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Multiple boundary blow-up solutions for nonlinear elliptic equations

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We consider the problem $\Delta u = \lambda f(u)$ in Ω, $u(x)$ tends to $+\infty$ as x approaches $\partial\Omega$. Here, Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 1$ and λ is a positive parameter. In this paper, we are interested in analysing the role of the *sign changes* of the function f in the number of solutions of this problem. As a consequence of our main result, we find that if Ω is star-shaped and f behaves like $f(u) = u(u-a)(u-1)$ with $\frac{1}{2} < a < 1$, then there is a solution bigger than 1 for all λ and there exists $\lambda_0 > 0$ such that, for $\lambda < \lambda_0$, there is no positive solution that crosses 1 and, for $\lambda > \lambda_0$, at least two solutions that cross 1. The proof is based on *a priori* estimates, the construction of barriers and topological-degree arguments.

1. Introduction and main results

Let Ω be a bounded regular domain in $\mathbb{R}^N$, $N \geqslant 1$. This paper deals with the study of multiple solutions for the boundary-value problem

$$\left.\begin{aligned} \Delta u &= \lambda f(u) \quad \text{in } \Omega, \\ u(x) &\to +\infty \quad \text{as } d(x) \to 0, \end{aligned}\right\} \tag{P_λ}$$

where $d(x)$ denotes the distance of x to $\partial\Omega$ and λ is a positive parameter. The study of this type of problem dates back to 1916, with the work of Bieberbach [7], who established existence and uniqueness for $f(u) = e^u$ and $N = 2$. This result was later improved to $N = 3$ by Rademacher [20].

Different aspects have then been addressed by various authors: the existence of solutions, the uniqueness (or multiplicity) and the asymptotic behaviour close to the boundary.

The question of existence of positive solutions of (P_λ) was first studied by Keller [13] and Osserman [18]. They gave a sufficient condition on f for the existence of

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positive solutions,

$$f \text{ is locally Lipschitz continuous and non-decreasing on } [0,\infty),$$
$$f(0) \geqslant 0 \text{ and } \int^{\infty} F^{-1/2} < \infty, \text{ where } F(u) = \int_0^u f(t)\,\mathrm{d}t.$$

This includes the case where $f(u) = \mathrm{e}^u$, which corresponds to results in electro hydrodynamics.

The question of uniqueness has been studied related to that of asymptotic behaviour near the boundary. It was analysed by Loewner and Nirenberg [16] for the special case $f(u) = u^{(N+2)/(N-2)}$ and $N > 2$, which appears in geometrical problems. Bandle and Marcus [3, 4] and Lazer and McKenna [15] extended the results of [16] to a much larger class of nonlinearities, including $f(u) = u^p$, $p > 1$, and convex nonlinearities. For other works concerning uniqueness, see [10, 14].

In this paper, we deal with the question of multiplicity of positive solutions for a nonlinearity $f(u)$, which is not monotone. A study of multiple solutions was made in [1] for functions f behaving like $|u|^p$ when u tends to $-\infty$: the existence of a positive and a sign-changing solution using a topological-degree argument was proved when $f(0) = 0$. Nevertheless, if f has no zero, then there are two solutions for small λ and no solution for large λ. When f has several zeros and $N = 1$, the existence of multiple solutions was investigated in [2] using ordinary differential equations techniques, and bifurcation curves were drawn. When the boundary condition is the Dirichlet condition $u = 0$ and λ is large, the study of the number of solutions depending on the zeros of f was made by several authors (see [8,9,11,12,17]). Their techniques do not apply directly to our problem, because of the infinite boundary condition. Here, we are interested in the case of multiple solutions according to the behaviour of f and the value of λ. The proof is based on *a priori* bounds, sub and supersolutions and topological-degree arguments.

To illustrate our main result, let us consider the model cubic nonlinearity $f(u) = u(u-a)(u-1)$, with $\frac{1}{2} < a < 1$, so that the area of the positive bump is bigger than that of the negative bump. By the result of Keller and Osserman, there is a solution $u > 1$ for all λ, and the question is whether other solutions exist. The following result holds: problem (P_λ) has no solution with minimum value less than 1 for small λ, while it has at least two such solutions for all sufficiently large λ. It turns out that when Ω is star-shaped, the sets of number λ where one solution and at least three solutions exist are connected.

More precisely, we make the following hypotheses.

