{aligned} \tag{8.56}$$

with similar computations in the set A_2.

In what comes next, we derive the system that governs the location of the interfaces, namely a system of PDE's that will guarantee that

$$c_l\big(x'\big) = 0, \quad l = 1,2.$$

Since the error $S(\mathrm{w})$ is also axially symmetric and even in the x_3-variable, we easily verify that $c_2(x') = -c_1(x') = c(x')$.

In order to determine the function $c(x')$, for $l = 1$, we multiply Eq. (8.51) by w and integrate in t to get that at main order

$$-\int_{\mathbb{R}} S_1(\mathrm{w})w(t)\,dt - \mathcal{O}\big(\alpha^4\big(1+|\alpha x'|\big)^{-3}\big) = c\big(x'\big)\int_{\mathbb{R}} w'^{\,2}\,dt.$$

This can be done since in inequality (8.45) as τ_1 approaches to 1, the constant ϱ goes to zero, while the constant $C > 0$ remains uniformly bounded.

Hence using Lemma 8.2, and setting

$$c^* := \int_{\mathbb{R}} \big|w'(t)\big|^2\,dt, \qquad a_0 = \big\|w'\big\|^{-2}_{L^2(\mathbb{R})}\int_{\mathbb{R}}\big(1-w^2(t)\big)w'(t)e^{-\sqrt{2}t}\,dt$$

we find that

$$\begin{aligned} c\big(x'\big) = {} & c^*\Delta_{\mathbb{R}^2}\mathrm{v}_\alpha + c^* 2\sqrt{2}a_0 e^{-2\sqrt{2}\mathrm{q}_\alpha}\mathrm{v}_\alpha \\ & + \underbrace{\Delta_{\mathbb{R}^2}\mathrm{f}_\alpha\int_{\mathbb{R}}\zeta_{1,2}w'(t-2\mathrm{f}_\alpha)w'(t)\,dt}_{A} + \underbrace{|\nabla\mathrm{f}_\alpha|^2\int_{\mathbb{R}}\zeta_{1,2}w''(t-2\mathrm{f}_\alpha)w'(t)\,dt}_{B} \\ & + \underbrace{e^{-4\sqrt{2}\mathrm{f}_\alpha}\int_{\mathbb{R}}\zeta_{1,2}\big[6\big(1-w^2(t)\big)+2\big(1-w(t)\big)\big]e^{2\sqrt{2}t}w'(t)\,dt}_{C} \\ & + c^* a_0 e^{-2\sqrt{2}\mathrm{q}_\alpha}\big[e^{-2\sqrt{2}\mathrm{v}_\alpha}-1\big]\int_{\mathbb{R}} 6\big(1-w^2(t)\big)e^{\sqrt{2}t}w'(t)(\zeta_{1,2}-1)\,dt \\ & - c^* a_0 e^{-2\sqrt{2}\mathrm{q}_\alpha}\big[e^{-2\sqrt{2}\mathrm{v}_\alpha}-1+2\sqrt{2}\mathrm{v}_\alpha\big] + \mathcal{O}\big(\alpha^4\big(1+|\alpha x'|\big)^{-3}\big) \end{aligned} \tag{8.57}$$

and using Lemma 8.1 one finds that

$$c\big(x'\big) + c^*\Delta_{\mathbb{R}^2}\mathrm{v}_{l\alpha}\big(x'\big) + 2\sqrt{2}c^* a_0 e^{-\sqrt{2}(\mathrm{q}_{2\alpha}-\mathrm{q}_{1\alpha})}\mathrm{v}_\alpha$$

is Lipschitz in the parameter function v_α. Actually it is not hard to check from Lemma 8.2 that its Lipschitz constant is of order $\mathcal{O}(\alpha^{2+\tau})$, for some $0 < \tau < \tau_1$ small. Hence we see that making $c(x') = 0$ is equivalent to a nonlinear and nonlocal equation of the form

$$\Delta v + 2\sqrt{2}a_0 e^{-2\sqrt{2}q} v = G(v), \quad \text{in } \mathbb{R}^2, \tag{8.58}$$

where we conclude from (8.57) that $A + B + C$ is the leading order term in the expression for $G(v)$.

In order to give a more precise expression for the nonlinear term $G(v)$, we recall that $w(s)$, the heteroclinic solution to

$$w'' + F(w) = 0, \qquad w(\pm\infty) = \pm 1, \quad w' > 0$$

has the asymptotic behavior

$$\begin{aligned} w(s) &= 1 - 2e^{-\sqrt{2}s} + 2e^{-2\sqrt{2}s} + \mathcal{O}\big(e^{-2\sqrt{2}|s|}\big), \quad s > 0, \\ w(s) &= -1 + 2e^{\sqrt{2}s} - 2e^{-2\sqrt{2}s} + \mathcal{O}\big(e^{-3\sqrt{2}|s|}\big), \quad s < 0, \end{aligned} \tag{8.59}$$

and these relations can be differentiated.

Since in the set A_1

$$w'(t - 2\mathrm{f}_\alpha) = 2\sqrt{2}e^{\sqrt{2}t} e^{-2\sqrt{2}\mathrm{f}_\alpha} + \mathcal{O}\big(e^{-2\sqrt{2}|t-2\mathrm{f}_\alpha|}\big)$$

we obtain that

$$w'(t - 2\mathrm{f}_\alpha)w'(t) = \begin{cases} 8e^{-2\sqrt{2}\mathrm{f}_\alpha} + \mathcal{O}(e^{-2\sqrt{2}|t|}e^{-2\sqrt{2}\mathrm{f}_\alpha}), & t > 0 \\ 8e^{2\sqrt{2}t}e^{-2\sqrt{2}\mathrm{f}_\alpha} + \mathcal{O}(e^{-3\sqrt{2}|t|}e^{-2\sqrt{2}\mathrm{f}_\alpha}), & t < 0. \end{cases}$$

Hence it is direct to check that

$$A = 8\Delta \mathrm{f}_\alpha e^{-2\sqrt{2}\mathrm{f}_\alpha}\big(\mathrm{f}_\alpha + \mathcal{O}(1)\big).$$

Proceeding in the same fashion, we obtain in the set A_1 that

$$w''(t - 2\mathrm{f}_\alpha)w'(t) = \begin{cases} 8\sqrt{2}e^{-2\sqrt{2}\mathrm{f}_\alpha} + \mathcal{O}(e^{-2\sqrt{2}|t|}e^{-2\sqrt{2}\mathrm{f}_\alpha}), & t > 0 \\ 8\sqrt{2}e^{2\sqrt{2}t}e^{-2\sqrt{2}\mathrm{f}_\alpha} + \mathcal{O}(e^{-3\sqrt{2}|t|}e^{-2\sqrt{2}\mathrm{f}_\alpha}), & t < 0 \end{cases}$$

so that

$$B = 8\sqrt{2}|\nabla \mathrm{f}_\alpha|^2 e^{-2\sqrt{2}\mathrm{f}_\alpha}\big(\mathrm{f}_\alpha + \mathcal{O}(1)\big).$$

Finally, we directly check using again (8.59) that

$$\big[6\big(1 - w^2(t)\big) + 2\big(1 - w(t)\big)\big]e^{2\sqrt{2}t}w'(t) = \begin{cases} 96\sqrt{2} + \mathcal{O}(e^{-\sqrt{2}|t|}), & t > 0, \\ \mathcal{O}(e^{-5\sqrt{2}|t|}), & t < 0, \end{cases}$$

from where

$$C = 96\sqrt{2}e^{-4\sqrt{2}\mathrm{f}_\alpha}\big(\mathrm{f}_\alpha + \mathcal{O}(1)\big).$$

Hence, we obtain that

$$-\alpha^2 G(v) = e^{-2\sqrt{2}\mathrm{f}_\alpha}\mathrm{f}_\alpha\big[8\Delta\mathrm{f}_\alpha + 8\sqrt{2}|\nabla\mathrm{f}_\alpha|^2 + 96\sqrt{2}e^{-2\sqrt{2}\mathrm{f}_\alpha}\big] + \mathcal{O}\big(\alpha^4\big(1 + |\alpha x'|\big)^{-3}\big).$$

From this expression, we obtain that

$$G(0) = \frac{\alpha^2}{2\sqrt{2}a_0}\left(\frac{1}{2\sqrt{2}}\log\left(\frac{1}{\alpha^2}\right) + q(x')\right)R_0(x') + \mathcal{O}\big(\alpha^4\big(1 + |\alpha x'|\big)^{-3}\big),$$

where

$$R_0(x') = -\big[8\Delta q + 8\sqrt{2}|\nabla q|^2 + 96\sqrt{2}e^{-2\sqrt{2}q}\big]e^{-2\sqrt{2}q} = -\frac{4|x'|^2 + 4 + 3\sqrt{2}}{(1 + |x'|^2)^4}.$$

Since, so far the scheme involves the same estimates as those in Proposition 6.2, we find that the function G satisfies

$$\big\|G(v)\big\|_{\infty,3} \leqslant K\alpha^2\big|\log(\alpha)\big|$$
$$\big\|G(v) - G(\tilde{v})\big\|_{\infty,3} \leqslant C\alpha^\tau\|v - \tilde{v}\|_{2,p,\beta}.$$

A direct application of Lemma 8.1 and the Banach fixed point theorem completes the construction of the solutions predicted in Theorem 1. We leave further details to the reader.

Using the integral formula (8.20) and the last remarks, and since

$$\psi_1(r) = \frac{1}{\sqrt{2}}\big(\log(r) - 1\big) + \mathcal{O}\big(r^{-2}\log(r)\big), \quad \text{as } r \to \infty$$

we can actually describe the asymptotic behavior for the function $v(x')$ as $|x'| \to \infty$, namely

$$v(x') = \frac{\alpha^2}{2a_0}\left[\log\left(\frac{1}{\alpha^2}\right)\beta_0 + \mathcal{O}(1)\right]\log\big(|x'|\big) + \mathcal{O}\big(\alpha^2\log(\alpha)\big|x'\big|^{-2}\log\big(|x'|\big)\big) \tag{8.60}$$

and

$$\beta_0 := \int_0^\infty \zeta\psi_2 R_0\, d\zeta = \frac{1}{12}(3 + 2\sqrt{2}\,) > 0.$$

Next, we study the smooth dependence of this family of solutions respect to the parameter α in order to obtain useful information about some elements of the kernel of the linear equation

$$\Delta_{\mathbb{R}^3}\phi + F'(u_\alpha)\phi = 0, \quad \text{in } \mathbb{R}^3. \tag{8.61}$$

This information is collected in the following proposition:

Proposition 8.2. *For every $\alpha > 0$ small, the functions $\partial_{x_i}u_\alpha(x', x_3)$, for $i = 1, 2, 3$ are bounded solutions to Eq. (8.61). Besides, u_α is smooth respect to α and the following asymptotic formulae hold true*

$$\partial_{x_i'}u_\alpha(x', z) = \alpha\partial_{x_i}q(\alpha x')\big[w'(x_3 - \mathrm{f}_{1\alpha}) + w'(x_3 - \mathrm{f}_{2\alpha})\big] + \alpha^2\mathcal{O}\left(\sum_{l=1}^2 e^{-\varrho|x_3 - \mathrm{f}_{l\alpha}|}\right), \quad i = 1, 2$$

$$\partial_{x_3}u_\alpha(x', x_3) = \big[w'(x_3 - \mathrm{f}_{1\alpha}) - w'(x_3 - \mathrm{f}_{2\alpha})\big] + \alpha^2\mathcal{O}\left(\sum_{l=1}^2 e^{-\varrho|x_3 - \mathrm{f}_{l\alpha}|}\right),$$

$$\partial_\alpha u_\alpha(x', x_3) = \partial_\alpha(\mathrm{q}_\alpha + \mathrm{v}_\alpha)\big[w'(x_3 - \mathrm{f}_{1\alpha}) + w'(x_3 - \mathrm{f}_{2\alpha})\big] + \alpha\mathcal{O}\left(\sum_{l=1}^2 e^{-\varrho|x_3 - \mathrm{f}_{l\alpha}|}\right).$$

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Proof. From the smoothness of these solutions we readily check the first two equalities. So, we only need to take care of the last assertion.

