

ON THE SHORT-TIME BEHAVIOR OF THE FREE BOUNDARY OF A POROUS MEDIUM EQUATION

CARMEN CORTÁZAR, MANUEL DEL PINO,
AND MANUEL ELGUETA

1. Introduction and preliminaries. Let Ω be a domain in \mathbf{R}^N , $N \geq 2$, with a bounded, smooth boundary and $m > 1$. In this paper, we study the following initial-boundary value problem for the porous medium equation,

$$\begin{aligned} u_t &= \Delta u^m \quad \text{in } \Omega \times (0, +\infty) \\ u(x, t) &= 1 \quad \text{in } \partial\Omega \times (0, +\infty), \\ u(x, 0) &= 0 \quad \text{in } \Omega. \end{aligned} \tag{1.1}$$

Problem (1.1) models the flow of a gas into a porous container shaped as Ω . Initially, the container is empty, and then the density of the gas, represented by u , is kept constant and equal to one at the boundary.

An important feature of the solution to (1.1), whose existence and uniqueness is guaranteed by the standard theory for the porous medium equation, is that it propagates with finite speed. By this we mean that if x_0 is an interior point of Ω , then there exists a positive time $T(x_0)$ such that $u(x_0, t) = 0$ for $0 < t < T(x_0)$ and $u(x_0, t) > 0$ for $T(x_0) < t$. See [A].

A natural question is that of estimating the value of $T(x_0)$. Of course, this quantity may depend strongly on the geometry of the domain, and it would be hard to provide a general precise estimate for it. This question was considered in [CE], where Neumann rather than Dirichlet boundary conditions were imposed, and a general upper estimate for $T(x_0)$ was derived in the case when Ω is bounded and convex. The estimate in [CE], involving certain integral quantities depending globally on the domain and the boundary condition, is however not sharp.

Our purpose in this paper is to find a precise estimate for $T(x_0)$ when the point x_0 lies sufficiently close to the boundary of Ω . When this is the case, it is natural to expect an answer that depends only on the local geometry of Ω near x_0 .

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It is reasonable to think that the curvature of $\partial\Omega$ near x_0 should play a role in the estimate. Indeed, one would expect $T(x_0)$ to be larger in the case when the domain is “negatively curved” near x_0 than when this curvature is positive, since in the former case “the gas has more space to escape.” This heuristic remark takes precise form in Theorem 1.1 below, our main result in this paper. To state it, we consider a point $\bar{x} \in \partial\Omega$ and denote by $n_{\bar{x}}$ the unit inner normal at \bar{x} , so that for all $\varepsilon > 0$ sufficiently small, the point

$$x_0 = \bar{x} + \varepsilon n_{\bar{x}} \quad (1.2)$$

lies in Ω .

THEOREM 1.1. *If x_0 is given by (1.2), then*

$$T(x_0) = \varepsilon^2 \{T_0 - H(\bar{x})T_1\varepsilon + o(\varepsilon)\} \quad (1.3)$$

where T_0 and T_1 are positive constants depending only on m and $H(\bar{x})$ denotes $(N-1)$ times the mean curvature of the boundary at \bar{x} .

The constants T_0 and T_1 are characterized precisely in terms of the solutions of certain ordinary differential equations, as we will see in the course of this paper.

Let us observe an interesting intuitive implication of the above result. In the two-dimensional case, the boundary of the support of $u(\cdot, t)$, sometimes called the free boundary, tends to “convexify” for small times since it advances faster the bigger the curvature is. In particular, this seems to imply that if Ω is convex, then the free boundary is a convex curve for all small times. It would be interesting to study whether this property indeed holds for small and larger times.

In related issues, we should mention that Angenent and Aronson have studied in [AA1] and [AA2] the way the support “closes” at the origin in the radially symmetric case, the so-called *focusing problem*.

Regularity and behavior of the support when the domain is the whole of \mathbf{R}^N and the initial data is compactly supported have been studied in [CF] and [CVW].

The proof of Theorem 1.1 is based on the careful construction of a super- and a subsolution for the problem, for which the asymptotic estimate (1.3) holds. This construction is based on finding a suitable formal first approximation to the solution after conveniently rescaling the parameter ε in it. It should be mentioned that in these arguments, the fact that the boundary condition is constant is not crucial, and we just assume this for simplicity. In fact, a similar formula can be derived in case the boundary condition is a smooth positive function whose value at \bar{x} equals one. A positive Neumann boundary condition can also be treated with entirely similar methods. Concerning the regularity of the boundary, it will be apparent from the arguments in the rest of the paper that C^4 is what we will actually be using, even though this may not be crucial.

In most of what follows, it will be more convenient to express (1.1) in terms of the *normalized pressure*, which is the quantity v defined as

$$v(x, t) = \frac{m}{2} u^{m-1}(x, t).$$

It is easily checked that u satisfies (1.1) if and only if v satisfies

$$\begin{aligned} v_t &= \Delta v^2 - \lambda |\nabla v|^2 \quad \text{in } \Omega \times (0, +\infty) \\ v(x, t) &= C \quad \text{in } \partial\Omega \times (0, +\infty), \\ v(x, 0) &= 0 \quad \text{in } \Omega, \end{aligned} \tag{1.4}$$

where $\lambda = 2(m-2)/(m-1)$ and $C = m/2 > 0$.

Since we are interested in the behavior of v at the point given by (1.2), and in light of the natural scaling of space-time for this equation, it seems natural to consider the function

$$v_\varepsilon(\bar{x}, s, t) = v(\bar{x}, s, t, \varepsilon) \equiv v(\bar{x} + \varepsilon s n_{\bar{x}}, \varepsilon^2 t),$$

where $\bar{x} \in \partial\Omega$ and $s > 0$. Then

$$T(\bar{x} + \varepsilon n_{\bar{x}}) = \varepsilon^2 T_\varepsilon(\bar{x}),$$

where $T_\varepsilon(\bar{x}) = \inf\{t | v_\varepsilon(\bar{x}, 1, t) > 0\}$. Thus establishing Theorem 1.1 amounts to showing the validity of the expansion

$$T_\varepsilon(\bar{x}) = T_0 - H(\bar{x}) T_1 \varepsilon + o(\varepsilon) \tag{1.5}$$

for appropriate constants T_0 and T_1 depending only on m . In the rest of this section, we will identify these constants at the formal level and define the elements necessary to build a first-order approximation in ε for v_ε .