(H1) $f : [0,\infty) \to \mathbb{R}$ is locally Lipschitz continuous.

(H2) f is positive, non-decreasing on $(1,+\infty)$, $f(1) = 0$ and $\int^{\infty} F^{-1/2} < \infty$, where $F(u) = \int_0^u f(t)\,\mathrm{d}t$.

(H3) There exists $0 < a < 1$ such that $f(0) = f(a) = 0$, $f > 0$ in $(0,a)$, $f < 0$ in $(a,1)$ and $F(1) > 0$.

Assumptions (H1) and (H2) ensure, after the works of Keller and Osserman, the existence of at least one solution, whose minimum value is larger than 1. The hypothesis $F(1) > 0$ is needed to ensure the existence of solutions that cross 1. It already appeared in the Dirichlet problem [9, 11, 12].

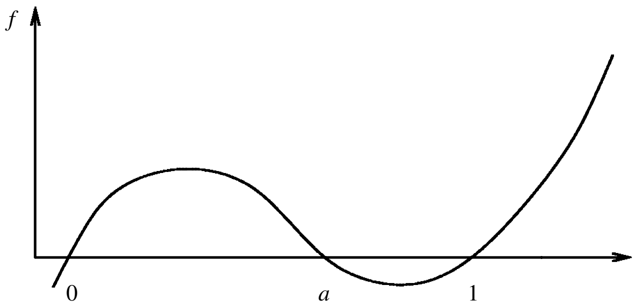

Figure 1. The function $f(s)$.

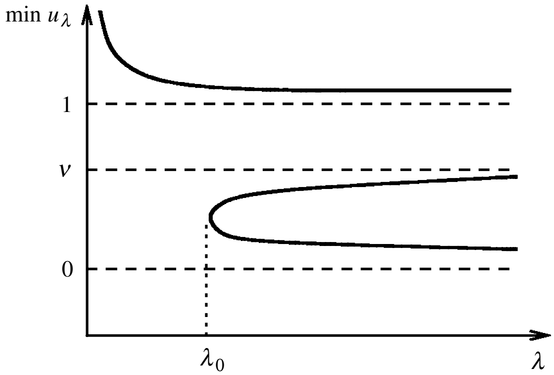

Figure 2. The bifurcation curve.

A picture of the behaviour of f is illustrated in figure 1. We want to justify the corresponding bifurcation diagram for solutions (figure 2). Our main result is the following.

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary of class C^2. Assume that f satisfies (H1)–(H3). Then, for any $\lambda > 0$, there exists a solution $\bar{u}_\lambda$, with*

$$\min_{x \in \Omega} \bar{u}_\lambda(x) > 1,$$

which satisfies

$$\lim_{\lambda \to +\infty} \bar{u}_\lambda(x) = 1 \quad \text{and} \quad \lim_{\lambda \to 0} \bar{u}_\lambda(x) = +\infty$$

uniformly on compact subsets of Ω.

Moreover, there exist numbers $0 < \lambda_0 \leqslant \lambda_1$ such that the following hold.

(i) *For $0 < \lambda < \lambda_0$, only solutions with minimum value greater than 1 exist.*

(ii) *For $\lambda > \lambda_1$, there are at least two positive solutions, $\underline{u}_\lambda$ and u_λ, whose minimum value is less than 1. $\underline{u}_\lambda$ is the minimal positive solution of (P_λ) and it satisfies*

$$\lim_{\lambda \to +\infty} \underline{u}_\lambda(x) = 0$$

uniformly on compact subsets of Ω. The minimum value of u_λ stays uniformly away from 0 when $\lambda \to +\infty$.

(iii) *When Ω is star-shaped, we have $\lambda_0 = \lambda_1$. Moreover, for this value of λ, at least one positive solution with minimum value less than 1 exists.*

A generalization of theorem 1.1 to a nonlinearity f having several zeros holds. Let us consider the alternative assumption.

(H3′) There exist numbers (b_i, a_i), $1 \leqslant i \leqslant n$, with $b_i < a_i < b_{i-1}$, $b_0 = 1$, such that $f(b_i) = f(a_i) = 0$ and $f > 0$ in (b_i, a_i), $f < 0$ in (a_i, b_{i-1}). Moreover, $\int_{b_i}^{b_{i-1}} f(s)\,\mathrm{d}s > 0$.

For simplicity, we only state the result in the star-shaped case.