Smoothness respect to $\alpha > 0$ small is a direct consequence of the Implicit Function Theorem. We remark that following step by step the construction and taking into account the dependence on $\rho \sim 1$ of this family of smooth solutions we have the asymptotic behavior

$$\begin{aligned} u_{\alpha\rho}(x) &= w\big(x_3 - \mathrm{f}_{1\alpha\rho}(x')\big) - w\big(x_3 - \mathrm{f}_{2\alpha\rho}(x')\big) - 1 + \phi_{1,0}\big(x', x_3 - \mathrm{f}_{1\alpha\rho}(x')\big) - \phi_{2,0}\big(x', x_3 - \mathrm{f}_{2\alpha\rho}(x')\big) \\ &\quad + \alpha^{2+\tau_1} \sum_{l=1}^{2} \mathcal{O}\big(\big(1 + |\alpha\rho x'|\big)^{-2} e^{-\varrho|x_3 - \mathrm{f}_{l\alpha\rho}(x')|}\big) \end{aligned} \tag{8.62}$$

with $0 < \varrho < \sqrt{2}$.

Provided ρ is taken in a small, bounded and fixed interval around one, we can recast the fact that the functions $\mathrm{v} = (v_1, v_2)$, $\Phi = (\phi_1, \phi_2)$ and ψ in (8.48) yield a solution to Eq. (1.1) as a system for $(\rho, \mathrm{v}, \Phi, \psi)$ of the form

$$\Phi - \Pi(\rho, \mathrm{v}, \Phi, \psi) = 0, \qquad \psi - \tilde{\Pi}(\rho, \mathrm{v}, \Phi, \psi) = 0,$$
$$\mathrm{v} - T(\rho, \mathrm{v}, \Phi, \psi) = 0,$$

where smooth dependence on each one of the variables, in the respective topologies described in (8.36), (8.39) and (8.41), is readily check from the scheme of the construction of this family of solutions. Solvability theory of the linear problems implies that the derivative of this system respect to (v, Φ, ψ) is an isomorphism and consequently, we obtain a smooth dependence of the solution respect to ρ. Uniqueness from the fixed point argument in our proof guarantees that these solutions correspond to those ones given by the Implicit Function Theorem.

In order to find the asymptotics of $\partial_\alpha u_\alpha$, we first notice from (8.62) that at main order

$$\partial_\alpha u_\alpha\big(x', x_3\big) = \partial_\alpha U_1\big(x', x_3\big) + \partial_\alpha \phi_1 - \partial_\alpha \phi_2. \tag{8.63}$$

We need to find the size of $\partial_\alpha \phi_l$ in terms of $\alpha > 0$ in the sets

$$A_l = \left\{\big(x', x_3\big) \in \mathbb{R}^2 \times \mathbb{R}\colon\ |x_3 - \mathrm{f}_{l\alpha}| \leqslant \frac{1}{2}|\mathrm{f}_{2\alpha} - \mathrm{f}_{1\alpha}|\right\}, \quad l = 1, 2$$

and to fix ideas, let us localize $\partial_\alpha \varphi$ in A_l. Consider cut-off functions $\tilde{\zeta}_l$ supported in the set A_l.

Set $x_3 = t + \mathrm{f}_{l\alpha}$

$$L_*(\partial_\alpha \phi_l) = \partial_{tt}\partial_\alpha \phi_l + \Delta_{\mathbb{R}^2}\partial_\alpha \phi_l + F'\big(w(t)\big)\partial_\alpha \phi_l + \mathrm{B}(\partial_\alpha \phi_l),$$

where $\mathrm{B}(\partial_\alpha \phi_l)$ is a small differential operator in $\partial_\alpha \phi_l$. We find that inside A_l, $\partial_\alpha \phi_l$ solves at main order an equation of the form

$$L_*(\partial_\alpha \phi_l) + \ \mathrm{B}_l(\partial_\alpha \phi_l) = E_{l\alpha} \quad \text{in } B_{\alpha^{-1}R}(0) \times \mathbb{R},$$

where

$$E_{l\alpha} =: -\tilde{\zeta}_l\big(\Delta \partial_\alpha U_1 + F'(u_\alpha)\partial_\alpha U_1\big).$$

Since we have symmetry respect to the z-axis we only focus the developments for the set $\tilde{A}_1$, where Notice for instance that in A_1

$$\begin{aligned}
&\tilde{\zeta}_1\big(\Delta\partial_\alpha U_1 + F'(u_\alpha)\partial_\alpha U_1\big)\\
&\quad= -\Delta_{\mathbb{R}^2}\partial_\alpha \mathrm{v}_{l\alpha} w'(t) - 6\big(1-w^2(t)\big)\partial_\alpha\big(e^{\sqrt{2}(-1)^{l-1}t}e^{-\sqrt{2}(\mathrm{q}_{2\alpha}-\mathrm{q}_{1\alpha})}\big[e^{-\sqrt{2}(\mathrm{v}_{2\alpha}-\mathrm{v}_{1\alpha})}-1\big]\big)\\
&\qquad+\partial_\alpha\big(2\nabla \mathrm{q}_{l\alpha}\cdot \mathrm{v}_{l\alpha} + |\nabla \mathrm{v}_{l\alpha}|^2\big)w''(t) + \mathcal{O}\big(\alpha^{2+\tau_1}\big(1+|\alpha x'|^{2+\frac{\varrho}{\sqrt{2}}-\varepsilon}\big)^{-1}e^{-\varrho|t|}\big).
\end{aligned} \tag{8.64}$$

From (8.26) and (8.64) we observe that

$$\big|\partial_\alpha \mathrm{v}_\alpha(x')\big| \leqslant C\alpha\log\left(\frac{1}{\alpha}\right)\log\big(2+|\alpha x'|\big)$$

so that and from (8.32) it is direct to check that

$$\big\|\tilde{\zeta}_1\big(\Delta\partial_\alpha U_1 + F'(u_\alpha)\partial_\alpha U_1\big)\big\|_{p,2-\beta,\varrho} \leqslant C\alpha^{1+\tau_1}.$$

Consider functions k_1, k_2 defined by the integrals

$$\int_{\mathbb{R}} \partial_\alpha\phi_1(x',t)w'(t)\,dt = k_1(x')\int_{\mathbb{R}}\big(w'(t)\big)^2\,dt + k_2(x')\int_{\mathbb{R}}\tilde{\zeta}_1 w(t+\mathrm{f}_{1\alpha}-\mathrm{f}_{2\alpha})w'(t)\,dt$$

$$\int_{\mathbb{R}} \partial_\alpha\phi_2(x',t)w'(t) = k_2(x')\int_{\mathbb{R}}\big(w'(t)\big)^2\,dt + k_1(x')\int_{\mathbb{R}}\tilde{\zeta}_2 w(t+\mathrm{f}_{2\alpha}-\mathrm{f}_{1\alpha})w'(t)\,dt$$

so that in the set A_1 we have the decomposition

$$\tilde{\phi}_1 = k_1(x')w'(t) + \tilde{\zeta}_1 k_2(x')w'(t+\mathrm{f}_{1\alpha}-\mathrm{f}_{2\alpha}) + \varphi_1$$
$$\int_{\mathbb{R}}\varphi_1 w'(t)\,dt = 0.$$

Analogously, in A_2, we have

$$\tilde{\phi}_2 = k_2(x')w'(t) + \tilde{\zeta}_2 k_1(x')w'(t+\mathrm{f}_{2\alpha}-\mathrm{f}_{1\alpha}) + \varphi_2$$
$$\int_{\mathbb{R}}\varphi_2 w'(t)\,dt = 0.$$

We recall that

$$\int_{\mathbb{R}}\phi_l(x',t)w'(t)\,dt = 0, \quad l=1,2,$$

and taking derivative respect to α in these orthogonality condition for ϕ_l, keeping in mind that $t = x_3 - \mathrm{f}_{1\alpha}$, we obtain

$$\int_{\mathbb{R}}\partial_\alpha\phi_l w'(t)\,dt = -\partial_\alpha \mathrm{f}_\alpha(x')\int_{\mathbb{R}}\phi_l w''(t)\,dt + \mathcal{O}\big(\alpha^{1+\tau}\big(1+|\alpha x'|\big)^{-2}\big), \quad \tau > \tau_1,$$

so that

$$\left|\int_{\mathbb{R}}\partial_\alpha\phi_l w'(t)\,dt\right| = \mathcal{O}\big(\alpha^{1+\tau_1}\big(1+|\alpha x'|\big)^{-2}\log\big(2+|\alpha x'|\big)\big).$$

It is clear that the functions $k_i(x')$, $\varphi_i's$ are smooth and bounded up to their second derivatives and actually for any $\beta > 0$ small

$$\left\|D^2 k_l\right\|_{p,2-\beta} + \|Dk_l\|_{\infty,2-\beta} + \|k_l\|_{\infty,2-\beta} \leqslant C\alpha^{1+\tau_1}, \quad l = 1, 2.$$

Dropping the subindexes we have that the equation for the functions φ_l have the form

$$L_*(\varphi) + B(\varphi) = E_\alpha - S_{*,0} + B\big(kw'\big)$$

in $B_{\alpha^{-1}R}(0) \times \mathbb{R}$, where for instance in A_1

$$\begin{aligned} S_{*,0} := {} & \underbrace{\Delta_{\mathbb{R}^2} k_1 w' + |\nabla \mathrm{q}_{1\alpha}|^2 k_1 w'''}_{\hat{Q}_1} \\ & + \underbrace{F''\big(w(t)\big)\big[-\big(2e^{\sqrt{2}t} + \psi_0(t)\big)e^{-\sqrt{2}(\mathrm{q}_{2\alpha} - \mathrm{q}_{1\alpha})} + |\nabla \mathrm{q}_{1\alpha}|^2 \psi_2(t)\big] k_1 w'}_{\hat{Q}_2} \\ & + \underbrace{\sqrt{2}\big(F'(w) - F'(1)\big)e^{\sqrt{2}t} e^{-\sqrt{2}(\mathrm{q}_{2\alpha} - \mathrm{q}_{1\alpha})} k_2}_{\hat{Q}_3} - \underbrace{w''[\Delta_{\mathbb{R}^2} \mathrm{f}_{1\alpha} k_1 + 2\nabla_{x'} \mathrm{f}_{1\alpha} \nabla_{x'} k_1]}_{\hat{Q}_4} \\ & + \underbrace{\mathcal{O}\big(\alpha^{2+\tau_1}\big(1 + |\alpha y|^{2+\frac{\varrho}{\sqrt{2}}-\varepsilon}\big)^{-1} e^{-\varrho|t|}\big)}_{\hat{Q}_5} \end{aligned} \tag{8.65}$$

for some $\tau > \tau_1 > 0$ small enough. Let us write $\varphi = \bar{\varphi}_1 + \bar{\varphi}_2$, where

$$L_*(\bar{\varphi}_1) = \partial_\alpha\big(2\nabla \mathrm{q}_{l\alpha} \cdot \nabla \mathrm{v}_{l\alpha} + |\nabla \mathrm{v}_{l\alpha}|^2\big) w''(t) + \hat{Q}_4$$

with

$$\int_{\mathbb{R}} \bar{\varphi}_1 w'(t)\, dt = 0.$$

Then, we obtain the estimate

$$\left\|D^2 \bar{\varphi}_1\right\|_{p,2-\beta,\varrho} + \|D\bar{\varphi}_1\|_{\infty,2-\beta,\varrho} + \|\bar{\varphi}_1\|_{\infty,2-\beta,\varrho} \leqslant C\alpha^{1+\tau}, \quad \tau > \tau_1.$$