To begin with, we observe that v_ε satisfies an equation of the form

$$\frac{\partial v_\varepsilon}{\partial t} = L_\varepsilon(v_\varepsilon) \equiv A_\varepsilon(v_\varepsilon) + \varepsilon^2 R_\varepsilon(v_\varepsilon) \tag{1.6}$$

in a region of the form $D = \{(\bar{x}, s, t) | \bar{x} \in \partial\Omega, s \in (0, 2), \text{ and } t \in (0, \bar{T})\}$, where \bar{T} is independent of ε for ε small.

We proceed now to show how to compute $A_\varepsilon(v_\varepsilon)$ and $\varepsilon^2 R_\varepsilon(v_\varepsilon)$ in local coordinates. Let us fix a point $\bar{x} \in \partial\Omega$. After a rotation and a translation, we may assume that \bar{x} is the origin, and that in a neighborhood of \bar{x} , Ω is described as the set of points for which $x_N > \psi(x')$, $x' = (x_1, \dots, x_{N-1})$, where ψ is a smooth function such that $\psi(0) = 0$, $\nabla\psi(0) = 0$, and $\Delta\psi(0) = H(\bar{x})$. Then, using the

change of variable $(y', \varepsilon y_n) = (x', x_n - \psi(x'))$ and the fact that the coordinate y_n coincides with s when $y' = 0$, we get that, at a point of the form $(0', s)$, with $s \in (0, 2)$, one has

$$A_\varepsilon(v_\varepsilon)(0', s) = \frac{\partial^2 v_\varepsilon^2}{\partial s^2}(0', s) - \varepsilon H(\bar{x}) \frac{\partial v_\varepsilon^2}{\partial s}(0', s) - \lambda \left| \frac{\partial v_\varepsilon}{\partial s} \right|^2(0', s) \quad (1.7)$$

and

$$R_\varepsilon(v_\varepsilon)(0', s) = \sum_{i=1}^{N-1} \left(\frac{\partial^2 \tilde{v}_\varepsilon^2}{\partial x_i^2} - \lambda \left| \frac{\partial \tilde{v}_\varepsilon}{\partial x_i} \right|^2 \right) \Big|_{(y', y_n) = (0', s)}, \quad (1.8)$$

where

$$\tilde{v}_\varepsilon(y', y_n, t) = v_\varepsilon(\bar{x}(y', y_n), s(y', y_n), t).$$

Here $\bar{x}(y', y_n), s(y', y_n)$ are defined implicitly by the relation

$$(y', \varepsilon y_n) = \bar{x} + \varepsilon s n_{\bar{x}}.$$

Now, letting formally $\varepsilon \rightarrow 0$ in equation (1.6) and recalling the initial and boundary conditions, we are left with a problem independent of \bar{x} of the form

$$\begin{aligned} (v_0)_t &= (v_0^2)_{ss} - \lambda((v_0)_s)^2 \\ v_0(s, 0) &\equiv 0 \\ v_0(0, t) &\equiv C, \end{aligned} \quad (1.9)$$

that is, with a one-dimensional porous-medium equation, which is well known to possess a unique solution $v_0(s, t)$. Thus, assuming $v_\varepsilon(\bar{x}, s, t) \rightarrow v_0(s, t)$ in some appropriate sense, we should also have that $T_\varepsilon(\bar{x}) \rightarrow T_0$, where T_0 is the time it takes the support of v_0 to reach the point $s = 1$, a number dependent on m only.

Next we find the second term in the ε -expansion of $T_\varepsilon(\bar{x})$. To do this, we differentiate implicitly the relation

$$v_\varepsilon(\bar{x}, 1, T_\varepsilon(\bar{x})) = 0$$

with respect to ε to get

$$\frac{\partial T_\varepsilon(\bar{x})}{\partial \varepsilon} \Big|_{\varepsilon=0} = - \frac{(\partial v_\varepsilon / \partial \varepsilon)(\bar{x}, 1, T_0)|_{\varepsilon=0}}{(v_0)_t(1, T_0)}. \quad (1.10)$$

Differentiating (1.6) with respect to ε and setting $\varepsilon = 0$, we obtain that

$$\frac{\partial v_\varepsilon}{\partial \varepsilon}(\bar{x}, s, t)|_{\varepsilon=0} = H(\bar{x})z(s, t),$$

where z satisfies

$$\begin{aligned} z_t &= 2(v_0 z)_{ss} - 2\lambda(v_0)_s z_s - (v_0^2)_s \\ z(s, 0) &= 0 \\ z(0, t) &= 0. \end{aligned} \tag{1.11}$$

We observe that the equation in (1.11) is meaningful only where v_0 does not vanish. In fact, we will see in §3 that this equation possesses a unique solution which behaves reasonably near the boundary of the support of v_0 , namely, that its extension by zero away from the support of v_0 (the only reasonable extension, since v_ε should vanish identically there for small ε) becomes a global distributional solution. In fact, we shall prove the desirable fact that $z(1, T_0) > 0$, so that the term T_1 , dependent only on m in the expansion (1.3) for $T_\varepsilon(\bar{x})$, becomes

$$T_1 = \frac{z(1, T_0)}{(v_0)_t(1, T_0)}.$$

Note then that the ε -derivative of v_ε at $\varepsilon = 0$ becomes a discontinuous function. The above formal computations are based, of course, on the possibility of taking the function $v_0(s, t) + \varepsilon H(\bar{x})z(s, t)$ as a “good” approximation to v_ε near the point $(\bar{x}, 1, T_0)$, a rather delicate issue because of the discontinuity of z .