THEOREM 1.2. *Let f satisfy (H1), (H2) and (H3′) and additionally that Ω is star-shaped. Then there exists λ_i such that, for $\lambda < \lambda_i$, there is no solution that crosses b_i. For $\lambda = \lambda_i$, there is a solution that crosses b_i and stays above b_{i+1}. For $\lambda > \lambda_i$, there are at least two solutions that cross b_i and stay above b_{i+1}. The lower solution $\underline{u}_{\lambda,i}$ is the minimal solution bigger than b_{i+1} and is decreasing with λ. When λ tends to ∞, $\underline{u}_{\lambda,i}$ converges to b_{i+1} uniformly on every compact subset of Ω.*

In order to complete the bifurcation diagram, one has to take into account the behaviour of f at $-\infty$. If f is negative for $s < b_n$, then there is no other solution than the ones mentioned above. If we assume that

(H4) there exists p in $(1, N^*)$ such that $0 < \lim_{s\to-\infty}(f(s)/|s|^p) < \infty$,

then it follows from [1] that, for all λ, there is a solution that crosses b_n and $\min u_\lambda$ tends to $-\infty$ as λ tends to 0. For other behaviours of f, the problem is open.

It is worthwhile mentioning that, in the case $f(u) = u(u-a)(u-1)$, $\frac{1}{2} < a < 1$ as $\lambda \to +\infty$, but, for finite boundary data, a solution exhibiting a single spike and tending to zero elsewhere, except near the boundary, can be constructed via the mountain-pass lemma. It was found in [9, 17] that the concentration point is at a maximal distance from the boundary. It seems an interesting question to see whether such a peak solution can be found in our setting. In the language of theorem 1.1, such a behaviour would correspond to the solution u_λ approaching asymptotically 1 almost everywhere, but with a spike shape at the minimum value. The proofs in the above-mentioned work do not seem to extend to our setting, since one would need to find estimates independent of the boundary value.

The rest of this paper is devoted to the proof of the main results. Section 2 is devoted to the study of the solution bigger than 1. Then we prove in § 3 the non-existence of solutions that cross 1, when λ is small. In § 4, we study the minimal solution for λ large. Finally, a topological-degree argument allows us, in § 5, to get existence of two solutions for λ large.

2. Preliminaries and the solution $\bar{u}_\lambda$

In this section, we assume that f satisfies (H1), (H2). We will construct a solution bigger than 1 for all $\lambda > 0$ and obtain some of its properties. We define the approximate problems $(P_{\lambda,c})$, where the infinite boundary condition in (P_λ) is replaced by the Dirichlet boundary condition $u = c$.

LEMMA 2.1. *Let $c > 1$. There is a unique solution $\bar{u}_{\lambda,c}$ of $(P_{\lambda,c})$, which is bigger than 1, and is increasing with c.*

Proof. If u is a solution of $(P_{\lambda,c})$, then, by the maximum principle, $u \leqslant c$ in Ω. We have $u \equiv 1$ as a subsolution and $u \equiv c$ as a supersolution, so $(P_{\lambda,c})$ has a maximal solution bigger than 1. Moreover, for $c_1 < c_2$, $\bar{u}_{\lambda,c_1}$ is a subsolution of (P_{λ,c_2}) and c_2 is a supersolution. Thus the maximal solution $\bar{u}_{\lambda,c_2}$ is bigger than $\bar{u}_{\lambda,c_1}$.

Since f is increasing, the maximum principle holds for the equation for the difference of two solutions. Hence it implies uniqueness. □

LEMMA 2.2. *Let ϕ_α be the solution of*

$$\left.\begin{aligned} \phi_\alpha'' &= \frac{\lambda}{N} f(\phi_\alpha), \\ \phi_\alpha(0) = \alpha, &\qquad \phi_\alpha'(0) = 0, \end{aligned}\right\} \tag{2.1}$$

where $\alpha \geqslant 1$. Then there is a maximal interval $(0, R_\alpha)$, where ϕ_α exists,

$$\lim_{x \to R_\alpha} \phi_\alpha(x) = \infty,$$

with

$$R_\alpha = \int_\alpha^\infty \frac{\mathrm{d}s}{\sqrt{(2\lambda/N)(F(s) - F(\alpha))}}.$$

R_α decreases with α,

$$\lim_{\alpha \to 1} R_\alpha = \infty \quad \textit{and} \quad \lim_{\alpha \to \infty} R_\alpha = 0.$$