Next, we observe that

$$L_*(\bar{\varphi}_2) + B(\bar{\varphi}_2) = g\big(x', t\big) + c\big(x'\big) w'(t),$$

where

$$g\big(x', t\big) = E_\alpha + \partial_\alpha\big(2\nabla \mathrm{q}_{l\alpha} \cdot \nabla \mathrm{v}_{l\alpha} + |\nabla \mathrm{v}_{l\alpha}|^2\big) w''(t) - S_{*,0} - (\hat{Q}_1 + \hat{Q}_2 + \hat{Q}_3 + \hat{Q}_5) - B(\bar{\varphi}_1)$$

and observe that $\|B(\bar{\varphi}_1)\|_{p,3-\tau,\varrho} \leqslant C\alpha^{2+\tau}$. Using the size of E_α we obtain that

$$\|g\|_{p,2-\beta,\varrho} \leqslant C\alpha^{1+\tau_1}$$

for some $\tau > 0$ small enough.

Since,

$$\int_{\mathbb{R}} \bar{\varphi}_2 w'(t)\, dt = 0$$

we obtain that

$$\left\|D^2\bar{\varphi}_2\right\|_{p,2-\beta,\varrho} + \|D\bar{\varphi}_2\|_{\infty,2-\beta,\varrho} + \|\bar{\varphi}_2\|_{p,2-\beta,\varrho} \leqslant C\alpha^{1+\tau}.$$

Hence from (8.63), we can write in the set A_1

$$\partial_\alpha u_\alpha = -\partial_\alpha \mathrm{f}_{1\alpha} w'(t) + \partial_\alpha \mathrm{f}_{2\alpha} w(t - 2\mathrm{f}_\alpha)$$
$$\alpha z_1(\alpha x')w'(t) + \alpha z_2(\alpha x')w(t-2\mathrm{f}_\alpha) + \mathcal{O}\left(\alpha^{1+\tau_1}\left(1+\left|\alpha x'\right|\right)^{-2+\beta} e^{-\varrho|t|}\right)$$

with

$$\|z_l\|_{\infty,2-\beta} \leqslant C$$

and this completes the proof of the proposition, since the same procedure yields an analogous expansion in the set A_2. □

9. Morse index of the solutions in Theorems 1 and 2

In this section we provide information about the Morse index of the solutions found in Theorems 1 and 2. Most of the developments we carry out in this part are motivated by those in Section 11 of [12]. Hence, we simply remark the key points of the scheme, referring the reader to Sections 10 and 11 in [12] for more details.

Let us consider the eigenvalue problem

$$\Delta_M h + (\sigma + \lambda)|A_M|^2 h = 0, \quad \text{in } M,\ h \in L^\infty(M) \tag{9.1}$$

with $\sigma \sim \log(\alpha^{-1})$. Using the stereographic projection $\theta = \arctan(y)$, we can recast this problem as

$$\Delta_{S^2}\tilde{h} + 2(\sigma+\lambda)\tilde{h} = 0, \quad \tilde{h} \in L^\infty\left(S^2\right).$$

By standard spectral theory on the sphere, we know that

$$\lambda_k = \frac{1}{2}k(k+1) - \sigma, \quad k \in \mathbb{N}$$

are the eigenvalues to problem (9.1), so that there are at least $\mathcal{O}(\sqrt{\sigma}\,)$ negative eigenvalues for this problem.

Next, let us consider another related eigenvalue problem, namely

$$\Delta_{\mathbb{R}^2} h + \frac{8+\lambda}{(1+|x'|^2)^2} h = 0, \quad \text{in } \mathbb{R}^2,\ h \in L^\infty\left(\mathbb{R}^2\right). \tag{9.2}$$

Using the stereographic projection $\theta = \arctan(\frac{r^2-1}{r^2+1})$, we can transform (9.2) into the problem

$$\Delta_{S^2}\tilde{h} + (2+\hat{\lambda})\tilde{h} = 0, \quad \tilde{h} \in L^\infty\left(S^2\right)$$

from where it is also direct to check that problem (9.2) has exactly one negative eigenvalue. On the other hand, using Fourier decomposition and maximum principle, in Proposition 1 of [6], it was shown that the graph of the function described in (8.24) is non-degenerate, in the sense that the space of bounded solutions to (9.2) for the case $\lambda = 0$ is spanned only by the functions described in polar coordinates by

$$\bar{z}_0 = \frac{-1+r^2}{1+r^2}, \qquad \bar{z}_1 \frac{r}{1+r^2}\cos(\theta), \qquad \bar{z}_2 \frac{r}{1+r^2}\sin(\theta), \quad r>0,\ \theta \in (0,2\pi). \tag{9.3}$$

As in [12], we define the Morse index of u_α, $m(u_\alpha)$, to be the largest dimension of a vector space E of compactly supported functions for which the quadratic form

$$Q(\psi,\psi) := \int_{\mathbb{R}^3} |\nabla\psi|^2 - F'(u_\alpha)\psi^2, \quad \psi \in E - \{0\} \tag{9.4}$$

is strictly negative.

In this part, we sketch briefly the proof of the inequalities $m(u_\alpha) \geqslant c_0\sqrt{\sigma}$ for the solutions in Theorem 1, and $m(u_\alpha) \geqslant 1$ for the solutions in Theorem 2.

To prove both inequalities, we follow the scheme developed in the proof of Theorem 2 in [12], getting information about the eigenvalue problem

$$\Delta_{\mathbb{R}^3}\phi + F'(u_\alpha)\phi + \lambda p(\alpha x)\phi = 0, \quad \text{in } \mathbb{R}^3,\ \phi \in L^\infty(\mathbb{R}^3), \tag{9.5}$$

where $p(x)$ is a function such that

$$p(x) := |A_M|^2, \quad x \in \mathcal{N},$$
$$\frac{a}{1+|x'|^4} \leqslant p(x) \leqslant \frac{b}{1+|x'|^4}, \quad x \in \mathbb{R}^3 - \mathcal{N}$$

for the case in Theorem 1 and

$$p(x) := \frac{8}{(1+|x'|^2)^2}, \quad x \in \mathbb{R}^3$$

for the case in Theorem 2.

In any case, a useful characterization of $m(u_\alpha)$ is given through the following eigenvalue problem:

$$\Delta_{\mathbb{R}^3}\phi + F'(u_\alpha)\phi + \lambda p(\alpha x)\phi = 0, \quad \text{in } C_R, \qquad \phi = 0, \quad \text{on } \partial C_R, \tag{9.6}$$

where C_R is the cylinder

$$C_R := \{(x', x_3):\ |x'| < R\alpha^{-1},\ |x_3| < R\alpha^{-1}\}.$$

Let $m_R(u_\alpha)$ denote the number of negative eigenvalues to (9.6), counting multiplicities. Then, as in [12] it is straightforward to check that

$$m(u_\alpha) = \sup_{R>0} m_R(u_\alpha). \tag{9.7}$$

9.1. Estimates on the Morse index for solutions in Theorem 2

Regarding solutions of Theorem 1 and to keep computations as clear as possible we consider only the case of two transitions, namely $m = 2$. We also recall the definition of the sets

$$A_l = \left\{X_\alpha(y,\theta,z)\colon \left|z - f_l(\alpha y)\right| \leqslant \frac{1}{2}\left[\sigma + \sqrt{2}\left(1 - \frac{1}{\sigma} - M\sigma^{-\frac{5}{4}}\right)\log\left(1+(\alpha y)^2\right)\right]\right\}.$$

We remark that the solutions we have found in Theorem 1 have the asymptotic expansion for $x = X_\alpha(y,\theta,z) \in \mathcal{N}_\alpha$

$$\begin{aligned} u_\alpha(x) &= w\big(z - f_1(\alpha y)\big) - w\big(z - f_2(\alpha y)\big) - 1 + \phi_{1,0}\big(y, z - f_1(\alpha y)\big) - \phi_{2,0}\big(y, z - f_2(\alpha y)\big) \\ &\quad + \mathcal{O}\big(\alpha^{2+\tau_1}\big(1 + |\alpha y|^2\big)^{-1} e^{-\varrho|t|}\big), \end{aligned} \tag{9.8}$$

where for $l = 1,2$ and $t = z - f_l(\alpha y)$

$$\phi_{l,0}(y,t) = e^{-\sqrt{2}(h_2-h_1)}\psi_0\big((-1)^{l+1}t\big) + \alpha^2\big[h_l'\big]^2\psi_1(t) + \alpha^2\big|A_M(\alpha y)\big|^2\psi_2(t) \tag{9.9}$$

and the functions ψ_i are those described in (5.19), (5.21) and (5.20).

Using (9.8)–(9.9), we find for instance in the set A_1 and in the coordinates X_{α,f_1} that

$$\begin{aligned} F'(u_\alpha)w'(t) &= F'(w)w' + F''(w)w'\big[\big(-2e^{\sqrt{2}t} + \psi_0(t)\big)e^{-\sqrt{2}(h_2-h_1)} + \alpha^2\big[h_1'\big]^2\psi_1(t) + \alpha^2|A_M|^2\psi_2(t)\big] \\ &\quad + \mathcal{O}\big(\alpha^{2+\tau_1}\big(1+|\alpha y|^2\big)^{-1}e^{-\varrho|t|}\big). \end{aligned} \tag{9.10}$$

Since $F''(w) = -6w$, taking derivatives in the equations that the functions $\psi_i(t)$ solve, and integrating against $w'(t)$, we can easily check that

$$\int_{\mathbb{R}} F''(w)\big(w'\big)^2\big(-2e^{\sqrt{2}t} + \psi_0(t)\big)\,dt = \sqrt{2}\int_{\mathbb{R}} 6\big(1-w^2\big)e^{\sqrt{2}t}w'\,dt = \sqrt{2}a_0\int_{\mathbb{R}}\big(w'(t)\big)^2\,dt$$

$$\int_{\mathbb{R}} F''(w)\big(w'\big)^2\psi_1(t)\,dt = -\int_{\mathbb{R}} tw'w''\,dt = \frac{1}{2}\int_{\mathbb{R}}\big(w'\big)^2\,dt$$

$$\int_{\mathbb{R}} F''(w)\big(w'\big)^2\psi_2(t)\,dt = -\int_{\mathbb{R}}\big(w''\big)^2\,dt.$$