The rest of this paper will be devoted to giving an actual proof of formula (1.3).

In §2 we study some properties of the solution v_0 of (1.9), which is well known to be self-similar, obtained from a corresponding ordinary differential equation.

We devote §3 to the study of equation (1.11), which inherits the self-similarity of v_0 ; this eventually leads to our desired unique solution z in a certain class.

In §4, we construct a super- and a subsolution of (1.6), enclosing the actual solution. This construction is based on the formal first approximation of v_ε and the precise properties of v_0 and z collected in the previous sections.

2. The first-order term. It is known that the solution v_0 of problem (1.9) is of the form

$$v_0(s, t) = \begin{cases} f\left(\frac{s}{\sqrt{t}}\right) & \text{if } 0 \leq s \leq \xi_0 \sqrt{t} \\ 0 & \text{if } \xi_0 \sqrt{t} \leq s, \end{cases}$$

where f satisfies

$$(f^2(\xi))'' - 2\frac{m-2}{m-1}(f'(\xi))^2 + \frac{\xi}{2}f'(\xi) = 0 \text{ for } \xi \in (0, \xi_0) \quad (2.1)$$

$$f(\xi_0) = 0, (f^{m/(m-1)})'(\xi_0) = 0 \text{ and } f(0) = C.$$

Moreover, $f(\xi) > 0$ for $\xi \in [0, \xi_0]$.

We note that, since $f(\xi) > 0$, (2.1) is equivalent to

$$(f^{m/(m-1)}(\xi))'' + m\frac{\xi}{4}(f^{1/(m-1)})'(\xi) = 0 \text{ for } \xi \in (0, \xi_0) \quad (2.2)$$

$$f(\xi_0) = 0, (f^{m/(m-1)})'(\xi_0) = 0 \text{ and } f(0) = C.$$

We review now some properties of the function f that will be used later. Integrating the equation in (2.2) from ξ to ξ_0 , we get

$$f'(\xi) = -\frac{m-1}{4} \left[\xi + \frac{1}{f^{1/(m-1)}(\xi)} \int_{\xi}^{\xi_0} f^{1/(m-1)}(s) ds \right]. \quad (2.3)$$

From (2.3) we get at once the following lemma.

LEMMA 2.1. *We have*

- (a) $-\frac{(m-1)\xi_0}{4} \leq f'(\xi) < -\frac{m-1}{4C^{1/(m-1)}} \int_0^{\xi_0} f^{1/(m-1)}(s) ds$ for all $\xi \in [0, \xi_0]$;
- (b) $f''(\xi)$ is bounded and negative for all $\xi \in [0, \xi_0]$;
- (c) $f'(\xi_0) = -\frac{(m-1)}{4}\xi_0$ and $f''(\xi_0) = -\frac{(m-1)}{4m}$.

We will also need the following.

LEMMA 2.2. *Let $h(\xi) = (f'(\xi) + ((m-1)/4)\xi)/f(\xi)$. Then*

- (a) $h \leq 0$ in $(0, \xi_0)$;
- (b) the function h is bounded in $(0, \xi_0)$;
- (c) the function h' is bounded in $(0, \xi_0)$.

Proof. From (2.3) we get

$$h(\xi) = -\frac{(m-1)}{4f^{m/(m-1)}(\xi)} \int_{\xi}^{\xi_0} f^{1/(m-1)}(s) ds \quad (2.4)$$

and hence (a). Part (b) follows by an application on the second mean value theorem.

As for part (c), we differentiate (2.4) to get

$$h'(\xi) = \frac{(m-1)}{4} \frac{f^{m/(m-1)}(\xi) + \frac{m}{m-1} f'(\xi) \int_{\xi}^{\xi_0} f^{1/(m-1)}(s) ds}{f^{(2m-1)/(m-1)}(\xi)},$$

and we apply again the second mean value theorem.

3. The second-order term. We study now problem (1.11). We seek a similar solution, so we set

$$z(s, t) = \sqrt{t} g\left(\frac{s}{\sqrt{t}}\right)$$

and observe that z satisfies (1.11) if g satisfies

$$2(f(\xi)g(\xi))'' - 4\frac{m-2}{m-1}f'(\xi)g'(\xi) + \frac{1}{2}(\xi g(\xi))' - g(\xi) = (f^2)'(\xi) \text{ for } \xi \in (0, \xi_0)$$

$$g(0) = 0. \quad (3.1)$$

Moreover, we impose the additional boundary condition

$$\lim_{\xi \rightarrow \xi_0} \left[2\left(f^{1/(m-1)}(\xi)g(\xi)\right)' + \frac{1}{2}\left(\xi \frac{g(\xi)}{f^{(m-2)/(m-1)}(\xi)}\right) \right] = 0. \quad (3.2)$$

Since $f(\xi) > 0$ in $(0, \xi_0)$, using (2.2) we have that (3.1), (3.2) is equivalent to

$$2(f^{1/(m-1)}(\xi)g(\xi))'' + \frac{1}{2}\left(\xi \frac{g(\xi)}{f^{(m-2)/(m-1)}(\xi)}\right)' - \frac{g(\xi)}{f^{(m-2)/(m-1)}(\xi)}$$

$$= 2\frac{(m-1)}{m}(f^{m/(m-1)})'(\xi) \text{ for } \xi \in (0, \xi_0) \quad (3.3)$$

$$g(0) = 0, \quad \lim_{\xi \rightarrow \xi_0} \left[2\left(f^{1/(m-1)}(\xi)g(\xi)\right)' + \frac{1}{2}\left(\xi \frac{g(\xi)}{f^{(m-2)/(m-1)}(\xi)}\right) \right] = 0.$$

It is a straightforward calculation to check that g is a C^2 solution of (3.3) if and only if g satisfies

$$g(\xi) + \frac{1}{2}e^{-H(\xi)} \int_0^{\xi} \frac{e^{H(r)}}{f^{1/(m-1)}(r)} \int_r^{\xi_0} \frac{g(s)}{f^{(m-2)/(m-1)}(s)} ds dr$$

$$= \frac{(m-1)}{m} e^{-H(\xi)} \int_0^{\xi} e^{H(s)} f(s) ds \quad (3.4)$$

where

$$H(\xi) = \frac{1}{m-1} \int_0^\xi h(s) ds,$$

with h as in Lemma 2.2.