Moreover, ϕ_α is increasing and

$$\Delta \phi_\alpha \leqslant \lambda f(\phi_\alpha) \quad \textit{in } B_{R_\alpha}. \tag{2.2}$$

Proof. There is a maximal interval $(0, R_\alpha)$ in which the solution ϕ_α exists, it is increasing in this interval, and ϕ_α' and ϕ_α'' are also increasing, since $f(s)$ is increasing for $s > 1$. This yields

$$\Delta \phi_\alpha = \phi_\alpha'' + \frac{N-1}{r} \phi_\alpha' \leqslant N \phi_\alpha''.$$

In particular, equation (2.2) is satisfied.

Multiplying the equation by ϕ_α' and integrating gives

$$\phi_\alpha'^2 = \frac{2\lambda}{N}(F(\phi_\alpha) - F(\alpha)),$$

where F is such that $F' = f$. Hence we get the expression of R_α. The properties of R_α are proved in [1] (see also [2]). □

LEMMA 2.3. *Given $\bar{\lambda} > 0$, $\delta > 0$ and any $0 < \lambda < \bar{\lambda}$, there exists a constant $M = M(\delta, \bar{\lambda})$ such that, for any $c > 1$ and any solution u of $(P_{\lambda,c})$ or of (P_λ), we have*

$$\sup_{d(x) > \delta} u(x) \leqslant M.$$

Proof. This result is essentially due to Keller [13]. Let u be a solution of $(P_{\lambda,c})$. Let $x_0 \in \Omega$ be such that $d(x_0) > \delta$ and let $\rho < \delta$. Thus $\overline{B_\rho(x_0)} \subset \Omega$. Let us choose $\alpha > 1$ so large that $R_\alpha \leqslant \rho$. This is possible, since $R_\alpha \to 0$ as $\alpha \to +\infty$. Since $f(s)$ is non-decreasing for $s > \alpha$, we can apply the maximum principle in the set $\{x_0 \in B_{R_\alpha}(x_0)/u(x) > \alpha\}$ using the previous lemma and derive $u(x) \leqslant \phi_\alpha(|x-x_0|)$ in $B_{R_\alpha}(x_0)$. A compactness argument then yields the desired result. □

As it may be expected, the previous result implies that, as $c \to +\infty$, a sequence of solutions of $(P_{\lambda,c})$ converges up to extraction of a subsequence, to a solution of (P_λ).

Proposition 2.4. *Let λ be fixed and $u_{\lambda,c}$ be a sequence of positive solutions of $(P_{\lambda,c})$. Then, as c tends to infinity, $u_{\lambda,c}$ converges up to the extraction of a subsequence to a solution u of (P_λ).*

Proof. We recall the proof in [1]. Let $\bar{u}_{\lambda,c}$ be the solution of $(P_{\lambda,c})$ constructed above. Let $u_{\lambda,c}$ be any other positive solution. We are going to prove that

$$u_{\lambda,c} \geqslant \bar{u}_{\lambda,c} - A, \tag{2.3}$$

where A is some constant and c is large enough. Since f is increasing for $s > 1$, there exists $A > 1$ such that $f(s) \leqslant f(A)$ for $s \in (0, A)$, so that $f(s) \leqslant f(s + A)$ $\forall s \geqslant 0$. We let $v_c = u_{\lambda,c} + A$. Then we have $\Delta v_c \leqslant \lambda f(v_c)$ and v_c is in the range where f is increasing. The maximum principle then yields $\bar{u}_{\lambda,c} \leqslant v_c$, which is precisely (2.3).

Now, by lemma 2.3, $\bar{u}_{\lambda,c}$ and $u_{\lambda,c}$ are uniformly bounded above on compact subsets of Ω, hence they are bounded in $C^{1,\alpha}$ and converge locally uniformly to $\bar{u}_\lambda$ and u_λ, which are solutions of the equation. The blow-up boundary condition is satisfied for $\bar{u}_\lambda$ because $\bar{u}_{\lambda,c}$ is increasing with c and for $u_{\lambda,c}$ because of relation (2.3). □

By virtue of the previous proposition, we may define a solution $\bar{u}_\lambda$ of (P_λ) as

$$\bar{u}_\lambda(x) \equiv \lim_{\lambda \to +\infty} \bar{u}_{\lambda,c}(x),$$

where $\bar{u}_{\lambda,c}$ is the unique solution of $(P_{\lambda,c})$, $c > 1$, whose minimum value is greater than 1. We observe that $\bar{u}_\lambda$ is then the minimal solution of (P_λ) with the property $\min \bar{u} > 1$. Let us establish limiting properties of this solution.