On the other hand,

$$\begin{aligned} F'(u_\alpha)w'(t+f_1-f_2) &= F'\big(w(t+f_1-f_2)\big)w'(t+f_1-f_2) \\ &\quad + \sqrt{2}\big(F'(w) - F'(1)\big)e^{\sqrt{2}t}e^{-\sqrt{2}(h_2-h_1)} + \mathcal{O}\big(\alpha^{2+\tau_1}e^{-\varrho}|t|\big(1+|\alpha y|\big)^{-4+\beta}\big), \end{aligned}$$

and since $F'(w) - F'(1) = 6(1-w^2)$, we obtain that

$$\sqrt{2}\int_{\mathbb{R}} 6\big(1-w^2(t)\big)e^{\sqrt{2}t}w'(t)\,dt = \sqrt{2}a_0\int_{\mathbb{R}}\big(w'(t)\big)^2\,dt.$$

With the previous remarks, let us now consider a test function $v(x)$ in the region $\mathcal{N}_\alpha$, defined in local coordinates $x = X_\alpha(y,\theta,z)$ as

$$v(x) := k_1(y,\theta)w'\big(z - f_1(\alpha y)\big) - k_2 w'\big(z - f_2(\alpha y)\big).$$

Using Lemma 2.1 together with (2.3), (9.8) and (9.9) and carrying out computations similar to those in Lemma 11.1 in [12], we obtain, for instance in A_1, the validity of the following expression:

$$\begin{aligned}
\Delta_{X_{\alpha,f_1}} v + F'(u_\alpha)v = {} & \underbrace{\Delta_{M_\alpha} k_1 w' - \alpha^2 |A_M|^2 k_1 t w'' + \alpha^2 \left[h_1'\right]^2 k_1 w''' + \alpha a_{1,0} f_1 \partial_y k_1 w'}_{Q_1} \\
& + \underbrace{F''\big(w(t)\big)\big[\big(-2e^{\sqrt{2}t} - \psi_0(t)\big)e^{-\sqrt{2}(h_2-h_1)} + \alpha^2 [h_1]^2 \psi_1(t) + \alpha^2 |A_M|^2 \psi_2(t)\big] k_1 w'}_{Q_2} \\
& \times \underbrace{\sqrt{2}\big(F'(w) - F'(1)\big) e^{\sqrt{2}t} e^{-\sqrt{2}(h_2-h_1)} k_2}_{Q_3} \\
& - \underbrace{w''\big[\alpha^2 \mathcal{J}_M(f_1) k_1 + 2\alpha f_1' \partial_y k_1 + \alpha^2 a_{1,0} f_1 \big(f_1' \partial_y k_1 + f_1'' k_1\big)\big]}_{Q_4} \\
& \times \underbrace{\alpha t w' [a_{1,0} \partial_{yy} k_1 + \alpha b_{1,0} \partial_y k_1]}_{Q_5} \underbrace{\alpha^2 (t + f_1)^2 a_{1,1} \left[\partial_{yy} k_1 w' - 2 f_1' \partial_y k_1 w''\right]}_{Q_6} \\
& + \underbrace{\mathcal{O}\big(\alpha^{2+\tau_1} \big(1 + |\alpha y|^{2+\frac{\varrho}{\sqrt{2}}-\varepsilon}\big)^{-1} e^{-\varrho|t|}\big)}_{Q_7}.
\end{aligned} \tag{9.11}$$

Observe also that

$$\int\limits_{|t|<\rho_\alpha} Q_i w'(t)\,dt = \int\limits_{\mathbb{R}} Q_i w'(t)\,dt + \mathcal{O}\big(\alpha^{2+\tau_1}\big(1 + |\alpha y|^{2+\frac{\varrho}{\sqrt{2}-\varepsilon}}\big)^{-1}\big),$$

where

$$\rho_\alpha(y) := \frac{1}{2}\left[\sigma_\alpha + \sqrt{2}\left(1 - \frac{1}{\sigma_\alpha} - M\sigma^{-\frac{5}{4}}\right)\log\big(1 + (\alpha y)^2\big)\right].$$

We notice also that

$$\int\limits_{\mathbb{R}} \sum_{i=1}^{5} Q_i w'(t)\,dt = \big(\Delta_{M_\alpha} k_1 + \alpha^2 |A_M|^2 k_1 + \sqrt{2} a_0 e^{-\sqrt{2}(h_2-h_1)}(k_1 + k_2) + \alpha a_{1,0} f_1 \partial_{yy} k_1\big) \int\limits_{\mathbb{R}} w'(t)^2\,dt$$

and

$$\int\limits_{\mathbb{R}} (Q_6 + Q_7) w'(t)\,dt = \mathcal{O}\big(\alpha^2 r(\alpha y)^{-2}\big)\partial_{yy} k_1 + \mathcal{O}\big(\alpha^3 r(\alpha y)^{-3}\big)\partial_y k_1.$$

Combining these computations, we have in the set A_1 that

$$\begin{aligned}
\int\limits_{|t|\leqslant\rho_\alpha} \big(\Delta v + F'(u_\alpha)v\big) w'(t)\,dt = {} & \big(\Delta_{M_\alpha} k_1 + \alpha^2 |A_M|^2 k_1 + \alpha a_{1,0} f_1 \partial_{yy} k_1\big) \int\limits_{\mathbb{R}} w'(t)^2\,dt \\
& + \sqrt{2} a_0 e^{-\sqrt{2}(h_2-h_1)}(k_1 + k_2) \int\limits_{\mathbb{R}} w'(t)^2\,dt + \mathcal{O}\big(\alpha^2 r(\alpha y)^{-2}\big)\partial_{yy} k_1 \\
& + \mathcal{O}\big(\alpha^3 r(\alpha y)^{-3}\big)\partial_y k_1 + \mathcal{O}\big(\alpha^{2+\tau_1} r(\alpha y)^{2+\beta}\big) k_1.
\end{aligned}$$

Regarding computations in the set A_2 in the coordinates X_{α,f_2} and using again (9.8) (9.9), we find in the set A_2 that

$$F'(u_\alpha)w' = F'(w)w' + F''(w)w'\big[\big(2e^{-\sqrt{2}t} - \psi_0(-t)\big)e^{-\sqrt{2}(h_2-h_1)} + \alpha^2\big[h_2'\big]^2\psi_1(t) + \alpha^2|A_M|^2\psi_2(t)\big]$$
$$+ \mathcal{O}\big(\alpha^{2+\tau_1}\big(1+|\alpha y|^2\big)^{-1}e^{-\varrho|t|}\big) \tag{9.12}$$

and the interaction term this time is

$$F'(u_\alpha)w'(t+f_2-f_1) = F'\big(w(t+f_2-f_1)\big)w'(t+f_2-f_1)$$
$$- \sqrt{2}\big(F'(w)-F'(1)\big)e^{-\sqrt{2}t}e^{-\sqrt{2}(f_2-f_1)} + \mathcal{O}\big(\alpha^{2+\tau_1}e^{-\varrho}|t|\big(1+|\alpha y|\big)^{-4+\beta}\big).$$

Consequently when testing against $w'(t)$ we obtain

$$\int_{\mathbb{R}} F''(w)\big(w'\big)^2\big(2e^{-\sqrt{2}t} - \psi_0(-t)\big)\,dt = \sqrt{2}\int_{\mathbb{R}} 6\big(1-w^2\big)e^{\sqrt{2}t}w'\,dt = \sqrt{2}a_0$$

so that in the set A_2

$$\int_{|t|\leqslant\rho_\alpha} \big(\Delta v + F'(u_\alpha)v\big)w'(t)\,dt = \big(\Delta_{M_\alpha}k_2 + \alpha^2|A_M|^2k_2 + \alpha a_{1,0}f_1\partial_{yy}k_2\big)\int_{\mathbb{R}} w'(t)^2\,dt$$
$$+ \sqrt{2}a_0e^{-\sqrt{2}(h_2-h_1)}(k_1+k_2)\int_{\mathbb{R}} w'(t)^2\,dt$$
$$+ \mathcal{O}\big(\alpha^2r(\alpha y)^{-2}\big)\partial_{yy}k_2 + \mathcal{O}\big(\alpha^3r(\alpha y)^{-3}\big)\partial_yk_2 + \mathcal{O}\big(\alpha^{2+\tau_1}r(\alpha y)^{2+\beta}\big)k_2.$$

Hence, choosing functions $k_1 = -k_2 = k$, with k bounded and using the fact that

$$dx = \sqrt{1+(\alpha y)^2}\big(1-\alpha^2(t+f_1)^2|A_M|^2\big)\,dy\,dt$$

we observe that in the region

$$W_R = \big\{x \in \mathcal{N}_\alpha:\ r(\alpha x) < R\big\}$$

it holds that

$$Q(v,v) = \iint_{W_R} |\nabla v|^2 - F'(u_\alpha)v^2\,dx$$
$$= 2\int_{\mathbb{R}} w'(t)^2\,dt\int_{M_\alpha^R} |\nabla_{M_\alpha}k|^2 - \alpha^2|A_M|^2k^2 + 2\sqrt{2}a_0e^{-\sqrt{2}(h_2-h_1)}k^2\,dV_{M_\alpha}$$
$$+ \mathcal{O}\Bigg(\alpha\int_{M_\alpha^R} |\nabla_Mk|^2 + \alpha^2\big(1+(\alpha y)\big)^{2+\beta}k^2\,dV_{M_\alpha}\Bigg).$$

On the other hand, since

$$e^{-\sqrt{2}(h_2-h_1)} = \alpha^2\sigma\big|A_M(\alpha y)\big|^2 + \mathcal{O}\big(\alpha^2\sigma^{-1}\big(1+r(\alpha y)\big)^{-4}\log\big(r(\alpha y)\big)^2\big)$$

we find that

$$Q(v,v) = 2\int_{\mathbb{R}} \left(w'(t)\right)^2 dt \int_{r(\alpha y)<R} |\nabla_{M_\alpha} k|^2 - \alpha^2(2\sigma+1)\left|A_M(\alpha y)\right|^2 k^2(y)\, dV_{M_\alpha}$$
$$+ \alpha^{\tau_1}\mathcal{O}\left(\int_{r(\alpha y)<R} |\nabla_{M_\alpha} k|^2 + \alpha^2\sigma\left|A_M(\alpha y)\right|^2 k^2\right).$$

Taking $k(y) = z(\alpha y)$, with $z \in C^2(M)$ is an eigenfunction associated to a negative eigenvalue of the problem (9.1) and taking $R \to \infty$, we obtain that

$$Q(v,v) = \alpha^2\lambda \int_M |A_M|^2 z^2\, dV + \mathcal{O}\left(\alpha^{2+\tau_1}\int_M |\nabla_M z|^2 + \sigma|A_M|^2 z^2\right).$$

Since we can take at least $\mathcal{O}(\sqrt{\sigma}\,)$ of these eigenfunctions, we conclude that $m(u_\alpha) \geqslant \tilde{c}\sqrt{\sigma}$.