We now define the operator $T: C([0, \xi_0]) \rightarrow C([0, \xi_0])$ by

$$(Tg)(\xi) = \frac{1}{2} e^{-H(\xi)} \int_0^\xi \frac{e^{H(r)}}{f^{1/(m-1)}(r)} \int_r^{\xi_0} \frac{g(s)}{f^{(m-2)/(m-1)}(s)} ds dr,$$

and we set

$$F(\xi) = \frac{(m-1)}{m} e^{-H(\xi)} \int_0^\xi e^{H(s)} f(s) ds.$$

With this notation, (3.4) reads

$$g(\xi) + (Tg)(\xi) = F(\xi). \quad (3.5)$$

As we have seen in §2, the function h is bounded. Therefore the operator $T: C([0, \xi_0]) \rightarrow C([0, \xi_0])$ is continuous and compact when $C([0, \xi_0])$ is endowed with the supremum norm.

We need now the following lemma.

LEMMA 3.1. *The operator $\text{Id} + T: C([0, \xi_0]) \rightarrow C([0, \xi_0])$ is injective.*

Proof. Let $g_1, g_2 \in C([0, \xi_0])$ be such that $(\text{Id} + T)(g_1) = (\text{Id} + T)(g_2)$. Then the function $p = g_1 - g_2$ belongs to C^2 and satisfies

$$2(f(\xi)p(\xi))'' - 4\frac{m-2}{m-1}f'(\xi)p'(\xi) + \frac{1}{2}(\xi p(\xi))' - p(\xi) = 0 \quad \text{for } \xi \in (0, \xi_0). \quad (3.6)$$

We claim that p cannot attain a positive maximum at a point $\xi_1 \in (0, \xi_0)$. Indeed, if so, at this point, one has $p(\xi_1) > 0$, $p'(\xi_1) = 0$, and $p''(\xi_1) \leq 0$. Evaluating (3.6) in ξ_1 , one gets a contradiction, and the claim is proved.

In a similar fashion, one proves that p cannot attain a negative minimum in $(0, \xi_0)$. Since $p(0) = 0$, it follows that p does not have a zero in $(0, \xi_0)$ unless $p \equiv 0$.

We observe that p also satisfies

$$2(f^{1/(m-1)}(\xi)p(\xi))'' + \frac{1}{2} \left(\xi \frac{p(\xi)}{f^{(m-2)/(m-1)}(\xi)} \right)' - \frac{p(\xi)}{f^{(m-2)/(m-1)}(\xi)} = 0 \quad \text{for } \xi \in (0, \xi_0)$$

$$\lim_{\xi \rightarrow \xi_0} \left[2(f^{1/(m-1)}(\xi)p(\xi))' + \frac{1}{2} \left(\xi \frac{p(\xi)}{f^{(m-2)/(m-1)}(\xi)} \right) \right] = 0 \quad (3.7)$$

$$p(0) = 0.$$

Now, integrating the equation in (3.7) from ξ to ξ_0 , using the boundary conditions, and multiplying the resulting equation by $f^{(m-2)/(m-1)}$, one gets

$$2f'(\xi)p(\xi) + \frac{2}{m-1}f(\xi)p'(\xi) + \frac{1}{2}\xi p(\xi) + f^{(m-2)/(m-1)}(\xi) \int_{\xi}^{\xi_0} \frac{p(s)}{f^{(m-2)/(m-1)}(s)} ds = 0. \quad (3.8)$$

If $p > 0$ in $(0, \xi_0)$, then, since f' is bounded and $f(0) = C > 0$, letting $\xi \rightarrow 0$ in (3.8), one obtains that there exists $\xi_2 \in (0, \xi_0)$ so that $p'(\xi_2) < 0$. This contradicts the fact that p cannot attain a positive maximum in $(0, \xi_0)$ since $p(0) = 0$. It is proved analogously that the assumption that $p < 0$ in $(0, \xi_0)$ leads to a contradiction. Consequently, $p \equiv 0$, and the lemma follows. ■

In view of the Fredholm's alternative and Lemma 3.1, we have the following theorem.

THEOREM 3.1. *Problem (3.1), (3.2) has a unique solution.*

In the remaining part of this section, we study some properties of the solution of problem (3.1), (3.2). So from now on we will denote by g the solution of this problem.

LEMMA 3.2. *We have $g \geq 0$ in $[0, \xi_0]$.*

Proof. We claim first that g cannot attain a negative relative minimum in $(0, \xi_0)$. Indeed, assume negative minimum is attained at $\xi_1 \in (0, \xi_0)$. Then $g'(\xi_1) = 0$, and from (3.1) one gets

$$2f''(\xi_1)g(\xi_1) + 2f(\xi_1)g''(\xi_1) + \frac{1}{2}(\xi g(\xi_1))' - g(\xi_1) = (f^2)'(\xi_1). \quad (3.9)$$

Since $g(\xi_1) < 0$ and $g''(\xi_1) \geq 0$, the left-hand side of (3.9) is positive. On the other hand, the right-hand side of (3.9) is negative by Lemma 2.1. This contradiction proves the claim.