Proposition 2.5. *Let $\bar{u}_\lambda$ be the minimal solution of (P_λ) with minimum value greater than 1. Then*

$$\lim_{\lambda \to 0} \min_{\Omega} \bar{u}_\lambda = \infty, \qquad \lim_{\lambda \to \infty} \max_{K} \bar{u}_\lambda = 1$$

for any compact subset K of Ω.

Proof. Limit when $\lambda \to \infty$. Assume that the ball B is contained in Ω. Let $\phi_{\alpha(\lambda)}(s)$ be the solution of (2.1) that blows up on ∂B. In B, we have $\Delta \phi_{\alpha(\lambda)} \leqslant \lambda f(\phi_{\alpha(\lambda)})$, so that elliptic comparisons imply that $\bar{u}_\lambda \leqslant \phi_{\alpha(\lambda)}$ in B.

When λ tends to infinity, $\alpha(\lambda) = \phi_{\alpha(\lambda)}(0)$ tends to 1. As a consequence, $\min \bar{u}_\lambda$ tends to 1 as well.

Let K be a compact subset and B a ball of radius less than $\operatorname{dist}(K, \partial\Omega)$. Thus K can be covered with such balls B and the previous argument yields $\bar{u}_\lambda \leqslant \phi_{\alpha(\lambda)}$ in B. In particular, for all x in K, x is the centre of a ball B, so, by moving around the ball, we get $\bar{u}_\lambda(x) \leqslant \min \phi_{\alpha(\lambda)}$, so that $\bar{u}_\lambda$ tends to 1 uniformly in K.

Limit when $\lambda \to 0$. Let B be a ball containing Ω. Let $w_{\lambda,c}$ be the maximal solution of $(P_{\lambda,c})$ in B. Then $w_{\lambda,c}$ is a subsolution of $(P_{\lambda,c})$ in Ω. Hence $w_{\lambda,c} \leqslant \bar{u}_{\lambda,c}$ and

$$w_\lambda = \lim_{c\to\infty} w_{\lambda,c} \leqslant \bar{u}_\lambda \quad \text{in } \Omega_\lambda.$$

Assume by contradiction that when λ tends to 0, $\min w_\lambda$ is bounded. Let ϕ_α be the solution of (2.1) with $\alpha = w_\lambda(0)$. It follows from the Sturm comparison principle that ϕ_α blows up before w_λ. If $w_\lambda(0)$ is bounded when λ tends to 0, it implies that α is bounded, and hence R_α tends to ∞. But R_α is less than the radius of B, which provides a contradiction. □

3. Non-existence of solutions with least value less than 1 for λ small

In this section we prove the following proposition, which contains part of the statement of theorem 1.1.

PROPOSITION 3.1. *There is a number* $\lambda_0 > 0$ *such that, for any* $0 < \lambda < \lambda_0$, *there is no solution of* (P_λ) *with minimum value less than* 1.

Proof. Assume that there is a solution u to (P_λ) such that $\min_\Omega u < 1$ and consider the set $\omega = \{x \mid u(x) < 1\}$. Then u satisfies

$$\left.\begin{aligned} \Delta u &= \lambda f(u) && \text{in } \omega, \\ u &= 1 && \text{on } \partial\omega. \end{aligned}\right\} \tag{3.1}$$

Let $h = 1 - u \geqslant 0$. Then h is a solution of $\Delta h + \lambda c(x) h = 0$ with 0 boundary value, where

$$c(x) = -\frac{f(u) - f(1)}{u - 1}.$$

Let φ be a positive first eigenfunction of the Laplacian in ω. Then

$$\int_\omega (\lambda_1(\omega) - \lambda c(x)) h\varphi = 0.$$

Let $\lambda_0 = \lambda_1(\Omega)/|c|_\infty$. Since $h, \phi > 0$, if $\lambda < \lambda_0$, this provides a contradiction, since $\lambda_1(\Omega) < \lambda_1(\omega)$. □