9.2. Estimates on the Morse index for solutions in Theorem 1

As for the solutions described in Theorem 1, we have the asymptotic expansion

$$u_\alpha(x) = w(x_3 - \mathrm{f}_{1\alpha}) - w(x_3 - \mathrm{f}_{2\alpha}) - 1 + \phi_{1,0}\left(x', x_3 - \mathrm{f}_{1\alpha}\right) - \phi_{2,0}\left(x', x_3 - \mathrm{f}_{2\alpha}\right)$$
$$+ \mathcal{O}\left(\alpha^{2+\tau_1}\left(1 + \left|\alpha x'\right|^2\right)^{-1} \sum_{l=1,2} e^{-\varrho|x_3 - \mathrm{f}_{l\alpha}|}\right), \tag{9.13}$$

where for $l = 1,2$

$$(-1)^l\phi_{l,0}\left(x',t\right) = (-1)^{l+1} e^{-\sqrt{2}(\mathrm{q}_{2\alpha}-\mathrm{q}_{1\alpha})}\psi_0\left((-1)^{l+1}t\right) + |\nabla \mathrm{q}_{l\alpha}|^2\psi_1(t) \tag{9.14}$$

and the functions ψ_0, ψ_1 are again those described in (5.19) and (5.20). We also recall the sets

$$A_l := \left\{x = \left(x', x_3\right):\ \left|x_3 - \mathrm{f}_{j\alpha}\left(x'\right)\right| \leqslant \frac{1}{2}\left(\mathrm{f}_{2\alpha}\left(x'\right) - \mathrm{f}_{1\alpha}\left(x'\right)\right)\right\}, \quad l = 1,2.$$

We use a test function $v(x', x_3)$ of the form

$$v\left(x', x_3\right) := k_1\left(x'\right)w'\left(x_3 - \mathrm{f}_{1\alpha}\left(x'\right)\right) + k_2\left(x'\right)w'\left(x_3 - \mathrm{f}_{2\alpha}\left(x'\right)\right),$$

and proceed as before to obtain in the set A_1

$$\Delta v + F'(u_\alpha)v = \underbrace{\Delta_{\mathbb{R}^2} k_1 w' + |\nabla \mathrm{q}_{1\alpha}|^2 k_1 w'''}_{Q_1}$$
$$+ \underbrace{F''\left(w(t)\right)\left[-\left(2e^{\sqrt{2}t} + \psi_0(t)\right)e^{-\sqrt{2}(\mathrm{q}_{2\alpha}-\mathrm{q}_{1\alpha})} + |\nabla \mathrm{q}_{1\alpha}|^2\psi_2(t)\right]k_1 w'}_{Q_2}$$
$$+ \underbrace{\sqrt{2}\left(F'(w) - F'(1)\right)e^{\sqrt{2}t}e^{-\sqrt{2}(\mathrm{q}_{2\alpha}-\mathrm{q}_{1\alpha})}k_2}_{Q_3} - \underbrace{w''\left[\Delta_{\mathbb{R}^2}\mathrm{f}_{1\alpha}k_1 + 2\nabla_{x'}\mathrm{f}_{1\alpha}\nabla_{x'}k_1\right]}_{Q_4}$$
$$+ \underbrace{\mathcal{O}\left(\alpha^{2+\tau_1}\left(1 + \left|\alpha x'\right|^{2+\frac{\varrho}{\sqrt{2}}-\varepsilon}\right)^{-1} e^{-\varrho|t|}\right)}_{Q_5}. \tag{9.15}$$

Observe also that

$$\int\limits_{|t|<\rho_\alpha} Q_i w'(t)\,dt = \int\limits_{\mathbb{R}} Q_i w'(t)\,dt + \mathcal{O}\big(\alpha^{2+\tau_1}\big(1+|\alpha y|^{2+\frac{\varrho}{\sqrt{2}-\varepsilon}}\big)^{-1}\big),$$

where this time

$$\rho_\alpha(y) := \frac{1}{2}\big(\mathrm{f}_{2\alpha}(x') - \mathrm{f}_{1\alpha}(x')\big).$$

We conclude that in the set A_1

$$\int\limits_{|t|\leqslant\rho_\alpha} \big(\Delta v + F'(u_\alpha)v\big)w'(t)\,dt = \big(\Delta_{\mathbb{R}^2}k_1 + \sqrt{2}a_0 e^{-\sqrt{2}(\mathrm{q}_{2\alpha}-\mathrm{q}_{1\alpha})}(k_1+k_2)\big)\int\limits_{\mathbb{R}} w'(t)^2\,dt$$
$$+ \mathcal{O}\big(\alpha^2 r(\alpha x')^{-2}\big)D^2k_1 + \mathcal{O}\big(\alpha^3 r(\alpha x')^{-3}\big)\nabla k_1 + \mathcal{O}\big(\alpha^{2+\tau_1} r(\alpha x')^{2+\beta}\big)k_1.$$

As before, a similar estimate holds estimate holds for the region A_2, namely

$$\int\limits_{|t|\leqslant\rho_\alpha} \big(\Delta v + F'(u_\alpha)v\big)w'(t)\,dt = \big(\Delta_{M_\alpha}k_2 + \sqrt{2}a_0 e^{-\sqrt{2}(\mathrm{q}_{2\alpha}-\mathrm{q}_{1\alpha})}(k_1+k_2)\big)\int\limits_{\mathbb{R}} w'(t)^2\,dt$$
$$+ \mathcal{O}\big(\alpha^2 r(\alpha y)^{-2}\big)D^2k_2 + \mathcal{O}\big(\alpha^3 r(\alpha y)^{-3}\big)\nabla k_2 + \mathcal{O}\big(\alpha^{2+\tau_1} r(\alpha y)^{2+\beta}\big)k_2,$$

so that for the test function

$$v(x',z) = k(x')\big[w'\big(x_3 - \mathrm{f}_{1\alpha}(x')\big) - w'\big(x_3 - \mathrm{f}_{2\alpha}(x')\big)\big]$$

it holds that

$$Q(v,v) = \int\limits_{\mathbb{R}^3}\big(|\nabla v|^2 - F'(u_\alpha)v^2\big)\,dx'\,dx_3 = 2\int\limits_{\mathbb{R}^2}\bigg(|\nabla k|^2 - \frac{8}{(1+|\alpha x'|^2)^2}k^2\bigg)\,dx'$$
$$+ \mathcal{O}\bigg(\alpha^\tau \int\limits_{\mathbb{R}^2}\bigg(|\nabla k|^2 + \frac{1}{(1+|\alpha x'|^2)^2}k^2\bigg)\,dx'\bigg).$$

Taking $k(y) = z(\alpha y)$, with $z \in C^2(\mathbb{R}^2)$ an eigenfunction associated to a negative eigenvalue of the problem (9.2) and taking $R \to \infty$, we obtain that

$$Q(v,v) = \alpha^2\lambda \int\limits_{\mathbb{R}} w'(t)^2\,dt \int\limits_{\mathbb{R}^2} p(x')z^2\,dV + \mathcal{O}\bigg(\alpha^{2+\tau_1}\int\limits_{\mathbb{R}^2} |\nabla_{\mathbb{R}^2} z|^2 + p(x')z^2\bigg).$$

This last expression implies that $m(u_\alpha) \geqslant 1$, since the problem (9.2) has exactly one negative simple eigenvalue.

Let us next prove the following lemma involving the size of negative eigenvalues to problem (9.5).

Lemma 9.1. *There exists a universal constant $\mu > 0$ such that for any eigenvalue $\lambda < 0$ for the problem (9.6) and any $R > 0$ large enough*

$$\lambda \geqslant -\mu\alpha^2. \tag{9.16}$$

Proof. To prove this claim, let us consider sets $\Omega_l := A_l \cap C_R$, $l = 1, 2$, where we recall that

$$A_l = \left\{ (x', t)\colon\ |t| \leqslant \frac{1}{2}\left|\mathrm{f}_{2\alpha}(x') - \mathrm{f}_{1\alpha}(x')\right| \right\}, \quad z = t + \mathrm{f}_{1\alpha}(x').$$

Observe that it is enough to prove that

$$Q_l(\psi, \psi) = \int_{\Omega_l} \left(|\nabla\psi|^2 - F'(u_\alpha)\psi^2\right) dx'\,dz \geqslant -\mu\alpha^2 \int_{\Omega_l} p(\alpha x)\psi^2\,dx'\,dz, \quad l = 1, 2.$$

As in [12], consider the eigenvalue problem

$$\Delta_{\mathbb{R}^3}\psi + F'(u_\alpha)\psi + \lambda p(\alpha x)\psi = 0, \quad \text{in } \Omega_1 \cup \Omega_2$$
$$\psi = 0, \quad \text{on } \left|\alpha x'\right| = R, \qquad \partial_n\psi = 0, \qquad \left|z - \mathrm{f}_{l\alpha}(x')\right| = \frac{1}{2}\left|\mathrm{f}_{2\alpha}(x') - \mathrm{f}_{1\alpha}(x')\right|, \quad l = 1, 2. \tag{9.17}$$

For a solution ψ to (9.17), we write in Ω_l

$$\psi(x', t) = \zeta_{l,1} k_l(x') w'(t) + \psi_l^\perp$$

and where we can choose the functions k_l so that

$$\int_{|t| \leqslant \frac{1}{2}|\mathrm{f}_{2\alpha} - \mathrm{f}_{1\alpha}|} \psi_l^\perp w'(t)\,dt = 0. \tag{9.18}$$

We write

$$Q_l(\psi, \psi) = Q_l(\zeta_{l,1}k_l, \zeta_{l,1}k_l) + 2Q_l\left(\zeta_{l,1}k_l, \psi_l^\perp\right) + Q_l\left(\psi_l^\perp, \psi_l^\perp\right) = I_l + II_l + III_l.$$

By a series of lengthy calculations similar to those performed in Section 9.1, we obtain that

$$\begin{aligned} I_l = {} & \int_{\mathbb{R}} w'(t)^2\,dt \int_{\mathbb{R}^2} \left(|\nabla k_l|^2 - \frac{4\alpha^2}{(1 + |\alpha x'|^2)^2} k_l^2 \right) dx' \\ & + \alpha^\tau \mathcal{O}\left(\int_{\mathbb{R}^2} \left(\left|\nabla k_l(x')\right|^2 + \frac{\alpha^2}{(1 + |\alpha x'|^2)^2} k_l^2(x') \right) dx' \right). \end{aligned} \tag{9.19}$$

Since, $\psi_l^\perp$ satisfies the same boundary conditions as ψ, we obtain that

$$\begin{aligned} III_l &= \int_{|\alpha x'| < R} \int_{|t| \leqslant \frac{1}{2}|\mathrm{f}_{2\alpha} - \mathrm{f}_{1\alpha}|} \left|\partial_t\psi_l^\perp\right|^2 + \left|\nabla_{x'}\psi_l^\perp\right|^2 - F'(u_\alpha){\psi_l^\perp}^2\,dx'\,dt \\ &= \int_{|\alpha x'| < R} \int_{|t| \leqslant \frac{1}{2}|\mathrm{f}_{2\alpha} - \mathrm{f}_{1\alpha}|} \left|\nabla\psi_l^\perp\right|^2 - \psi_l^\perp\left[\partial_{tt}\psi_l^\perp + F'(u_\alpha)\psi_l^\perp - \nabla_{x'}(\mathrm{f}_{2\alpha} - \mathrm{f}_{1\alpha}) \cdot \nabla\psi_l^\perp\right] \\ &\geqslant \left(\frac{\gamma}{2} \iint \left|\partial_t\psi_l^\perp\right|^2 + \left|\nabla_{x'}\psi_l^\perp\right|^2 + \left|\psi_l^\perp\right|^2 \right) + \alpha^{2-\tau}\mathcal{O}\left(\iint \left(1 + |\alpha x'|\right)^{-4+\beta} {\psi_l^\perp}^2 \right) \\ &\geqslant \tilde{\mu} \iint \left(\left|\partial_t\psi_l^\perp\right|^2 + \left|\nabla\psi_l^\perp\right|^2 + \left|\psi_l^\perp\right|^2 \right). \end{aligned} \tag{9.20}$$