Now if it is not true that $g \geq 0$ in $(0, \xi_0)$, then there exists $\xi_2 \in [0, \xi_0)$ so that $g(\xi_2) = 0$ and $g'(\xi) \leq 0$ for $\xi \in (\xi_2, \xi_0)$. Integrating (3.3) from ξ_2 to ξ_0 , using the boundary conditions, we get

$$0 \leq \int_{\xi_2}^{\xi_0} \frac{g(s)}{f^{(m-2)/(m-1)}(s)} ds - 2 \frac{(m-1)}{m} (f^{m/(m-1)})(\xi_2). \quad (3.10)$$

This is a contradiction, and the lemma is proved. ■

LEMMA 3.3. *We have $g(\xi_0) > 0$.*

Proof. By integration in (3.3), one gets

$$2g'(\xi) = 2 \frac{(m-1)}{m} f(\xi) - \frac{2}{m-1} h(\xi)g(\xi) - \frac{1}{f^{1/(m-1)}(\xi)} \int_{\xi}^{\xi_0} \frac{g(s)}{f^{(m-2)/(m-1)}(s)} ds. \quad (3.11)$$

To see that $g(\xi_0) > 0$, assume for contradiction that $g(\xi_0) = 0$. Since $h \leq 0$, from (3.11) we get

$$2g'(\xi) \geq 2 \frac{(m-1)}{m} f(\xi) - \frac{1}{f^{1/(m-1)}(\xi)} \int_{\xi}^{\xi_0} \frac{g(s)}{f^{(m-2)/(m-1)}(s)} ds.$$

Or, using Lemma 2.1(a), there exist positive constants a and A such that

$$g'(\xi) \geq A(\xi_0 - \xi) - a \frac{1}{(\xi_0 - \xi)^{1/(m-1)}} \int_{\xi}^{\xi_0} \frac{g(s)}{(\xi_0 - s)^{(m-2)/(m-1)}} ds,$$

which implies $g'(\xi) > 0$ for ξ near ξ_0 since $g(\xi_0) = 0$. This is a contradiction, and the lemma is proved. ■

LEMMA 3.4. *We have*

$$\lim_{\xi \rightarrow \xi_0} g'(\xi) \text{ exists and } \lim_{\xi \rightarrow \xi_0} g'(\xi) = -\frac{2m-1}{m\xi_0} g(\xi_0).$$

Proof. The proof consists of taking the limit as $\xi \rightarrow \xi_0$ in (3.11). ■

LEMMA 3.5. *We have $g''(\xi)$ is bounded in $(0, \xi_0)$.*

Proof. Differentiating (3.11) we get

$$\begin{aligned} 2g''(\xi) &= 2 \frac{m-1}{m} f'(\xi) - \frac{2}{m-1} [h(\xi)g'(\xi) + h'(\xi)g(\xi)] \\ &\quad + \frac{f^{1/(m-1)}(\xi)g(\xi) + \frac{f'(\xi)}{m-1} \int_{\xi}^{\xi_0} \frac{g(s)}{f^{(m-2)/(m-1)}(s)} ds}{f^{m/(m-1)}(\xi)}. \end{aligned}$$

Therefore it suffices to show that

$$\frac{f^{1/(m-1)}(\xi)g(\xi) + \frac{f'(\xi)}{m-1} \int_{\xi}^{\xi_0} \frac{g(s)}{f^{(m-2)/(m-1)}(s)} ds}{f^{m/(m-1)}(\xi)}$$

is bounded. This can be done by an application of the second mean value theorem, and the lemma follows. ■

4. The main result. We would like to start by giving the idea behind the construction of a supersolution of (1.6) for small values of ε . If we formally expand the function $v(\bar{x}, s, t, \varepsilon) \equiv v_\varepsilon(\bar{x}, s, t)$ in a Taylor series in ε at $\varepsilon = 0$, we

obtain

$$v(\bar{x}, s, t, \varepsilon) = v_0(s, t) + H(\bar{x})z(s, t)\varepsilon + O(1)\varepsilon^2.$$

Therefore the function $w(\bar{x}, s, t, \varepsilon) = v_0(s, t) + H(\bar{x})z(s, t)\varepsilon$ should satisfy

$$w_t - L_\varepsilon(w) = O(1)\varepsilon^2,$$

where L_ε is given by (1.6). Now, if it did happen that $w_t > c > 0$, then $\bar{w} = w(\bar{x}, s, t(1 + \varepsilon^{5/4}), \varepsilon)$ satisfies

$$\bar{w}_t - L_\varepsilon(\bar{w}) = c\varepsilon^{5/4} + O(1)\varepsilon^2,$$

and its positive part \bar{w}_+ should be a supersolution for ε small.

Unfortunately, it is only true that $w_t \geq 0$. Moreover, in the case that $H(\bar{x}) > 0$, an extra difficulty arises. That is, at such a point the function w , which is only defined for $0 \leq s \leq \xi_0\sqrt{t}$, does not vanish there. This last problem is overcome (for each fixed \bar{x} and t) by extending w in s beyond $\xi_0\sqrt{t}$ as a suitable parabola.

The rest of this section is devoted to the construction of the supersolution. The construction of the subsolution is the same with some obvious changes. Finally, we would like to warn the reader that the function w defined below is a slight modification of the one defined above. This seems necessary to overcome some technical difficulties.

We start with the observation that, by a comparison argument, all the times t involved in the arguments that follow are bounded above by a time \bar{T} independent of ε . Therefore, for the rest of this section, we will always have that $1/\sqrt{t} = O(1/t)$.