4. The minimal solution $\underline{u}_\lambda$

PROPOSITION 4.1. *Let f satisfy assumptions (H1)–(H3). Then there exists a number* $\bar{\lambda} > 0$ *such that, for all* $\lambda \geqslant \bar{\lambda}$, *there is a minimal positive solution* $\underline{u}_\lambda$ *of* (P_λ) *whose minimum value is less than* 1. *Moreover,*

$$\lim_{\lambda\to+\infty} \max_K \underline{u}_\lambda(x) = 0 \tag{4.1}$$

for any compact subset K of Ω. If Ω is star-shaped, then the set of values $\lambda > 0$ such that there exists a positive solution with minimum value less than 1 *exists is a closed unbounded interval* $[\lambda_0, +\infty)$, *with* λ_0 *given by proposition 3.1.*

Proof. We will construct a minimal solution of the problem that has minimum value less than 1, provided that λ is sufficiently large.

We begin by considering the problem in a ball $B = B_\delta$, $\delta > 0$. Let us extend f by setting $f(s) = 0$ for $s \leqslant 0$. Let us fix a number $c_0 > 1$ and consider the problem of minimizing the functional

$$J(u) = \int_B |\nabla u|^2 + \lambda F(u)$$

over all functions in $H^1(B)$ with boundary value c_0. Here, we have denoted $F(u) = \int_0^u f(s)\,\mathrm{d}s$. Note that, since (H3) holds, we have $J(u) \geqslant 0$. Moreover, J is coercive and weakly lower semicontinuous, and therefore has a minimizer in the corresponding class. Let u_* be one minimizer of $J(u)$ with boundary value c_0, which we also choose to be radially symmetric. We are going to build a test function in order to prove that $\min u_*$ tends to 0 as $\lambda \to \infty$. We choose a test function that is equal to 0 in $B_{\delta-1/\lambda^{1/2}}$ and extended linearly to reach the value c on the boundary. This yields

$$0 \leqslant J(u_*) < D\lambda^{1/2},$$

where D depends on c_0 and δ. It follows that

$$0 \leqslant \int_B F(u_*) \leqslant D\lambda^{-1/2} \to 0$$

as $\lambda \to \infty$, and hence $u_* \to 0$ almost everywhere in B. In particular, the minimum of u_*, which is at the centre, goes to 0. From now on, we assume that λ is sufficiently large so that $\min u_* < 1$.

Let us take an arbitrary $c > c_0$. In $B_{2\delta}$, we are going to build a supersolution of $(P_{\lambda,c})$. We let $w(x) = u_*$ in B_δ and $w(x) = \psi(|x|)$ in $B_{2\delta} \setminus B_\delta$, where ψ is a one-dimensional solution of $\psi'' = \lambda/N f(\psi)$ with $\psi(\delta) = c_0$ and $\psi'(\delta) = 0$. We know that ψ blows up at some number R_λ. When λ is large enough, R_λ tends to 0 and hence it is less than 2δ. Hence $\min(w, c)$ is a supersolution of $(P_{\lambda,c})$ for all $\lambda > \bar{\lambda}$, where $\bar{\lambda}$ does not depend on $c > c_0$. Since 0 is a subsolution of the problem, we obtain the existence of a minimal positive solution $\underline{u}_{\lambda,c}$ of $(P_{\lambda,c})$ that is less than u_* in B_δ, and hence has a minimum value less than 1.

We deduce that $\underline{u}_{\lambda,c} \leqslant u_*$ in B_δ. Let us consider now a compact set $K \subset \Omega$ whose distance to $\partial\Omega$ is larger than δ. Then moving the centre of the ball B_δ around K implies that

$$\sup_K \underline{u}_{\lambda,c} \leqslant \inf_{B_\delta} u_*.$$

In particular, $\sup_K \underline{u}_{\lambda,c}$ converges to 0 uniformly in K and in $c \geqslant c_0$.

Given δ and $\bar{\lambda}$, let us consider any number $\lambda > \bar{\lambda}$. Proposition 2.4 allows us to pass to the limit in c,

$$\underline{u}_\lambda(x) = \lim_{c\to+\infty} \underline{u}_{\lambda,c}(x), \tag{4.2}$$

is the minimal solution of problem (P_λ), since any solution of (P_λ) is a supersolution of $(P_{\lambda,c})$ and, by definition, must lie above $\underline{u}_{\lambda,c}$. Moreover, equation (4.2) implies (4.1).