As for II_l, we proceed as follows. For instance in Ω_1 it holds that

$$L\big(k_1\big(x'\big)w'(t)\big) := \Delta k_1\big(x'\big)w'(t) + F'(u_\alpha)k_1\big(x'\big)w'(t) = \Delta_{\mathbb{R}^2}k_1w' + |\nabla \mathrm{q}_{1\alpha}|^2 k_1 w'''$$

$$F''\big(w(t)\big)\big[-\big(2e^{\sqrt{2}t} + \psi_0(t)\big)e^{-\sqrt{2}(\mathrm{q}_{2\alpha}-\mathrm{q}_{1\alpha})} + |\nabla \mathrm{q}_{1\alpha}|^2\psi_2(t)\big]k_1w' - w''[\Delta_{\mathbb{R}^2}\mathrm{f}_{1\alpha}k_1 + 2\nabla_{x'}\mathrm{f}_{1\alpha}\nabla_{x'}k_1]$$
$$+ \mathcal{O}\big(\alpha^{2+\tau}\big(1+\big|\alpha x'\big|^{2+\frac{\varrho}{\sqrt{2}}-\varepsilon}\big)^{-1}e^{-\varrho|t|}\big)$$

and this implies that

$$II_1 = -\int \zeta_{1,1}L\big(k_1\big(x'\big)w'(t)\big)\psi_1^\perp + \int 2\nabla\zeta_{1,1}\cdot\nabla\big(k_1\big(x'\big)w'(t)\big)\psi_1^\perp + \Delta\zeta_{1,1}w'(t)k_1\big(x'\big)\psi_1^\perp.$$

Since $\psi_1^\perp$ satisfies condition (9.18), using Eq. (9.17), we obtain that

$$\begin{aligned} II_1 &= -\int_{\Omega} L\big(k_1\big(x'\big)w'(t)\big)\psi_1^\perp\,dx'\,dz - \int_{\Omega_1}(1-\zeta_{1,1})L\big(k_1\big(x'\big)w'(t)\big)\psi_1^\perp\,dx'\,dz + \theta \\ &= -\int_{\Omega_1}\big(6w(t)w'(t) + 3w(t)\psi_0(t)\big)e^{\sqrt{2}t}e^{-\sqrt{2}(\mathrm{f}_{2\alpha}-\mathrm{f}_{1\alpha})}k_1\big(x'\big)\psi_1^\perp\,dx'\,dz + \theta, \end{aligned}$$

where

$$\theta = o(1)\int_{\mathbb{R}^2}\big(\big|\nabla k_1\big(x'\big)\big|^2 + \alpha^2 p(\alpha x)k_1^2\big(x'\big)\big)\,dx' + o(1)\int_{\Omega_1}\big(\big|\psi_1^\perp\big|^2 + \big|\nabla\psi_1^\perp\big|^2\big)\,dx'\,dz.$$

So, we obtain that

$$|II_1| \leqslant C\nu^{-1}\alpha^2\int_{\mathbb{R}^2}\big(1+\big|\alpha x'\big|\big)^{-4}k_i^2\big(x'\big)\,dx' + \nu\int_{\Omega_i}\big(1+\big|\alpha x'\big|\big)^{-4}\big|\psi_i^\perp\big|^2\,dx'\,dz. \tag{9.21}$$

Putting together estimate (9.19)–(9.21) we get the estimate

$$Q_{\Omega_1}(\psi,\psi) \geqslant -\mu_1\alpha^2\int\big(1+\big|\alpha x'\big|\big)^{-4}k_1^2\big(x'\big)\,dx'.$$

Then inequality

$$Q(\psi,\psi) \geqslant -\mu\alpha^2\int p(\alpha x)\psi^2$$

follows since a similar procedure can be applied in the region A_2. □

9.3. *The proof of inequality $m(u_\alpha) \leqslant 1$ for solutions in Theorem 1*

We begin this subsection by proving that eigenvalues to problem (9.6) that are close to zero are actually positive and we give a precise estimate on their size. This information is collected in the following lemma, whose proof proceeds as in Section 11 of [12], but that we include here for the sake of completeness.

Lemma 9.2. *Assume that* $\phi_{\alpha,R}$ *and* $\lambda_{\alpha,R} \neq 0$ *are respectively an eigenfunction and eigenvalue for problem* (9.6) *such that*

$$\|\phi_{\alpha,R}\|_{L^\infty(\mathbb{R}^3)} = 1, \qquad |\lambda_{\alpha,R}| \leqslant M\alpha^2 \tag{9.22}$$

for some $M \to 0$ *as* $\alpha \to 0$*. Then there exists a positive universal constant* $\hat{\beta}$ *such that for every* $\alpha > 0$ *small and* R *large enough*

$$\lambda_\alpha := \lim_{R\to\infty} \lambda_{\alpha,R} = \alpha^3 \log\left(\frac{1}{\alpha}\right)\hat{\beta} + \mathcal{O}(\alpha^3)$$

and

$$\phi_{\alpha,R}(x', x_3) = Z(\alpha x')\left[w'(x_3 - \mathrm{f}_{1\alpha}) - w'(x_3 - \mathrm{f}_{2\alpha})\right] + \mathcal{O}\left(\sum_{l=1,2} e^{-\varrho|x_3 - \mathrm{f}_{l\alpha}|}\right),$$

where $Z(x')$ *is a scalar multiple of the function* $\bar{z}_0(x')$ *described in* (9.3)*.*

Proof. Let us consider a solution ϕ to the problem (9.6). Using assumption (9.22) and a sub and super solutions scheme, it can be proven that

$$\left|\phi(x', z)\right| \leqslant C \sum_{j=1}^{2} e^{-\varrho|z - \mathrm{f}_{j\alpha}(x')|}$$

for $|\alpha x'| \geqslant R_0$ and R_0 large enough. This inequality basically states that any solution to problem (9.6) can have values bounded away from zero only in the regions A_l.

From inequality (9.22) we can write

$$\lambda = \lambda_{\alpha,R} = \mu_{\alpha,R}\alpha^2, \qquad \mu_{\alpha,R} \to \mu_\alpha, \quad \text{as } R \to \infty.$$

We consider the sets

$$\tilde{A}_l = \left\{(x', x_3) \in \mathbb{R}^2 \times \mathbb{R}\colon\ |x_3 - \mathrm{f}_{l\alpha}| \leqslant \theta[\mathrm{f}_{2\alpha} - \mathrm{f}_{1\alpha}]\right)\}, \quad \theta \in \left(\frac{1}{2}, 1\right),\ l = 1, 2$$

and consider a cut-off function $\tilde{\zeta}_l$, supported in the set $\tilde{A}_l$.

We consider a solution to the eigenvalue problem

$$\Delta_{\mathbb{R}^3}\phi + F'(u_\alpha)\phi + \alpha^2\mu p(\alpha x')\phi = 0, \quad \text{in } C_{\mathbb{R}}$$
$$\phi = 0, \quad \text{on } \partial C_R$$

and to fix ideas, let us localize ϕ in $\tilde{A}_1$ by setting

$$\tilde{\phi}_1 = \zeta_1\phi$$

which implies that $\tilde{\phi}_1$ must solve the equation

$$\Delta_{\mathbb{R}^3}\tilde{\phi}_1 + F'(u_\alpha)\tilde{\phi}_1 + \alpha^2\mu p(\alpha x')\tilde{\phi}_1 = 2\nabla_{x'}\zeta_1 \cdot \nabla_{x'}\phi_1 + \phi_1\Delta\zeta_1 =: E_{1\alpha}.$$

Since in the set $\tilde{A}_1$

$$|D\phi| + |D\phi| + |\phi| \leqslant Ce^{-\varrho|t|}, \quad z = t + \mathrm{f}_{1,\alpha}, \quad 0 < \varrho < \sqrt{2} \tag{9.23}$$

we find that

$$|E_{1\alpha}| \leqslant Ce^{-\varrho\varepsilon|t|}e^{-(1-\varepsilon)\varrho|t|} \leqslant C\big[\alpha^2\big(1+|\alpha x'|\big)^{-4+\beta}\big]^{\frac{\varrho\theta}{\sqrt{2}}(1-\varepsilon)}e^{-\hat{\varrho}|t|}$$

from where we conclude that

$$|E_{1\alpha}| \leqslant C\alpha^{1+\tau}\big(1+|\alpha x'|\big)^{-2(1+\tau)}e^{-\hat{\varrho}|t|}, \quad \text{in } \tilde{A}_1.$$

for some $\hat{\varrho} > 0$, $\frac{1}{2} < \theta < 1$ and $\tau > 0$ small.

Setting $x_3 = t + \mathrm{f}_{1\alpha}$, we write inside $\tilde{A}_1$

$$L_*(\tilde{\phi}_1) = \partial_{tt}\tilde{\phi}_1 + \Delta_{\mathbb{R}^2}\tilde{\phi}_1 + F'\big(w(t)\big)\tilde{\phi}_1 + B(\tilde{\phi}_1),$$

where

$$B(\tilde{\phi}_1) := -\Delta_{\mathbb{R}^2}\mathrm{f}_{1\alpha}\partial_t\tilde{\phi}_1 - 2\nabla\mathrm{f}_{1\alpha}\nabla_{x'}\partial_t\tilde{\phi}_1 + [\nabla\mathrm{f}_{1\alpha}]^2\partial_{tt}\tilde{\phi}_1 + \big[F'(u_\alpha) - F'(w)\big]\tilde{\phi}_1.$$

Hence, $\tilde{\phi}_1$ solves the equation

$$L_*(\tilde{\phi}_1) + \alpha^2\mu p\big(\alpha x'\big)\tilde{\phi}_1 + B_1(\tilde{\phi}_1) = E_{1\alpha} \quad \text{in } B_{\alpha^{-1}R}(0) \times \mathbb{R}.$$

Proceeding in the same fashion, localizing ϕ in $\tilde{A}_2$, we find that

$$L_*(\tilde{\phi}_2) + \alpha^2\mu p\big(\alpha x'\big)\tilde{\phi}_2 + B_2(\tilde{\phi}_2) = E_{2\alpha} \quad \text{in } B_{\alpha^{-1}R}(0) \times \mathbb{R},$$

where

$$E_{2\alpha} := 2\nabla_{x'}\zeta_2 \cdot \nabla_{x'}\phi_2 + \phi_2\Delta\zeta_2$$

and

$$B_2(\tilde{\phi}_2) := -\Delta_{\mathbb{R}^2}\mathrm{f}_{2\alpha}\partial_t\tilde{\phi}_2 - 2\nabla\mathrm{f}_{2\alpha}\nabla_{x'}\partial_t\tilde{\phi}_2 + [\nabla\mathrm{f}_{2\alpha}]^2\partial_{tt}\tilde{\phi}_2 + \big[F'(u_\alpha) - F'(w)\big]\tilde{\phi}_2.$$

Consider functions k_1, k_2 defined by the integrals

$$\int_{\mathbb{R}} \tilde{\phi}_1\big(x',t\big)w'(t)\,dt = k_1\big(x'\big)\int_{\mathbb{R}}\big(w'(t)\big)^2\,dt + k_2\big(x'\big)\int_{\mathbb{R}}\tilde{\zeta}_1 w(t+\mathrm{f}_{1\alpha}-\mathrm{f}_{2\alpha})w'(t)\,dt$$
$$\int_{\mathbb{R}} \tilde{\phi}_2\big(x',t\big)w'(t) = k_2\big(x'\big)\int_{\mathbb{R}}\big(w'(t)\big)^2\,dt + k_1\big(x'\big)\int_{\mathbb{R}}\tilde{\zeta}_2 w(t+\mathrm{f}_{2\alpha}-\mathrm{f}_{1\alpha})w'(t)\,dt$$

so that in the set $\tilde{A}_1$ we have the decomposition

$$\tilde{\phi}_1 = k_1\big(x'\big)w'(t) + \tilde{\zeta}_1 k_2\big(x'\big)w'(t+\mathrm{f}_{1\alpha}-\mathrm{f}_{2\alpha}) + \varphi_1,$$
$$\int_{\mathbb{R}}\varphi_1 w'(t)\,dt = 0.$$