We redefine now the function w . For any $(\bar{x}, s, t) \in D$, let us set

$$w(\bar{x}, s, t, \varepsilon) = v_0(s, t) + \alpha z(s, t), \quad (4.1)$$

where

$$\alpha = \alpha(\bar{x}, \varepsilon) = \varepsilon H(\bar{x})(1 + \varepsilon^{1/2}H(\bar{x})).$$

Recalling that $v_0(s, t) = f(s/\sqrt{t})$ and $z(s, t) = \sqrt{t}g(s/\sqrt{t})$, w can be written as

$$w(\bar{x}, s, t, \varepsilon) = f\left(\frac{s}{\sqrt{t}}\right) + \alpha\sqrt{t}g\left(\frac{s}{\sqrt{t}}\right). \quad (4.2)$$

With the notation $\xi = s/\sqrt{t}$, one has

$$w_t = -\frac{1}{2t}\xi f'(\xi) + \frac{\alpha}{2\sqrt{t}}g(\xi) - \frac{\alpha}{2\sqrt{t}}\xi g'(\xi). \quad (4.3)$$

Using the equations and the properties of the functions f and g , a rather lengthy but straightforward calculation shows that, with our choice of α ,

$$w_t - A_\varepsilon(w) = -\varepsilon^{3/2} H^2(\bar{x}) (f^2)'(\xi) \frac{1}{\sqrt{t}} + \varepsilon^2 O(1). \quad (4.4)$$

Moreover, since $H \in C^2$, we have

$$R_\varepsilon(w) = \varepsilon^2 g(\xi) O(1). \quad (4.5)$$

We set $\bar{t} = t(1 + \varepsilon^{5/4})$, $\bar{\xi} = s/\sqrt{\bar{t}}$ and define

$$\bar{w}(\bar{x}, s, t, \varepsilon) = w(\bar{x}, s, \bar{t}, \varepsilon)$$

at points where $\bar{\xi} \leq \xi_0$. We can state now the following lemma.

LEMMA 4.1. *There exists ε_0 so that for any $0 < \varepsilon \leq \varepsilon_0$, the function \bar{w} satisfies*

$$\bar{w}_t - L_\varepsilon(\bar{w}) \geq 0,$$

at points where $\bar{\xi} \leq \xi_0$.

Proof. From (4.3), (4.4), and (4.5), we get

$$\begin{aligned} \bar{w}_t - L_\varepsilon(\bar{w}) &= \varepsilon^{5/4} \left(-\frac{1}{2\bar{t}} \bar{\xi} f'(\bar{\xi}) + \frac{\alpha}{2\sqrt{\bar{t}}} g(\bar{\xi}) - \frac{\alpha}{2\sqrt{\bar{t}}} \bar{\xi} g'(\bar{\xi}) \right) + \\ &\quad - \varepsilon^{3/2} H^2(\bar{x}) (f^2)'(\xi) \frac{1}{\sqrt{t}} + \varepsilon^2 O(1) + \varepsilon^2 g(\xi) O(1). \end{aligned} \quad (4.6)$$

We recall that $f' \leq -b < 0$ on $[0, \xi_0]$ for some $b > 0$. Pick a fixed ξ_1 so that $0 < \xi_1 < \xi_0$.

For $\bar{\xi} \in [\xi_1, \xi_0]$ we obtain from (4.6) that

$$\bar{w}_t - L_\varepsilon(\bar{w}) = -\varepsilon^{5/4} \frac{1}{2\bar{t}} \bar{\xi} f'(\bar{\xi}) + o(\varepsilon^{5/4}) O\left(\frac{1}{\bar{t}}\right); \quad (4.7)$$

therefore, $\bar{w}_t - L_\varepsilon(\bar{w}) \geq 0$ for ε small enough.

As for the range $\bar{\xi} \in (0, \xi_1]$, we observe that since $g(0) = 0$ and g' is bounded, there exists a constant K so that $|g(\bar{\xi})| \leq K\bar{\xi}$. Therefore,

$$-\frac{1}{2\bar{t}} \bar{\xi} f'(\bar{\xi}) + \frac{\alpha}{2\sqrt{\bar{t}}} g(\bar{\xi}) - \frac{\alpha}{2\sqrt{\bar{t}}} \bar{\xi} g'(\bar{\xi}) + \varepsilon^2 g(\xi) O(1) \geq 0$$

for small values of ε . Consequently,

$$\bar{w}_t - L_\varepsilon(\bar{w}) \geq -\varepsilon^{3/2} H^2(\bar{x})(f^2)'(\xi) \frac{1}{\sqrt{t}} + \varepsilon^2 O(1) \quad (4.8)$$

and, as $(f^2(\xi))' > c > 0$ for some $c > 0$ for $\xi \in [0, \xi_1]$, we have again $\bar{w}_t - L_\varepsilon(\bar{w}) \geq 0$ for ε small. The lemma is proved. ■

So far we have constructed a supersolution to our differential equation in the region $D_\varepsilon = \{(\bar{x}, s, t)/0 < s < \xi_0\sqrt{t}\}$. To have a supersolution for the boundary value problem, we need to extend this function beyond D_ε . This will be achieved, for any fixed \bar{x} and t , by fitting a suitable parabola in the direction of s .

To be more precise, let us define

$$\begin{aligned} p = p(\bar{x}, s, t, \varepsilon) = & -\frac{(m-1)}{8mt} (s - \xi_0\sqrt{t})^2 \\ & + \left(\frac{1}{\sqrt{t}} f'(\xi_0) + \alpha g'(\xi_0) \right) (s - \xi_0\sqrt{t}) + \alpha\sqrt{t}g(\xi_0). \end{aligned}$$

Recalling that $f'(\xi_0) = -((m-1)/4)\xi_0$ and $g'(\xi_0) = -((2m-1)/m)(g(\xi_0)/\xi_0)$, p can be written in the form

$$\begin{aligned} p(\bar{x}, s, t) = & -\frac{(m-1)}{8m} \frac{(s - \xi_0\sqrt{t})^2}{t} \\ & - \left(\frac{(m-1)}{4} \xi_0 \frac{1}{\sqrt{t}} + \alpha \frac{(2m-1)}{m} \frac{g(\xi_0)}{\xi_0} \right) (s - \xi_0\sqrt{t}) + \alpha\sqrt{t}g(\xi_0). \end{aligned} \quad (4.9)$$