Now we want to establish the following fact. If Ω is star-shaped, then the set of values $\lambda > 0$ such that a positive solution with minimum value less than 1 exists is a closed unbounded interval. We let $\Omega_\lambda = \sqrt{\lambda}\Omega$ and $v_\lambda(x) = u(x/\sqrt{\lambda})$. The function v_λ solves

$$\left.\begin{aligned} \Delta v_\lambda &= f(v_\lambda) && \text{in } \Omega_\lambda, \\ v_\lambda(x) &\to \infty && \text{as } x \to \partial\Omega_\lambda. \end{aligned}\right\} \tag{Q_λ}$$

We observe that if Ω is star-shaped around the origin, then $\Omega_{\lambda_1} \subset\subset \Omega_{\lambda_2}$ for $\lambda_1 < \lambda_2$.

Let μ be such that there is a positive solution of (P_μ) with minimum value less than 1, which we call $\underline{u}_\mu$. We are going to prove that the property holds for any $\lambda > \mu$. Let $v_\mu(x) = \underline{u}_\mu(\mu^{-1/2}x)$. It is a solution of (Q_μ). It then follows that, for any $c > 1$, the function $\min\{v_\mu, c\}$, extended by c on $\Omega_\lambda \setminus \Omega_\mu$, is a supersolution of $(Q_{\lambda,c})$ for any $\lambda > \mu$. Since 0 is a subsolution, there exists of a minimal solution whose minimum value is less than that of v_μ, and is thus less than 1. By rescaling to Ω, this provides a solution $\underline{u}_{\lambda,c}$ of $(P_{\lambda,c})$ that crosses 1 and, passing to the limit in c, a solution $\underline{u}_\lambda$ of (P_λ) that crosses ‘. In fact, one can show that the minimum value of $\underline{u}_\lambda$ is a decreasing function of λ. Let λ_0 be the infimum of the λ such that there is a positive solution with minimum value less than 1. Then $\lambda_0 > 0$ by proposition 3.1. We claim that, for $\lambda = \lambda_0$, there is a solution of (P_{λ_0}). Let λ_n be a decreasing sequence tending to λ_0. Then there is a solution $v_{n,c}$ of $(Q_{\lambda_n,c})$. As λ_n decreases to λ_0, $v_{n,c}$ converges locally uniformly to $v_{0,c}$ on every compact subset of Ω_{λ_0} by proposition 2.3. Moreover, since $\Omega_\lambda = \sqrt{\lambda}\Omega$, on $\partial\Omega_{\lambda_0}$, $v_{n,c}$ converges to c. Hence $v_{0,c}$ is a solution of $(Q_{\lambda_0,c})$. It follows that $\min v_{\lambda_0,c} \leqslant 1$. Passing to the limit as in proposition 2.4 yields the result. □

5. A second solution with minimum value less than 1

The remaining part of theorem 1.1 is given by the following result.

PROPOSITION 5.1. *There exists a number $\lambda_1 > 0$ such that, for all $\lambda > \lambda_1$, there exist at least two positive solutions of (P_λ) with minimum value less than 1. If Ω is star-shaped, we have $\lambda_1 = \lambda_0$, where λ_0 is the number found in proposition 3.1.*

Proof. We will use a topological-degree argument that is similar to that used in [1] (see also [5, 6, 19]). Let $c > 1$ be fixed and let us choose a number $\mu_0 > 0$ such that $(P_{\mu_0,c})$ has no solution with minimum value less than 1.

We introduce the operators F_t, $0 \leqslant t \leqslant 1$, as follows. For

$$v \in C^c(\bar{\Omega}) = \{v \in C(\bar{\Omega}),\ v = c \text{ on } \partial\Omega\},$$

we define $w = F_t v$ to be the unique solution of the problem

$$\left.\begin{aligned} \Delta w - \mu w &= (t\mu_0 + (1-t)\lambda)f(v)) - \mu v && \text{in } \Omega, \\ w &= c && \text{on } \partial\Omega, \end{aligned}\right\} \tag{5.1}$$

where $\mu = \sup_{[0,c]} |f'|$. Then F_t defines a compact operator from $(t,v) \in [0,1] \times C^c(\bar{\Omega})$ into $C^c(\bar{\Omega})$. Since the function $s \mapsto t\mu_0 + (1-t)\lambda f(s) - \mu s$ is non-decreasing in $[0,c]$, F_t is order preserving on functions v whose values lie in $[0,c]$.