Analogously, in $\tilde{A}_2$, we have

$$\tilde{\phi}_2 = k_2(x')w'(t) + \tilde{\zeta}_2 k_1(x')w'(t + \mathrm{f}_{2\alpha} - \mathrm{f}_{1\alpha}) + \varphi_2,$$
$$\int_{\mathbb{R}} \varphi_2 w'(t)\,dt = 0.$$

From (9.23), it is clear that the functions are smooth and bounded up to their second derivatives. We perform the subsequent developments for $\tilde{\phi}_1$ only, since for $\tilde{\phi}_2$ is the procedure is the same. Dropping again the subindexes we have that the equations for the function φ have the form

$$L_*(\varphi) + \alpha^2 \mu p(\alpha x')\varphi + B(\varphi) = S_{*,\mu} + E_\alpha + B(kw'), \quad \text{in } B_{\alpha^{-1}R}(0)\times\mathbb{R},$$

where

$$\begin{aligned} S_{*,\mu} = {} & \underbrace{\Delta_{\mathbb{R}^2} k_1 w' + |\nabla \mathrm{q}_{1\alpha}|^2 k_1 w'''}_{Q_1} \\ & + \underbrace{F''(w(t))\big[-\big(2e^{\sqrt{2}t} + \psi_0(t)\big)e^{-\sqrt{2}(\mathrm{q}_{2\alpha}-\mathrm{q}_{1\alpha})} + |\nabla \mathrm{q}_{1\alpha}|^2\psi_2(t)\big]k_1 w'}_{Q_2} \\ & + \underbrace{\sqrt{2}\big(F'(w) - F'(1)\big)e^{\sqrt{2}t}e^{-\sqrt{2}(\mathrm{q}_{2\alpha}-\mathrm{q}_{1\alpha})}k_2}_{Q_3} - \underbrace{w''[\Delta_{\mathbb{R}^2}\mathrm{f}_{1\alpha}k_1 + 2\nabla_{x'}\mathrm{f}_{1\alpha}\nabla_{x'}k_1]}_{Q_4} \\ & + \underbrace{\mathcal{O}\big(\alpha^{2+\tau_1}\big(1+|\alpha y|^{2+\frac{\varrho}{\sqrt{2}}-\varepsilon}\big)^{-1}e^{-\varrho|t|}\big)}_{Q_5} \end{aligned} \tag{9.24}$$

for some $\tau > 0$ small enough. Observe that $k_1(x'), k_2(x')$ are bounded in C^2-norm, in $B_{\alpha^{-1}R}(0)$.

Testing this equation against w', we observe that

$$\begin{aligned} & \Delta_{\mathbb{R}^2} k_1 + \sqrt{2}a_0 e^{-\sqrt{2}(\mathrm{q}_{2\alpha}-\mathrm{q}_{1\alpha})}(k_1 + k_2) + \alpha^2\mu p(\alpha x')k_1 \\ & \quad = \tilde{B} + \mathcal{O}\big(\alpha^2 r(\alpha x')^{-2}\big)D^2 k_1 + \mathcal{O}\big(\alpha^3 r(\alpha x')^{-3}\big)\nabla k_1 + \mathcal{O}\big(\alpha^{2+\tau_1} r(\alpha x')^{2+\beta}\big)k_1, \end{aligned}$$

where

$$\tilde{B} = \frac{1}{\int_{\mathbb{R}} w'(t)^2\,dt}\int_{\mathbb{R}} B(\varphi).$$

We will prove that $\tilde{B} \sim \mathcal{O}(\alpha^{2+\tau})$ for some $\tau > 0$ small enough.

Let us write $\varphi = \bar{\varphi}_1 + \bar{\varphi}_2$, where

$$L_*(\bar{\varphi}_1) + \alpha^2\mu p(\alpha x')\bar{\varphi}_1 = Q_4, \qquad \int_{\mathbb{R}} \bar{\varphi}_1 w'(t)\,dt = 0.$$

Then, we obtain the estimate

$$\big\|D^2\bar{\varphi}_1\big\|_{p,1,\varrho} + \|D\bar{\varphi}_1\|_{\infty,1,\varrho} + \|\bar{\varphi}_1\|_{\infty,1,\varrho} \leqslant C\alpha.$$

Next, we observe that

$$L_*(\bar{\varphi}_2) + \alpha^2\mu p(\alpha x')\bar{\varphi}_2 + B(\bar{\varphi}_2) = g(x',t) + c(x')w'(t),$$

where

$$g(x',t) = E_\alpha - \alpha^2 \mu p(\alpha x') k_1 w'(t) - (Q_1 + Q_2 + Q_3 + Q_5) - B(\bar{\varphi}_1),$$

and

$$\int_{\mathbb{R}} \bar{\varphi}_2 w'(t)\, dt = 0.$$

Observe that $\|B(\bar{\varphi}_1)\|_{p,2-\tau,\varrho} \leqslant C\alpha^2$. Using the size of E_α we obtain that

$$\|g\|_{p,2-\tau,\varrho} \leqslant C\alpha^{1+\tau}$$

for some $\tau > 0$ small enough, so that we conclude

$$\|D^2\bar{\varphi}_2\|_{p,2-\tau,\varrho} + \|D\bar{\varphi}_2\|_{p,2-\tau,\varrho} + \|\bar{\varphi}_2\|_{p,2-\tau,\varrho} \leqslant C\alpha^{1+\tau}$$

and consequently

$$\|B(\bar{\varphi}_2)\|_{p,3-\tau,\varrho} \leqslant C\alpha^{2+\tau}.$$

So, we decompose

$$\tilde{B} = \tilde{B}_1 + \tilde{B}_2, \qquad \tilde{B}_l = \frac{1}{\alpha^2\|w'\|^2_{L^2(\mathbb{R})}} \int_{\mathbb{R}} B(\bar{\varphi}_l),$$

where

$$|\tilde{B}_1| \leqslant C, \qquad |\tilde{B}_2| \leqslant C\alpha^\tau.$$

Even more, keeping into account the procedure for φ_2 and setting $z_l(\alpha x') = k_l(x')$, for $l = 1, 2$, and using elliptic estimates in the system of equations for z_1, z_2 we find that

$$\|D^2 z\|_{p,2-\tau} + \|(1 + |x'|)^{1-\tau} Dz\|_\infty \leqslant C\alpha$$

from which

$$\|Q_4\|_{p,2-\tau,\varrho} \leqslant C\alpha^2$$

and so

$$\|\bar{\varphi}_1\|_{p,2-\tau,\varrho} \leqslant C\alpha^2, \qquad \|\tilde{B}_1\|_{p,3-\tau} \leqslant C\|\tilde{B}\|_{p,2-\tau} \leqslant C\alpha^3$$

and so

$$|\tilde{B}| \leqslant C\alpha^{2+\tau}.$$

At this point we recall that $\phi = \phi_{\alpha,R}$ has a uniform C^1 bound and that

$$\phi_{\alpha,R}(x',x_3) = k_{1,\alpha,R}(x') w'(x_3 - \mathrm{f}_{1\alpha}) + k_{2,\alpha,R}(x') w'(x_3 - \mathrm{f}_{2\alpha}) + \mathcal{O}\left(\alpha \sum_{j=1}^{2} e^{-\varrho|x_3 - \mathrm{f}_{j\alpha}|}\right) \tag{9.25}$$

so that

$$\phi_{\alpha,R} \to \phi_\alpha, \quad \text{as } R \to \infty$$

uniformly on compact sets and

$$\Delta_{\mathbb{R}^3}\phi_\alpha + F'(u_\alpha)\phi_\alpha = 0, \quad \text{in } \mathbb{R}^3$$

with

$$\phi_\alpha(x', x_3) = k_{1,\alpha}(x')w'(x_3 - \mathrm{f}_{1\alpha}) + k_{2,\alpha}(x')w'(x_3 - \mathrm{f}_{2\alpha}) + \mathcal{O}\left(\alpha \sum_{j=1}^{2} e^{-\varrho|x_3 - \mathrm{f}_{j\alpha}|}\right).$$

Observe also that $z_{l,\alpha,R}(x') = k_{l,\alpha,R}(\frac{x'}{\alpha})$ satisfies

$$\Delta_{\mathbb{R}^2} z_{1,\alpha,R} + \sqrt{2}e^{-\sqrt{2}(q_2-q_1)}(z_{2,\alpha,R} - z_{1,\alpha,R}) + \mu_{\alpha,R}p(x')z_{1,\alpha,R} = \mathcal{O}\big(\alpha^\tau \big(1 + |x'|\big)^{-2-\beta}\big),$$
$$\Delta_{\mathbb{R}^2} z_{2,\alpha,R} + \sqrt{2}e^{-\sqrt{2}(q_2-q_1)}(z_{2,\alpha,R} + z_{1,\alpha,R}) + \mu_{\alpha,R}p(x')z_{2,\alpha,R} = \mathcal{O}\big(\alpha^\tau \big(1 + |x'|\big)^{-2-\beta}\big)$$

so that, after passing to the limit $R \to \infty$, we obtain the estimates

$$\|z_{1,\alpha} \pm z_{2,\alpha}\|_{L^\infty(\mathbb{R}^2)} \leqslant C\big[\|z_{1,\alpha} \pm z_{2,\alpha}\|_{L^\infty(|x'|<R_0)} + \mathcal{O}\big(\alpha^\tau\big)\big] \tag{9.26}$$

or equivalently

$$\|k_{1,\alpha} \pm k_{2,\alpha}\|_{L^\infty(\mathbb{R}^2)} \leqslant C\big[\|k_{1,\alpha} \pm k_{2,\alpha}\|_{L^\infty(|\alpha x'|<R_0)} + \mathcal{O}\big(\alpha^\tau\big)\big].$$

From (9.27) we know that $k_{1,\alpha} \pm k_{2,\alpha}$ cannot be simultaneously zero. Then we obtain the limit system

$$\Delta_{\mathbb{R}^2} z_1 + \sqrt{2}e^{-\sqrt{2}(q_2-q_1)}(z_2 + z_1) = 0,$$
$$\Delta_{\mathbb{R}^2} z_2 + \sqrt{2}e^{-\sqrt{2}(q_2-q_1)}(z_2 + z_1) = 0,$$

hence, for every $\alpha > 0$ small and R large enough we have the asymptotics

$$\phi_{\alpha,R}(x', x_3) = z_1(\alpha x')w'(x_3 - \mathrm{f}_{1\alpha}) + z_2(\alpha x')w'(x_3 - \mathrm{f}_{2\alpha}) + \mathcal{O}\left(\alpha \sum_{j=1}^{2} e^{-\varrho|x_3 - \mathrm{f}_{j\alpha}|}\right)$$

and the functions $z = z_1 + z_2$, $\hat{z} = z_1 - z_2$ are bounded, no simultaneously zero and solve the system

$$\Delta_{\mathbb{R}^2} z + 2\sqrt{2}e^{-\sqrt{2}(q_2-q_1)}z = 0, \qquad \Delta\hat{z} = 0, \quad \text{in } \mathbb{R}^2.$$