By an elementary computation, we get

$$p_t = \frac{(m-1)}{8m} \frac{(\xi - \xi_0)^2}{t} + \frac{(m^2-1)}{8m} \xi_0 \frac{(\xi - \xi_0)}{t} + \frac{(3m-1)}{2m} \frac{\alpha}{\sqrt{t}} g(\xi_0) + \frac{(m-1)}{8} \frac{\xi_0^2}{t}, \quad (4.10)$$

and straightforward computation gives

$$\begin{aligned} p_t - \left[(p^2)_{ss} - 2 \frac{(m-2)}{(m-1)} (p_s)^2 \right] \\ = \frac{1}{16} \frac{(m-1)^2}{m^2} \frac{(\xi - \xi_0)^2}{t} - \frac{(m-1)(2m-1)}{2m^2} \frac{\alpha}{\sqrt{t}} \frac{g(\xi_0)}{\xi_0} (\xi - \xi_0) \\ - \frac{2(2m-1)^2}{(m-1)m^2} \frac{g^2(\xi_0)}{\xi_0^2} \frac{\alpha^2}{t}. \end{aligned} \quad (4.11)$$

We define

$$\bar{p} = \bar{p}(\bar{x}, s, t, \varepsilon) = p(\bar{x}, s, \bar{t}, \varepsilon).$$

LEMMA 4.2. *There exists ε_0 so that for any $0 < \varepsilon \leq \varepsilon_0$ the function \bar{p} satisfies*

$$\bar{p}_t - L_\varepsilon(\bar{p}) \geq 0$$

in the domain $\{(\bar{x}, s, t) | \xi \geq \xi_0 \text{ and } \bar{p}(\bar{x}, s, t, \varepsilon) > 0\}$.

Proof. We observe first that if $\xi \geq \xi_0$ and $\bar{p}(\bar{x}, s, t, \varepsilon) > 0$, then $|\bar{\xi} - \xi_0| < K\varepsilon$ for some K independent of ε . Therefore, using (4.10), (4.11), and the definition of L_ε , we get

$$\bar{p}_t - L_\varepsilon(\bar{p}) = \varepsilon^{5/3} \frac{\xi_0^2}{8t} + \frac{O(1)}{t} \varepsilon^2,$$

which proves the lemma. ■

We extend the function \bar{w} as follows:

$$\bar{w}(\bar{x}, s, t, \varepsilon) = \begin{cases} \bar{w}(\bar{x}, s, t, \varepsilon) & \text{if } \bar{\xi} \leq \xi_0 \\ \bar{p}(\bar{x}, s, t, \varepsilon) & \text{if } \xi_0 \leq \bar{\xi}. \end{cases}$$

We observe that the function \bar{w} , so extended, is continuous, and also the partial first derivatives $\partial \bar{w} / \partial s$ and $\partial \bar{w} / \partial x_j$ for $j = 1, \dots, N-1$ are continuous.

We study now the set where \bar{w} is positive. This is contained in the next lemma.

LEMMA 4.3. *There exists ε_0 so that for any $0 < \varepsilon \leq \varepsilon_0$, the function $s \rightarrow \bar{h}(s) = \bar{w}(\bar{x}, s, t, \varepsilon)$ vanishes exactly at one point $s = \bar{\eta}(\bar{x}, \varepsilon)\sqrt{\bar{t}}$.*

Moreover,

$$\bar{\eta}(\bar{x}, \varepsilon) = \xi_0 - \alpha\sqrt{\bar{t}} \frac{g(\xi_0)}{f'(\xi_0)} + o(\varepsilon).$$

Proof. We have $\bar{h}(0) > 0$ and $\bar{h}(s) < 0$ for s large enough. Also for ε small, the function \bar{h} is strictly decreasing. Hence, for ε small, \bar{h} vanishes exactly once.

Let $\bar{\eta}\sqrt{\bar{t}} = \eta(\bar{x}, \varepsilon)\sqrt{\bar{t}}$ be the zero of \bar{h} . We distinguish three cases. First assume $H(\bar{x}) < 0$. In this case $\bar{\eta} < \xi_0$. We have

$$\alpha\sqrt{\bar{t}}g(\xi_0) = \bar{h}(\xi_0\sqrt{\bar{t}}) - \bar{h}(\bar{\eta}\sqrt{\bar{t}}) = (f'(\xi_2) + \alpha\sqrt{\bar{t}}g'(\xi_2))(\xi_0 - \bar{\eta})$$

for some $\xi_2 \in (\bar{\eta}, \xi_0)$. As there exists $b > 0$ so that $|f'| \geq b$, we get, after recalling the definition of α , that $|\xi_0 - \bar{\eta}| = O(\varepsilon)$. Taking one more term in the Taylor

expansion, we get

$$\alpha\sqrt{\bar{t}}g(\xi_0) = (f'(\xi_0) + \alpha\sqrt{\bar{t}}g'(\xi_0))(\xi_0 - \bar{\eta}) + (f''(\xi_3) + \alpha\sqrt{\bar{t}}g''(\xi_3))(\xi_0 - \bar{\eta})^2$$

for some $\xi_3 \in (\bar{\eta}, \xi_0)$, and consequently,

$$\bar{\eta} = \xi_0 - \alpha\sqrt{\bar{t}} \frac{g(\xi_0)}{f'(\xi_0)} + o(\varepsilon).$$

The case when $H(\bar{x}) > 0$, and hence $\bar{\eta} > \xi_0$, is dealt with in a similar way. Actually, in this case, $\bar{\eta}$ can be explicitly computed from the formula of the parabola. In the case when $H(\bar{x}) = 0$, we get $\bar{\eta} = \xi_0$. ■

We summarize the previous results of this section in the following proposition.