Let us consider the following sets,

$$\mathcal{B} = \Big\{ v \in C^c(\bar{\Omega}) / v < c \text{ in } \Omega,\ \min_{\Omega} v \in (0,1) \Big\},$$
$$\mathcal{O} = \{ v \in \mathcal{B},\ v < \psi_0 \text{ in } \Omega \},$$

where ψ_0 is a supersolution of $(P_{\lambda,c})$, defined as follows. Fix a number $\bar{\lambda}$ (the existence follows from proposition 4.1) such that there exists $\underline{u}_{\bar{\lambda}}$, a minimal solution of $(P_{\bar{\lambda}})$. Assuming that $0 \in \Omega$, then we choose λ_1 sufficiently large so that, for $\lambda > \lambda_1$, $(\lambda^{-1}\bar{\lambda})^{1/2}\Omega \subset \Omega$. We observe that if Ω is star-shaped around zero, then we may choose $\bar{\lambda} = \lambda_1 = \lambda_0$, where λ_0 is the number in proposition 3.1. Then the function $u_0(x) = \underline{u}_{\bar{\lambda}}(\bar{\lambda}^{-1}\lambda)^{1/2}x)$,

$$\Delta u_0 = \lambda f(u_0) \quad \text{in } (\lambda^{-1}\bar{\lambda})^{1/2}\Omega \subset \Omega.$$

Then we set $\psi_0 = \min\{u_0, c\}$, extended by c to all of Ω. Thus ψ_0 is a supersolution of $(P_{\lambda,c})$.

We are going to prove that $d(I - F_0, \mathcal{B}, 0) = 0$ and $d(I - F_0, \mathcal{O}, 0) = +1$, where d denotes the Leray–Schauder degree. This will imply

$$d(I - F_0, \mathcal{B} \setminus \bar{\mathcal{O}}, 0) = d(I - F_0, \mathcal{B}, 0) - d(I - F_0, \mathcal{O}, 0) = -1, \tag{5.2}$$

and hence there is a solution with minimum value less than 1, which is somewhere larger than the supersolution ψ_0, so that it is different from the minimal solution $\underline{u}_{\lambda,c}$. The result of the proposition then follows by taking the limit (up to subsequences) $c \to +\infty$. The limiting functions $\underline{u}_\lambda$ and u_λ are different since $\underline{u}_\lambda \leqslant u_0$ and u_λ crosses u_0.

To prove the above, let us observe that $v = F_t(v)$ means that v solves

$$\left.\begin{aligned} \Delta v &= (t\mu_0 + (1-t)\lambda) f(v) && \text{in } \Omega, \\ v &= c && \text{on } \partial\Omega. \end{aligned}\right\} \tag{5.3}$$

Then, since $c > 1$, the strong maximum principle prevents v belonging to $\partial\mathcal{B}$. Hence $d(I - F_t, \mathcal{B}, 0)$ is well defined for all $t \in [0,1]$ and hence it is constant in t. But since, by the choice of μ_0, F_1 cannot have any fixed point, we get

$$d(I - F_0, \mathcal{B}, 0) = d(I - F_1, \mathcal{B}, 0) = 0.$$

Let us now show that $d(I - F_0, \mathcal{O}, 0) = +1$.

We have already proved the existence of a solution $\bar{v}_{\lambda,c}$ in $\mathcal{O}$. Let us define

$$H_t = tF_0 + (1-t)\bar{v}_{\lambda,c}, \quad t \in [0,1].$$

Let us suppose that v in $\bar{\mathcal{O}}$ is such that $v = H_t v$. Then $\|v\|_{C^1(\bar{\Omega})} < M$. Recall that F_0 is order preserving in $\mathcal{O}$ by the definition of μ. By the strong maximum principle, we have, in fact, that $v \in \mathcal{O}$. Thus $d(I - H_t, \mathcal{O}, 0)$ is well defined and hence constant in t. As $\bar{v}_{\lambda,c}$ is in $\mathcal{O}$, we get

$$d(I - F_0, \mathcal{O}, 0) = d(I - \bar{v}_{\lambda,c}, \mathcal{O}, 0) = +1.$$

This yields (5.2) and the result of the proposition follows. □

Proof of theorems 1.1 and 1.2. The result of theorem 1.1 is contained in those in propositions 2.4–5.1. For the proof of theorem 1.2, one just has to replace the interval $(0, 1)$ by (b_{i+1}, b_i). □

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