Since the bounded kernel of the operator

$$\Delta_{\mathbb{R}^2} + 2\sqrt{2}e^{-\sqrt{2}(q_2-q_1)}$$

is described in polar coordinates in (9.3)

$$\bar{z}_0 := \frac{-1 + r^2}{1 + r^2}, \qquad \bar{z}_1 := \frac{r}{1 + r^2}\cos(\theta), \qquad \bar{z}_2 := \frac{r}{1 + r^2}\sin(\theta), \quad r > 0,\ \theta \in (0, 2\pi),$$

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we find that

$$z(x') = \sum_{i=0}^{2} \bar{\beta}_i \bar{z}_i(x'), \quad \bar{\beta}_i \in \mathbb{R}.$$

Since we are assuming $\lambda \neq 0$, we may also assume from spectral theory that

$$\int_{C_R} p(\alpha x')\phi_{\alpha,R} \cdot \bar{\phi}\, dx'\, dz = 0$$

for every bounded $\bar{\phi}$ solving

$$\Delta\bar{\phi} + F'(u_\alpha)\bar{\phi} = 0, \quad \text{in } C_{\mathbb{R}}, \qquad \bar{\phi} = 0, \quad \text{on } \partial C_R,$$

and from Proposition 8.2 we know that the functions

$$\partial_{x_1'} u_\alpha, \qquad \partial_{x_2'} u_\alpha, \qquad \partial_z u_\alpha$$

are bounded solutions to the equation

$$\Delta\phi + F'(u_\alpha)\phi = 0, \quad \text{in } \mathbb{R}^3.$$

Passing to the limit, we obtain that

$$\int_{\mathbb{R}^3} p(\alpha x')\phi_{2,\alpha}(x', x_3) Z(x', x_3)\, dx'\, dx_3 = 0$$

for any Z having the form

$$Z = \sum_{i=1}^{3} \beta_i \partial_{x_i} u_\alpha, \quad \beta_i \in \mathbb{R},\ i = 1, 2, 3.$$

From the asymptotic expansion

$$\partial_{x_3} u_\alpha(x', x_3) = w'(x_3 - \mathrm{f}_{1\alpha}) - w'(x_3 - \mathrm{f}_{2\alpha}) + \mathcal{O}\left(\alpha\big(1 + |\alpha x'|\big)^{-2} \sum_{j=1}^{2} e^{-\varrho|x_3 - \mathrm{f}_{j\alpha}|}\right)$$

we can pass to the limit as $\alpha \to 0$, in the orthogonality condition respect to $\partial_z u_\alpha$, to obtain that

$$\int_{\mathbb{R}^2} p(x')\hat{z}\, dx' = 0$$

so that from Liouville theorem we get that $\hat{z}_2 = 0$. This implies that

$$\phi_{\alpha,R}(x', x_3) = \frac{1}{2} z(\alpha x')\big[w'(x_3 - \mathrm{f}_{1\alpha}) + (\alpha x')w'(x_3 - \mathrm{f}_{2\alpha})\big] + \mathcal{O}\left(\alpha \sum_{j=1}^{2} e^{-\varrho|x_3 - \mathrm{f}_{j\alpha}|}\right).$$

Proceeding similarly, but this time using the asymptotic expansions

$$\partial_{x_i'} u_\alpha(x', z) = \alpha \partial_{x_i} q(\alpha x')\big[w'(z - \mathrm{f}_{1\alpha}) - w'(z - \mathrm{f}_{2\alpha})\big] + \mathcal{O}(\alpha^2), \quad i = 1, 2$$

and orthogonality conditions respect to $\partial_{x_i} u_\alpha$, we find that

$$\int_{\mathbb{R}^2} p(x') z(x') \bar{z}_i(x')\, dx' = 0, \quad i = 1, 2.$$

Consequently, $z(x')$ must be a scalar multiple of $\bar{z}_0(x')$ and with no loss of generality we write

$$\phi_{\alpha,R}(x', x_3) = \bar{z}_0(\alpha x')\big[w'(x_3 - \mathrm{f}_{1\alpha}) + (\alpha x') w'(x_3 - \mathrm{f}_{2\alpha})\big] + \mathcal{O}\left(\alpha \sum_{j=1}^{2} e^{-\varrho |x_3 - \mathrm{f}_{j\alpha}|}\right).$$

To finish the proof of the lemma, let us consider again the sets $\Omega_l = A_l \cap C_R$ defined in Lemma 9.1 and notice that

$$\begin{aligned}
\alpha^2 \mu_\alpha \int_{\Omega_{1,R} \cup \Omega_{2,R}} p(\alpha x') \alpha \partial_\alpha u_\alpha \cdot \phi_\alpha \, dx'\, dx_3 &= \alpha \int_{\Omega_{1,R} \cup \Omega_{2,R}} \nabla \phi_\alpha \cdot \nabla \partial_\alpha u_\alpha - F'(u_\alpha) \phi_\alpha \cdot \partial_\alpha u_\alpha \\
&= \alpha \int_{\partial(\Omega_{1,R} \cup \Omega_{2,R})} \phi_\alpha \partial_n(\partial_\alpha u_\alpha)\, dS.
\end{aligned}$$

Observe first that

$$\begin{aligned}
\alpha^3 \mu_\alpha \int_{\Omega_{1,R} \cup \Omega_{2,R}} p(\alpha x') \phi_\alpha \cdot \partial_\alpha u_\alpha \, dx'\, dx_3 &= 2\alpha^2 \mu_\alpha \int_{|\alpha x'| \leqslant R,\ |t| \leqslant \frac{1}{2}(\mathrm{f}_{2\alpha} - \mathrm{f}_{1\alpha})} p(\alpha x') \bar{z}_0(\alpha x')^2 w'(t)^2 \, dx'\, dt + \mathcal{O}(\alpha^\tau) \\
&= \mu_\alpha \|w'\|_{L^2}^2 \int_{|x'| \leqslant R} p(x') \bar{z}_0^2 \, dx' + \mathcal{O}(\alpha^\tau) = c_0 \mu_{2,\alpha} + \mathcal{O}(\alpha^\tau), \quad c_0 > 0.
\end{aligned}$$

On the other hand,

$$\alpha \int_{\partial(\Omega_{1,R} \cup \Omega_{2,R})} \phi_{2,\alpha} \partial_n(\partial_\alpha u_\alpha)\, dS = \underbrace{\int_{|\alpha x'| = R, |t| \leqslant \frac{1}{2}(\mathrm{f}_{2\alpha} - \mathrm{f}_{1\alpha})}}_{I} + \underbrace{\int_{|\alpha x'| \leqslant R, |t| = \frac{1}{2}(\mathrm{f}_{2\alpha} - \mathrm{f}_{1\alpha})} \phi_{2,\alpha} \partial_n(\partial_\alpha u_\alpha)}_{II}.$$

Clearly, the largest contribution in this integral comes from the first term, which from the asymptotic formula (8.60), yields that

$$\begin{aligned}
I &= 2\pi \alpha^{-1} R \|w'\|_{L^2}^2 \bar{z}_0(R) [\alpha \partial_{r,\alpha} q_\alpha + \alpha \partial_{r,\alpha} \mathrm{v}_\alpha]_{|x'| = \alpha^{-1} R} + \mathcal{O}(\alpha^{1+\tau}) \\
&= \tilde{\beta}_0 \alpha \log\left(\frac{1}{\alpha}\right) + \mathcal{O}(\alpha)
\end{aligned}$$

with $\tilde{\beta}_0 > 0$. Hence, taking $R \to \infty$, we find that $\mu_\alpha \sim \alpha \log(\alpha) \hat{\beta}$ for some $\hat{\beta} > 0$ and this completes the proof of the lemma. □

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9.4. Proof of inequality $m(u_\alpha) \leqslant 1$ for solutions in Theorem 1

To sketch the proof of inequality $m(u_\alpha) \leqslant 1$ we proceed as in the proof of Lemma 9.2. From the characterization of $m(u_\alpha)$ in (9.7), we can take an eigenfunctions $\phi_{\alpha,R}$, associated to strictly negative eigenvalue $\lambda_{\alpha,R} < 0$, which from the variational characterization of the eigenvalues can be chosen to be decreasing in R. We also may assume that

$$\|\phi_{\alpha,R}\|_\infty = 1, \qquad \int_{\mathbb{R}^3} p(x')\phi_{\alpha,R}\bar{\phi}_{\alpha,R}\,dx'\,dz = 0 \tag{9.27}$$

for $\bar{\phi}_{\alpha,R}$ an eigenfunction to problem (9.6) associated to a different eigenvalue. From inequality (9.16) we can write

$$\lambda_{\alpha,R} = \alpha^2\mu_{\alpha,R}, \qquad \mu_{\alpha,R} \to \mu_\alpha < 0, \quad \text{as } R \to \infty.$$

Proceeding as above, we find the asymptotics for $\phi_{\alpha,R}$

$$\phi_{\alpha,R}(x',x_3) = z_1(\alpha x')w'(x_3 - \mathrm{f}_{1\alpha}) + z_2(\alpha x')w'(x_3 - \mathrm{f}_{2\alpha}) + \mathcal{O}\left(\alpha\sum_{j=1}^{2} e^{-\varrho|x_3 - \mathrm{f}_{j\alpha}|}\right)$$

with

$$\Delta_{\mathbb{R}^2} z_1 + \sqrt{2}e^{-\sqrt{2}(q_2-q_1)}(z_2+z_1) + \mu p(x')z_1 = 0,$$
$$\Delta_{\mathbb{R}^2} z_2 + \sqrt{2}e^{-\sqrt{2}(q_2-q_1)}(z_2+z_1) + \mu p(x')z_2 = 0,$$

where $\mu \leqslant 0$.

The case $\mu = 0$ is discarded with the help of Lemma 9.2, which states that there are not strictly negatives eigenvalues close to zero. Hence, $\mu < 0$ and we observe that the equation for the difference $\hat{z} = z_1 - z_2$, reads as

$$\Delta_{\mathbb{R}^2}\hat{z} + \mu p(x')\hat{z} = 0, \quad \|\hat{z}\|_\infty < \infty. \tag{9.28}$$

Since the eigenspace associated to the eigenvalue in (9.28) μ is spanned by exactly one simple and positive eigenfunction and using as in the proof of Lemma 9.2 the orthogonality condition against $\partial_{x_3}u_\alpha$, we find that

$$\int_{\mathbb{R}^3} p(x')\hat{z}\,dx' = 0$$

which implies that $\hat{z} = 0$. So we have the asymptotic expansion

$$\phi_{\alpha,R}(x',x_3) = z(\alpha x')\left[w'(x_3 - \mathrm{f}_{1\alpha}) + w'(x_3 - \mathrm{f}_{2\alpha})\right] + \alpha^2\mathcal{O}\left(\sum_{j=1}^{2} e^{-\varrho|x_3 - \mathrm{f}_{j\alpha}|}\right),$$

where

$$\Delta_{\mathbb{R}^2} z + 2\sqrt{2}e^{-\sqrt{2}(q_2-q_1)}z + \mu p(x')z = 0, \quad \|z\|_{L^\infty(\mathbb{R}^3)} < \infty.$$

From condition (9.27) for eigenfunctions associated to the same eigenvalue and since there is exactly one negative eigenvalue for problem (9.6), we conclude that this eigenvalue must be simple so that $m(u_\alpha) \leqslant 1$ and this concludes the proof of Theorem 1.

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