PROPOSITION 4.1. *There exists a function $\bar{w}(\bar{x}, s, t, \varepsilon)$ and $\varepsilon_0 > 0$ so that for any $0 < \varepsilon \leq \varepsilon_0$, its positive part $\bar{w}_+(\bar{x}, s, t, \varepsilon)$ is a weak supersolution of $w_t = L_\varepsilon(w)$ in Ω . Moreover, $\bar{w}(\bar{x}, 0, t, \varepsilon) = C$, and for its positivity set we have*

$$\{(\bar{x}, s, t)/\bar{w}(\bar{x}, s, t, \varepsilon) > 0\} = \{(\bar{x}, s, t)/0 \leq s < \bar{\eta}(\bar{x}, \varepsilon)\sqrt{\bar{t}}\},$$

where

$$\bar{\eta}(\bar{x}, \varepsilon) = \xi_0 - \varepsilon H(\bar{x})\sqrt{\bar{t}} \frac{g(\xi_0)}{f'(\xi_0)} + o(\varepsilon).$$

Proof. We have already observed that at points where \bar{w} is positive and $\bar{\xi} \neq \xi_0$, one has $\bar{w}_t - L_\varepsilon(\bar{w}^2) \geq 0$. On the other hand, at points where \bar{w} is positive and $\bar{\xi} = \xi_0$, the function and the partial derivatives $\partial w/\partial s$ and $\partial w/\partial x_j$ for $j = 1, \dots, N-1$ are continuous. Moreover, at the free boundary, $s = \bar{\eta}(\bar{x}, \varepsilon)\sqrt{\bar{t}}$, $\bar{w}_+(\bar{x}, s, t, \varepsilon)$ is continuous, and the partial derivatives $\partial w^2/\partial s$ and $\partial w^2/\partial x_j$ for $j = 1, \dots, N-1$ are also continuous and equal to zero, due to the fact that $\partial w/\partial s$ and $\partial w/\partial x_j$ exist and are bounded at such a point. This is all that is needed to check that \bar{w}_+ is a weak supersolution.

The statement about the positivity set is just a rephrasing of Lemma 4.3. ■

If in all the previous arguments, we use for α the value $\alpha = \varepsilon H(\bar{x}) \cdot (1 - \varepsilon^{5/4} H(\bar{x}))$ (notice the change of sign) and $\underline{t} = t(1 - \varepsilon^{5/4})$ instead of \bar{t} , we can construct a function \underline{w} that provides us with a subsolution for small values of ε . We state this in the next proposition, whose proof is omitted.

PROPOSITION 4.2. *There exists a function $\underline{w}(\bar{x}, s, t, \varepsilon)$ and $\varepsilon_0 > 0$ so that for any $0 < \varepsilon \leq \varepsilon_0$, its positive part $\underline{w}_+(\bar{x}, s, t, \varepsilon)$ is a weak subsolution of $w_t = L_\varepsilon(w)$ in Ω . Moreover, $\underline{w}(\bar{x}, 0, t, \varepsilon) = C$, and for its positivity set we have*

$$\{(\bar{x}, s, t)/\underline{w}(\bar{x}, s, t, \varepsilon) > 0\} = \{(\bar{x}, s, t)/0 \leq s < \underline{\eta}(\bar{x}, \varepsilon)\sqrt{\underline{t}}\}$$

where

$$\underline{\eta}(\bar{x}, \varepsilon) = \xi_0 - \varepsilon H(\bar{x}) \sqrt{\frac{g(\xi_0)}{f'(\xi_0)}} + o(\varepsilon).$$

Since we can now compute the constants T_0 and T_1 in (1.3), we rephrase and prove now our main result, Theorem 1.1.

THEOREM 4.1. *If x_0 is given by (1.2), then*

$$T(x_0) = \varepsilon^2 \left\{ \frac{1}{\xi_0^2} - H(\bar{x}) \frac{8g(\xi_0)}{(m-1)\xi_0^5} \varepsilon + o(\varepsilon) \right\}.$$

Proof. It follows, by a comparison argument, from Propositions 4.1 and 4.2 that

$$\underline{w}_+(\bar{x}, s, t, \varepsilon) \leq v(\bar{x}, s, t, \varepsilon) \leq \bar{w}_+(\bar{x}, s, t, \varepsilon).$$

Now the time $T_{\bar{w}_+}(\bar{x}, \varepsilon)$ that it takes the support of \bar{w}_+ to reach the point $(\bar{x}, 1)$ can be directly computed to get

$$T_{\bar{w}_+}(\bar{x}, \varepsilon) = \frac{1}{\xi_0^2} + H(\bar{x}) \frac{2g(\xi_0)}{\xi_0^4 f'(\xi_0)} \varepsilon + o(\varepsilon).$$

Analogously,

$$T_{\{w_+\}}(\bar{x}, \varepsilon) = \frac{1}{\xi_0^2} + H(\bar{x}) \frac{2g(\xi_0)}{\xi_0^4 f'(\xi_0)} \varepsilon + o(\varepsilon).$$

Since $T_{\bar{w}_+}(\bar{x}, \varepsilon) \leq T_\varepsilon(\bar{x}) \leq T_{w_+}(\bar{x}, \varepsilon)$, $f'(\xi_0) = -((m-1)/4)\xi_0$, (1.5) follows with $T_0 = 1/\xi_0^2$ and $T_1 = 8g(\xi_0)/(m-1)\xi_0^5$. This proves the theorem. ■

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CORTÁZAR AND ELGUETA: FACULTAD DE MATEMÁTICAS, UNIVERSIDAD CATÓLICA, CASILLA 306, CORREO 22, SANTIAGO, CHILE

DEL PINO: DEPARTAMENTO DE MATEMÁTICAS, FACULTAD DE CIENCIAS, UNIVERSIDAD DE CHILE, CASILLA 653, SANTIAGO, CHILE